Homework 8, Honors Calculus II

11/1/2018

This home work continues the discussion of integration of rational functions (see HW 7). After long division and decomposition of the denominator polynomial into linear an quadratic factors (which in most cases one cannot to explicitly), we may assume that our function has the form

$$f(x) = \frac{P(x)}{(x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_r)^{n_r}(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_lx+c_l)^{m_l}}$$

where $P(x)$ is a polynomial whose degree is strictly less than the degree of the denominator polynomial, which we read off to be $n = n_1+\cdots+n_r+2m_1+\cdots+2m_l$. We already dealt with the case when there are none of the quadratic terms.

The next problems guide you to deal with the quadratic terms in the denominator. The general case is then just a combination of the techniques from HW 7 and what follows. The partial fraction decomposition for the quadratic denominator case works similar—with a slight variation in the numerator terms—as follows:

$$\frac{P(x)}{(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_lx+c_l)^{m_l}} =$$

$$= \frac{A_1x+B_1}{x^2+b_1x+c_1} + \frac{A_2x+B_2}{(x^2+b_1x+c_1)^2} + \cdots + \frac{A_{n_1}x+B_{n_1}}{(x^2+b_1x+c_1)^{n_1}} + \text{same for the other quadratic factors}$$

Notice the linear terms in the numerator when compared to the decomposition given in HW 7.

Problem 1. Find the partial fraction decomposition of $\frac{1}{x^4+1}$. Hint: $x^4+1 = (x^2+ax+1)(x^2+bx+1)$ for some (which?) $a, b \in \mathbb{R}$.

Using the partial fraction decomposition, we only need to calculate integrals of the form

$$\int \frac{Ax+B}{(x^2+ax+b)^k}\,dx$$

Problem 2. Show that by completing the square followed by a substitution the integral

$$\int \frac{Ax+B}{(x^2+ax+b)^k}\,dx$$

can be written as

$$\int \frac{Au+B}{(u^2+a^2)^k}\,du$$

where A, B, a stand for new constants to be computed form the old A, B, a, b. Notice the term $a^2 > 0$, which is positive (!), a fact we will use later.

If you have difficulty doing this in the abstract, do it first for the example

$$\int \frac{x+2}{(x^2+2x+3)^k}\,dx$$

Problem 3. Show that the integral $\int \frac{Ax+B}{(x^2+a^2)^k}\,dx$ can be decomposed into a sum (with some coefficients) of two terms of the form

$$\int \frac{2x}{(x^2+a^2)^k}\,dx = I_1$$

and

$$\int \frac{1}{(x^2+a^2)^k}\,dx = I_2$$

Calculate the integral I_1.

Use the technology so far to calculate

$$\int \frac{1}{x^4+1}\,dx = ?$$

To finish our discussion (which at this stage became the complete theory of integration of rational functions), we need to calculate the integral I_2.

Problem 4. First show that it suffices to calculate I_2 when $a = 1$ (why?). Let's assume this and rename I_2 to indicate its dependence on k by

$$J_k = \int \frac{1}{(x^2+1)^k}\,dx$$

Verify the following recursion formula:

$$J_{k+1} = \frac{1}{2k}\frac{x}{(1+x^2)^k} + \frac{2k-1}{2k}J_k$$

Hint: integration by parts first and then the often used "complification" to express zero as a difference of two equal numbers, in our case $0 = 2-2$. Calculate J_1, J_2 and J_3.

Problem 5. Calculate the integral of the rational function

$$\frac{2x^6 - 3x^5 - 9x^4 + 23x^3 + x^2 - 44x + 39}{x^5 + x^4 - 5x^3 - x^2 + 8x - 4}$$

Hint: the denominator polynomial has $x = 1$ and $x = -2$ as its zeros.

We now switch over to the new material of infinite sums etc.

Problem 6. Use the FPof $\mathbb{R}$ to show that the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

has a limit. Hint: $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = ?$, think of partial fractions, and then of telescoping series...

Problem 7. Investigate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$$

for their convergency depending on the value of the real parameter $\alpha \in \mathbb{R}$, that is, show:

(i) If $\alpha > 1$ the series converges. Hint: Riemann sum for which function (draw a picture)? An older HW combined with the FPof $\mathbb{R}$.

(ii) If $\alpha \leq 1$ the series does not converge, that is, the sequence of partial sums is unbounded. Hint: Riemann sum for which function (draw a picture)?

Problem 8. Provide a value (preferably small of course) for $n \in \mathbb{N}$, so that the finite sum $\sum_{k=0}^{n} \frac{1}{k!}$ agrees with the value of $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ to five decimals. At this stage we don't know the exact value of e of course, and the question is how we can find the value of e up to any prescribed accuracy, in this case 10^{-5}. As check of your answer carry out the finite sum (up to your predicted lenghth n) using your calculator to see if this number agrees with e (which you read off your calculator) up to five decimals.

Problem 9. Use the Newton method to find zeros to construct a sequence $\{s_n\}$ which converges to $\sqrt{2}$:

(i) Let $f(x) = x^2 - 2$ for $x \geq 0$, so its zero satisfies $x^2 = 2$. Choose some $x_0 > 0$. Find the intersection point x_1 of the tangent line through $(x_0, f(x_0))$ with the x-axis. Repeating this construction, one obtains a sequence $\{x_n\}$ via the prescription that x_{n+1} is the intersection with the x-axis of the tangent line through the point $(x_n, f(x_n))$ (draw a picture). You should find the formula for $x_{n+1} = g(x_n)$ expressing x_{n+1} in terms of x_n.

(ii) Starting at $x_0 = 2$, show that the sequence $\{x_n\}$ is decreasing and bounded from below by $M = 0$. Apply the FPof $\mathbb{R}$ (in the version of a decreasing, bounded from below sequence) to deduce that this sequence has a limit L.

(iii) Use your recursion formula $x_{n+1} = g(x_n)$ and take $n \to \infty$ on both sides, keeping in mind that $x_n \to L$, to find out what L is (it should be $\sqrt{2}$.

(iv) Calculate the first six sequence elements $x_0, \ldots, x_5$ and compare x_5 to the value of $\sqrt{2}$. How good/bad is the approximation?