HOMEWORK 7, HONORS CALCULUS II
10/31/2019

Problem 1. Consider the curve $\gamma(t) = \cos(2t)(\cos t, \sin t)$ for $t \in [0, 2\pi]$. Draw an accurate picture of this curve and label the points on the curve for $t = \frac{n\pi}{4}$, $n = 0, \ldots, 8$. What is the length of this curve? Write down the correct integral. You will not be able to explicitly integrate the curve length integral, but you can do it numerically.

Integration of rational functions II. We continue the (admittedly lengthy) discussion of integration of rational functions (see HW 5)—most of the work is parsing the text of the problem....After long division and decomposition of the denominator polynomial into linear an quadratic factors (which in most cases one cannot to explicitly), we may assume that our function has the form

$$f(x) = \frac{P(x)}{(x-x_1)^{n_1}(x-x_2)^{n_2}\cdots(x-x_r)^{n_r}(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_lx+c_l)^{m_l}}$$

where $P(x)$ is a polynomial whose degree is strictly less than the degree of the denominator polynomial, which we read off to be $n = n_1+\cdots+n_r+2m_1+\cdots+2m_l$. We already dealt with the case when there are none of the quadratic terms.

The next problems guide you to deal with the quadratic terms in the denominator. The general case is then just a combination of the techniques from HW 5 and what follows. The partial fraction decomposition for the quadratic denominator case works as follows:

$$\frac{P(x)}{(x^2+b_1x+c_1)^{m_1}\cdots(x^2+b_lx+c_l)^{m_l}} =$$
$$= \frac{A_1x+B_1}{x^2+b_1x+c_1} + \frac{A_2x+B_2}{(x^2+b_1x+c_1)^2} + \cdots + \frac{A_{n_1}x+B_{n_1}}{(x^2+b_1x+c_1)^{n_1}} + \text{same for the other quadratic factors}$$

Notice the linear terms in the numerator when compared to the decomposition given in HW 5 (those guarantee that the unknown coefficients can be solved for)

Problem 2. Find the partial fraction decomposition of $\frac{1}{x^4+1}$. Hint: $x^4+1 = (x^2+ax+1)(x^2+bx+1)$ for some (which?) $a, b \in \mathbb{R}$.

Using the partial fraction decomposition, we only need to calculate integrals of the form

$$\int \frac{Ax+B}{(x^2+ax+b)^k}\,dx$$

for some non-negative integer $k \in \mathbb{N}$.

Problem 3. Show that by completing the square followed by a substitution the integral

$$\int \frac{Ax+B}{(x^2+ax+b)^k}\,dx$$

can be written as

$$\int \frac{\tilde{A}u+\tilde{B}}{(u^2+\tilde{a}^2)^k}\,du$$

where $\tilde{A}, \tilde{B}, \tilde{a}$ are new constants to be computed form the old A, B, a, b. Notice the term $\tilde{a}^2 > 0$, which is positive due to the fact that x^2+ax+b has no real zeros (check this assertion!), a fact we will use later.

If you have difficulty doing this in the abstract, do it first for the example

$$\int \frac{x+2}{(x^2+2x+3)^k}\,dx$$

Problem 4. Show that the integral $\int \frac{Ax+B}{(x^2+a^2)^k}\,dx$ can be decomposed into a sum (with some constant coefficients in front of the integrals) of two terms of the form

$$\int \frac{2x}{(x^2+a^2)^k}\,dx = I_1$$

and

$$\int \frac{1}{(x^2+a^2)^k}\,dx = I_2$$

Calculate the integral I_1.

Use the technology so far to calculate

$$\int \frac{1}{x^4+1}\,dx = ?$$

To finish our discussion (which at this stage became the complete theory of integration of rational functions...), we need to calculate the integral I_2.

Problem 5. First show that it suffices to calculate I_2 when $a = 1$ (why?). Let's assume this and rename I_2 to indicate its dependence on k by

$$J_k = \int \frac{1}{(x^2+1)^k}\,dx$$

Verify the following recursion formula:

$$J_{k+1} = \frac{1}{2k}\frac{x}{(1+x^2)^k} + \frac{2k-1}{2k}J_k$$

Hint: integration by parts first and then the often used "complification" to express zero as a difference of two equal numbers, in our case $0 = 2 - 2$. Calculate J_1, J_2 and J_3.

Problem 6. Calculate the integral of the rational function

$$\frac{2x^6 - 3x^5 - 9x^4 + 23x^3 + x^2 - 44x + 39}{x^5 + x^4 - 5x^3 - x^2 + 8x - 4}$$

Hint: the denominator polynomial has $x = 1$ and $x = -2$ as its zeros.

Problem 7. Calculate the volume of the spindle formed by revolving $y = \sin x$, $x \in [0, \pi]$, around the x-axis.

Problem 8. How much water does the following clay bowl hold? The bowl is made on a pottery wheel and has the shape of revolving the parabola $y = x^2 - 1$ around the y-axis,. The bottom of the bowl is the disk at $y = 0$ and the bowl has height $h > 0$ to its top. Draw a picture before calculating. The length unit is inches.

Problem 9. A "cooling tower" of height $2h > 0$ is obtained by revolving the hyperbola $x^2 - y^2 = 1$ around the y-axis between $-h \le y \le h$. Draw a picture. How much volume does it enclose?