# Homework 4, Honors Calculus II
due Tuesday 10/8/19

*Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top. All integrals have to be computed without using symbolic calculators. You may use a calculator only to verify a result and for numerical calculations which you cannot to on paper or in your head.*

**Problem 1.** Apply the $d$-operator to the following functions to calculate their differentials:

(i) $f(x) = \ln(\cos\sqrt{x})$
(ii) $f(x) = \frac{\tan(x)}{x^3}$
(iii) $f(x) = \frac{\sinh(\ln(x))}{x}$
(iv) $f(x) = (\sin x)e^{4\cos x^2}$
(v) $f(x) = \frac{1}{\sqrt{1+\ln x}}$

**Problem 2.** Find an anti-derivative for each of the following functions:

(i) $f(x) = \frac{\cos(\sqrt{x})}{\sqrt{x}}$
(ii) $f(x) = \frac{\ln(x)}{x^3}$
(iii) $f(x) = \frac{\sinh(\ln(x))}{x}$
(iv) $f(x) = (\sin x)e^{4\cos x}$
(v) $f(x) = \frac{1}{\sqrt{1+x^2}}$

**Problem 3.** An ellipse, centered at the origin, is given by the equation

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$$

where $a \geq b > 0$ are the half axes length. For instance, if $a = b$ then the ellipse becomes a circle of radius $a$. Calculate the area enclosed by the ellipse. In particular, this gives a formula for the area enclosed by a circle.

**Problem 4.** Calculate the area bounded between the graphs $y = \sin x$ and $y = \cos x$ between their first and second intersection points on the positive $x$-axis. Again, draw a picture of the region first.

**Problem 5.** In class we defined the hyperbolic trig functions

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$

Draw accurate graphs of both of these functions. Why is $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$? Notice, that compared to the usual trig functions, the hyperbolic trig functions have *no* periodicity. Give the largest domain and codomains on which sinh and cosh are bijective functions, and thus can be inverted. We call $\sinh^{-1}$ and $\cosh^{-1}$ the inverse hyperbolic trig functions, in other words

$$\sinh(\sinh^{-1}(y)) = y \quad \text{and} \quad \sinh^{-1}(\sinh(x)) = x$$

and likewise for cosh.

Now verify the the following formulas (and compare them to the corresponding formulas for the trig functions):

(i) $\cosh^2(x) - \sinh^2(x) = 1$.
(ii) $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$.

(iii) $\sinh(2x) = 2\sinh(x)\cosh(x)$ and $\cosh(2x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1$.
(iv) Putting $\tanh(x) := \frac{\sinh(x)}{\cosh(x)}$ calculate $\tanh'(x) = ?$.
(v) Calculate the derivative of the inverse hyperbolic trig functions

$$\frac{d}{dx}\sinh^{-1}(x) = ? \quad \frac{d}{dx}\cosh^{-1}(x) = ? \quad \frac{d}{dx}\tanh^{-1}(x) = ?$$

Note, this gives us three more anti-derivatives to use when trying to calculate integrals.

**Problem 6.** In class we calculated the area under the graph (of one branch) of the hyperbola $x^2 - y^2 = 1$ over an interval $[1, b]$ using the hyperbolic sine and cosine functions (see previous problem) via substitution. We also started a calculation following a student's suggestion using integration by parts. In class we did not pursue this further, but one can get somewhere by using a "trick", namely adding zero to an expression in an intelligent way, usually by thinking $0 = 1 - 1$. Fill in the calculation below:

$$\int \sqrt{x^2 - 1}\,dx = x\sqrt{x^2 - 1} - \int \frac{x^2}{\sqrt{x^2 - 1}}dx = \ldots ???$$

Now apply the "adding zero trick" in the last integrant numerator, continue the above calculation, and show that

$$\int \sqrt{x^2 - 1}\,dx = x\sqrt{x^2 - 1} - \int \sqrt{x^2 - 1}\,dx + \int \frac{dx}{\sqrt{x^2 - 1}}$$

Conclude from this that

$$\int \sqrt{x^2 - 1}\,dx = \frac{1}{2}\left(x\sqrt{x^2 - 1} + \int \frac{dx}{\sqrt{x^2 - 1}}\right)$$

Now integrate $\int \frac{1}{\sqrt{x^2-1}}\,dx$ using the hyperbolic trig substitution, which gives an easier integral than the original integral with the reciprocal integrant.

**Problem 7.** Consider the hyperbola $x^2 - y^2 = 1$. Calculate the area bounded by the two branches of the hyperbola and the two horizontal lines $y = \pm 1$.

**Problem 8.** Combining integration with some elementary series (e.g. the geometric series), one can obtain methods to calculate values of various functions, e.g. the natural logarithm in this example. Here how this could work:

(i) Prove the formula for the finite geometric series

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

(ii) Now provide an argument that for numbers $|x| < 1$ one can in fact add up infinitely many terms and obtain

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

(iii) Now use the partial fraction decomposition to integrate $\int \frac{dx}{1-x^2}$ and combine this with the geometric series, to show that

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} = \frac{1}{2}\ln\frac{1+x}{1-x}$$

(iv) Next verify that the function $y = \frac{1+x}{1-x}$ maps the interval $[0, 1)$ to the interval $[1, \infty)$. Use this and the previous formula (iii) to calculate $\ln(2)$ (what value of $x$ does $y = 2$ correspond to ?) to three decimals accuracy (compared to the value you get on the calculator). Do the same for $\ln(5)$.

(v) **Bonus**: comparing to the calculator is of course cheating, since the calculator cannot compare its calculation to itself, so it has to have a method to decide whether its calculation of $\ln(2)$ is accurate to whatever precision is demanded. So can you find a method to decide whether your approximation is accurate up to three decimals without comparison to the calculator? Here a hint for that: the series

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

probably converges pretty fast. What you would need to have is an estimate how close to the true value of the series you are when you sum only a finite number (say $n$) of terms, i.e. you need to estimate the "tail" of the infinite series:

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} - \sum_{k=0}^{n} \frac{x^{2k+1}}{2k+1} = \sum_{k=n+1}^{\infty} \frac{x^{2k+1}}{2k+1}$$

One way to do this (not super sophisticated, but a first step) is to look at the series tail for a given $0 \leq x < 1$ (justify each step in the calculation below):

$$\sum_{k=n+1}^{\infty} \frac{x^{2k+1}}{2k+1} = \frac{x^{2n+3}}{2n+3} + \frac{x^{2n+5}}{2n+5} + \cdots \leq \frac{x^{2n+3}}{2n+3}(1 + x^2 + x^4 + \dots) = \frac{x^{2n+3}}{2n+3} \frac{1}{1-x^2}$$

So you get a worst case scenario how big your error is for a given $x$ when you just sum the first $n$ many terms. Try to apply this to the case at hand.

You may wonder why we started with the function $\frac{1}{1-x^2}$ rather than the simpler function $\frac{1}{1-x}$ (one would not have to do that partial fraction stunt). Any ideas?