HOMEWORK 3, HONORS CALCULUS II
DUE TUESDAY 10/1/19

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Find an anti-derivative of each of the following functions:

(i) $f(x) = \cos^2(x)$
(ii) $f(x) = e^x + 3\cos(x) + \frac{\pi}{x}$
(iii) $f(x) = \frac{\ln(x)}{x}$
(iv) $f(x) = x^2 e^x$
(v) $f(x) = \tan x$

Problem 2. Show that the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k+n} := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k+n} = \log 2\,.$$

We always use logarithms to base e unless specifically indicated. Two hints: think of a Riemann sum for which function? What is the area under the hyperbola $y = 1/x$ over the interval $[1, 2]$?

Problem 3. Find an anti-derivative of $f(x) = \log x$ and calculate the area under the graph $y = \log x$ over the interval $[1, 2]$.

Problem 4. Calculate the indefinite integrals (that is, find anti-derivatives)

$$\int \sin^n(x)dx = ? \quad \text{and} \quad \int \cos^n(x)dx = ?$$

Think of $\sin^n(x) = \sin(x)\sin^{n-1}(x)$. Then use integration by parts. Then try to find some kind of recursion pattern as n decreases. If you don't succeed with that, do the cases $n = 2, 3, 4$ to get an idea what happens for general n. Same ideas work for the $\cos(x)$ integral.

Problem 5. Find the area under of the domain bounded by the parabola $y = x^2 - 1$ and the line passing through the points $(-2, 3)$ and $(3, 8)$. Draw a picture of the region to scale and with labels before attempting the calculation. Also keep in mind that the integral calculates "signed" areas, that is, regions below the x-axis have a negative integral—but areas are always positiv.

Problem 6. A function $f\colon [-a, a] \to \mathbb{R}$ is called *even*, respectively *odd*, if $f(-x) = f(x)$, respectively $f(-x) = -f(x)$, for all $x \in [-a, a]$. Draw the graphs of an even and odd function.

(i) Give two examples of an even and two examples of an odd function.
(ii) Show that for an even (continuous) function $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$. Interpret this in terms of areas.
(iii) Show that for an odd (continuous) function $\int_{-a}^{a} f(x)\,dx = 0$. Interpret this in terms of areas.
(iv) Give an example of a function which is *not* even, but for which (ii) holds. Give an example of a function which is *not* odd, but for which (iii) holds.

Problem 7. Show that

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \end{cases}$$

where $n, m = 0, 1, 2\ldots$ are non-negative integers.

Problem 8. Find an anti-derivative of

$$f(x) = \frac{1}{x^2 - a^2}$$

for arbitrary $a \geq 0$ (you can take $a = 3$ as an example if you dislike working with arbitrary a). This is the first instance of the more general "partial fraction decomposition", which is found in letter exchanges between Bernoulli and Leibniz around 1700. Use the difference of squares formula, and use this to write the fraction with the quadratic denominator as a sum of two fractions with linear denominators.