HOMEWORK 1, HONORS CALCULUS II
DUE 9/20

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. In class we defined the length $L(\overline{AB})$ of a line segment with endpoints $A, B \in \mathbb{R}^2$ in the plane. Provide a mathematical proof/argument/calculation of the following statements:

(i) L is translation invariant, i.e., if you translate the line segment by some vector $T \in \mathbb{R}^2$, then its length does not change.

(ii) L is rotation invariant, i.e., if you rotate the line segment by some angle α around some point in the plane, then its length does not change.

Here some thoughts/hints to item (ii): once you know that L is translation invariant, you can (without loss of generality, as mathematicians say, meaning that using item (i) you can reduce to this case) safely assume that the segment is rotated around one of the endpoints of the line segment which you could even assume to be the origin (why?). Now find a formula for the coordinates of the point $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2$ obtained by rotating by angle α in anti-clockwise direction a point $(x, y) \in \mathbb{R}^2$. So you aim for a formula

$$\tilde{x} = \text{ some expression in } x,\ y \text{ and } \alpha$$

and the same for $\tilde{y}$. If you already know some vector/matrix notation, you could apply this here. If not, you just find the formula. Draw pictures to get the correct formulas from trigonometry. You should do this for general T and α and points A, B, in other words, you do not choose explicit numbers for those quantities, since then you would not have verified a general fact, but just a special case.

Problem 2. Give the formula for the length of a line segment from point $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in 3-dimensional space. Explain how you arrived at this formula.

Problem 3. Assuming that we know that the area of a circle of radius R is $R^2\pi$ (which can be shown by elementary geometry), calculate an approximation of π by using the function

$$f\colon [0,1] \to \mathbb{R}\,, \quad f(x) = \sqrt{1 - x^2}$$

whose graph is a quarter-circle of radius $R = 1$. Use equidistant partition into five subintervals and choose as your evaluation points the *left* end points in each subinterval (in class we used the right end points of the subintervals), and calculate the Riemann sum for that partition.

Problem 4. In class we discussed Gauss' trick to calculate the sum $\sum_{k=1}^{n} k$ of the first n integers and what this had to to with calculating the integral under the graph of the line $y = x$ over $0 \le a \le x \le b$.

In this problem try to calculate the area under the graph of $y = x^2$ using Riemann sums. You will notice that you have to calculate the sum of the squares $\sum_{k=1}^{n} k^2$ of the first n integers. Here a way to do that:

(i) Use the binomial formula $(k+1)^3 = \ldots$ to deduce $(k+1)^3 - k^3 = \ldots$ (fill in the correct expressions for the dots) and then sum this last identity over $k = 1, \ldots, n$. If you get confused, do this for small n (like $n = 3$ or $n = 4$) first and see what happens....

(ii) **extra points for the die hard** Use the previous idea to calculate the sum of cubes $\sum_{k=1}^{n} k^3$. Can you see a method to calculate $\sum_{k=1}^{n} k^4$ and etc.?

Problem 5. Find the area under the cubic $y = x^3$ over the interval $[0, 1]$. You will need to find a formula for the sum of the cubes $\sum_{k=1}^{n} k^3$ of the first n numbers to do that (see previous problem).

Problem 6. Here a function for which the integral does not exist. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = 1$ if x is a rational number, and $f(x) = 0$ if x is an irrational number (try to graph this function....). Show that the Riemann sum for f does not converge (that is, its limit depends on the choice of partition) by choosing a sequence of partitions and evaluation points so that the Riemann sum converges to zero, and then choose another sequence of partitions and evaluation points so that the Riemann sum converges to one.