Homework 11, Honors Calculus II

11/29/2018

Problem 1. Provide an argument why the alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \ln(2)\,.$$

Problem 2. Consider the following function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for} \quad x \neq 0 \\ 0 \quad \text{for} \quad x = 0 \end{cases}$$

defined on all of $\mathbb{R}$. Verify the following statements:

(i) The left and right derivatives at $x_0 = 0$ of all orders are zero, i.e. $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

(ii) The Taylor series of $f(x)$ centered at $x_0 = 0$ has infinite convergency radius, that is, converges for all $x \in \mathbb{R}$.

(iii) The Taylor series of $f(x)$ centered at $x_0 = 0$ evaluated at any $a \neq 0$ is not equal to the function value $f(a) = e^{-1/a^2}$.

This example shows that even if the Taylor series of a function converges, it may fail to be equal to the function. But you notice that to construct an example of such a function, we needed to "splice together" two well-behaved functions, both of which individually would have Taylor series expansion presenting them.

To verify that the Taylor series of a function $f(x)$ on its interval of convergency actually equals the function on that interval, requires to check whether the remainder term in the Taylor polynomial expansion tends to zero as the degree of the Taylor polynomial tends to infinity.

Problem 3. Calculate the Taylor series expansion centered at $x_0 = 0$ of the function $\sin^{-1}(x)$ in two ways: first, by using the integral formula for $\sin^{-1}(x)$ and expanding the integrant into a a power series; second, by using the definition of the Taylor series.

Problem 4. Calculate the Taylor series expansion of $\cos(x)$ centered at $x_0 = 0$ and show that it has infinite convergency radius. Also verify that the remainder term tends to zero for any $x \in \mathbb{R}$, thus the Taylor series does present the $\cos(x)$ and can be used to calculate $\cos(x)$.

Problem 5. Derive the formula

$$\int_0^1 x^x\,dz = 1 - 1/2^2 + 1/3^3 - 1/4^4 + 1/5^5 - \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^k}$$

by using the following hints: $x^x = e^{x\ln(x)}$; the exponential series; term by term integration; $\int_0^1 x^n(\ln x)^m\,dx$ via integration by parts.