HOMEWORK 10, HONORS CALCULUS II
12/3/2019

Problem 1. Determine and provide a proof whether the following infinite series converge or diverge:

(i) $\sum_{k=1}^{\infty} n^3 2^{-n}$
(ii) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$
(iii) $\sum_{k=1}^{\infty} \frac{n^2+3n-5}{1+n^2}$
(iv) $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$
(v) Write $0.6666666\ldots$ periodic as an infinite series and show that this series converges to $2/3$. How about 0.43434343434343.... =? as a fraction of integers?
(vi) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$
(vii) $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)(2n^2+5)}$
(viii) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

Problem 2. Use the following ideas to derive the value for the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$:

(i) Write $\ln(1-x) = \int ?\, dx$.
(ii) Expand the integrant ? in a geometric series and then integrate term by term to obtain

$$\ln(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

for some specific numbers a_k which you will find when carrying out those steps.

Problem 3. Here an idea of Newton's who wrote: "All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded mathematics and philosophy more than at any other time since". For a real number $\alpha > 0$ consider the power series

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where, in agreement with the coefficient of the binomial formula, we define

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

Show the following:

(i) If $\alpha = n$ is a positive integer, then the above series is a finite sum (Identify this sum as an expression you know well.
(ii) If $\alpha > 0$ is *not a natural number* (the sum will have infinitely many terms in this case) show that the power series converges for $0 \leq x < 1$.
(iii) Extrapolating from (i), it was clear to Newton that

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

holds for any real number α. This *generalized binomial formula* is engraved on his tomb in London. Use this formula to approximate $\sqrt{3/2}$ by summing the first 4 terms of the series and compare this to the "actual" value of $\sqrt{3/2}$ given by the calculator.

Problem 4. Show that the generalized Newton binomial series

$$(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k$$

converges for all $x \in (-1, 1]$. Use this to calculate an approximate value for $\sqrt{2}$ by summing the first 5 terms of the series. How good is this approximation (compare to the value of $\sqrt{2}$ the calculator provides).

Problem 5. Determine and provide proof of whether the following series of numbers converge or diverge:

(i) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$
(ii) $\sum_{k=1}^{\infty} (-1)^k \frac{k^4}{4^k}$
(iii) $\sum_{k=1}^{\infty} \tan(1/k)$
(iv) $\sum_{n=0}^{\infty} n^2 e^{-n^2}$
(v) $\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n+1}}$
(vi) $\sum_{n=1}^{\infty} \frac{n \ln n}{(n+1)^3}$
(vii) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$
(viii) $\sum_{n=1}^{\infty} n \sin(1/n)$
(ix) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+3}$

Problem 6. Determine *all* values for x such that the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

converges.

Problem 7. Find a power series expansion for

$$\tan^{-1} x = \sum_{k=0}^{\infty} a_k x^k$$

Hint: $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$ and expand the integrant into a series, integrate term by term, and determine what the constant c has to be. Find all x for which this series converges.

Problem 8. The curve $\gamma\colon [0, 2\pi] \to \mathbb{R}^2$ given by $\gamma(t) = (a\cos t, b\sin t)$ traces out an ellipse. We assume $0 < b < a$, so that a is the main axis.

(i) Write down the integral which gives you a quarter of the length of the ellipse's circumference. This is called an *elliptic integral* and people have long tried to find an antiderivative without success. In fact, the antiderivative is a new type of function, called an *elliptic function*, which I believe was first properly understood by Riemann. So don't waste time trying to solve the integral by our methods.

(ii) Putting the eccentricity $k = \frac{a^2-b^2}{b^2}$ (so a cricle has $k = 0$), you should have obtained $L = b\int_0^{\pi/2} \sqrt{1 + k(\sin t)^2}\, dt$ for the quarter length of the ellipse. Now use Newton's generalized binomial formula (see previous problem) to expand the integrant into a series and then integrate term by term. There will be a restriction on k for this series to converge (what is this restriction?).

(iii) Choose $a = 2$ and $b = 3/2$ (then the series will converge, why?), and calculate an approximate quarter length L by summing the first 3 terms of your series expansion. Can you get a sense how the eccentricity influences the circumference of an ellipse?