132H FINAL PROJECTS

1. Project: Logarithm and Exponentials revisited

This project takes a different route (even though all routes lead to Rome...) to obtain the logarithm and the exponential function. So the starting point is that you erase everything you know about log and exponential functions, i.e., you are not allowed to use *anything* you have learned so far about these functions—they do not yet exist until you have worked through the problem (that's why we notate them with different letters for this problem).

We noticed that $(x^n)' = nx^{n-1}$ does not provide an anti-derivative for the function $1/x$. To remedy this, we define the new function

$$L\colon (0,\infty) \to \mathbb{R}\,, \qquad L(x) := \int_1^x \frac{dt}{t}$$

an anti-derivative of $1/x$, which measures the area under the graph of $f(t) = \frac{1}{t}$ between 1 and x. Proof the following assertions:

(i) $L(1) = 0$ and $L(ab) = L(a) + L(b)$. Interpret the latter in terms of areas under the graph of $f(t) = \frac{1}{t}$. Provide a picture.

(ii) L is strictly increasing and concave up everywhere on $(0, \infty)$.

(iii) $\lim_{x\to\infty} L(x) = +\infty$ and $\lim_{x\to 0} L(x) = -\infty$.

(iv) $L\colon (0,\infty) \to \mathbb{R}$ is bijective. Denote the inverse function L^{-1} by

$$E\colon \mathbb{R} \to (0,\infty)$$

(v) $E'(x) = E(x)$ and $E(0) = 1$.

(vi) $E(a + b) = E(a)E(b)$.

(vii) Expand $\frac{1}{t}$ in a power series around $t = 1$ (note $t = 1 + (t - 1)$) and use this to derive a power series expansion of $L(1 + x)$ around $x = 0$ via term by term integration. Find the largest interval on which the power series of $L(1 + x)$ converges.

(viii) Using the identity (i) calculate a power series expansion of $L(\frac{1+x}{1-x})$. Find the largest interval on which this power series converges. Verify that the expression $\frac{1+x}{1-x}$ ranges over *all* positive real numbers when $x \in (-1, 1)$.

(ix) Calculate the first 6 partial sums of the series for $L(2)$ using the power series for $L(1 + x)$ and then for $L(\frac{1+x}{1-x})$. How accurate (to how many decimals) do you get the value of $L(2)$ in each of the two cases by inspection?

(x) Show that it suffices to know $L(p)$ for any prime number p to calculate $L(a)$ for any positive rational number $a = n/m$. Since irrational numbers can be approximated to arbitrary precision by rational numbers, this allows for accurate calculations of $L(a)$ for positive real numbers a.

(xi) Here the beginning of a systematic way to calculate $L(p)$ for larger and larger primes. We already know $L(2)$ to fairly high accuracy from (ix). To calculate $L(3)$ write $3 = 3/2 \times 2$ and solve for a in $3/2 = \frac{1+a}{1-a}$. Then, use the power series for $L(\frac{1+x}{1-x})$ to calculate $L(3/2)$ to "good" accuracy, and therefore $L(3) = L(2) + L(3/2)$ can be calculated. Now apply a similar procedure to calculate $L(5)$ and $L(7)$ to "good" accuracy (maybe to 3 decimals). For example, $5 = 5/4 \times 4$... You can use your calculator but only to do additions and multiplications!

2. Project: New Functions

2.1. Euler's Gamma Function. Consider der following integral

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$$

and verify the following statements:

(i) $\Gamma(x)$ is a finite number for all $x > 0$, i.e., the indefinite integral is convergent for all $x > 0$. Note that the integral is improper because of an infinite domain of integration and, for $0 < x < 1$, also improper at $t = 0$ since the integrant has a vertical asymptote.
(ii) $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.
(iii) $\Gamma(n+1) = n!$ for integers $n > 0$.
(iv) $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\Gamma(\frac{1}{2})$.

2.2. Riemann Zeta Function. This problem requires knowledge of complex numbers. We discussed the series $\sum_{k=1}^\infty \frac{1}{k^\alpha}$ for real numbers α and found that the series converges for $\alpha > 1$ and diverges for $\alpha \leq 1$ by the integral test. Riemann, in his study of the distribution of prime numbers, noticed that it is important to study what is now called the *Riemann Zeta Function*

$$\zeta(z) := \sum_{k=1}^\infty \frac{1}{k^z}$$

where $z \in \mathbb{C}$ is a complex number. For example, we can evaluate the zeta function at the positive integers n to get $\zeta(n) = \sum_{k=1}^\infty \frac{1}{k^n}$, in particular $\zeta(2) = \sum_{k=1}^\infty \frac{1}{k^2}$.

(i) Show that the series $\sum_{k=1}^\infty \frac{1}{k^z}$ converges absolutely for $\mathrm{Re}(z) > 1$. Draw a picture of this domain in the complex number plane $\mathbb{C}$. Thus $\zeta(z)$ is a well defined function on this domain with values in $\mathbb{C}$.
(ii) Show that the zeta function can be expressed as an infinite product over primes (hinting towards a connection between primes and the Zeta Function)

$$\zeta(z) = \Pi_{p\,\mathrm{prime}} \frac{1}{1-p^{-z}} = (\frac{1}{1-2^{-z}})(\frac{1}{1-3^{-z}})(\frac{1}{1-5^{-z}})(\frac{1}{1-7^{-z}})\cdots$$

Hint: Use the fact that every integer number $n = p_1^{k_1} \cdots p_m^{k_m}$ is a finite product of powers of primes in a unique way (if one orders the primes $p_1 < p_2 < \cdots < p_m$), e.g. $40 = 2^3 \cdot 5^1$, and use the geometric series.
(iii) Show that there are infinitely many primes. *Hint:* Use the identity in (ii) for a suitable choice of z, and not Euclid's classical proof.

3. Project: Curvature of curves in the plane: A

Looking at a curve $M \subset \mathbb{R}^2$ in the plane, we may have some intuitive sense how curved it is at any given point $p \in M$ on the curve. The problem is how to quantify this intuitive notion. For instance, we surely would say that a straight line has zero curvature at all of its points. How about a circle C of radius $R > 0$? First, its curvature should be the same at any point $p \in C$ on the circle, so whatever the curvature is, it should be constant along the circle. Second, if we draw a circle and enlarge it, the curvature visibly gets smaller, and the rate by which is gets smaller seems to be linear in the scaling. In other words, the curvature of a circle of radius R should be proportional to $1/R$. Since there is nothing to determine the proportionality factor, we set it to 1, and are led to *define*:

The curvature of a circle of radius $R > 0$ is $K = 1/R$.

Now let M be a curve in the plane. The problem is to assign to each point $p \in M$ on the curve a real number, the curvature $K(p)$ of the curve M at $p \in M$. This way we would have a *curvature function*

$$K : M \to \mathbb{R}$$

measuring the curvature of the curve M at any of its points. Since we know what the curvature of a circle is, the idea is to attach to each point $p \in M$ of the curve $M \subset \mathbb{R}^2$ a "best fitting circle" $C(p)$ and define $K(p) := 1/R(p)$ as the reciprocal of the radius $R(p)$ of this best fitting circle $C(p)$. What is a "best fitting circle"? First, it should touch the curve at $p \in M$, that is, the circle $C(p)$ should go through the point $p \in M$ and have the same tangent line $L(p)$ than M at the point p. Thus, the center of $C(p)$ must lie on the normal line $L^{\perp}(p)$ (perpendicular to $L(p)$) of the curve M through the point $p \in M$. But any point $c \in L^{\perp}(p)$ gives rise to circle with center c touching the curve at $p \in M$. Draw a picture so you understand the situation.

Among all those touching circles, we have to single out a "best" one, somehow reflecting our notion how curved M is at $p \in M$. To do so, we take a nearby point $q \in M$ and intersect its normal line $L^{\perp}(q)$ with $L^{\perp}(p)$, i.e. we look at the point $c = L^{\perp}(p) \cap L^{\perp}(q)$. We then let $q \to p$ and track the point c through this process. The limiting position of $c \in L^{\perp}(p)$ is then our candidate for the center of the "best fitting" circle $C(p)$. Draw a picture of this situation.

(i) Carry out the above ideas in the case when the curve $M \subset \mathbb{R}^2$ is the graph of a function $y = f(x)$ and derive a formula for the curvature $K(p) = K(a)$ of the curve M at an arbitrary point $p = (a, f(a)) \in M$. *Hint:* find the equation of the normal line through $(a, f(a))$; intersect this normal line with the normal line through $(a + h, f(a + h))$ and calculate the limit of the intersection point as $h \to 0$.

(ii) Calculate the curvature function of a straight line $y = mx + b$ and a circle $x^2 + y^2 = R^2$ to check if your formula gives what we based our theory on. Then calculate the curvature for the parabola $y = x^2$ and the ellipse/hyperbola $(\frac{x}{a})^2 \pm (\frac{y}{b})^2 = 1$. At which points is the curvature largest/smallest?

(iii) Assume now that the curve $M \subset \mathbb{R}^2$ is traced out by a parametrization $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $t \in I$ where $I \subset \mathbb{R}$ is an interval. Calculate a formula for the curvature $K(t)$ at a point $p = \gamma(t) \in M$. *Hint:* regard $\gamma_1(t) = x(t)$ and $\gamma_2(t) = f(x(t))$ and use the previous formula for a graph together with the chain rule:

$$\frac{df}{dx}(x(t)) = \frac{\frac{df}{dt}}{\frac{dx}{dt}} = \frac{\gamma_2'(t)}{\gamma_1'(t)}$$

and similarly for 2nd derivatives (which will appear in your formula in (i)).

(iv) Carry out "sanity checks": if the curve is given by the graph parametrization $\gamma(t) = (t, f(t))$, then the formula for the curvature from (iii) should become the formula for the curvature from (i) (with x replaced by t). Use the formula in (iii) to calculate the curvatures of a line parametrized by $\gamma(t) = tv + p$ with $v, p \in \mathbb{R}^2$; of a circle of radius $R > 0$ parametrized by $\gamma(t) = (R\cos(t), R\sin(t))$; of an ellipse parametrized by $\gamma(t) = (a\cos(t), b\sin(t))$ with $a \geq b > 0$. Verify that for a proper ellipse (not a circle) the curvature function $K(t)$ has exactly 4 critical points (i.e. points where the curvature function satisfies $K'(t) = 0$), called *vertices*. Which ones are they? And which of them are maxima/minima for the curvature function?

A classical theorem (Four Vertex Theorem) states that every *closed* curve in the plane without self-intersections must have at least 4 vertices, i.e. points where the curvature function has vanishing derivative. Draw a couple of pictures of curves with more than 4 vertices; can you have an odd number of vertices?

(v) Try to find all curves of constant curvature, i.e., curves such the curvature at each point is the same number $K \in \mathbb{R}$. You already know that straight lines and circles have constant curvature. You need to show (it it is true), that those are the only possibilities. *Hint:* use any of the two formulas for the curvature function and set it to be constant K. This gives a differential equation. Maybe treat the cases $K = 0$ and $K \neq 0$ separately. You can assume a fact you will learn in M331, namely that differential equations have unique solutions by given initial condition. How does this help?

4. Project: Curvature of curves in the plane: B

Here is another way to compute the curvature of a curve $M \subset \mathbb{R}^2$ traced out by a parametrization $\gamma = (\gamma_1, \gamma_2)\colon I \to \mathbb{R}^2$ motivated by physics: assume you drive along a curve $M \subset \mathbb{R}^2$. Whenever you turn, you will feel some force since you change direction and perhaps speed–you may slow down, or speed up—which, by Newton's Law, is proportional to the acceleration $\gamma''(t)$. Experience shows that the sharper the turn the stronger the force (acceleration), so somehow the curvature should be related to the acceleration $\gamma''(t)$. The problem is, that one can traverse a given curve M in many different ways, slowly, or with greater speed along straighter parts and slowing down at sharp curves etc. so that simply taking the magnitude $||\gamma''(t)||$ of the acceleration as measure of the curvature will not work.

(i) As an example take a circle of radius 1, which should have curvature $K = 1$, and drive around it with speed $||\gamma'(t)|| = 2$. What parametrization $\gamma(t)$ of the circle has speed 2? And what is the magnitude of acceleration $||\gamma''(t)||$ in this case?

(ii) To fix this ambiguity, we demand that the driver drives along the curve with constant speed $||\gamma'(t)|| = 1$ for all $t \in I$, no matter how sharp the turns are. If the curve you are supposed to follow is reasonable, and speed 1 means very slow, then this surely should be possible. The mathematical statement, which you should proof, is the following

Given a parametrized curve $\gamma = (\gamma_1, \gamma_2)\colon [a, b] \to \mathbb{R}^2$ whose speed never vanishes, then one can find a bijective function $\varphi\colon [0, L] \to [a, b]$, where L is the length of the curve, so that

$$\tilde{\gamma} = \gamma \circ \varphi\colon [0, L] \to \mathbb{R}^2$$

has speed $||\tilde{\gamma}'(s)|| = 1$ for all $s \in [0, L]$. Note that $\tilde{\gamma}$ traces out the same curve in the plane as γ does, the only difference is that the curve is traversed with constant speed 1, i.e. parametrized via arclength $s = \lambda(t)$.

Hint: consider the arclength function $\lambda(t) := \int_a^t ||\gamma'(x)||dx$ measuring the length of the curve between a and t. Verify that λ is strictly increasing (use the 1st derivative criterion) and maps the interval $[a, b]$ bijectively to $[0, L]$. The inverse (not reciprocal!) function $\varphi = \lambda^{-1}$ will do the required job (Chain Rule).

(iii) Let $\gamma = (\gamma_1, \gamma_2)\colon [0, L] \to \mathbb{R}^2$ be a curve of constant speed $||\gamma'(s)|| = 1$ (so γ is arclength parametrized, L is the length of the curve). Show that the acceleration (vector) $\gamma''(s)$ is perpendicular to the velocity (vector) $\gamma'(s)$ for all $s \in [0, L]$.

Let $N(s)$ be the unique vector of length 1 normal to the velocity (tangent) vector $\gamma'(s)$ (which has length 1 since the speed of γ is 1) obtained by 90° counter clockwise rotation from $\gamma'(s)$. Draw a picture. Show that there exists a unique function $K\colon [0,L] \to \mathbb{R}$ so that

$$\gamma''(s) = K(s)N(s) \tag{1}$$

for all $s \in [0,L]$. The function $K(s)$ measures the *curvature* of γ at the point $\gamma(s)$. Calculate a formula for $K(s)$ only involving (the components of) $\gamma'(s)$ and $\gamma''(s)$.

(iv) Parametrize a straight line in direction $v \in \mathbb{R}^2$ through $p \in \mathbb{R}^2$ by arclength and calculate its curvature. Do the same for a circle of radius $R > 0$ centered at some point $p \in R^2$. This is some kind of "sanity check" whether your formulas give answers we already know.

(v) Calculate a formula for the curvature function $K(t)$ for an arbitrarily parameterized curve $\gamma = (\gamma_1, \gamma_2)\colon [a,b] \to \mathbb{R}^2$. *Hint:* from (ii) we can obtain the speed 1 curve $\tilde{\gamma}(s)$ where $s = \lambda(t)$ is the arclength function The curvature at the point $\gamma(t) = \tilde{\gamma}(\lambda(t))$ is calculated via the relation (1) applied to $\tilde{\gamma}$. Thus you have to calculate the normal $\tilde{N}(s)$ and the acceleration $\tilde{\gamma}''(s)$ and relate them to $N(t)$ and $\gamma(t)$ (Chain Rule debauch). To not get confused it helps to keep track of the variable w.r.t. which one differentiates: use $\frac{d}{dt}$ and $\frac{d}{ds}$ instead of just prime. Compare to the formula obtained in Project 2 (iii).

5. Project: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and Fourier series

This project provides two different ways to calculate the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

5.1. **Euler's approach.**

(i) Let $x_1, \dots, x_n \in \mathbb{R}$ be non-zero numbers. Verify that

$$P(x) := (1 - \frac{x}{x_1}) \cdots (1 - \frac{x}{x_n})$$

is a degree n polynomial with zeros at $x_1, \dots, x_n$ of the form

$$P(x) = a_n x^n + a_{n+1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0 = 1$ and $a_1 = -\sum_{k=1}^{n} \frac{1}{x_k}$. Calculate also a_2 and a_n.

(ii) Now consider the infinite polynomial

$$P(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

which is just the power series expansion of $\sin x$ divided by x. In analogy to the finite degree case (i) write

$$P(x) = (1 - \frac{x}{x_1})(1 - \frac{x}{x_2})(1 - \frac{x}{x_3}) \cdots$$

as an infinite product where x_k are the infinitely many zeros (what are they?) of $\frac{\sin x}{x}$.

(iii) Use the ideas of (i) and (ii) to show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

To this day there are no known formulas for the sums of the odd power reciprocals of the integers, e.g. $\sum_{k=1}^{\infty} \frac{1}{n^3}$. One does not even know whether they are irrational in general, though there are conjectures.

5.2. **Fourier Series.** If you pluck a string (on a viola, guitar etc.) with end points at $x = \pm 1$ it vibrates at a certain basic frequency, say $\sin(\pi x)$. But generally one can also hear overtones, which are twice $\sin(2\pi x)$, three $\sin(3\pi x)$ the frequency, and so on. One could of course have done the same with $\cos(n\pi x)$. The idea of Fourier series is that an arbitrary vibration of the string can be expressed as a superposition of such basic $\sin(\pi n x)$ and $\cos(\pi n x)$ frequencies where $n \geq 0$ are integers. Translating this idea into mathematics, one may ask whether a given a function $f\colon [-1,1] \to \mathbb{R}$ can be expressed by an infinite series (superposition) of the form

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi x) + b_n \sin(n\pi x)\right) \tag{2}$$

for some coefficients $a_n, b_n \in \mathbb{R}$ which will be determined by $f(x)$. Such a series is called a *Fourier series.*

(i) Draw the graphs of $\sin(n\pi x)$ for $x \in [-1,1]$ say for $n = 0,1,2,3,4$ in different colors in the same coordinate grid to get an idea what those functions look like. Draw a large enough interval $[-1,1]$ to get a good picture. The zeros of $\sin(n\pi x)$ are called the "nodes". Count how many nodes $\sin(n\pi x)$ has on the interval $[-1,1]$.

(ii) Given a function $f\colon [-1,1] \to \mathbb{R}$ find a formula for the coefficients a_n, b_n o the Fourier series (2) of $f(x)$. *Hints:* First verify the formulas

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} dx = \int_{-1}^{1} \cos(n\pi x) \cdot \cos(n\pi x) dx = \int_{-1}^{1} \sin(n\pi x) \cdot \sin(n\pi x) dx = 1$$

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot \cos(n\pi x) dx = \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot \sin(n\pi x) dx = \int_{-1}^{1} \cos(n\pi x) \cdot \sin(m\pi x) dx = 0$$

$$\int_{-1}^{1} \cos(n\pi x) \cdot \cos(m\pi x) dx = \int_{-1}^{1} \sin(n\pi x) \cdot \sin(m\pi x) dx = 0 \text{ for } n \neq m$$

What happens if you multiply (2) on both sides by say $\frac{1}{\sqrt{2}}$, or $\sin(2\pi x)$, or $\cos(\pi x)$ etc. and integrate over $[-1,1]$?

(iii) Consider $f(x) = x^2$ on the interval $[-1,1]$ and determine the Fourier series for this function.

(iv) How can you use the Fourier series for $f(x) = x^2$ to get the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

(v) How can you use the Fourier series for $f(x) = x^2$ to get the value of the alternating sum $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$?

6. Project: The beginnings of variational calculus: shortest paths

We know from experience that the shortest path between two points in the plane is the straight line segment connecting the two points. Is there a mathematical way to verify this using some sort of calculus?

Consider two fixed points $A \neq B$ in the plane $\mathbb{R}^2$. The idea is to consider all possible curves γ connecting those two points, calculate their lengths $L(\gamma)$, and find the curve of smallest length among them. Draw a picture of this situation. This may sound familiar to you from Calculus I, namely, finding the minimum of a function, only that this time the inputs for the function, the length function L, consists of curves γ connecting A with B (and not real numbers). So in some sense this is a Calculus I problem albeit on an "infinite dimensional" domain—people call this *variational calculus.*

How do we go about this? Let us assume that we have a curve γ connecting A with B which has smallest length among all curves connecting A and B. We want to show that γ is a straight line segment. Imagining the curve γ as some race track with start A and finish B, we surely can traverse it (by car, running, bicycling, etc.) with constant speed, say speed 1. In other words, we may assume that the parametrized curve

$$\gamma = (\gamma_1, \gamma_2)\colon [0, b] \to \mathbb{R}^2\,, \qquad \gamma(0) = A\,, \quad \gamma(b) = B$$

has speed $||\gamma'(t)|| = \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} = 1$ for all values of t. Since the length of γ is calculated by

$$L(\gamma) = \int_0^b ||\gamma'(t)||\, dt = \int_0^b 1\, dt = b\,,$$

we see that b is the length of the curve γ, and we have assumed that b is the smallest length among all curves connecting A with B.

Now comes the crucial idea of *variational calculus*: we wiggle our shortest curve γ around keeping the beginning and endpoints A and B fixed. Draw a picture of this "wiggling". Mathematically, we model this "wiggling" by a family of curves

$$\gamma_x = (\gamma_{x,1}, \gamma_{x,2})\colon [0, b] \to \mathbb{R}^2\,, \qquad \gamma_x(0) = A\,, \quad \gamma_x(b) = B$$

parametrized by real numbers x varying in some interval around $x = 0$ where $\gamma_0 = \gamma$ is our original shortest curve. So for each fixed x we have a curve γ_x which connects A to B and for $x = 0$ we get the original curve γ. Draw a picture of this situation.

(i) Show that the function $\lambda(x) := L(\gamma_x)$ has a minimum at $x = 0$.

(ii) Show that $\dot{\lambda}(0) = 0$. Here $\dot{(\,)}$ will denote differentiation w.r.t. x (as not to confuse it with differentiation w.r.t. t which we denote by $(\,)'$.

(iii) Show that $\dot{\gamma}_x(0) = 0$ and $\dot{\gamma}_x(b) = 0$ for any x.

(iv) Calculate $\dot{\lambda}(0)$ by differentiating

$$\lambda(x) = L(\gamma_x) = \int_0^b ||\gamma_x'(t)||\, dt = \int_0^b \sqrt{\gamma_{x,1}'(t)^2 + \gamma_{x,2}'(t)^2}\, dt$$

under the integral sign with respect to x. A word of caution: even though the original curve $\gamma = \gamma_0$ has speed 1, the curves γ_x for $x \neq 0$ will generally not have speed 1, so you cannot assume $||\gamma_x'(t)|| = 1$ for all t if $x \neq 0$!

Hints: You will have to use that one can interchange t and x derivatives (it does not matter in which order they are taken), integration by parts, and the fact that for all x the curves γ_x have the same beginning and end points A and B.

(v) The formula you should eventually arrive at is

$$\dot{\lambda}(0) = -\int_0^b \left(\dot{\gamma}_{x,1}(t)_{|x=0}\gamma_1''(t) + \dot{\gamma}_{x,2}(t)_{|x=0}\gamma_2''(t)\right)\, dt$$

Show that this implies $\gamma''(t) = 0$ for all t.

Hints: recall that the curves γ_x are obtained from $\gamma = \gamma_0$ by "wiggling" γ keeping the beginning and end points fixed. So any wiggling is allowed, which means that the functions $\dot{\gamma}_{x,1}(t)_{|x=0}$ and $\dot{\gamma}_{x,2}(t)_{|x=0}$ for $t \in [0, b]$ can be chosen arbitrarily subject only to the conditions in (iii). How can you use this? Think of the following problem: given a continuous function f on an interval $[a, b]$ such that $\int_a^b f(t)g(t)\, dt = 0$ for all continuous functions $g\colon [a, b] \to \mathbb{R}$ with $g(a) = g(b) = 0$, then $f(t) = 0$ for all t. Can you verify this? And then apply it to the case at hand?

(vi) Show that a curve γ satisfying $\gamma''(t) = 0$ for all t has to be a straight line. This means, that after all this work, we have verified that the curve of shortest length is indeed the straight line segment.

The approach outlined is rather universal and can be used to characterized the curves of shortest lengths on arbitrarily shaped objects: for example, on the surface of a round sphere, an ellipsoid, hyperboloid, or the surface of a doughnut, or the surface of an airplane wing etc. One can also do the same problem for curves in higher dimensional shapes. More generally, one can replace the length function by some other function, for instance, the energy of a particle traveling from A and B in some force field etc. and then calculate the path the particle takes in order to minimize the energy.