A GENERALIZATION OF THE BALA-CARTER THEOREM FOR NILPOTENT ORBITS

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Abstract. Let G be a connected simple algebraic group over the complex numbers with Lie algebra $\mathfrak{g}$. Let N be a nilpotent element in $\mathfrak{g}$ and let $Z_G(N)$ be the centralizer in G of N. When G is of adjoint type, we give a unified description of the conjugacy classes in the component group of $Z_G(N)$, generalizing the Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$. Our result turns out to be enough in all cases to determine the component group.

1. Introduction

Let G be a connected simple algebraic group defined over the complex numbers with Lie algebra $\mathfrak{g}$. Assume that G is of adjoint type. For a nilpotent element N in $\mathfrak{g}$, one often needs to compute the finite group $A(N) := Z_G(N)/Z_G^0(N)$, where $Z_G(N)$ is the centralizer of N in G under the adjoint action and $Z_G^0(N)$ is its identity component. For example, these groups play an essential role in the Springer correspondence [Sp1]. The computation of $A(N)$ appears in the literature [Al], [Mi], [S-S] and in three texts on the subject [Ca], [C-M], [Hu], but without a satisfactory unified approach.

In this paper, we generalize the Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$ in order to determine the groups $A(N)$. We thus briefly recall the Bala-Carter classification [B-C]. First, one enumerates the distinguished nilpotent orbits in any reductive Lie algebra $\mathfrak{l}$. These are the nilpotent orbits in $\mathfrak{l}$ which consist of elements whose Lie algebra centralizer in $\mathfrak{l}$ contains no non-central semisimple elements. For example, the regular nilpotent orbit is always distinguished and in $\mathfrak{sl}(n, \mathbf{C})$, this is the only distinguished nilpotent orbit. However, in other Lie algebras, there can be other distinguished orbits and their enumeration was carried out by Bala and Carter. The second step in the Bala-Carter classification is to attach to any nilpotent orbit in $\mathfrak{g}$ a distinguished nilpotent orbit in a Levi subalgebra of $\mathfrak{g}$ (up to G-conjugacy). More precisely, given a nilpotent element $N \in \mathfrak{g}$, there is a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ (unique up to $Z_G^0(N)$-conjugacy) such that N is distinguished in $\mathfrak{l}$. Associating $\mathfrak{l}$ to N, one obtains a bijection between nilpotent orbits in $\mathfrak{g}$ and G-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O})$, where $\mathcal{O}$ is a distinguished nilpotent orbit in the Levi subalgebra $\mathfrak{l}$. It is also important to note that for such a pair $(\mathfrak{l}, \mathcal{O})$, the G-orbit through $\mathcal{O}$ intersects $\mathfrak{l}$ precisely in $\mathcal{O}$. Hence, if we list the Levi subalgebras of $\mathfrak{g}$ up to G-conjugacy (these correspond to subsets of the simple roots, up to W-conjugacy) and we

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list the distinguished nilpotent orbits in each Levi subalgebra, we obtain a list of the nilpotent orbits in $\mathfrak{g}$.

In Section 3, we generalize the Bala-Carter classification. Here is a sketch of our result. First, we recall that for a semisimple element $x \in G$, the centralizer of x in $\mathfrak{g}$, denoted $Z_{\mathfrak{g}}(x)$, is not in general a Levi subalgebra, but rather corresponds, up to G-conjugacy, to a subset of the extended simple roots (that is, the usual simple roots together with the lowest root) in the same way that a Levi subalgebra corresponds, up to G-conjugacy, to a subset of the usual simple roots. We call a subalgebra of the form $Z_{\mathfrak{g}}(x)$ a pseudo-Levi subalgebra. Second, let C be a conjugacy class in $A(N)$. We prove that there exists a pseudo-Levi subalgebra $\mathfrak{l}$ in $\mathfrak{g}$ (unique up to $Z_G(N)$-conjugacy) with the property that N is distinguished in $\mathfrak{l}$ and any element x which generates the component group of the center of L (where L is the connected subgroup of G with Lie algebra $\mathfrak{l}$) is a representative of C in $Z_G(N)$. This turns out to yield a bijection between G-conjugacy classes of pairs (N, C), where N is a nilpotent element in $\mathfrak{g}$ and C is a conjugacy class in $A(N)$, and G-conjugacy classes of pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a pseudo-Levi subalgebra in $\mathfrak{g}$ and N is a distinguished nilpotent element in $\mathfrak{l}$. Our bijection also gives information about the order of elements in each conjugacy class in $A(N)$ and it turns out that this is sufficient in all cases to determine $A(N)$.

As an example, consider $\mathfrak{sp}_4(\mathbf{C})$. There are four conjugacy classes of Levi subalgebras in $\mathfrak{sp}_4(\mathbf{C})$, namely $0, A_1 = C_1, \tilde{A}_1, C_2$ (the tilde corresponds to the short simple root). Each of these Levi subalgebras has a single distinguished nilpotent orbit, namely the regular one. Hence, there are four nilpotent orbits in $\mathfrak{sp}_4(\mathbf{C})$. However, $\mathfrak{sp}_4(\mathbf{C})$ has a subalgebra of type $C_1 + C_1$ which is a pseudo-Levi subalgebra (but not a Levi subalgebra). The regular nilpotent orbit in $C_1 + C_1$ is the only distinguished orbit and any element in it is conjugate under G to a regular nilpotent element in $\tilde{A}_1$. Since there are no other pseudo-Levi subalgebras up to conjugacy, our result implies that $A(N)$ has 2 conjugacy classes when N is conjugate to a regular nilpotent element in $\tilde{A}_1$ (hence $A(N) \simeq \mathbb{Z}/2$) and has one otherwise (hence $A(N)$ is trivial).

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2. First Definitions

Let G be a connected simple algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$ and let $T \subset B \subset G$ be a maximal torus of dimension n contained in a Borel subgroup in G. Let $X^*(T)$ denote the characters of T, that is, the free abelian group of rank n of homomorphisms of algebraic groups $\chi : T \to \mathbf{C}^*$. We will often write $\hat{\mathcal{L}}$ for $X^*(T)$. We will use the next lemma (see [O-V, Theorem 3.2.5])

Lemma 1. *There is a bijection φ between the (algebraic) subgroups of T and the subgroups of $X^*(T)$, which to a subgroup $Y \subset X^*(T)$ assigns the subgroup*

$$T^Y = \{x \in T |\ \chi(x) = 1 \ \text{ for all } \ \chi \in Y\} \subset T.$$

The root system Φ of G is a subset of $X^*(T)$. The choice of Borel subgroup determines a set of positive roots Φ^+ and a set of simple roots $\Pi = \{\alpha_i\}$ in Φ. We label the simple roots as in [Dy, Table 1] except in E_8 the labels $1, 2, ..., 7$ are flipped with respect to Dynkin's notation about the vertical axis.

Any $\alpha \in \Phi^+$ can be expressed in $\hat{\mathcal{L}}$ as $\sum_{i=1}^n a_i\alpha_i$ where the a_i are non-negative integers. The height of α, denoted $\mathrm{ht}(\alpha)$, equals $\sum a_i$. Since G is simple, there exists a unique root $\theta \in \Phi^+$, called the highest root, for which $\mathrm{ht}(\theta) \geq \mathrm{ht}(\alpha)$ for all $\alpha \in \Phi$. Set $\alpha_0 = -\theta$ and let $\tilde{\Pi} = \Pi \cup \{\alpha_0\}$. Define the coefficients c_i of θ from the equation $\theta = \sum_{i=1}^n c_i\alpha_i$ and set $c_0 = 1$.

For any subset J of $\tilde{\Pi}$ (always assumed to be proper), let $\mathcal{L}_J$ be the lattice in $\hat{\mathcal{L}} = X^*(T)$ generated by J. Instead of $\mathcal{L}_\Pi$, we will write $\mathcal{L}$ for the lattice generated by Π (the root lattice). The intersection $\Phi \cap \mathcal{L}_J$ is an abstract root system, denoted by Φ_J. It is easy to check that J is a set of simple roots for Φ_J. For $J \subsetneq \tilde{\Pi}$, let d_J be the greatest common divisor of those c_i for which $\alpha_i \notin J$. Define $\tau_J \in \mathcal{L}$ to be

$$\tau_J := \frac{1}{d_J} \sum_{\alpha_i \in J} c_i\alpha_i = -\frac{1}{d_J} \sum_{\alpha_i \in \tilde{\Pi} - J} c_i\alpha_i. \qquad (1)$$

Then the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$ is isomorphic to $\mathbb{Z}/d_J$ and is generated by the image of τ_J.

Let $W = N_G(T)/T$ be the Weyl group of G. The Weyl group acts on T and hence also on $X^*(T)$. For $w \in W$ denote by $\dot{w} \in N_G(T)$ any representative of w. For $\alpha \in \Phi$, let s_α be the unique Weyl group element which acts as the identity on the kernel of α. For $J \subsetneq \tilde{\Pi}$, let $S_J = \{s_\alpha |\ \alpha \in J\}$ and let W_J be the subgroup of W generated by the elements in S_J. So W_J is the Weyl group of the root system Φ_J.

2.1. **Pseudo-Levi subalgebras.** Let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras of $T \subset B \subset G$. Denote the adjoint action of G on itself and on $\mathfrak{g}$ by Ad. By abuse of notation, we also view the characters of T as elements of $\mathfrak{t}^*$ (these are the differentials of the characters of T). In particular, Φ is also a subset of $\mathfrak{t}^*$. For $\alpha \in \Phi$, there exists an isomorphism x_α from $\mathbf{C}$ onto a unique closed subgroup U_α of G such that $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$ where $t \in T$ and $u \in \mathbf{C}$ [Sp2, 9.3.6]. Let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{t}\}.$$

The Lie algebra of U_α is $\mathfrak{g}_\alpha$.

For $S \subset G$ or $S \subset \mathfrak{g}$, we denote by $Z_G(S)$ the centralizer in G of S and $Z_\mathfrak{g}(S)$ the centralizer in $\mathfrak{g}$ of S, under the appropriate action. When S is a subset of T, the Lie algebra of $Z_G(S)$ is $Z_\mathfrak{g}(S)$ [Sp2, 4.4.7] and the identity component of $Z_G(S)$ is generated by T and those U_α with $\alpha(S) = 1$ [Ca, Theorem 3.5.3]. Also $Z_{\mathfrak{g}}(S)$ is spanned by $\mathfrak{t}$ and those $\mathfrak{g}_\alpha$ with $\alpha(S) = 1$.

For $J \subsetneq \tilde{\Pi}$, we define G_J to be the subgroup of G generated by T and those U_α with $\alpha \in \Phi_J$. Then G_J is a connected reductive algebraic subgroup of G with root system Φ_J. The Lie algebra of G_J is

$$\mathfrak{g}_J := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha.$$

Since G_J is reductive, the center Z of G_J is contained in T. Hence Z consists of the elements of T which lie in the kernel of all the roots in Φ_J. In other words, $Z = T^{\mathcal{L}_J}$. Thus by Lemma 1, the character group of Z is isomorphic to $\hat{\mathcal{L}}/\mathcal{L}_J$. It follows that the identity component Z^0 of Z is isomorphic to a torus whose dimension is the rank of $\hat{\mathcal{L}}/\mathcal{L}_J$ and the character group of Z/Z^0 is isomorphic to the torsion subgroup of $\hat{\mathcal{L}}/\mathcal{L}_J$. When G is of adjoint type (meaning $\hat{\mathcal{L}} = \mathcal{L}$), we have that Z/Z^0 is cyclic of order d_J.

When $J \subset \Pi$, then G_J is a Levi subgroup of G and $\mathfrak{g}_J$ is a Levi subalgebra of $\mathfrak{g}$. For lack of better terminology when $J \subsetneq \tilde{\Pi}$, we call any G-conjugate of G_J a pseudo-Levi subgroup of G and any G-conjugate of $\mathfrak{g}_J$ a pseudo-Levi subalgebra of $\mathfrak{g}$. We call G_J (resp. $\mathfrak{g}_J$) a standard pseudo-Levi subgroup (resp. subalgebra). A Levi subgroup is characterized by the fact that it is the centralizer of a torus in G (and so it is automatically connected). By choosing an element as regular as possible in the torus, the Levi subgroup is the connected centralizer of that element. More generally, we have the following proposition which is reasonably well-known. See [Lu, 5.5] for a proof.

Proposition 2. *Pseudo-Levi subgroups are the subgroups of G of the form $Z^0_G(x)$ where x is a semisimple element in G. Pseudo-Levi subalgebras are the subalgebras of $\mathfrak{g}$ of the form $Z_{\mathfrak{g}}(x)$ where x is a semisimple element in G.*

Remark 3. In fact, when G is of adjoint type, the center Z of $Z^0_G(x)$ has the property that Z/Z^0 is cyclic and the image of x generates Z/Z^0. Moreover, if S is a torus in $Z_G(x)$, then $Z^0_G(x, S)$ is a pseudo-Levi subgroup of G and the image of x still generates the component group of the center of $Z^0_G(x, S)$.

A standard argument implies that $\mathfrak{g}_J$ is G-conjugate to $\mathfrak{g}_{J'}$ for $J, J' \subsetneq \tilde{\Pi}$ if and only if J is W-conjugate to J'. Therefore, to understand the conjugacy classes of pseudo-Levi subgroups, we need to understand when two proper subsets of $\tilde{\Pi}$ are conjugate under W.

2.2. **Equivalence classes of subsets of $\tilde{\Pi}$.** Assume G is simply-connected in this subsection. We need a way to determine the equivalence classes of subsets of $\tilde{\Pi}$, where J is equivalent to J' if and only if $J = wJ'$ for some $w \in W$. These were first determined by Dynkin in [Dy].

Let $\Lambda = \hat{\mathcal{L}}/\mathcal{L}$ (the character lattice modulo the root lattice). For $J \subsetneq \tilde{\Pi}$, let $(w_0)_J$ be the longest element of the Weyl group W_J. We write w_0 instead of $(w_0)_\Pi$. Let $W_\Lambda = \{w \in W \mid w\tilde{\Pi} = \tilde{\Pi}\}$. The proof of the following proposition can be found in [I-M].

Proposition 4 (Iwahori-Matsumoto)**.** *The non-identity elements in W_Λ are of the form $w_0(w_0)_J$ where J is a maximal proper subset of Π and $c_i = 1$ for $\alpha_i \in \Pi - J$. Moreover W_Λ is isomorphic to Λ and acts simply-transitively on those $\alpha_i \in \tilde{\Pi}$ with $c_i = 1$.*

Now given $J \subsetneq \tilde{\Pi}$, we would like to find all subsets of $\tilde{\Pi}$ which are W-conjugate to J. By utilizing the elements in W_Λ both for $\tilde{\Pi}$ itself and for the extended Dynkin diagram associated to any proper subset of $\tilde{\Pi}$, we can quickly find many subsets of $\tilde{\Pi}$ which are equivalent to J. On the other hand, we can distinguish equivalences classes by the isomorphism type of Φ_J, the value of d_J, the torsion subgroup of $\hat{\mathcal{L}}/\mathcal{L}_J$, and the length of the roots in J since these are all invariants of the equivalence class of J.

For types A_n, C_n, G_2, F_4, E_6, we find that the equivalence class of J is determined by the isomorphism type of Φ_J and the length of the roots in J. Hence, to specify an equivalence class we need only give the type of Φ_J and specify the summands of Φ_J which contain short roots. Following convention, we place a tilde over any summand of Φ_J containing only short roots. For B_n the isomorphism type of Φ_J and the length of roots in J determine the equivalence class of J in $\tilde{\Pi}$ if we distinguish between D_2 and $2A_1$ and between D_3 and A_3 (the values of d_J distinguishes the equivalence classes here).

In E_7 the isomorphism type of Φ_J determines the equivalence class of J except when Φ_J is of type A_5, $A_5 + A_1$, $4A_1$, $3A_1$, $A_3 + 2A_1$, or $A_3 + A_1$. In these cases, there are two equivalence classes with isomorphic Φ_J. When J is conjugate to a subset of $A_7 \subset E_7$, either the torsion subgroup of $\hat{\mathcal{L}}/\mathcal{L}_J$ or the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$ is different than when J is not conjugate to a subset of A_7. Following Dynkin, we give the root system corresponding to the former situation one prime and the latter, two primes.

In E_8 the isomorphism type of Φ_J determines the equivalence class of J except when Φ_J is of type A_7, $A_5 + A_1$, $4A_1$, $2A_3$, or $A_3 + 2A_1$. In these cases, there are two equivalence classes with isomorphic Φ_J. To distinguish equivalence classes, we use the value of d_J. When J is conjugate to a subset of $A_8 \subset E_8$, it turns out that $d_J = 1$, but when J is not conjugate to a subset of A_8, then $d_J = 2$. We give the root system corresponding to the former situation one prime and the latter, two primes.

In D_n the isomorphism type of Φ_J determines the equivalence class of J unless $\Phi_J \simeq A_{i_1} + A_{i_2} + \cdots + A_{i_k}$ where all i_j are odd and $\sum(i_j + 1) = n$ (the very even case). As in type B_n, we are distinguishing between the following pairs: D_2 and $2A_1$; D_3 and A_3. In the very even cases, however, there exists two subsets J', J'' (up to conjugacy) which are not conjugate but for which $\Phi_{J'} \simeq \Phi_{J''}$. We can detect that J', J'' are not conjugate because the images in $\hat{\mathcal{L}}/\mathcal{L} \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ of the torsion subgroups of $\hat{\mathcal{L}}/\mathcal{L}_{J'}$ and $\hat{\mathcal{L}}/\mathcal{L}_{J''}$ are different.

3. Component Group of the Centralizer of a Nilpotent Element

For the remainder of the paper, G is of adjoint type. Let $N \in \mathfrak{g}$ be a nilpotent element. Let $Z_G(N)$ be the elements of G which centralize N under the adjoint action and let $Z_G^0(N)$ be the identity component of $Z_G(N)$. We want to understand the component group $A(N) := Z_G(N)/Z_G^0(N)$. In this section we prove a generalization of the Bala-Carter theorem for nilpotent orbits in $\mathfrak{g}$. The generalization will allow us to determine each $A(N)$.

3.1. **Review of results about nilpotent orbits.** To each nilpotent orbit $\mathcal{O}$ there is a weighted Dynkin diagram which completely determines the orbit; we now recall how to construct the weighted Dynkin diagram.

For $N \in \mathcal{O}$, the Jacobson-Morozov theorem implies the existence of $M, H \in \mathfrak{g}$ such that $[H, N] = 2N$, $[H, M] = -2M$, and $[N, M] = H$. So $\{N, H, M\}$ generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{C})$ and we call $\{N, H, M\}$ an $\mathfrak{sl}_2$-triple for N. Since H is semisimple in $\mathfrak{g}$, we can conjugate $\{N, H, M\}$ via an element of G so that $H \in \mathfrak{t}$. By $\mathfrak{sl}_2(\mathbf{C})$-theory, it follows that $\alpha_i(H) \in \mathbb{Z}$ for $\alpha_i \in \Pi$. Conjugating $\{N, H, M\}$ via an element of $N_G(T)$, we can assume that $\alpha_i(H) \geq 0$ for all $\alpha_i \in \Pi$. Assigning $\alpha_i(H)$ (which turns out to be $0, 1$, or 2) to the α_i-node ($1 \leq i \leq n$) of the Dynkin diagram yields the weighted Dynkin diagram associated to $\mathcal{O}$. We refer to H in the $\mathfrak{sl}_2$-triple $\{N, H, M\}$ as the neutral element.

Definition 5. A nilpotent element N in a reductive Lie algebra $\mathfrak{g}'$ is called **distinguished** if the conditions $X \in \mathfrak{g}'$ semisimple and $[X, N] = 0$ imply that X is in the center of $\mathfrak{g}'$. We also call a nilpotent orbit distinguished if any (hence all) of its elements are distinguished.

Note that N is distinguished in $\mathfrak{g}$ if and only if $Z_G^0(N)$ is unipotent. In type A_n, only the regular nilpotent orbit is distinguished. For other simple $\mathfrak{g}$, the list of distinguished nilpotent orbits can be found in [Ca].

Remark 6. We observe that distinguished nilpotent orbits in a simple Lie algebra $\mathfrak{g}$ are invariant under any automorphism of $\mathfrak{g}$ since their weighted Dynkin diagrams are unchanged under any automorphism of the Dynkin diagram. This fact continues to hold for the reductive Lie algebras $\mathfrak{g}_J$, except for the cases in C_n and D_n where there are two isomorphic simple components of $\mathfrak{g}_J$ of type C_k ($k \geq 2$) and D_k ($k \geq 4$), respectively. In these cases, there exists an element $w \in W_\Lambda$ which induces an automorphism of $\mathfrak{g}_J$ and interchanges the two isomorphic simple components that are not of type A. We see that this automorphism will conjugate a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_J$ to a different nilpotent orbit if and only if $\mathcal{O}$ intersects the two non-type A simple components in different nilpotent orbits.

We conclude that G-conjugacy classes of pairs $(\mathfrak{l}, N)$, where N is a distinguished nilpotent element in a pseudo-Levi subalgebra $\mathfrak{l}$, are in bijection with W-equivalence classes of pairs (J, J_{dynkin}), where $J \subsetneq \tilde{\Pi}$ and J_{dynkin} is the weighted Dynkin diagram of a distinguished nilpotent orbit.

3.2. **Generalization of the Bala-Carter Theory.** The Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$ states that the nilpotent orbits in $\mathfrak{g}$ are in bijection with pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and N is a distinguished nilpotent element in $\mathfrak{l}$ (up to simultaneous conjugation of both factors by G) [B-C].

We will extend the Bala-Carter classification in order to understand the conjugacy classes in $A(N)$. More precisely, we will establish a bijection between pairs (N, C), where N is a nilpotent element in $\mathfrak{g}$ and C is a conjugacy class in $A(N)$ (up to conjugation by G), and pairs $(\mathfrak{l}, N)$, where N is a distinguished nilpotent element in the pseudo-Levi subalgebra $\mathfrak{l}$ (up to simultaneous conjugation of both factors by G).

We will need a series of propositions to establish the bijection.

Proposition 7. *For any $J \subsetneq \tilde{\Pi}$, there exists $w \in W$ such that $w(J) = J$ and the action of w on $\mathcal{L}/\mathcal{L}_J$ generates the automorphism group of the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$.*

Proof. We may assume $d_J \geq 3$, for otherwise there are no non-trivial automorphisms of the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$ (of course we always have $d_J \leq 6$ since no coefficient of θ exceeds 6). Note $\alpha_0 \in J$ since $c_0 = 1$ and otherwise we would have $d_J = 1$. The isomorphism types of Φ_J for the cases that arise are: A_2 in G_2; $A_2 + \tilde{A_2}$, $A_3 + \tilde{A_1}$ in F_4; $3A_2$ in E_6; $2A_3 + A_1$, $A_5 + A_2$, $3A_2$ in E_7; $2A_4$, $A_5 + A_2 + A_1$, $A_1 + A_7$, $D_5 + A_3$, $2A_3 + A_1$, A_8, $E_6 + A_2$, $A_2 + A_5$, $3A_2 + A_1$, $3A_2$ in E_8.

First, we observe that in these cases the longest element w_0 takes J to $-J$. This is trivial in all exceptional groups except E_6 since w_0 is just multiplication by -1 on $\mathcal{L}$. In E_6, the action of w_0 on $\mathcal{L}$ is multiplication by -1 followed by interchanging α_1 with α_5 and α_2 with α_4. There is only one subset J in E_6 for which $d_J \geq 3$, namely $\tilde{\Pi} - \{\alpha_3\}$, and so indeed $w_0(J) = -J$. Since $(w_0)_J$ maps J to $-J$, it follows that $w = w_0(w_0)_J$ has the property that $w(J) = J$.

Let $\alpha_j \in J$ be such that $w(\alpha_j) = \alpha_0$ and let $J' = J - \{\alpha_0\}$.

Next, consider the action of w on τ_J, the generator of the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$. We have from (1)

$$d_J \tau_J = \sum_{\alpha_i \in J, i \neq j} c_i \alpha_i \; + c_j \alpha_j. \tag{2}$$

Now $w(\alpha_i) \in \mathcal{L}_{J'}$ for all $\alpha_i \in J$ with $i \neq j$. Therefore, applying w to both sides of (2) yields

$$w(d_J \tau_J) \equiv c_j \alpha_0 \;\text{ modulo }\; \mathcal{L}_{J'}.$$

On the other hand, from (1) we see that $\alpha_0 \equiv d_J \tau_J$ modulo $\mathcal{L}_{J'}$ and thus

$$w(d_J \tau_J) \equiv c_j d_J \tau_J \;\text{ modulo }\; \mathcal{L}_{J'}$$

which means

$$d_J(w(\tau_J) - c_j \tau_J) \in \mathcal{L}_{J'}.$$

But $\mathcal{L}/\mathcal{L}_{J'}$ is torsion free, which implies

$$w(\tau_J) - c_j \tau_J \in \mathcal{L}_{J'}.$$

Hence $w(\tau_J) \equiv c_j \tau_J$ modulo $\mathcal{L}_{J'}$ (and also modulo $\mathcal{L}_J \supset \mathcal{L}_{J'}$).

Now in each case we find (by inspection) that c_j is congruent to -1 modulo d_J. In other words, w is an automorphism of order 2 of the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$. Thus the only case left unresolved is the one in E_8 where $J = \tilde{\Pi} - \{\alpha_4\}$. Here $d_J = 5$ and the automorphism group is isomorphic to $\mathbb{Z}/4$. To handle this, we consider the following permutation σ of the elements in J

$$\alpha_0 \to \alpha_1 \to \alpha_5 \to \alpha_8 \to \alpha_0$$
$$\alpha_2 \to \alpha_6 \to \alpha_3 \to \alpha_7 \to \alpha_2$$

and extend σ linearly to the real span of $\mathcal{L}$. Then

$$\sigma(\alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8,$$

which is a root; hence σ preserves Φ. One checks that σ is actually an automorphism of Φ and consequently σ coincides with the action of an element $w \in W$ since all automorphisms of E_8 come from W. Finally, because $w(\alpha_8) = \alpha_0$ and $c_8 = 3$, we see that w generates the automorphism group of the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$. □

For $J \subsetneq \tilde{\Pi}$ consider the standard pseudo-Levi subgroup G_J. As we noted in Section 2.1, the center Z of G_J coincides with $T^{\mathcal{L}_J}$ and Z/Z^0 is cyclic of order d_J. Let x be an element of Z whose image generates Z/Z^0. Any such $x \in T$ is thus characterized by the fact that $\alpha(x) = 1$ for $\alpha \in J$ and $\tau_J(x)$ is a primitive d_J-th root of unity. Let N be a distinguished nilpotent element in $\mathfrak{g}_J$. Clearly, $x \in Z_G(N)$ which means that the image of x defines an element in $A(N)$. Although there are many choices for x, we have the following proposition.

Proposition 8. *The image of x in $A(N)$ is well-defined up to conjugacy in $A(N)$.*

Proof. Suppose $x_1, x_2 \in Z$ both generate Z/Z^0. Then $x_2 \equiv x_1^l$ modulo Z^0 for some l prime to d_J. By Proposition 7 there exists $w^{-1} \in W$ such that $w^{-1}(J) = J$ and w^{-1} acts on the torsion subgroup of $\mathcal{L}/\mathcal{L}_J$ by multiplying each element by l. We thus have $\mathrm{Ad}(\dot{w})x_1 \equiv x_1^l \equiv x_2$ modulo Z^0.

In addition $\mathrm{Ad}(\dot{w})$ is an automorphism of $\mathfrak{g}_J$. By Remark 6, the distinguished nilpotent orbits in $\mathfrak{g}_J$ for $\mathfrak{g}$ exceptional (which is the case we are considering) are invariant under any automorphism of $\mathfrak{g}_J$. Hence N and $\mathrm{Ad}(\dot{w})N$ are in the same nilpotent orbit in $\mathfrak{g}_J$ and so there exists $g \in G_J$ such that $\mathrm{Ad}(g\dot{w})N = N$, i.e. $g\dot{w} \in Z_G(N)$. Since x_2 is in the center of G_J and $g \in G_J$, we have $\mathrm{Ad}(g\dot{w})x_1 \equiv x_2$ modulo Z^0. But $Z^0 \subset Z_G^0(N)$, thus x_1 and x_2 are conjugate in $A(N)$. □

Proposition 9. *Let K be a reductive algebraic subgroup of G. Let x, y be two semisimple elements in K whose images in K/K^0 are in the same conjugacy class. Let S be a maximal torus in the reductive group $Z_K^0(x)$.*

Then for some $g \in K$ and $s \in S$, we have $gyg^{-1} = xs$. In particular, $Z_{\mathfrak{g}}(x, S) \subset Ad(g)Z_{\mathfrak{g}}(y)$.

Proof. This is a result about the semisimple automorphisms of the connected reductive algebraic group K^0 and can be found in [O-V, Chapter 4.4]. □

Let $N \in \mathfrak{g}$ be a nilpotent element and let $\mathfrak{m} = \langle N, H, M \rangle$ be the span of an $\mathfrak{sl}_2$-triple for N. The centralizer $Z_G(\mathfrak{m})$ of $\mathfrak{m}$ in G is reductive and there is a decomposition $Z_G(N) = Z_G(\mathfrak{m})U_N$ where U_N is the unipotent radical of $Z_G(N)$. Moreover, the natural map from $Z_G(\mathfrak{m})/Z_G^0(\mathfrak{m})$ to $A(N)$ is an isomorphism since U_N is connected [B-V].

Definition 10. Let L be a pseudo-Levi subgroup with center Z and Lie algebra $\mathfrak{l}$. Given a conjugacy class C in $A(N)$, we say that $\mathfrak{l}$ has the **key property** for (N, C) if $N \in \mathfrak{l}$ and there exists $x \in Z$ such that

1. The image of x in Z/Z^0 generates Z/Z^0.
2. The image of x in $A(N)$ belongs to C.

Given a pair (N, C) as in the definition above, we will now locate a pseudo-Levi subalgebra $\mathfrak{l}$ with the key property for (N, C). Let $x \in Z_G(\mathfrak{m})$ represent an element in the conjugacy class C in $A(N)$. Let $x = x_s x_u$ be the Jordan decomposition of x in $Z_G(\mathfrak{m})$. Since x_u is unipotent, we have $x_u \in Z_G^0(\mathfrak{m})$ which means that the image of x_s in $A(N)$ coincides with the image of x. In other words, we can assume x is semisimple.

Let $K = Z_G(\mathfrak{m})$. Let S be a maximal torus in $Z_K^0(x)$. Then by Remark 3

$$\mathfrak{l} := Z_{\mathfrak{g}}(x, S) \tag{3}$$

is a pseudo-Levi subalgebra of $\mathfrak{g}$ containing N with corresponding group $L = Z_G^0(x, S)$ and x generates the component group of the center of L. Hence, $\mathfrak{l}$ has the key property for (N, C).

Proposition 11. *The subalgebra $\mathfrak{l}$ in* (3) *is minimal among the pseudo-Levi subalgebras with the key property for (N, C). Moreover, any other minimal pseudo-Levi subalgebra with the key property for (N, C) is conjugate to $\mathfrak{l}$ by an element in $Z_G(N)$.*

Proof. Suppose $\mathfrak{l}'$ is another pseudo-Levi subalgebra with the key property for (N, C) and L' its corresponding (connected) group. Then there exists x' in the center of L' whose image generates the component group of the center of L' and whose image in $A(N)$ belongs to C. Multiplying x' by an appropriate element in the identity component of the center of L', we can assume that $\mathfrak{l}' = Z_{\mathfrak{g}}(x')$.

Let $\mathfrak{m}' = \langle N, H', M' \rangle$ be the span of an $\mathfrak{sl}_2$-triple for N in $\mathfrak{l}'$. Clearly, x' centralizes $\mathfrak{m}'$. By Kostant's theorem there exists $g \in Z_G^0(N)$ such that $\mathrm{Ad}(g)(\mathfrak{m}') = \mathfrak{m}$. Conjugating $\mathfrak{l}', L', x'$ by g, we can assume that x' and x are semisimple elements in $Z_G(\mathfrak{m})$ and they represent the same conjugacy class in the component group of $Z_G(\mathfrak{m})$. Applying Proposition 9, there exists $g \in Z_G(\mathfrak{m})$ such that $Z_{\mathfrak{g}}(x, S) = \mathfrak{l}$ is contained in $\mathrm{Ad}(g)Z_{\mathfrak{g}}(x') = \mathrm{Ad}(g)\mathfrak{l}'$.

Now if $\mathfrak{l}'$ is also a minimal pseudo-Levi subalgebra with the key property for (N, C), then $\mathfrak{l} = \mathrm{Ad}(g)\mathfrak{l}'$. □

Proposition 12. *Let $\mathfrak{l} = Z_{\mathfrak{g}}(x)$ be a pseudo-Levi subalgebra with the key property for (N, C). Then $\mathfrak{l}$ is minimal for the key property if and only if N is distinguished in $\mathfrak{l}$.*

Proof. Let S be a maximal torus in $Z_G(x, N)$ and set $\mathfrak{l}' = Z_{\mathfrak{g}}(x, S)$. By Proposition 11, $\mathfrak{l}'$ is minimal for the key property for (N, C). Hence $\mathfrak{l}$ is minimal if and only if $\mathfrak{l} = \mathfrak{l}'$. This will occur if and only if $\mathfrak{s}$, the Lie algebra of S, lies in the center of $\mathfrak{l}$. Finally, $\mathfrak{s}$ belongs to the center of $\mathfrak{l}$ if and only if N is distinguished is $\mathfrak{l}$. □

With these results we can extend the Bala-Carter bijection.

Theorem 13. *There is a bijection ϕ between G-conjugacy classes of pairs $(\mathfrak{l}, N)$, where $\mathfrak{l}$ is a pseudo-Levi subalgebra in $\mathfrak{g}$ and N is a distinguished nilpotent element in $\mathfrak{l}$, and G-conjugacy classes of pairs (N, C), where N is a nilpotent element in $\mathfrak{g}$ and C is a conjugacy class in $A(N)$.*

Proof. Given the pair $(\mathfrak{l}, N)$, let x be any element in the center Z of L which generates Z/Z^0. Then ϕ maps $(\mathfrak{l}, N)$ to (N, C) where C is the conjugacy class of the image of x in $A(N)$. This is well-defined by Proposition 8.

Proof of surjectivity: the construction preceding Proposition 11, together with Propositions 11 and 12 give surjectivity.

Proof of injectivity: suppose $\phi((\mathfrak{l}, N)) = \phi((\mathfrak{l}', N'))$. Then there exists $g_1 \in G$ such that $\mathrm{Ad}(g_1)N' = N$. Now both $\mathrm{Ad}(g_1)\mathfrak{l}'$ and $\mathfrak{l}$ have the key property for the pair $\phi((\mathfrak{l}, N))$. Since N is distinguished in both of these subalgebras, they must both be minimal for the key property by Proposition 12. Hence by Proposition 11 there exists $g_2 \in Z_G(N)$ such that $\mathrm{Ad}(g_2 g_1)\mathfrak{l}' = \mathfrak{l}$. Since $\mathrm{Ad}(g_2 g_1)N' = N$, that completes the proof of injectivity. □

Consider the trivial conjugacy class C in $A(N)$. Let $\mathfrak{l}$ be a minimal Levi subalgebra containing N with corresponding group L. Since the center of L is connected, it follows that $\phi((\mathfrak{l}, N)) = (N, C)$ under the bijection. We call such a pair $(\mathfrak{l}, N)$ a Bala-Carter pair. The fact that the trivial conjugacy class C is always represented by a Levi subalgebra leads to two easy corollaries of the theorem.

Corollary 14. *Any $\mathfrak{g}_J$ with $d_J = 1$ is a Levi subalgebra of $\mathfrak{g}$.*

Corollary 15. *Any pair $(\mathfrak{g}_J, N)$ with $d_J > 1$ gets mapped under ϕ to a non-trivial conjugacy class in $A(N)$.*

Now suppose $\phi((\mathfrak{g}_J, N))$ equals (N, C) for some non-trivial conjugacy class C in $A(N)$. What can we say about the order of an element in C? By construction, we represented C by an element x in the center Z of G_J such that the image of x generates the cyclic group $Z/Z^0 \simeq \mathbb{Z}/d_J$. Therefore the image of x in $A(N)$ has order different from 1 and dividing d_J. If d_J is prime or N is distinguished in $\mathfrak{g}$, then elements in C have order exactly d_J.

3.3. **Computing the bijection.** Given a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, we would like to find all pairs $(\mathfrak{l}, N)$ up to G-conjugacy such that $N \in \mathcal{O}$ and N is a distinguished nilpotent element in the pseudo-Levi subalgebra $\mathfrak{l}$. Then for any $N \in \mathcal{O}$, we would know the number of conjugacy classes of $A(N)$ and some information about the order of elements in each conjugacy class.

Let Ψ denote the map from pairs $(\mathfrak{l}, N)$ appearing in the bijection (up to conjugacy) to nilpotent elements in $\mathfrak{g}$ (up to conjugacy) given by $\Psi((\mathfrak{l}, N)) = N$. To compute the fiber of Ψ above any element in the orbit $\mathcal{O}$, we only need to find all pairs $(\mathfrak{g}_J, N)$ up to conjugacy such that N is distinguished in $\mathfrak{g}_J$ and $N \in \mathcal{O}$. Moreover by Remark 6 the conjugacy classes of these pairs is in bijection with pairs (J, J_{dynkin}) up to equivalence under W, where J_{dynkin} is the weighted Dynkin diagram of N in $\mathfrak{g}_J$.

We now present an algorithm for computing Ψ. For each pseudo-Levi subalgebra $\mathfrak{g}_J$, we make a list of the weighted Dynkin diagrams of the distinguished nilpotent orbits in $\mathfrak{g}_J$ by looking at tables in [Ca]. Fix such a weighted Dynkin diagram for the distinguished nilpotent element $N \in \mathfrak{g}_J$. This tells us the values of $\alpha_i(H)$ for $\alpha_i \in J$, where $H \in \mathfrak{t}$ is the neutral element of an $\mathfrak{sl}_2$-triple for some conjugate of N. Moreover, H belongs to the semisimple part of $\mathfrak{g}_J$ and this fact uniquely determines the values of $\alpha_i(H)$ for all $\alpha_i \in \Pi$. Then we locate $w \in W$ such that $\alpha_i(\mathrm{Ad}(\dot{w})H) \geq 0$ for all $\alpha_i \in \Pi$. These positive integers yield the weighted Dynkin diagram of the nilpotent orbit in $\mathfrak{g}$ through N.

In fact, the location of w is not difficult. If $\alpha_i(H) < 0$ for some $\alpha_i \in \Pi$, then the number of positive roots α such that $\alpha(\mathrm{Ad}(\dot{s}_i)H) < 0$ is one less than the number with $\alpha(H) < 0$. We also have $\alpha_j(\mathrm{Ad}(\dot{s}_i)H) = s_i(\alpha_j)(H)$ for $\alpha_j \in \Pi$. The integers $\alpha_j(\mathrm{Ad}(\dot{s}_i)H)$ for $\alpha_j \in \Pi$ yield a new labeled diagram and we continue applying simple reflections in this manner until all nodes of the diagram are non-negative, arriving at the desired weighted Dynkin diagram.

3.4. **Exceptional groups.** We carried out the algorithm for the exceptional groups. We use the notation S_m for the symmetric group on m letters and Bala and Carter's notation for a distinguished nilpotent orbit in a semisimple Lie algebra. There are five cases that occur:

1. The fiber of Ψ consists of one element, namely a distinguished nilpotent element in a Levi subalgebra (the Bala-Carter pair). Thus $A(N)$ is trivial.
2. The fiber of Ψ consists of two elements. In addition to the Bala-Carter pair, there is another pair which contributes a conjugacy class to $A(N)$. Thus $A(N) \simeq S_2$ since $A(N)$ has only two conjugacy classes. We find that when $\mathfrak{g}_J$ is of type $2A_3 + A_1$ in E_7 or of type $2A_3 + A_1$, $A_1 + A_7$, or $D_5 + A_3$ in E_8, the regular nilpotent orbit in $\mathfrak{g}_J$ gives rise to an element of order 2 in $A(N)$ even though $d_J = 4$. These are the only three cases when a pair $(\mathfrak{g}_J, N)$ gives rise to an element in $A(N)$ whose order is different from d_J.
3. The fiber has three elements. In addition to the Bala-Carter pair, there is one conjugacy class with elements of order 2 and another with elements of order 3. It is not hard to see that $A(N) \simeq S_3$.
4. The distinguished nilpotent orbit $F_4(a_3)$ in F_4. Since $F_4(a_3)$ is distinguished, the conjugacy class in $A(N)$ corresponding to a pair $(\mathfrak{g}_J, N)$ contains elements of order exactly d_J. We find that there are four non-trivial conjugacy classes in $A(N)$ corresponding to $A_3 + \tilde{A}_1$, $A_2 + \tilde{A}_2$, $B_4(a_2)$, and $A_1 + C_3(a_1)$ consisting of elements of order 4, 3, 2, and 2, respectively.

There is a unique finite group with this characterization, namely S_4 (we omit the proof).

It remains to determine which of $B_4(a_2)$ or $A_1 + C_3(a_1)$ corresponds to a transposition in S_4 and which corresponds to a product of commuting transpositions. In other words, does the square of an element in the conjugacy class $A_3 + \tilde{A}_1$ belong to $B_4(a_2)$ or $A_1 + C_3(a_1)$. By the proof of Theorem 13, this amounts to checking whether $A_3 + \tilde{A}_1$ is a root subsystem of B_4 or $A_1 + C_3$. Indeed, $A_3 + \tilde{A}_1$ is a root subsystem of B_4, but not of $A_1 + C_3$. Hence, $B_4(a_2)$ corresponds to a product of commuting transpositions in S_4.

5. The distinguished nilpotent orbit $E_8(a_7)$ in E_8. There are six non-trivial conjugacy classes in $A(N)$ corresponding to $A_5+A_2+A_1$, $2A_4$, $D_5(a_1)+A_3$, $E_6(a_3) + A_2$, $D_8(a_5)$, and $E_7(a_5) + A_1$ consisting of elements of order 6, 5, 4, 3, 2, and 2, respectively. There is a unique finite group with this characterization, namely S_5.

   Since $A_5 + A_2 + A_1$ is a root subsystem of $E_7 + A_1$ (and not of D_8), it follows that $E_7(a_5) + A_1$ corresponds to (12) in S_5. And since $D_5 + A_3$ is a root subsystem of D_8 (and not of $E_7 + A_1$), it follows that $D_8(a_5)$ corresponds to (12)(34) in S_5.

The results for the bijection in the exceptional groups are listed in the tables in Chapter 4.

3.5. **Classical groups.** In type A_n, d_J always equals one and hence all $A(N)$ are trivial.

For the other classical groups, $d_J = 1$ or 2 and so every (non-identity) element in $A(N)$ has order two and therefore $A(N)$ must be an elementary abelian 2-group. Using the usual description of the classical groups and their nilpotent orbits, we will now describe the fiber of the map Ψ above a nilpotent element N.

Let $\varepsilon = 0, 1$. All congruences are modulo 2.

Consider a complex vector space V of dimension m (m is even if $\varepsilon = 1$) with basis $e_1, e_2, \ldots, e_m$ and an inner product $(\, , \,)$ satisfying $(e_i, e_j) = 0$ if $i + j \neq m + 1$ and $(e_i, e_{m+1-i}) = (-1)^{\varepsilon}(e_{m+1-i}, e_i) = 1$ if $1 \leq i \leq \lceil m/2 \rceil$.

Let G_1 be the identity component of the subgroup of $GL(V)$ which preserves $(\, , \,)$ and let $G = G_{ad}$ be the quotient of G_1 by its center. Their Lie algebra $\mathfrak{g}$ consists of the elements $X \in \mathfrak{gl}(V)$ for which

$$(X.v_1, v_2) + (v_1, X.v_2) = 0 \text{ for all } v_1, v_2 \in V.$$

We choose $\mathfrak{t}$ to be the diagonal matrices in $\mathfrak{g} \subset \mathfrak{gl}(V)$ and $\mathfrak{b}$ to be the upper triangular matrices in $\mathfrak{g} \subset \mathfrak{gl}(V)$. The rank of $\mathfrak{g}$ is $n = \lfloor m/2 \rfloor$. We now describe the simple components of the pseudo-Levi subalgebras in $\mathfrak{g}$ containing $\mathfrak{t}$.

For $1 \leq k < l \leq m/2$, let V_1 be the subspace of V spanned by $e_k, e_{k+1}, \ldots, e_l$ and V_2 be the subspace of V spanned by $e_{m+1-k}, e_{m-k}, \ldots, e_{m+1-l}$. Then

$$\{X \in \mathfrak{g} | \; X.V_1 \subset V_1, X.V_2 \subset V_2, \text{ and } X.e_i = 0 \text{ for all } e_i \notin V_1 + V_2\}$$

is isomorphic to $\mathfrak{gl}_{l-k+1}$ and it is a summand of $\mathfrak{g}_J$ where $J = \{\alpha_k, \ldots, \alpha_{l-1}\}$.

On the other hand, for $k = 1$ and $1 \leq l \leq m/2$, we have

$$\{X \in \mathfrak{g} |\ X.(V_1 + V_2) \subset V_1 + V_2, \text{ and } X.e_i = 0 \text{ for all } e_i \notin V_1 + V_2\}$$

is isomorphic to $\mathfrak{so}_{2l}$ if $\varepsilon = 0$ and is isomorphic to $\mathfrak{sp}_{2l}$ if $\varepsilon = 1$. This is a summand of $\mathfrak{g}_J$ where $J = \{\alpha_0, \ldots, \alpha_{l-1}\}$ (we assume $l > 1$ if $\varepsilon = 0$).

Finally, if $l = \lceil m/2 \rceil$ and $1 \leq k \leq l$, then

$$\{X \in \mathfrak{g} |\ X.(V_1 + V_2) \subset V_1 + V_2, \text{ and } X.e_i = 0 \text{ for all } e_i \notin V_1 + V_2\}$$

is isomorphic to $\mathfrak{so}_{m-2k+2}$ if $\varepsilon = 0$ and is isomorphic to $\mathfrak{sp}_{m-2k+2}$ if $\varepsilon = 1$. This is a summand of $\mathfrak{g}_J$ where $J = \{\alpha_k, \ldots, \alpha_n\}$ (we assume $k < l$ if $\varepsilon = 0$).

Each nilpotent element $N \in \mathfrak{g}$ has a Jordan normal form in $\mathfrak{gl}(V)$ and so we can associate to N a partition $[p_1 \geq p_2 \cdots \geq p_k]$ of m, abbreviated $[p_j]$. Let $\mu(i)$ be the number of times i appears in the partition. The only partitions which actually arise are the ones where $i \equiv \varepsilon$ implies $\mu(i)$ is even. The partition completely determines the nilpotent orbit in $\mathfrak{g}$ except for the very even orbits in D_n, n even, where all the parts in the partition are even (see Section 2.2).

The distinguished nilpotent orbits in $\mathfrak{g}$ correspond to partitions where $i \equiv \varepsilon$ implies $\mu(i) = 0$ and $i \not\equiv \varepsilon$ implies $\mu(i) = 1$.

Given a distinguished nilpotent element in a pseudo-Levi subalgebra $\mathfrak{g}_J$, corresponding to a partition in each simple component, which partition does it correspond to in $\mathfrak{g}$? The answer is clear from the above description of standard pseudo-Levi subalgebras in $\mathfrak{g}$: the parts in the partition for each $\mathfrak{gl}$ are doubled and taken together with the parts in the two simple components not of type A. These yield a partition of m after we tack on the appropriate number of 1's. This partition automatically satisfies the condition on parts imposed by $\mathfrak{g}$.

Conversely, given the partition $[p_j]$ for a nilpotent element $N \in \mathfrak{g}$, we will now describe the fiber of the map Ψ above N.

Define the following sets which depend on the partition $[p_j]$ and ε:

$$S_{odd} = \{i \in \mathbb{N} |\ i \not\equiv \varepsilon,\ \mu(i) \equiv 1\}$$
$$S_{even} = \{i \in \mathbb{N} |\ i \not\equiv \varepsilon,\ \mu(i) \equiv 0\}.$$

Let $s = \sum_{i \in S_{odd}} i$.

Choose $T_1 \subset S_{odd}$ and $T_2 \subset S_{even}$. Let $t_1 = \sum_{i \in T_1} i$ and $t_2 = \sum_{i \in T_2} i$. Define $a(i)$ as follows

$$a(i) = \begin{cases} \frac{\mu(i)}{2} & \text{if } i \equiv \varepsilon \text{ or } i \in S_{even} - T_2 \\ \frac{\mu(i)-1}{2} & \text{if } i \in S_{odd} \\ \frac{\mu(i)}{2} - 1 & \text{if } i \in T_2 \end{cases}$$

Type C_n

Here s, t_1, t_2 are automatically even. To the pair (T_1, T_2) we can associate the standard pseudo-Levi subalgebra with simple components $\mathfrak{sp}_{t_1+t_2}$, $\mathfrak{sp}_{t_2+s-t_1}$, and $a(i)$ copies of type A_{i-1} for each $i \in \mathbb{N}$. Choose a regular nilpotent element in each A_{i-1} component. Choose a distinguished nilpotent element in $\mathfrak{sp}_{t_1+t_2}$ corresponding to the partition whose parts are the elements of the set $T_1 \cup T_2$

and choose a distinguished nilpotent element in $\mathfrak{sp}_{t_2+s-t_1}$ corresponding to the partition whose parts are the elements of the set $T_2 \cup (S_{odd} - T_1)$.

For all such pairs (T_1, T_2), we obtain all the pairs $(\mathfrak{g}_J, N')$ in the fiber of Ψ above N. Note that interchanging T_1 and $S_{odd} - T_1$ yields conjugate pairs. This is the phenomenon discussed in Remark 6.

Type B_n

Here s is automatically odd. We require that $t_1 \equiv t_2$. To the pair (T_1, T_2) we can associate the standard pseudo-Levi subalgebra with simple components $\mathfrak{so}_{t_1+t_2}, \mathfrak{so}_{t_2+s-t_1}$ and $a(i)$ copies of type A_{i-1} for each $i \in \mathbb{N}$. Choose a regular nilpotent element in each A_{i-1}. Choose a distinguished nilpotent element in $\mathfrak{so}_{t_1+t_2}$ corresponding to the partition whose parts are the elements of the set $T_1 \cup T_2$ and choose a distinguished nilpotent element in $\mathfrak{so}_{t_2+s-t_1}$ corresponding to the partition whose parts are the elements of the set $T_2 \cup (S_{odd} - T_1)$.

For all such pairs (T_1, T_2), we obtain all the pairs $(\mathfrak{g}_J, N')$ in the fiber of Ψ above N.

Type D_n

Here s is automatically even. We require that $t_1 \equiv t_2$. To the pair (T_1, T_2) we can associate the standard pseudo-Levi subalgebra with simple components $\mathfrak{so}_{t_1+t_2}, \mathfrak{so}_{t_2+s-t_1}$ and $a(i)$ copies of type A_{i-1} for each $i \in \mathbb{N}$. Choose a regular nilpotent element in each A_{i-1}. Choose a distinguished nilpotent element in $\mathfrak{so}_{t_1+t_2}$ corresponding to the partition whose parts are the elements of the set $T_1 \cup T_2$ and choose a distinguished nilpotent element in $\mathfrak{so}_{t_2+s-t_1}$ corresponding to the partition whose parts are the elements of the set $T_2 \cup (S_{odd} - T_1)$.

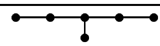
For all such pairs (T_1, T_2), we obtain all the pairs $(\mathfrak{g}_J, N')$ in the fiber of Ψ above N. As in type C_n, interchanging T_1 and $S_{odd} - T_1$ yields conjugate pairs. Note that when N is very even, only a Bala-Carter pair maps to N under Ψ.

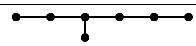
4. Tables

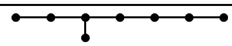
In the following tables, we show explicitly the bijection of Section 3.4 for the exceptional groups. We have listed only those N with non-trivial $A(N)$. The notation for a distinguished nilpotent element in a semisimple Lie algebra follows Bala and Carter.

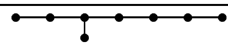
G_2			
	$A(N)$	$(\mathfrak{l}, N)$	Class in $A(N)$
2 0	S_3	$G_2(a_1)$ A_2 $A_1 + \tilde{A}_1$	1 (123) (12)

F_4			
	$A(N)$	$(\mathfrak{l}, N)$	Class in $A(N)$
0 0 0 1	S_2	$\tilde{A}_1$ $2A_1$	1 (12)
2 0 0 0	S_2	A_2 $2A_1 + \tilde{A}_1$	1 (12)
2 0 0 1	S_2	B_2 A_3	1 (12)
1 0 1 0	S_2	$C_3(a_1)$ $A_1 + B_2$	1 (12)
0 2 0 0	S_4	$F_4(a_3)$ $A_3 + \tilde{A}_1$ $A_2 + \tilde{A}_2$ $B_4(a_2)$ $A_1 + C_3(a_1)$	1 (1234) (123) (12)(34) (12)
0 2 0 2	S_2	$F_4(a_2)$ $A_1 + C_3$	1 (12)
2 2 0 2	S_2	$F_4(a_1)$ B_4	1 (12)

E_6			
	$A(N)$	$(\mathfrak{l}, N)$	Class in $A(N)$
0 0 0 0 0 2	S_2	A_2 $4A_1$	1 (12)
0 0 2 0 0 0	S_3	$D_4(a_1)$ $3A_2$ $A_3 + 2A_1$	1 (123) (12)
2 0 2 0 2 0	S_2	$E_6(a_3)$ $A_5 + A_1$	1 (12)

E_7			
	$A(N)$	$(\mathfrak{l}, N)$	Class in $A(N)$
2 0 0 0 0 0 0	S_2	A_2 $(4A_1)'$	1 (12)
1 0 0 0 1 0 0	S_2	$A_2 + A_1$ $5A_1$	1 (12)
0 2 0 0 0 0 0	S_3	$D_4(a_1)$ $3A_2$ $(A_3 + 2A_1)'$	1 (123) (12)
0 1 0 0 0 1 1	S_2	$D_4(a_1) + A_1$ $A_3 + 3A_1$	1 (12)
0 0 1 0 1 0 0	S_2	$A_3 + A_2$ $D_4(a_1) + 2A_1$	1 (12)
2 0 0 0 2 0 0	S_2	A_4 $2A_3$	1 (12)
1 0 1 0 1 0 0	S_2	$A_4 + A_1$ $A_1 + 2A_3$	1 (12)
2 0 1 0 1 0 0	S_2	$D_5(a_1)$ $D_4 + 2A_1$	1 (12)
0 2 0 0 2 0 0	S_2	$E_6(a_3)$ $(A_5 + A_1)'$	1 (12)
0 0 2 0 0 2 0	S_3	$E_7(a_5)$ $A_5 + A_2$ $A_1 + D_6(a_2)$	1 (123) (12)
2 0 2 0 0 2 0	S_2	$E_7(a_4)$ $A_1 + D_6(a_1)$	1 (12)
2 0 2 0 2 0 0	S_2	$E_6(a_1)$ A_7	1 (12)
2 0 2 0 2 2 0	S_2	$E_7(a_3)$ $A_1 + D_6$	1 (12)

E_8			
	$A(N)$	$(\mathfrak{l}, N)$	Class in $A(N)$
0 0 0 0 0 0 2 0	S_2	A_2 $(4A_1)''$	1 (12)
1 0 0 0 0 0 1 0	S_2	$A_2 + A_1$ $5A_1$	1 (12)
2 0 0 0 0 0 0 0	S_2	$2A_2$ $A_2 + 4A_1$	1 (12)
0 0 0 0 0 2 0 0	S_3	$D_4(a_1)$ $3A_2$ $(A_3 + 2A_1)''$	1 (123) (12)
0 0 0 0 0 1 0 1	S_3	$D_4(a_1) + A_1$ $3A_2 + A_1$ $A_3 + 3A_1$	1 (123) (12)
1 0 0 0 1 0 0 0	S_2	$A_3 + A_2$ $D_4(a_1) + 2A_1$	1 (12)
2 0 0 0 0 0 2 0	S_2	A_4 $(2A_3)''$	1 (12)
0 0 0 0 0 0 0 2	S_2	$D_4(a_1) + A_2$ $A_3 + A_2 + 2A_1$	1 (12)
1 0 0 0 1 0 1 0	S_2	$A_4 + A_1$ $A_1 + 2A_3$	1 (12)
1 0 0 0 1 0 2 0	S_2	$D_5(a_1)$ $D_4 + 2A_1$	1 (12)
0 0 1 0 0 0 1 0	S_2	$A_4 + 2A_1$ $D_4(a_1) + A_3$	1 (12)
0 0 0 0 0 0 2 2	S_2	$D_4 + A_2$ $D_5(a_1) + 2A_1$	1 (12)
2 0 0 0 0 2 0 0	S_2	$E_6(a_3)$ $(A_5 + A_1)''$	1 (12)
0 1 0 0 0 1 0 1	S_2	$D_6(a_2)$ $D_4 + A_3$	1 (12)
1 0 0 1 0 1 0 0	S_2	$E_6(a_3) + A_1$ $A_5 + 2A_1$	1 (12)
0 0 1 0 1 0 0 0	S_3	$E_7(a_5)$ $A_5 + A_2$ $A_1 + D_6(a_2)$	1 (123) (12)

E_8			
	$A(N)$	$(\mathfrak{l}, N)$	Class in $A(N)$
0 0 0 2 0 0 0 0	S_5	$E_8(a_7)$ $A_5 + A_2 + A_1$ $2A_4$ $D_5(a_1) + A_3$ $D_8(a_5)$ $E_7(a_5) + A_1$ $E_6(a_3) + A_2$	1 (123)(45) (12345) (1234) (12)(34) (12) (123)
0 1 0 0 0 1 2 1	S_2	$D_6(a_1)$ $D_5 + 2A_1$	1 (12)
0 0 1 0 1 0 2 0	S_2	$E_7(a_4)$ $A_1 + D_6(a_1)$	1 (12)
2 0 0 0 2 0 2 0	S_2	$E_6(a_1)$ $(A_7)''$	1 (12)
0 0 0 2 0 0 2 0	S_2	$D_5 + A_2$ $E_7(a_4) + A_1$	1 (12)
1 0 1 0 1 0 1 0	S_2	$D_7(a_2)$ $D_5 + A_3$	1 (12)
1 0 1 0 1 0 2 0	S_2	$E_6(a_1) + A_1$ $A_7 + A_1$	1 (12)
2 0 1 0 1 0 2 0	S_2	$E_7(a_3)$ $A_1 + D_6$	1 (12)
0 0 2 0 0 0 2 0	S_3	$E_8(b_6)$ $E_6(a_1) + A_2$ $D_8(a_3)$	1 (123) (12)
2 0 0 2 0 0 2 0	S_2	$D_7(a_1)$ $E_7(a_3) + A_1$	1 (12)
0 0 2 0 0 2 0 0	S_3	$E_8(a_6)$ A_8 $D_8(a_2)$	1 (123) (12)
0 0 2 0 0 2 2 0	S_3	$E_8(b_5)$ $E_6 + A_2$ $E_7(a_2) + A_1$	1 (123) (12)
2 0 2 0 0 2 0 0	S_2	$E_8(a_5)$ $D_8(a_1)$	1 (12)
2 0 2 0 0 2 2 0	S_2	$E_8(b_4)$ $E_7(a_1) + A_1$	1 (12)
2 0 2 0 2 0 2 0	S_2	$E_8(a_4)$ D_8	1 (12)
2 0 2 0 2 2 2 0	S_2	$E_8(a_3)$ $E_7 + A_1$	1 (12)

References

[Al] A. Alexeevsky, *Component group of centralizers for unipotent elements in semisimple algebraic groups* [in Russian], Trudy Tbilissi. Mat. Inst. Razmazde Akad. Nauk. Gruzin SSR 62, (1979), 5 -27.

[B-C] P. Bala and R. Carter, *Classes of unipotent elements in simple algebraic groups I*, Math. Proc. Camb. Phil. Soc. **79** (1976), 401-425; *Classes of unipotent elements in simple algebraic groups II*, Math. Proc. Camb. Phil. Soc. **80** (1976), 1-18.

[B-V] D. Barbasch and D. A. Vogan, Jr., *Unipotent representations of complex semisimple groups*, Ann. of Math. (2) **121** (1985), no. 1, 41-110.

[Ca] R. W. Carter, *Finite Groups of Lie Type.* John Wiley and Sons: Chichester, 1985.

[C-M] D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie algebras.* Van Nostrand Reinhold: New York, 1993.

[Dy] E. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. Ser. 2 **6** (1957), 111-245.

[Hu] J. E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups,* Mathematical Surveys and Monographs, Vol. 43. American Mathematical Society: Providence, 1995.

[I-M] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Etudes Sci. Publ. Math., Vol. 25 (1965), 5-48.

[Lu] G. Lusztig, *Classification of unipotent representations of simple p-adic groups*, I.M.R.N., no. 11 (1995), 517-589.

[Mi] K. Mizuno, *The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8,* Tokyo J. Math. **3** (1980), 391-461.

[O-V] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups.* Springer-Verlag: Berlin, 1990.

[Sp1] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173-207.

[Sp2] T. A. Springer, *Linear Algebraic Groups.* Progress in Mathematics, Vol. 9. Birkhäuser: Boston, 1981.

[S-S] T. A. Springer and R. Steinberg. Conjugacy Classes, in *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes in Math. **131**, Springer-Verlag: New York, 1970, 167-266.

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