Volumes of Root Zonotopes via the W-Laplacian

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for the virtual conference

Polytopics: Recent advances in Polytopes

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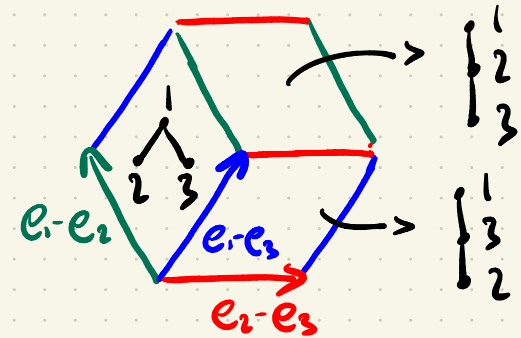
The unimodular root zonotope $Z_{A_{n-1}^+}$ and its volume.

The root zonotope Z_{Φ^+} is the Minkowski sum of the positive roots of the root system Φ.



The root zonotope $Z_{A_{n-1}^+}$ is unimodular and its (normalized) volume is given as $\mathrm{Vol}(Z_{A_{n-1}^+}) = n^{n-2}$

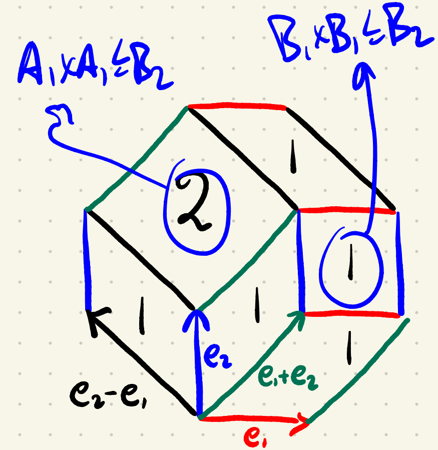
(it has a tiling indexed by trees)

Φ	A_{n-1}	B_2	B_3	B_4	D_3	D_4	F_4	...
$\mathrm{Vol}(Z_{\Phi^+})$	n^{n-2}	7	$3\cdot 29$	1553	2^4	$2\cdot 3\cdot 53$	$2\cdot 3\cdot 31\cdot 67$	...

? ? ? ? ? ? ? ? ?

Why not a product formula for all Weyl groups?

- The zonotopes Z_{Φ^+} are not unimodular
- Their tiles are less nice than trees.



[Shephard-McMullen formula]

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{\vec{r}} \underbrace{|\det(\vec{r})|}_{\text{volume of parallelepiped formed by } \vec{r}}$$

$\vec{r}$ ⇝ summing over all $\mathbb{C}$-bases of Z_{Φ^+}

$$\mathrm{Vol}(Z_{B_2^+}) = 7$$

[Baumeister-Wegener] The reflections associated to $\vec{r}$ generate the Weyl group W if and only if $\vec{r}$ and $\vec{\rho}$ are $\mathbb{Z}$-bases of Q and $\hat{Q}$

Q ↓ root lattice

$\hat{Q}$ ↓ coroot lattice

[Corollary]

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W' \leq W} \underbrace{|\mathrm{RGS}(W')|}_{\#\text{ of reflection generating sets of } W'} \cdot \mathrm{Vol}(Q')$$

$W' \leq W$ ← W' a max-rank reflection subgroup

$\mathrm{Vol}(Q')$ → volume of root lattice Q' of W'

The W-Laplacian and its determinant

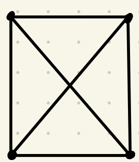
W is a Weyl group acting on $V \simeq \mathbb{R}^n$. It has set of reflections R, root system Φ, and reflection representation ρ_V. Its W-Laplacian L_W is:

Defn 1: $$GL(V) \ni L_W := \sum_{\tau \in R} \left(I_n - \rho_V(\tau) \right)$$

↳ the $n \times n$ identity matrix

Defn 2: $$L_W(v) := \sum_{\sigma \in \Phi^+} \langle v, \hat{\sigma} \rangle \cdot \sigma$$

TYPE-A PICTURE



K_4 ⟿ graph ⟿ Laplacian

$$\underbrace{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}}_{L(K_4)} = \sum_{i<j} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sum_{i<j} \left(I_4 - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

The W-Laplacian and its determinant

Defn 1: $GL(V) \ni L_W := \sum_{\tau \in R} (I_n - \rho_V(\tau))$

the $n \times n$ identity matrix

Defn 2: $L_W(v) := \sum_{\delta \in \Phi^+} \langle v, \hat{\delta} \rangle \cdot \delta$

<=> Defn of Lap. #s by Gordon-Griffeth

by Defn 1 and W-invariance: $\det(L_W) = h^n$ (h is the Coxeter # of W)

If W is reducible $\det(L_W) = \prod_{i=1}^{n} h_i(W)$ i.e. $\{h_i(A_2 \times B_3 \times F_4)\} = \{3, 3, 6, 6, 6, 12, 12, 12, 12\}$

The W-Laplacian sees both?

by Defn 2 and as a sum of rank-1 operators:

$$\det(L_W) = \sum_{\vec{r}} \det(\langle r_i, \hat{r}_j \rangle) = \sum_{W' \leq_{max} W} |RGS(W')| \cdot I(W')$$

connection index

Corollary: $$|RGS(W)| = \frac{1}{I(W)} \sum_{W' \leq_{max} W} \mu(W, W') \cdot \prod_{i=1}^{n} h_i(W')$$

Möbius function for lattice of reflection subgroups

Volumes $\mathrm{Vol}(Z_{\Phi^+})$ in terms of Coxeter #s and reflection subgroups

[after Shephard-McMullen] $\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W' \leq_{max} W} \underbrace{|RGS(W')|}_{\text{\# of reflection generating sets of } W} \cdot \mathrm{Vol}(Q')$

Applying the previous calculation of $|RGS(W)|$ will give:

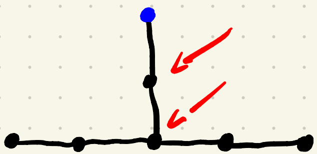
Theorem:

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W'' \leq_{max} W} \left(\prod_{i=1}^{n} h_i(W'') \right) \cdot \sum_{W'' \leq W' \leq W} \mu(W', W'') \cdot \frac{1}{\mathrm{Vol}(\hat{Q}')}$$

Corollary: To compute the Ehrhart polynomial of Z_{Φ^+} replace the interval $[W'', W]$ by $[W'', \text{parab. closure}(W'')]$

Working out the case for E_6

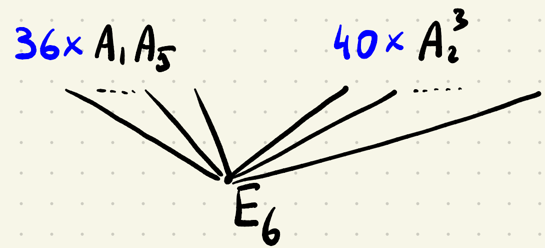
Borel - de Siebenthal Theory



Maximal Reflection subgroups:

- $A_1 A_5$
- A_2^3

Lattice of maximal rank reflection subgroups



$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W'' \leq_{max} W} \left(\prod_{i=1}^{n} h_i(W'') \right) \cdot \sum_{W'' \leq W' \leq W} \mu(W', W'') \cdot \frac{1}{\mathrm{Vol}(\hat{Q}')}$$

$$\mathrm{Vol}(E_6) = 12^6 \cdot \frac{1}{\sqrt{3}} + 36 \cdot (2 \cdot 6^5) \cdot \left(\frac{1}{\sqrt{12}} - \frac{1}{\sqrt{3}} \right) + 40 \cdot 3^6 \cdot \left(\frac{1}{\sqrt{27}} - \frac{1}{\sqrt{3}} \right)$$

$$= \sqrt{3} \cdot 895536$$

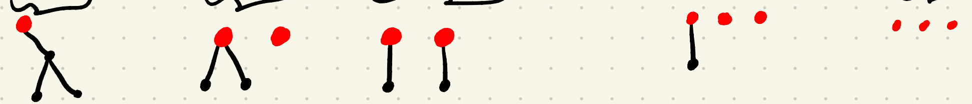
Are there W-trees?

Matrix Forest Theorem (For complete graph K_n, unweighted)

$$\boxed{(t+n)^{n-1} = \sum_{k=1}^{n} c_k \cdot t^{k-1}}$$

$c_k \to$ # of rooted forests on $[n]$ w/ k trees

Example: $(t+4)^3 = 4^3 + (4\cdot 3^2 + 3\cdot(2\cdot 2))\cdot t + 6\cdot 2\cdot t^2 + 1\cdot t^3$



[W-Laplacian char. polyn.]

$$(t+h)^n = \sum_{\substack{W_x \le W \\ \text{parabolic}}} \left(\prod_{i=1}^{\operatorname{codim}(x)} h_i(W_x) \right) \cdot t^{\dim(x)}$$

↳ Counts $\overbrace{\text{generating sets}}^{W'\text{-trees}}$ of $W' \le_{\max} W_x$ weighted by $I(W')$.

Summary / Advertisement

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Existing approaches

- Postnikov $\mathrm{Vol}(Z_{\Phi^+}) = \sum_{w \in W} \cdots$ ⇝ all group elements

- De Concini - Procesi $\mathrm{Vol}(Z_{\Phi^+}) = \sum_{f \in \mathcal{F}} \cdots$ ⇝ $n!$ flags on the simplex.

* The above generalize to arbitrary permutohedra.

- Ardila-Beck-McWhirter
  - combinatorics of trees
  - generating functions for Ehrhart polynomials
  - only classical types (A_n, B_n, C_n, D_n)

Our approach

- Debatably more explicit

$$\mathrm{Vol}(Z_{\Phi^+}) = \sum_{W' \leq_{max} W} \cdots$$

- Reveals connection with a general version of W-trees (i.e. reflection gen'g sets)

- The W-Laplacian plays a "higher-arithmetic" role:

$$\det(L_W) = \sum_{\vec{r}} \left| \det(\vec{r}) \cdot \det(\vec{\bar{r}}) \right|$$

Thank You

Max Planck

and (polytopi-)Friends