Recursions and Proofs in Coxeter-Catalan combinatorics

Arrangements and Symmetries seminar

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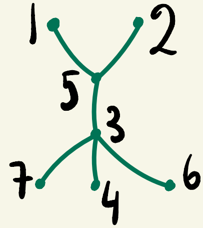
(w/ Guillaume Chapuy*)

Counting Trees

Cayley's Theorem 1889

There are n^{n-2} labeled trees on n vertices.



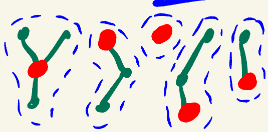
Matrix-Forest theorem (Folklore?)

The following expansion counts rooted forests on n vertices.

$$t\cdot(t+n)^{n-1} = n^{n-1}\cdot t + \cdots + \underbrace{\binom{n-1}{k}\cdot n^{n-1-k}}\cdot t^{k+1} + \cdots + t^{n}$$

$n\cdot n^{n-2}$

# of roots



$(k+1)$ connected components (trees) each rooted

Coming Soon! (in this talk): A version for reflection groups

Counting Factorizations

Hurwitz 1891

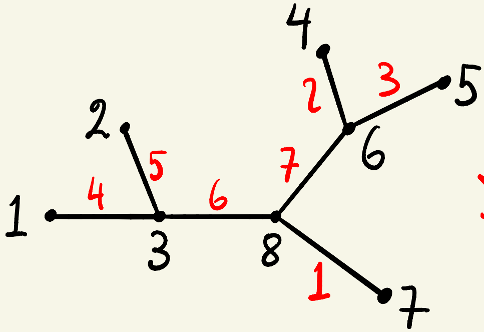
There are n^{n-2} shortest length Factorizations $t_1 \cdot t_2 \cdot \cdots \cdot t_{n-1} = (1\,2 \cdots n)$ in transpositions t_i.

Ex: $(1\,2)(2\,3) = (1\,2\,3)$ $\quad$ $(2\,3)(1\,3) = (1\,2\,3)$

$(1\,3)(1\,2) = (1\,2\,3)$

Dénes 1959

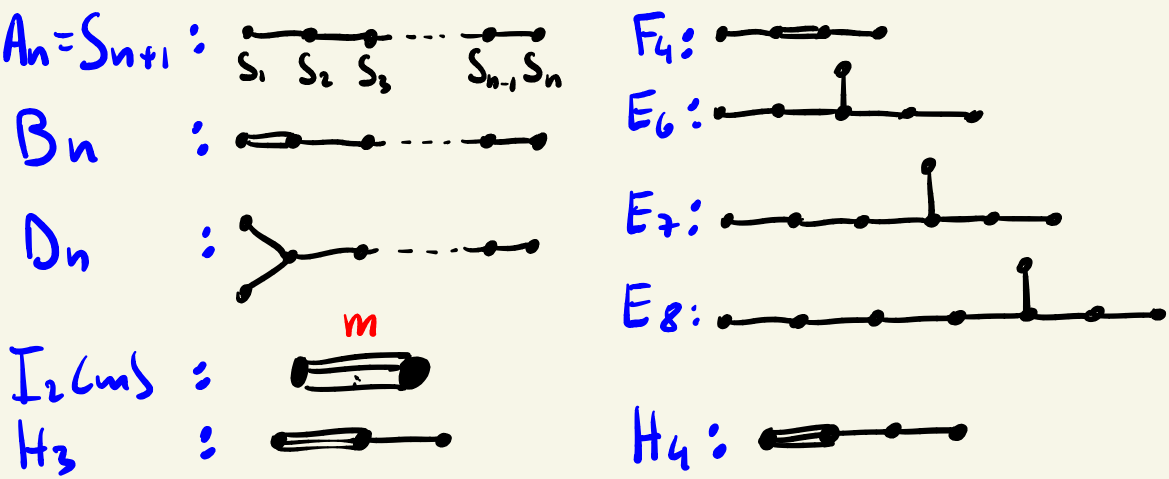
Factorizations really correspond to trees?



$\rightsquigarrow$ $(6\,8)(3\,8)(2\,3)(1\,3) \cdot (5\,6)(4\,6)(7\,8) = (1\,2\,6\,4\,5\,8\,7\,3)$

From symmetric groups to reflection groups

Reflection groups W are finite subgroups of $GL(\mathbb{R}^n)$ generated by (euclidean) reflections



a Coxeter element c is a product of all simple generators s_i in any order

The Coxeter number h of W is the order of any Coxeter element c.

Coxeter-Steinberg:

$$2 \cdot \#\{\text{reflections}\} = \overset{\text{Coxeter number}}{h} \cdot \overset{\text{rank}}{n}$$

From symmetric groups to reflection groups

[Looijenga-Deligne-Arnold-Chapoton-Bessis]

There are $\frac{h^n \cdot n!}{|W|}$ shortest length factorizations $t_1 \cdots t_n = c$ of a Coxeter element c in reflections t_i.

N.B. For S_N, $(h,n) = (N, N-1)$ so $\frac{N^{N-1} \cdot (N-1)!}{N!} = N^{N-2}$

	$S_n = A_{n-1}$	D_n	E_6	E_7	E_8
$\frac{h^n \cdot n!}{\lvert W \rvert}$	n^{n-2}	$2(n-1)^n$	$2^9 \cdot 3^4$	$2 \cdot 3^{12}$	$2 \cdot 3^5 \cdot 5^7$

Coming soon! (in this talk): A simple, uniform proof of this formula.

A Formula looking for a name

Looijenga (conj.): # Factorizations = degree of LL map

The Lyashko-Looijenga (LL) map is a quasi-homogeneous morphism which sends a function to its multiset of critical values.

Deligne (w/ Tits + Zagier): proves it

Arnold: observes uniform formula for degree of LL map

Chapoton: rediscovers the formula in the context of the noncrossing lattice

Bessis: Confirms it for all well-generated complex reflection groups.

A Formula is an over-achiever

Coxeter presentation:

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1,\ \underbrace{s_i s_j \cdots s_i}_{m_{ij}\text{ terms}} = \underbrace{s_j s_i \cdots s_j}_{m_{ij}\text{ terms}} \rangle$$

Artin presentation:

$$B(W) = \langle s_1, \ldots, s_n \mid \underbrace{s_i s_j \cdots s_i}_{m_{ij}\text{ terms}} = \underbrace{s_j s_i \cdots s_j}_{m_{ij}\text{ terms}} \rangle$$

Dual braid presentation: (Relies on $\frac{h^n n!}{|W|}$)

$$B(W) = \left\langle t_1, \ldots, t_N \;\middle|\; \begin{array}{l} t_1 \cdot \cdots \cdot t_n = \\ t_1' \cdot \cdots \cdot t_n' = \\ \cdots \\ \tilde{t}_1 \cdots \tilde{t}_n \end{array} \right\rangle$$

Last non-uniform ingredient in Paolini-Salvetti proof of $K(\pi,1)$ conj. for affine Artin groups

The Deligne-Reading recursion

$\mathrm{Hur}(W) = \#\{$ shortest length reflection factorizations $t_1 \cdots t_n = c\}$

Deligne ('74), Reading ('08)

$$\mathrm{Hur}(W) = \frac{h}{2} \cdot \sum_{s \in S} \mathrm{Hur}(W_{\langle s \rangle})$$

Idea: $(t_1, \ldots, t_n) \mapsto ({}^c t_1, \ldots, {}^c t_n)$ $\quad {}^c g := c g c^{-1}$

c-orbit ${}^{c^k} t_i$ always has a simple gen'or.

Recursively proves $\mathrm{Hur}(W) = \frac{h^n \cdot n!}{|W|}$ but separately for each family.

In B_n it comes down to Abel's identity

$$n^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot k^k \cdot (n-k)^{n-k-2}$$

The Noncrossing Lattice $NC(W)$

Define $\ell_R(w)$ as the minimum # of reflections needed to write $w = t_1 \cdots t_k$

Define $\leq_R$ in W via $u \leq_R v$ if and only if $\ell_R(u) + \ell_R(u^{-1} \cdot v) = \ell_R(v)$

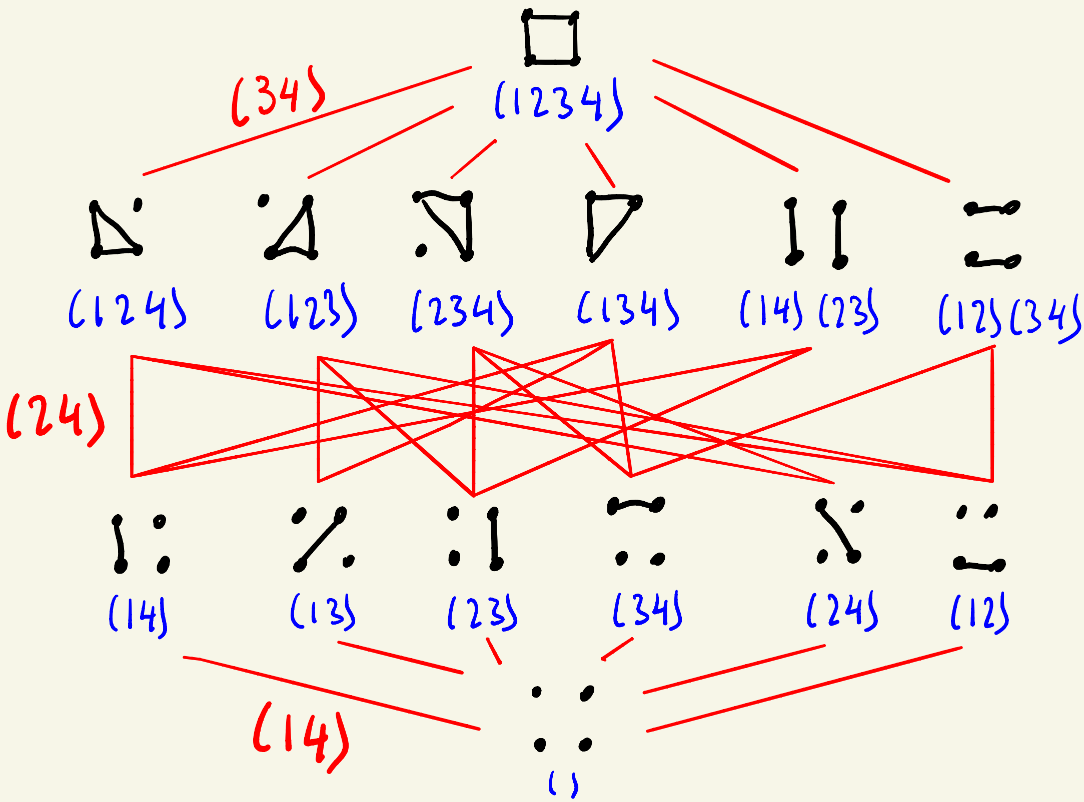
[Kreweras-Reiner-Biane-Brady-Watt-Bessis]
The noncrossing lattice $NC(W)$ is defined as the interval $[1, c]_{\leq_R}$

[Brady-Watt]

$NC(W)$ embedds in the intersection lattice of the reflection arrangement of W

$$[1, c]_{\leq_R} \ni g \to V^g$$

The Noncrossing Lattice NC(W)



Maximal chains in NC(W) correspond to shortest length reflection factorizations of c.

A different recursion

$$\mathrm{Hur}(W) = \sum_{L \in NC(W)\ \dim(L)=1} \mathrm{Hur}(W_L)$$

Idea: $\underbrace{t_1 \cdot t_2 \cdot \cdots t_{n-1}}\cdot t_n = c$

this product w is a noncrossing partition s.th. $\dim(V^w) = 1$

[Armstrong-Rhoades-Reiner] (uniformly)

There are $\dfrac{h}{[N(L):W_L]}$ noncrossing partitions in the W-orbit of L.

Implication: It is sufficient to prove that

$$\frac{h^n \cdot n!}{|W|} = \sum_{[L] \in \mathcal{L}/W} \frac{h}{[N(L):W_L]} \cdot \frac{(n-1)! \cdot \prod h_i(W_L)}{|W_L|}$$

or...

$$h^{n-1} \cdot n = \sum_{L \in \mathcal{L},\ \dim(L)=1} \prod_{i=1}^{n-1} h_i(W_L)$$

The W-Laplacian Recursion

For any real hyperplane arrangement $\mathcal{A}$ in some $V = \mathbb{R}^n$ define:

$$GL(V) \ni L_{\mathcal{A}} := \sum_{H \in \mathcal{A}} (\underbrace{I_n}_{\substack{n\times n \\ \text{identity} \\ \text{matrix}}} - \underbrace{S_H}_{\substack{\text{orthogonal} \\ \text{reflection} \\ \text{across } H}})$$

Lemma [Chapuy–D., Burman]

$$\det(L_{\mathcal{A}} + t) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \underbrace{\mathrm{pdet}}_{\text{pseudo-det}}(L_{\mathcal{A}_X}) \cdot t^{\dim(X)}$$

pseudo-det (product of non-zero eigenvalues)

Lemma [Coxeter–Steinberg–Gordon–Griffeth]

$$\det(L_{\mathcal{A}_W}) = \prod_{i=1}^{n} h_i(W)$$

COROLLARIES

- $\Rightarrow (h+t)^n = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} \prod_{i=1}^{\mathrm{codim}(X)} h_i(W_X) \cdot t^{\dim(X)}$

- $\Rightarrow \mathrm{Hur}(W) = h^n \cdot n! / |W|$

Modules of logarithmic derivations

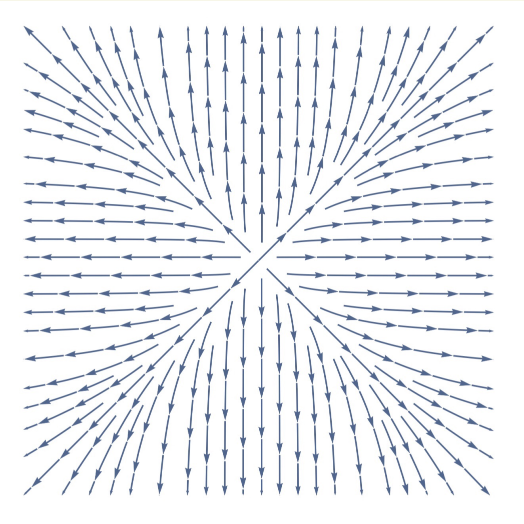
- $\mathcal{A}$ a hyperplane arrangement in V
- $S := \mathbb{C}[V]$ algebra of polynomials on V
- $\mathrm{Der}(S)$ algebra of derivations on S

(equiv. polynomial vector fields on V)

linear form defining H

$$D(\mathcal{A}) := \{ g \in \mathrm{Der}(S) : g \cdot \alpha_H \in S \cdot \alpha_H \quad \forall H \in \mathcal{A} \}$$

(equiv. g as a vector field is tangent to all $H \in \mathcal{A}$)



An element of $D(B_2)$

where B_2 is the arrangement $x \cdot y \cdot (x^2 - y^2)$

Multi-Arrangements

- $(\mathcal{A}, m)$ $\mathcal{A}$ arrangement
  $m : \mathcal{A} \to \mathbb{Z}_+$ multiplicity fnc.

$$D(\mathcal{A}, m) := \{ g \in \mathrm{Der}(S) : g \cdot \alpha_H \in S \alpha_H^{m_H} \}$$

polynomial vector fields that are tangent to the m_H-th degree on each $H \in \mathcal{A}$

[Saito, Terao, Ziegler]
If $D(\mathcal{A}, m)$ is a free S-module, we call the (multi-) arrangement $(\mathcal{A}, m)$ free.
The degrees of its S-generators are called the exponents of $(\mathcal{A}, m)$.

[Terao, characteristic polynomial]
If $m_H \equiv 1$, the characteristic polynomial of a free arrangement $\mathcal{A}$, with exponents e_i,
Factors as $\chi(\mathcal{A}, t) = \prod (t - e_i)$

[Abe, Terao, Wakefield]
There is an analog $\chi((\mathcal{A}, m), t)$

Multi-Reflection Arrangements

[Terao-Yoshinaga]
For a reflection arrangement $\mathcal{A}_W$ and multiplicity functions $m \equiv 2\cdot k$ or $m^* \equiv 2k+1$ both $(\mathcal{A}, m)$ and $(\mathcal{A}, m^*)$ are free.

Moreover,

$$\text{exponents}((\mathcal{A}, m)) = \{\overbrace{kh, kh, \ldots, kh}^{n \text{ times}}\}$$

$$\text{exponents}((\mathcal{A}, m^*)) = \{kh + e_1, \ldots, kh + e_n\}$$

where h is the Coxeter number of W and $e_i = d_i - 1$ with d_i being the invariant degrees of W.

Local-to-global identities

[Brieskorn's lemma]

$$\chi(\mathcal{A},t) = \sum_{X\in L_{\mathcal{A}}} \underbrace{[t^{\dim(X)}]}_{\text{"coefficient of..."}}\cdot\chi(\mathcal{A}_X,t)\cdot t^{\dim(X)}$$

[Abe-Terao-Wakefield]

$$\chi((\mathcal{A},m),t) = \sum_{X\in L_{\mathcal{A}}} [t^{\dim(X)}]\cdot\chi((\mathcal{A}_X,m_X),t)\cdot t^{\dim(X)}$$

[Implications]

- $$(h+t)^n = \sum_{X\in L_{\mathcal{A}_W}} \prod h_i(W_X)\cdot t^{\dim(X)}$$ [Recovers the W-Laplacian recursion]

- $$\prod_{i=1}^{n}(t+\kappa\cdot h+e_i) = \sum_{X\in L_{\mathcal{A}_W}} \prod\left(\kappa\cdot h_i(W_X)+e_i(W_X)\right)\cdot t^{\dim(X)}$$

Summary

Deligne-Reading Recursion:

$$\mathrm{Hur}(W) = \frac{h}{2} \sum_{s \in S} \mathrm{Hur}(W_{\langle s \rangle})$$

[Chapuy-D. recursion]

$$\mathrm{Hur}(W) = \sum_{\substack{L \in N\mathcal{C}(W) \\ \dim(L)=1}} \mathrm{Hur}(W_L)$$

The Chapuy-D. recursion uniformly recovers the $h^n \cdot n!/|W|$ number via the formula:

$$(h+t)^n = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} \prod_{i=1}^{\mathrm{codim}(X)} h_i(W_X) \cdot t^{\dim(X)}$$

which is proven either

- via the W-Laplacian

$$L_{\mathcal{A}_W} := \sum_{H \in \mathcal{A}_W} (I_n - S_H) \quad \text{or}$$

- via the [ATW] local-to-global identities

That's all Folks!

Thank you for sticking along