Recursions and Proofs in Coxeter-Catalan combinatorics

Theo Douvropoulos joint w/ Guillaume Chapuy

UMass Amherst Discrete Math Pre-Seminar

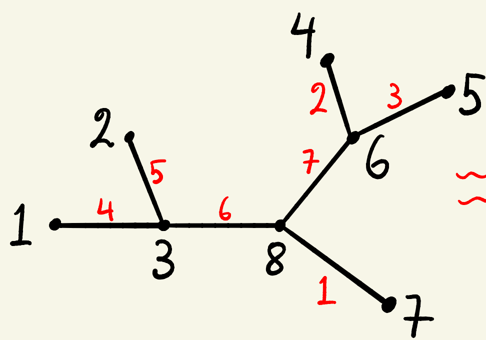
Sept. 4th 2020

Some enumerative results

• Hurwitz (1891) There are n^{n-2} minimal length factorizations of the long cycle of S_n in transpositions: $t_1 \cdot t_2 \cdot \cdots \cdot t_{n-1} = (1\,2 \cdots n)$

ex: $(13)\cdot(12) = (123)$ $\quad (12)(23) = (123)$ $\quad (23)(13) = (123)$ $\quad 3^{3-2} = 3$

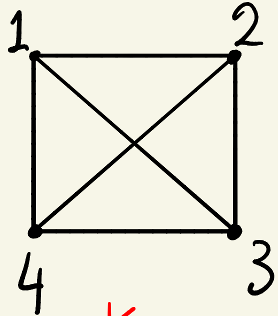
• Really these are all trees. [Dénes '59]



$\rightsquigarrow (68)(38)(23)(13)(56)(46)(78) =$

$= (12645873)$

What is *your* favorite way to count trees?



K_4

$$\xrightarrow[\text{Matrix}]{\text{Laplacian}} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Ex: What is the characteristic polynomial for $L(K_n)$?

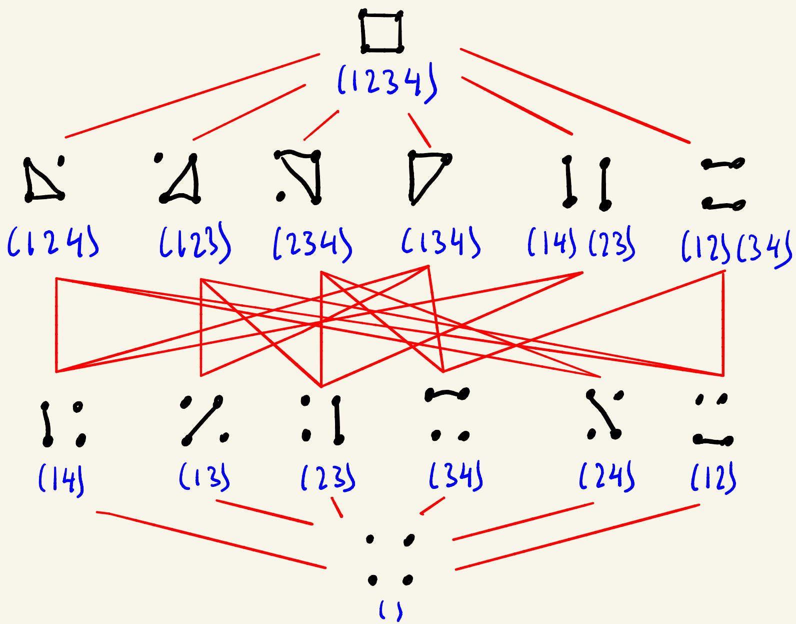
• [Matrix-Tree theorem]

The Laplacian of a graph G on n vertices counts spanning trees of G

$$\#\{\text{spann. trees of } G\} = \underbrace{\prod_{\lambda_i \neq 0} \lambda_i}_{\text{AKA the "pseudo-determinant" of } L_G}$$

$\lambda_i \rightarrow$ eigenvalues of L_G

The non-crossing partition lattice $NC(n)$



Do I stay sane or do I join the Catalan obsession?!

- It has 14 elements?!
- It has $\frac{1}{n+1}\binom{2n}{n}$ elements $=: Cat(n)$
- Posets have Zeta-polynomials

$$Z(NC(n),k) := \#\left\{\begin{matrix}\text{chains of}\\ \text{length } k\\ \text{in } NC(n)\end{matrix}\right\}$$

$$\overset{?!}{=} \frac{1}{n!}\cdot\prod_{i=2}^{n}(kn+i)$$

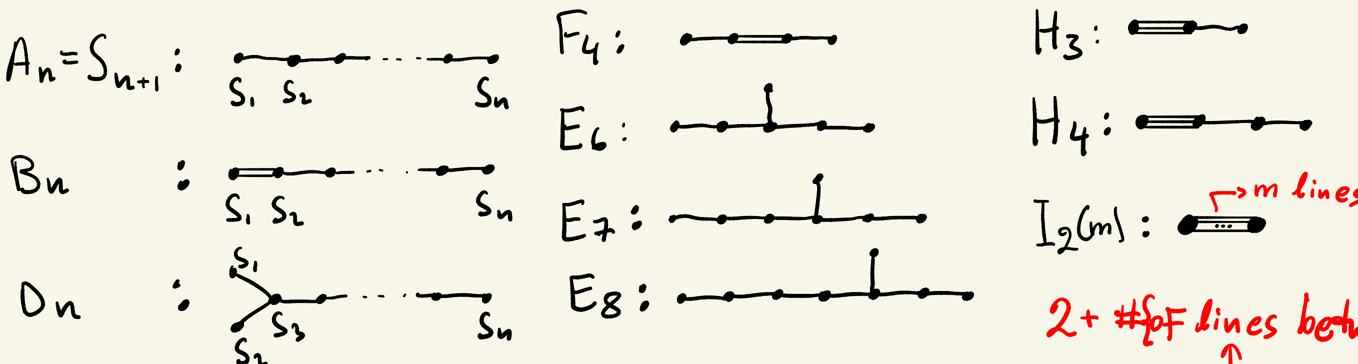
- It has n^{n-2} maximal chains

Symmetric groups and reflection groups

- Reflection groups W are:
  - Finite subroups of $GL(\mathbb{R}^n)$
  - generated by (euclidean) reflections

Moto: If you like it in S_n $\rightsquigarrow$ prove it for reflection groups

don't know how? : Use the classification:



2+ #{of lines betw. i,j}

Diagrams encode angles & presentation

$$W = \langle \underbrace{s_1, \cdots, s_n}_{\text{simple generators}} \mid s_i^2 = 1, (s_i \cdot s_j)^{m_{ij}} = 1 \rangle$$

Numerology ??

	Symmetric group S_n	Reflection group W (of rank n)
• invariant "degrees"	$2, 3, 4, \cdots, n$ ↳ sym. polys: $x_1^i + \cdots + x_n^i \ldots$	$d_1, d_2, \cdots, d_n$
• number of reflections	$\binom{n}{2} = 1 + 2 + \cdots + n-1$	$\lvert R \rvert = d_1 + \cdots + d_n - n$
• size	$\lvert S_n \rvert = n! = 2 \cdot 3 \cdots n$	$\lvert W \rvert = d_1 \cdot \cdots \cdot d_n$
• Statistics	$\sum_{\pi \in S_n} t^{n - \underbrace{c(\pi)}_{\#\text{cycles of } \pi}} = \prod_{i=1}^{n-1} (1 + t \cdot i)$	$\sum_{g \in W} t^{\ell_R(g)} = \prod_{i=1}^{n} (1 + t(d_i - 1))$
• Zeta polyn.	$\mathcal{Z}(NC(n)) = \frac{1}{n!} \prod_{i=2}^{n} (k \cdot n + i)$	$\mathcal{Z}(NC(W)) = \frac{1}{\lvert W \rvert} \cdot \prod (kh + d_i)$

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Some enumerative results

Hurwitz (1891) There are n^{n-2} minimal length factorizations of the long cycle of S_n in transpositions: $t_1 \cdot t_2 \cdot \cdots \cdot t_{n-1} = (12\cdots n)$

Symmetric group	long cycle	transpositions
$\Downarrow$	$\Downarrow$	$\Downarrow$
Reflection groups	Coxeter element	Reflections

[Deligne-Arnol'd-Chapoton-Bessis] There are $\dfrac{h^n \cdot n!}{|W|}$ minimal length reflection factorizations $t_1 \cdots t_n = c$ of a Coxeter element $c \in W$ (of order h) for any rank n reflection group W

$$\left[\left[\frac{h^n \cdot n!}{|W|} \xrightarrow[h \to n,\ n \to n-1]{\text{in } S_n} \frac{n^{n-1}\cdot (n-1)!}{n!} = n^{n-2} \right]\right]$$

The noncrossing partition lattice $NC(W)$

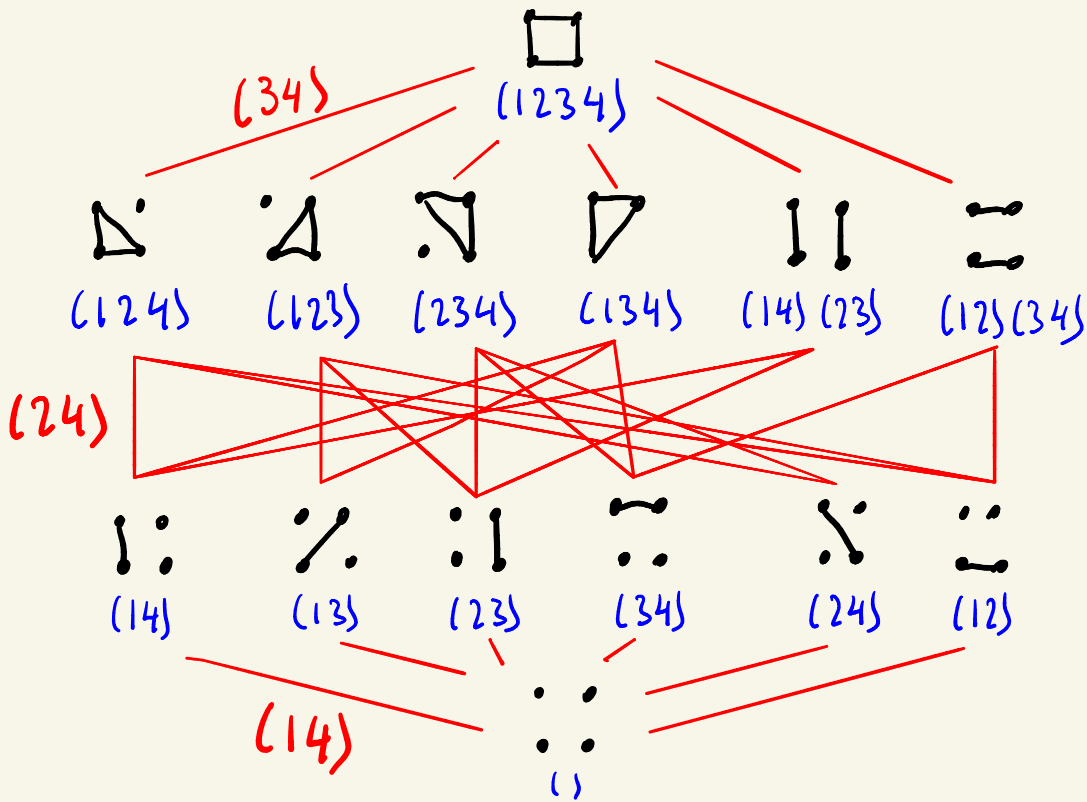
- The absolute reflection length $\ell_R(w)$ is the min. $\#k$ s.th. $\exists\, t_1 \cdots t_k = w$ (reflections)
- Define $\leq_R$: $u \leq_R v \iff \ell_R(u) + \ell_R(u^{-1}v) = \ell_R(v)$
- Define $NC(W) := [1, c]_{\leq_R}$ (Cox. elt.)

Numerology:

- The number $M(NC(W))$ of max'l chains in $NC(W)$ equals $\dfrac{h^n \cdot n!}{|W|}$
- The zeta polynomial of $NC(W)$ is $Z(NC(W), k) = \prod \dfrac{kh + d_i}{d_i}$

[[d_i's : the invariant degrees of W]]

The noncrossing lattice for S_4



Chains correspond to length-additive factorizations:

$(34)\cdot(24)\cdot(14) = (1234)$

Recursions for $M(NC(W))$

↳ # of max'l chains

- Deligne & Tits-Zagier '74 : $\boxed{M(NC(W)) = \frac{h}{2} \cdot \sum_{s \in S} M(NC(W_{\langle s \rangle}))}$
  Reading '08

- $W_{\langle s \rangle}$ might be reducible $\rightsquigarrow$ $M(NC(W')) = \frac{n!}{|W'|} \cdot \prod_{i=1}^{\mathrm{rank}(W')} h_i(W')$

where $\{\{h_i(W')\}\}$ is the multiset of Coxeter #s of W':

$F_4 \to \{\{h_i(F_4)\}\} = \{\{12, 12, 12, 12\}\}$ $\quad A_2 \times B_3 \to \{\{h_i(A_2 \times B_3)\}\} = \{\{3, 3, 6, 6, 6\}\}$

- Implication: Prove that $h^{n-1} \cdot \frac{n}{2} = \sum_{s \in S} [W : W_{\langle s \rangle}] \cdot \prod_{i=1}^{n-1} h_i(W_{\langle s \rangle})$

⚠ No case-free proof ⚠

"Recursion on Simples" - to - "Recursion on Flats"

- The Deligne-Reading recursion can be rewritten:

$$M(NC(W))\cdot |W| = h\cdot \sum_{L\in\mathcal{L}_W,\ \dim(L)=1} M(NC(W_L))\cdot |W_L|$$

Important ingredient [Armstrong-Rhoades-Reiner]

There are $\frac{h}{[N(W_L):W_L]}$ noncrossing lines in the W-orbit of L

- Implication: Prove that $h^{n-1}\cdot n = \sum_{L\in\mathcal{L}_W,\ \dim(L)=1} \prod_{i=1}^{n-1} h_i(W_L)$

- In Fact [Chapuy, D. '19]:

$$\boxed{(h+t)^n = \sum_{X\in\mathcal{L}_W} \prod_{i=1}^{\mathrm{codim}(X)} h_i(W_X)\cdot t^{\dim(X)}}$$

The $\mathcal{A}$-Laplacian matrix

- For any (real) hyperplane arrangement $\mathcal{A}$, in some $V \simeq \mathbb{R}^n$,

$$GL(V) \ni L_{\mathcal{A}} := \sum_{H \in \mathcal{A}} (I_n - S_H)$$

S_H: orth. refln across H

I_n: $n \times n$ id. matrix

$$\begin{bmatrix} \sum w_{1j} & -w_{12} & -w_{13} & -w_{14} \\ -w_{12} & \sum_{2j} & -w_{23} & -w_{24} \\ -w_{13} & -w_{23} & \sum w_{3j} & -w_{34} \\ -w_{14} & -w_{24} & -w_{34} & \sum w_{j4} \end{bmatrix} = \sum_{i<j} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & w_{ij} & -w_{ij} & 0 \\ 0 & -w_{ij} & w_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sum_{i<j} \left(I_4 - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

- Lemma [Chapuy-D. + Burman] (char. polyn. of $L_{\mathcal{A}}$)

$$\det(L_{\mathcal{A}} + t) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \operatorname{pdet}(L_{\mathcal{A}_X}) \cdot t^{\dim(X)}$$

The proof (briefly)

Needed to show:

$$\boxed{(h+t)^n = \sum_{X \in L_W} \prod_{i=1}^{\operatorname{codim}(X)} h_i(W_X) \cdot t^{\dim(X)}}$$

W-Laplacian Recursion:

$$\boxed{\det(L_{\mathcal{A}} + t) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \operatorname{pdet}(L_{\mathcal{A}_X}) \cdot t^{\dim(X)}}$$

FACT: $\operatorname{pdet}(L_{\mathcal{A}_W}) = \prod_{i=1}^{\operatorname{rank}(W)} h_i(W)$ $\Big\{ \underset{S_n}{\overset{I_n}{\rightsquigarrow}} \Big\}$ Matrix-Tree theorem

The Fomin-Reading recursion

- The cluster complex $\Delta(W)$ is a not-so-distant cousin of $NC(W)$
  - $\rightsquigarrow$ • Its facets are indexed by elts of $NC(W)$
    • Its h-vector is the rank-vector of $NC(W)$.
- Fomin-Reading '05: A rotation on the vertices of $\Delta(W)$

Forces: $\mathcal{Z}(NC(W), k+1) = \frac{kh+2}{2n} \cdot \sum_{s \in S} \mathcal{Z}(NC(W_{\langle s \rangle}), k+1)$

Q: Is there an analog here of the formula

$$(h+t)^n = \sum_{X \in \mathcal{L}_W} \prod_{i=1}^{\operatorname{codim}(X)} h_i(W_X) \cdot t^{\dim(X)}$$?

"Recursion on Simples" - to - "Recursion on Flats"

- The Fomin-Reading recursion is equivalent to:

$$\mathcal{Z}(NC(W), k+1) \cdot |W| = \frac{kh+2}{n} \cdot \sum_{L \in \mathcal{L}_W,\ \dim(L)=1} \mathcal{Z}(NC(W_L), k+1) \cdot |W_L|$$

- Implication: Prove that

$$\boxed{\prod_{i=1}^{n} (kh+d_i) = \frac{kh+2}{n} \cdot \sum_{L \in \mathcal{L}_W,\ \dim(L)=1} \prod_{i=1}^{n-1} \left(k \cdot h_i(W_L) + d_i(W_L)\right)}$$

- Fact [Biane, Josuat-Vergès, D.]

$$\binom{n}{s} \cdot \prod_{i=1}^{n} (kh+d_i) = \sum_{X \in \mathcal{L}_W,\ \dim(X)=s} \left(\prod_{i=1}^{\dim(X)} (kh+1+b_i^X)\right) \cdot \left(\prod_{i=1}^{\operatorname{codim}(X)} \left(kh_i(W_X) + d_i(W_X)\right)\right)$$

Relation w/ Free arrangements

- Solomon-Terao introduced a q-version of the characteristic polynomial of an arrangement $\mathcal{A}$, with $\ell = \dim(\mathcal{A})$

and Abe-Terao-Wakefield it extended it to multi-arrangements

Deriv. module

$$\Psi(\mathcal{A}, m, t, q) = \sum_{p=0}^{\ell} H(D^p(\mathcal{A}, m))(t(q-1)-1)$$

mult. function

Hilbert series

- ∫ "q-version": $\Psi(\mathcal{A}, 1, t, 1) = (-1)^{\ell} \cdot \chi(\mathcal{A}, t)$

Also, if $(\mathcal{A}, m)$ is free w/ exponents $e_1, \cdots, e_n$ then

$$\Psi(\mathcal{A}, m, t, q) = (-1)^{\ell} \cdot \prod (q^{e_i} \cdot t - [e_i]_q)$$

Recursion for characteristic polynomials

- Standard case: $\chi(\mathcal{A},t) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} [t^{\mathrm{bot}}]\, \chi(\mathcal{A}_X,t) \cdot t^{\dim(X)}$
- Multi-arrangements case implies for reflection arrangements:

$$\boxed{(h+t)^n = \sum_{X \in \mathcal{L}} \prod_{i=1}^{\mathrm{codim}(X)} h_i(W_X) \cdot t^{\dim(X)}}$$

$$\boxed{\prod (t + mh + d_i - 1) = \sum_{X \in \mathcal{L}} \prod_{i=1}^{\mathrm{codim}(X)} (mh_i(W_X) + d_i(W_X) - 1) \cdot t^{\dim(X)}}$$

- Any hope then for the Fomin-Reading recursion?

Thank You UMass!

(and Friends)