Geometric factorizations of a coxeter element.

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Combinatorics Seminar

MIT

Some enumeration formulas:

- Thm [Hurwitz, 1892]

$$\#\left\{\begin{array}{l}\text{shortest factorizations of an } n\text{-cycle} \\ \quad \underset{\downarrow}{c} = t_1 \cdot \dots \cdot t_{n-1} \text{ w/ } t_i \text{ transpositions}\end{array}\right\} = n^{n-2}$$

$(123\cdots n)$

$(12)(23) = (123)$
$(13)(12) = (123)$
$(23)(13) = (123)$

Now, given a coxeter element c in an irreducible, well-generated complex reflection group W of rank n, with $\mathrm{ord}(c) = h$:

- Thm [Bessis 2006-2016]

$$\#\{\text{shortest reflection factorizations } c = t_1 \cdots t_n\} = \frac{h^n \cdot n!}{|W|}$$

Some enumeration formulas:

- Thm [D. 2016] Given an intersection flat X,

$$\#\left\{\begin{array}{l}\text{shortest factorizations } c=\chi\cdot t_1\cdots t_\ell,\ \ell=\dim X,\\ \text{w/ } t_i \text{ reflections } \& \ V^{\chi} \text{ in the } W\text{-orbit of } X\end{array}\right\} = \frac{h^{\dim X}\cdot(\dim X)!}{[N_X : W_X]}$$

$X=V$

$$\left[\text{Bessis}: \frac{h^n\cdot n!}{|W|}\right] \xRightarrow{W=A_{n-1}} \left[\text{Hurwitz}: n^{n-2}\right]$$

...Some Invariant Theory

- It was already known to Gauss that any symmetric polynomial can be written in terms of the elementary symmetric polynomials.

S_3: $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3$

- Also known: The elementary symmetric polynomials are algebraically independent.

$\Rightarrow$ in other words: $\mathbb{C}[x_1,\dots,x_n]^{S_n} = \mathbb{C}[e_1,\dots,e_n]$

•) When we define the action abstractly:

for $\pi \in S_n$ and $v \in \mathbb{C}^n$, $\quad (\pi \cdot f)(v) = f(\pi^{-1} \cdot v)$

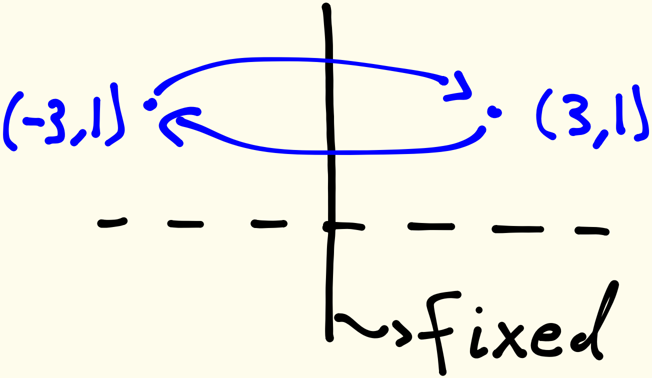
•) ... it makes sense to study, for any group $G \leq GL_n(V)$, the G-invariant polynomials:

$$\mathbb{C}[x_1, \dots, x_n]^G := \left\{ f \in \mathbb{C}[x_1, \dots, x_n] : \begin{array}{c} f(g^{-1}v) = f(v) \quad \forall\, v \in V \\ \forall\, g \in G \end{array} \right\}$$

... let's consider two simple examples:

$C_2 = \{id, (12)\} \curvearrowright \mathbb{R}^2$ via

$(x,y) \longmapsto (-x, y)$



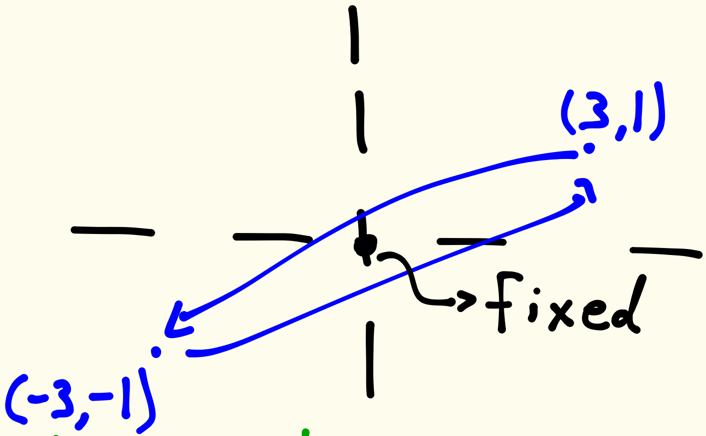
Invariant polynomials:

$f_1 = x^2, \ f_2 = y$

$* \ \deg(f_1) \cdot \deg(f_2) = |C_2|$

$C_2 = \{id, (12)\} \curvearrowright \mathbb{R}^2$ via

$(x,y) \longmapsto (-x, -y)$



Invariant polynomials:

$f_1 = x^2, \ f_2 = y^2, \ f_3 = xy$

$$\boxed{f_1 \cdot f_2 = f_3^2}$$

Theorem [Shephard-Todd-Chevalley, '54-'55]

For a finite group $G \leq GL_n(\mathbb{C})$, the ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^G$ is a <u>polynomial ring</u> (i.e. can be gen'd by algebraically independent polynomials) if and only if

G is a complex reflection group.

where "complex reflection group" means a $G \leq GL_n(\mathbb{C})$ generated by pseudoreflections.

i.e. elements $g \in GL_n(\mathbb{C})$ s.th. $\mathrm{Fix}(g)$ has codim 1 (is a hyperpane) and $\mathrm{ord}(g) < \infty$

$$g \mapsto \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & & & & \zeta \end{bmatrix}$$

... some (easy) Geometric Invariant Theory (GIT)

- The values $e_i(x_1,\dots,x_n)$ $i=1,\dots,n$ completely determine the set $\{x_1,\dots,x_n\}$.

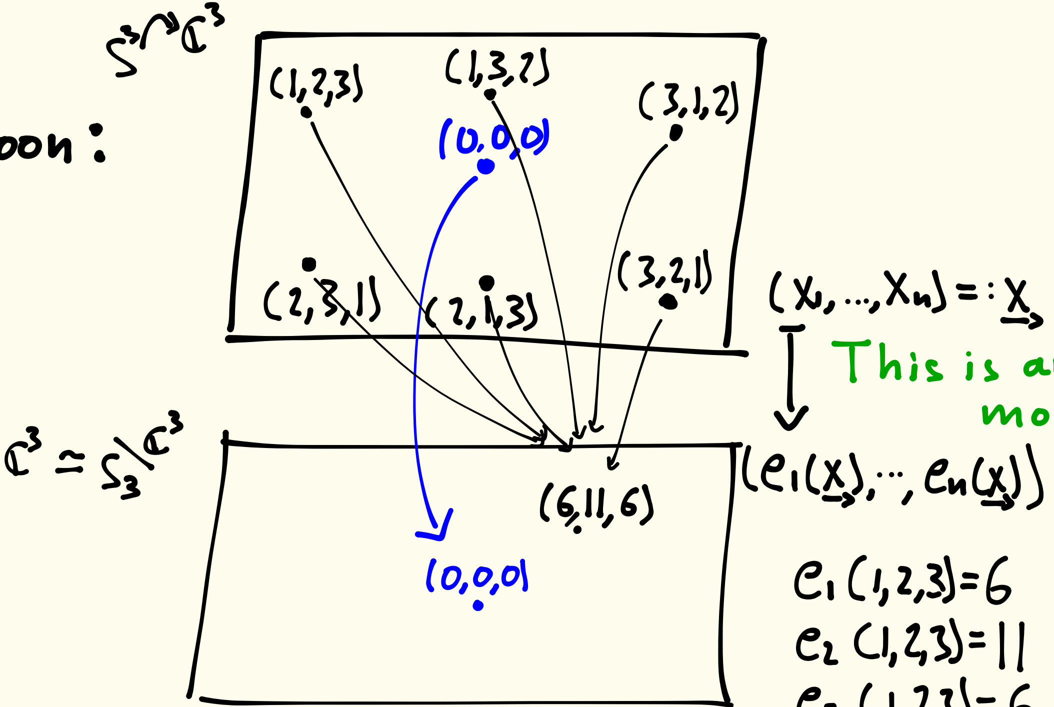
  Indeed, the x_i's are the roots of

$$f(t) = t^n - e_1(\underline{x})t^{n-1} + e_2(\underline{x})t^{n-2} + \cdots + (-1)^n e_n(\underline{x}).$$

- In fact the tuple $(e_1(\underline{x}),\dots,e_n(\underline{x}))$ completely determines the orbit of the tuple $(x_1,\dots,x_n)\in\mathbb{C}^n$ under the action of S_n.

GIT mantra: If a group G acts on the space X, the best candidates for coordinates of the quotient $G\backslash X$ are the invariants $\mathbb{C}[X]^G$.

Cartoon:



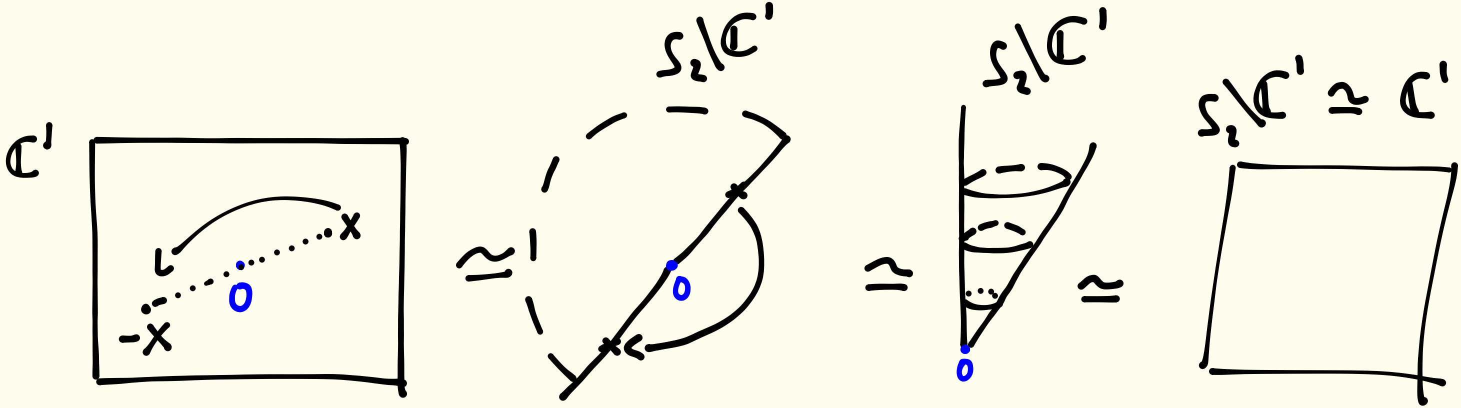
$e_1(1,2,3) = 6$

$e_2(1,2,3) = 11$

$e_3(1,2,3) = 6$

GIT mantra: If $G \le GL_n(\mathbb{C})$ is a cx reflection group and $\mathbb{C}[x_1,\dots,x_n]^G = \mathbb{C}[f_1,\dots,f_n]$, then the polynomial map: $\underline{x} : (x_1,\dots,x_n) \mapsto (f_1(\underline{x}),\dots,f_n(\underline{x}))$ realizes $\mathbb{C}^n$ as the topological quotient $G\backslash\mathbb{C}^n$.

- The fact that $G\backslash\mathbb{C}^n \simeq \mathbb{C}^n$ is very difficult to see topologically (already non-trivial for S_2):



... Discriminant Hypersurfaces

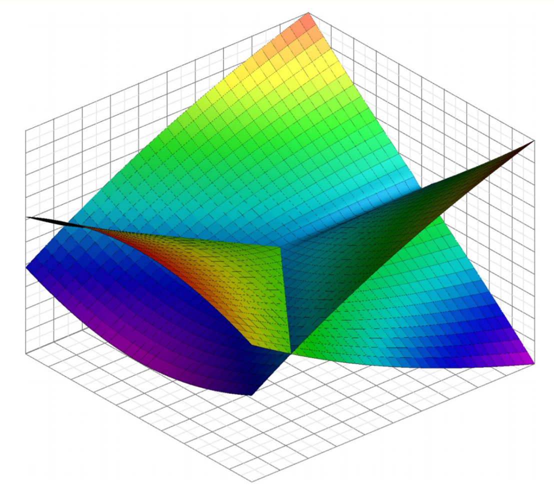
Q: How does the GIT map $\underline{f} = (f_1, \cdots, f_n)$ act on the reflecting hyperplanes?

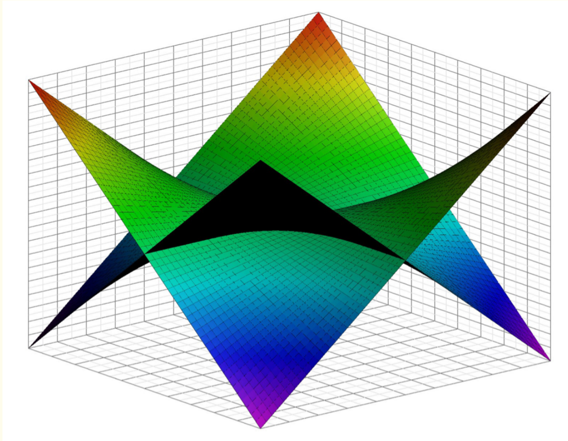
A: It "glues" them together in a hypersurface, called the **discriminant**.

In particular, if ℓ_H is a linear form that cuts H and e_H the order of the associated reflection, then

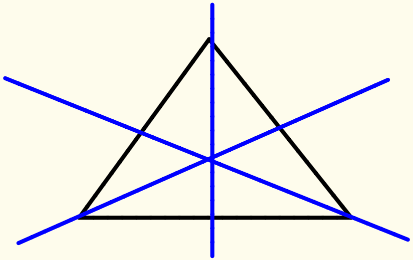
$D := \prod \ell_H^{e_H}$ is G-invt. (i.e. it is a polynomial in the f_i's)

... Discriminant hypersurfaces (example).

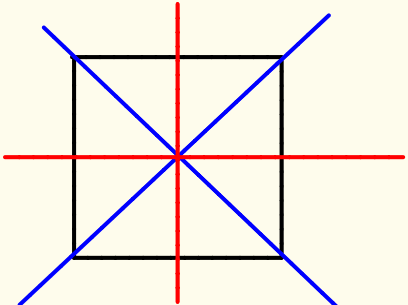




$D(I_2(3)) = f_2^2 - 4f_1^3$

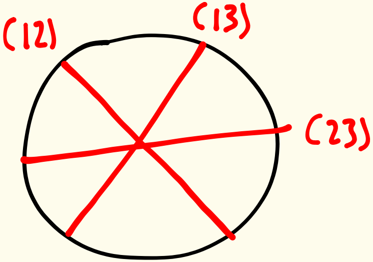


$D(I_2(4)) = f_2^2 - 4f_1^4$

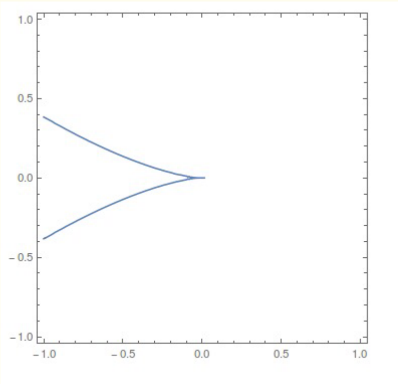


... Discriminant Hypersurfaces: The Swallow's Tail

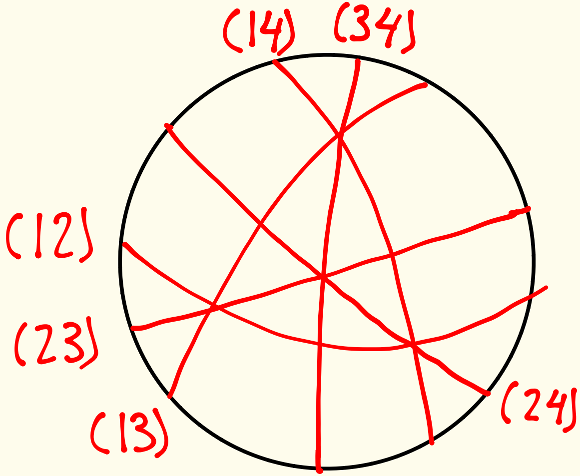
S_3:

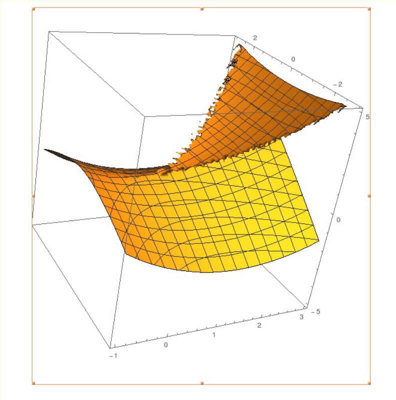


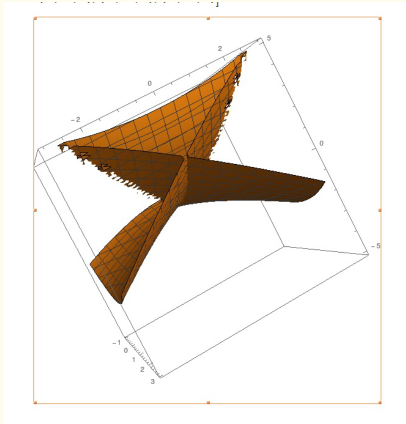
$$D(S_3) = 4c^3 + 27d^2$$
$$= \mathrm{Disc}(t^3 + bt^2 + ct + d)\big|_{b=0}$$



S_4:

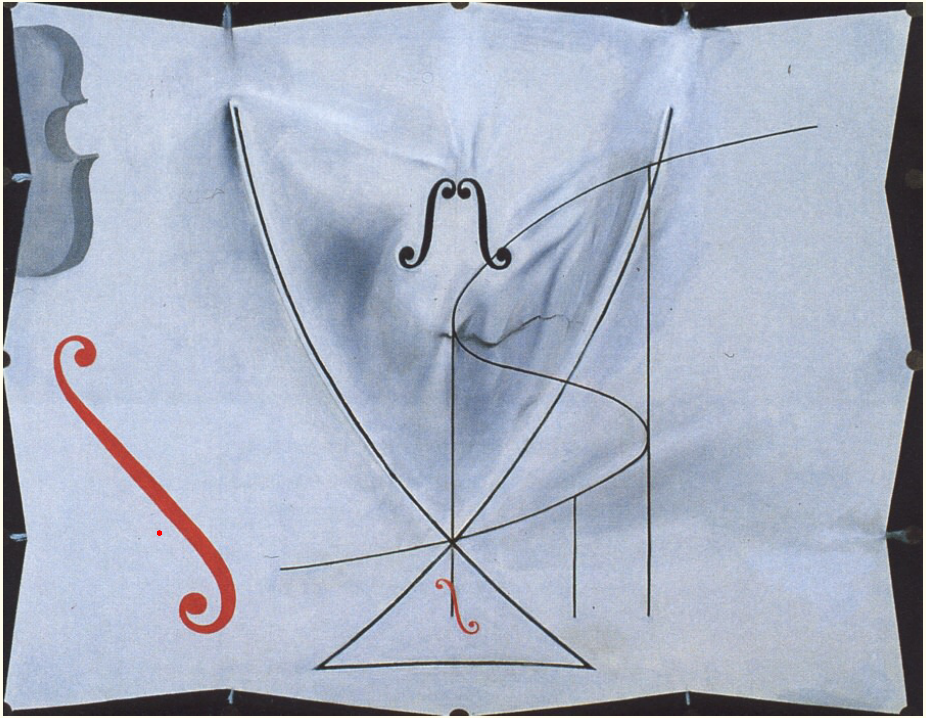






"The most beautiful aesthetic theory in the world"

Salvador Dali, For René Thom's Catastrophe Theory



... the Russian school might have a different opinion about René Thom's style! :

"Neither in 1965 nor later was I ever able to understand a word of Thom's own talks on catastrophes. He once described them to me (in French?) as "blah-blah-blah", when I asked him, in the early seventies, whether he had proved his announcements"

- V.I. Arnold

... but Arnold also loved the swallow-tail.

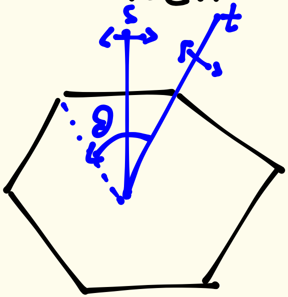
... Coxeter elements ... and their eigenvalues

- For the symmetric group S_n, $c=(123\cdots n)$

$(123\cdots n) * (\zeta^{n-1}, \zeta^{n-2}, \ldots, \zeta, 1)$
$= (1, \zeta^{n-1}, \zeta^{n-2}, \ldots, \zeta)$
$= \zeta \cdot (\zeta^{n-1}, \ldots, 1)$
for $\zeta^n = 1$

($X_i \neq X_j$ above)

- For a finite real reflection group, c is the product (in any order) of the reflections across walls of a chamber.



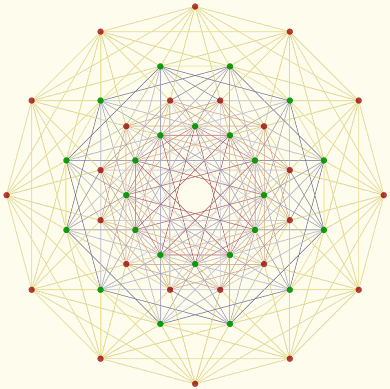
$c = st =$ rotation by θ.

rotation has order m for $I_2(m)$, so eigenvalues ζ, ζ^{-1} for $\zeta^m = 1$

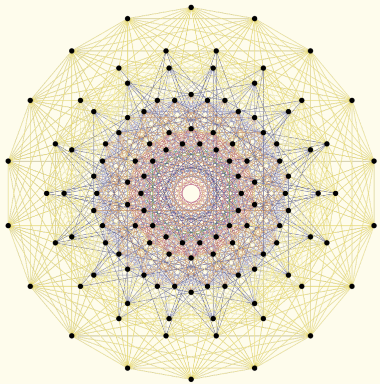
... Pictorial intimidation ...

$\boxed{\text{Thm}}$ Coxeter elts are characterized by having an eigenvector $\vec{v}$, which lies on no refl. hyperplane, with eigenvalue $\zeta = e^{\frac{2\pi i}{d_n}}$, $d_n = \deg(f_n)$.

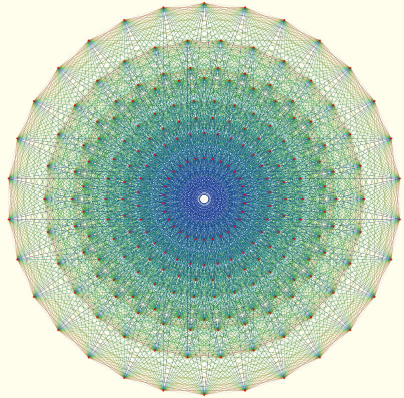
"Proof":



E_6



E_7

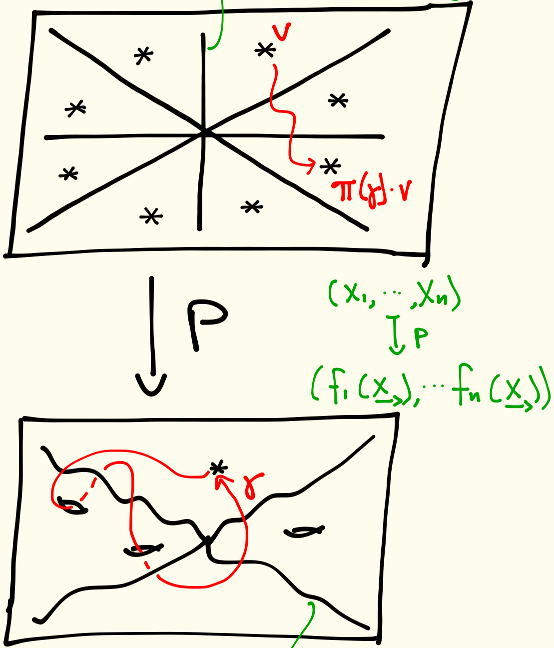


E_8

...towards a geometric construction of the coxeter element...

$W \curvearrowright V \simeq \mathbb{C}^n$

$\bigcup_{H \text{ hyp.}} H = Z\left(\prod_{H \text{ hyp.}} \ell_H\right)$



$W \backslash V \simeq \mathbb{C}^n$

$W \backslash \bigcup_{H \text{ hyp.}} H =: \mathcal{H}$

"discriminant hypersurface"

- **Steinberg:** W acts freely on $V - \bigcup_{H \text{ hyp.}} H$

$\Longrightarrow$ the map $\underline{x} \overset{p}{\longmapsto} \underline{f}(\underline{x})$ is a covering map at $V - \bigcup_{H \text{ hyp.}} H$.

- The covering map $V - \bigcup_{H \text{ hyp.}} H \xrightarrow{p} W \backslash V - \mathcal{H}$ induces maps:

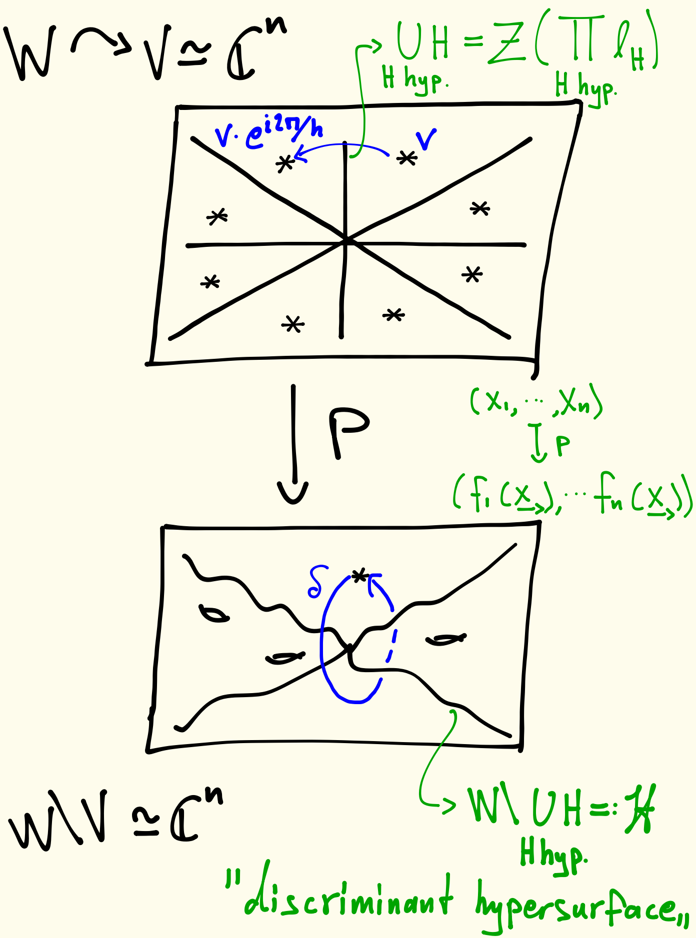
$$1 \longrightarrow \underset{\substack{\| \\ PB(W)}}{\pi_1(V - \cup H)} \longrightarrow \underset{\substack{\| \\ B(W)}}{\pi_1(W \backslash V - \mathcal{H})} \overset{\pi}{\twoheadrightarrow} W \longrightarrow 1$$

- **Saito's thm:** The discriminant equation is monic in f_n. It can be written as:

$$\mathcal{D}(W) = \underbrace{f_n^n + a_2 \cdot f_n^{n-2} + \cdots + a_n}_{\text{quasi-homogeneous, } \deg a_i = i \cdot h} \quad \text{w/ } a_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$$

ex: $A_1: x^2 + bx + c \mapsto \mathcal{D} = b^2 - 4c$

$A_2: x^3 + bx^2 + cx + d \mapsto \mathcal{D} = b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd$



- Consider a loop δ in $W \backslash V$ given by:
  $f_1 = \cdots = f_{n-1} = 0,\ f_n = e^{i\theta}$ $\quad \theta \in [0, 2\pi]$

  It lifts to a path $v \cdot e^{i\theta/h}$, $\theta \in [0, 2\pi]$

  for some v s.th. $f_1(v) = \cdots = f_{n-1}(v) = 0$.

  $\boxed{\text{Saito's thm} \Longrightarrow v \notin \bigcup_{H \text{ hyp.}} H \quad (!!)}$

  i.e. $\delta \in B(W)$. Therefore, $\exists\, c \in W$ $(c = \pi(\delta))$

  (via $1 \to PB(W) \to B(W) \xrightarrow{\pi} W \to 1$)

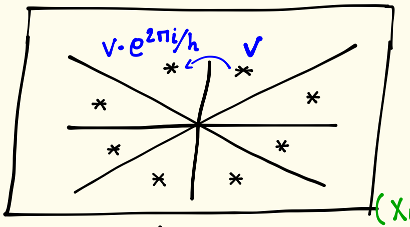
  s.th. $c \cdot v = e^{i2\pi/h} \cdot v$

- c is our coxeter element!

... and now, for something completely different...

Geometric Factorizations of the Coxeter element

$W \curvearrowright V \simeq \mathbb{C}^n$

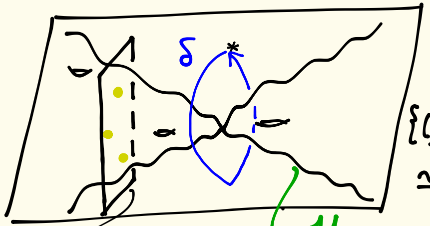


$(x_1, x_2, \dots, x_n)$

$\downarrow P$

$\downarrow P$

$(\underbrace{f_1, \dots, f_{n-1}}_{y}, \underbrace{f_n}_{t})$



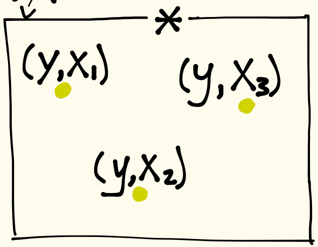
$\{(y,t)\} \simeq \mathbb{C}^{n-1} \times \mathbb{C}$
$\simeq \mathbb{C}^n \simeq W\backslash V$

$\mathcal{H}$ given by

$f_n^n + a_2 f_n^{n-2} + \dots + a_n = 0$

$a_i \in \mathbb{C}[f_1, \dots f_{n-1}]$

ZOOM IN



$(f_1, \dots, f_{n-1}) =$ fixed

$y \times \mathbb{C}$

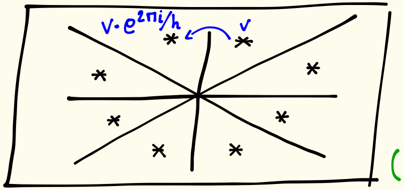
- Consider a slice $y \times \mathbb{C}$ in $W\backslash V$ given by $y = (f_1, \dots, f_{n-1}) =$ fixed and $f_n = t \in \mathbb{C}$ arbitrary.

- The complex line $(y \times \mathbb{C})$ intersects the discriminant hypersurface $\mathcal{H}$ at n-many pts (w/ mult.); call them $\{n_1 \cdot (y, x_1), \dots, n_k \cdot (y, x_k)\}$ w/ $n_i =$ mult. of (y, x_i). order $=$ $\mathbb{C}$-lexicographic

because of Saito + Bessis

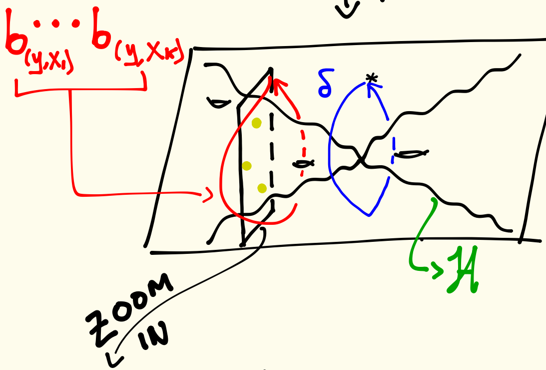
- These are exactly the roots of the monic polynomial $t^n + a_2(y) t^{n-2} + \dots + a_n(y)$

$W \curvearrowright V \simeq \mathbb{C}^n$

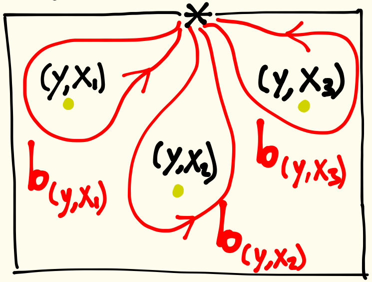


$$\begin{array}{c} (x_1, x_2, \cdots, x_n) \\ \downarrow P \\ (\underbrace{f_1, \cdots, f_{n-1}}_{y}, \underbrace{f_n}_{t}) \end{array}$$

$\downarrow P$



$\{(y,t)\} \simeq \mathbb{C}^{n-1} \times \mathbb{C}$
$\simeq \mathbb{C}^n \simeq W \backslash V$



$(f_1, \cdots, f_{n-1}) = \text{fixed}$
$\uparrow$
$y \times \mathbb{C}$

- Consider small loops $b_{(y,x_i)} \in B(W) = \pi_1(W \backslash V - \mathcal{H})$ that don't leave the slice $y \times \mathbb{C}$ and each surrounds counterclockwise only the point (y, x_i).

- Each such loop gives an element $c_{(y,x_i)} \in W$ via Galois action:

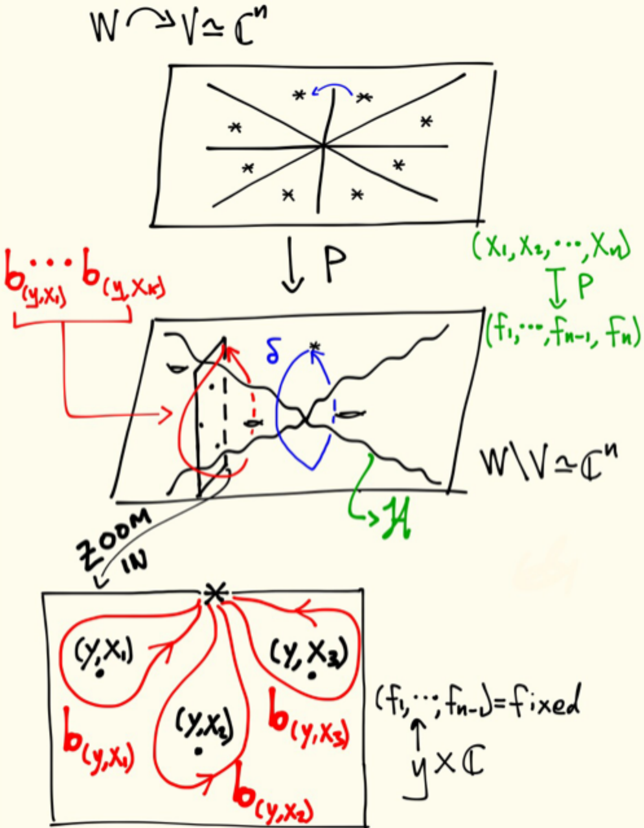
$$1 \to PB(W) \to B(W) \to W \to 1$$

- Their product "completely surrounds" $\mathcal{H}$. In fact, $b_{(y,x_1)} \cdots b_{(y,x_k)} = \delta$

Or, in W, $c_{(y,x_1)} \cdots\cdots c_{(y,x_k)} = c$

A factorization of the coxeter elt.

"Geometry is the art of reasoning well from badly drawn figures" H. Poincaré – Analysis situs



- "block factorizations". If $\ell_R(w)$ is the absolute refl. length (i.e. smallest k s.th. $\exists$ fact. $t_1 \cdot t_2 \cdot \cdots \cdot t_k = w$ w/ t_i reflections), then:

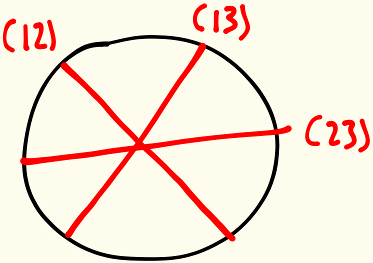
$$\sum \ell_R(c_{(y,x_i)}) = \ell_R(c) = n$$

- $c_{(y,x_i)}$ is a "Parabolic Coxeter element": there is some $v \in \underset{\text{flat}}{X} \subseteq \underset{H \text{ hyp}}{\bigcup H}$, $v \in p^{-1}((y,x_i))$ s.th. $c_{(y,x_i)}$ is a coxeter element in the parabolic subgroup

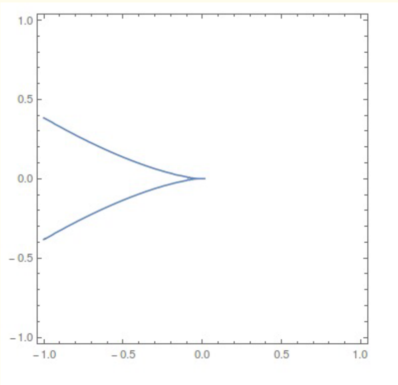
$$W_v := \{ w \in W : w \cdot v = v \}$$

...Discriminant Hypersurfaces: The Swallow's Tail

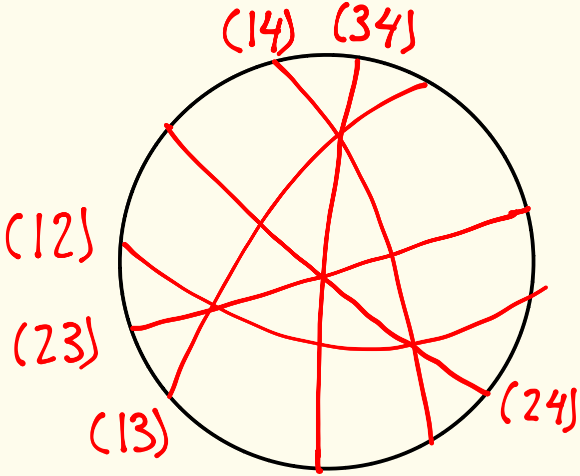
S_3:

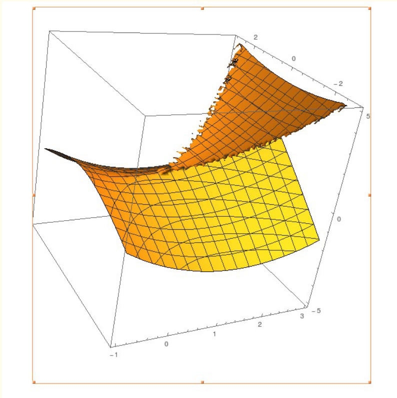


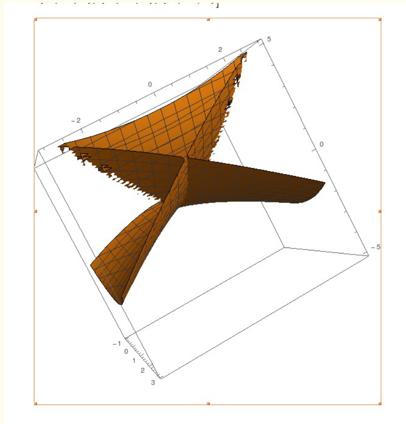
$$D(S_3) = 4c^3 + 27d^2$$
$$= \mathrm{Disc}(t^3 + bt^2 + ct + d)\big|_{b=0}$$

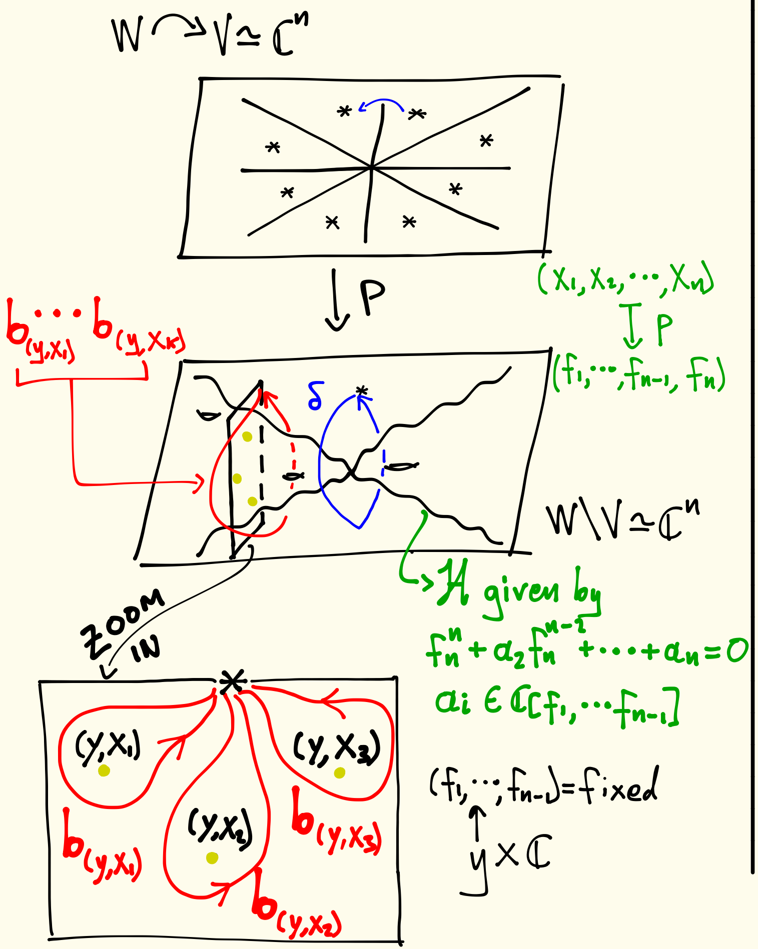


S_4:



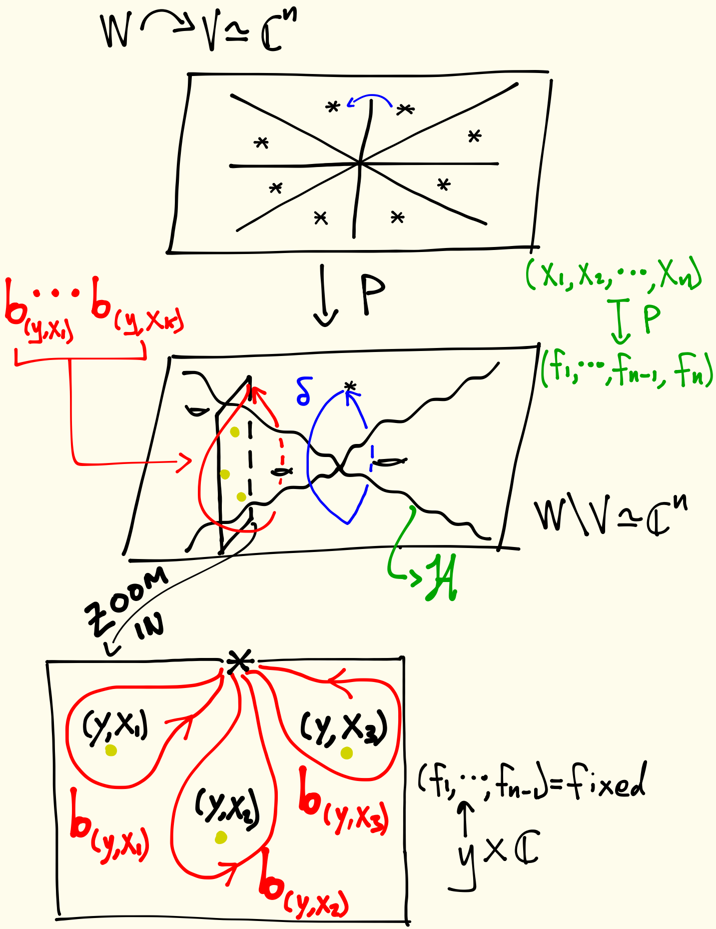






Q: What happens when we move the slice $y \times \mathbb{C}$?

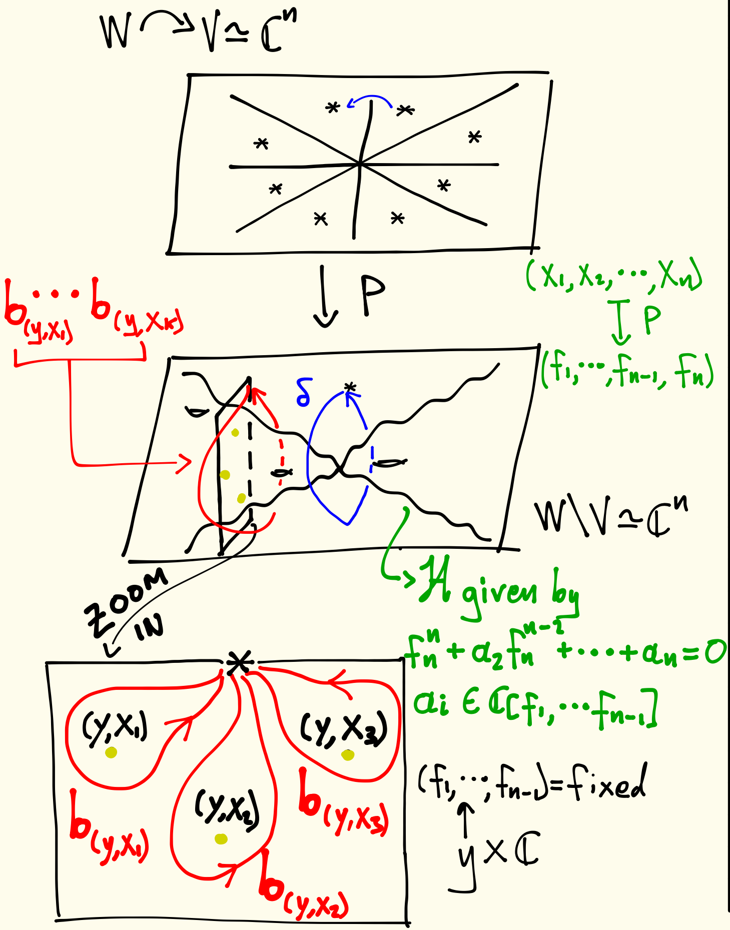
A [Bessis]: All block factorizations are attained "exactly once" !!!



Trivialization Theorem [Bessis]

For every block factorization $w_1 \cdots w_k = c$ and every configuration of pts $\{n_1 \cdot z_1, \cdots n_k \cdot z_k\}$ in $\mathbb{C}$ with the multiplicities satisfying $n_i = \ell_R(w_i)$,

there exists a unique $y = (f_1, \cdots, f_{n-1})$ $(y \in \mathbb{C}^{n-1})$ s.th.: $c_{(y,x_i)} = w_i$ and $x_i = z_i$



Important points in the proof:

- LL-morphism: The assignment

$$(f_1, \cdots, f_{n-1}) = y \overset{LL}{\longmapsto} \left\{ \begin{matrix} \text{intersection pts} \\ \text{of } y \times \mathbb{C} \cap \mathcal{H} \end{matrix} \right\}$$

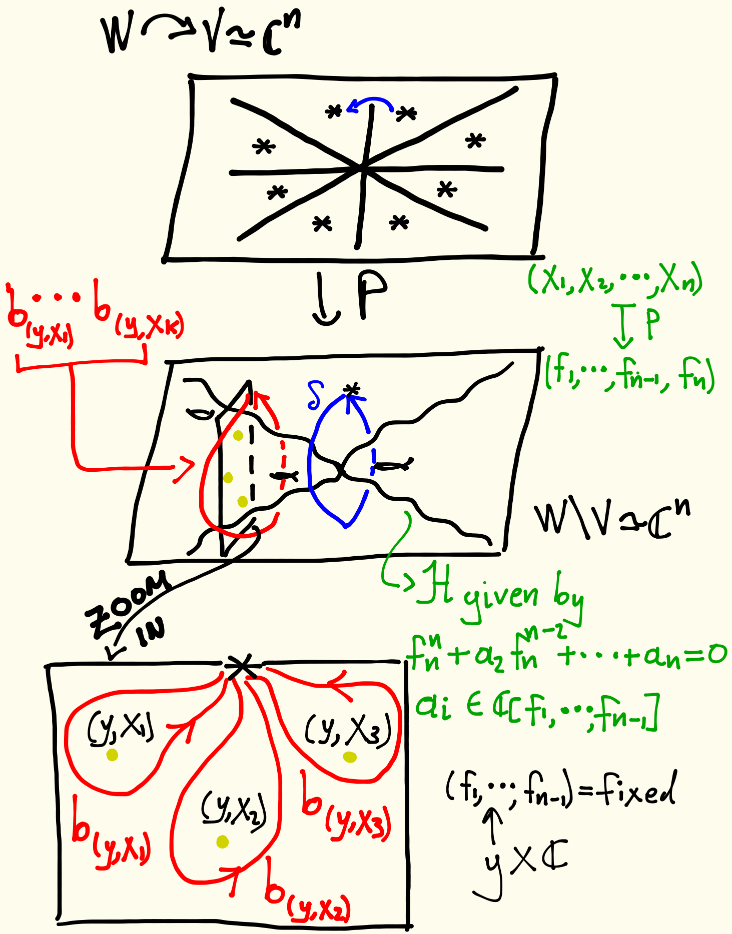
is an algebraic finite morphism.

Indeed $y \overset{LL}{\longmapsto} \left\{ \begin{matrix} \text{roots of} \\ t^n + a_2(y) t^{n-2} + \cdots + a_n(y) \end{matrix} \right\}$

becomes $y \overset{LL}{\longmapsto} (a_2(y), \cdots, a_n(y))$

$(LL : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1})$

Remark: This comes down to $LL^{-1}(\vec{0}) = \vec{0}$ which needs a geometric argument that depends on the fact that $\deg f_n \gneq \deg f_i$

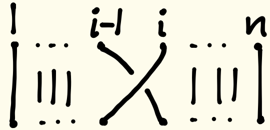


Important points in the proof:

- Transitivity of the Hurwitz action:

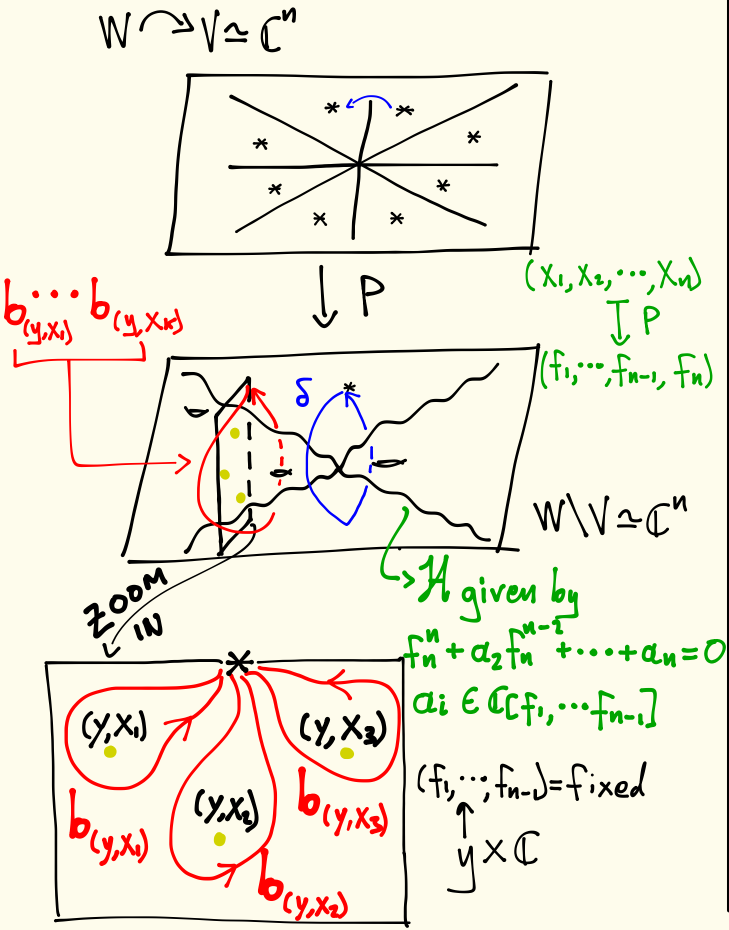
The braid group B_n acts naturally on reflection factorizations of c:

$$(t_1, \dots, t_{i-1}, t_i, \dots, t_n) \mapsto (t_1, \dots, t_i, t_i^{-1} t_{i-1} t_i, \dots, t_n)$$



The action is transitive on reduced reflection factorizations.

[Bessis, uniform in the real case]



Important points in the proof:

- The following numerological coincidence:

$$\deg LL = \frac{h^n \cdot n!}{|W|} = \mathrm{Red}_R(c)$$

!!!

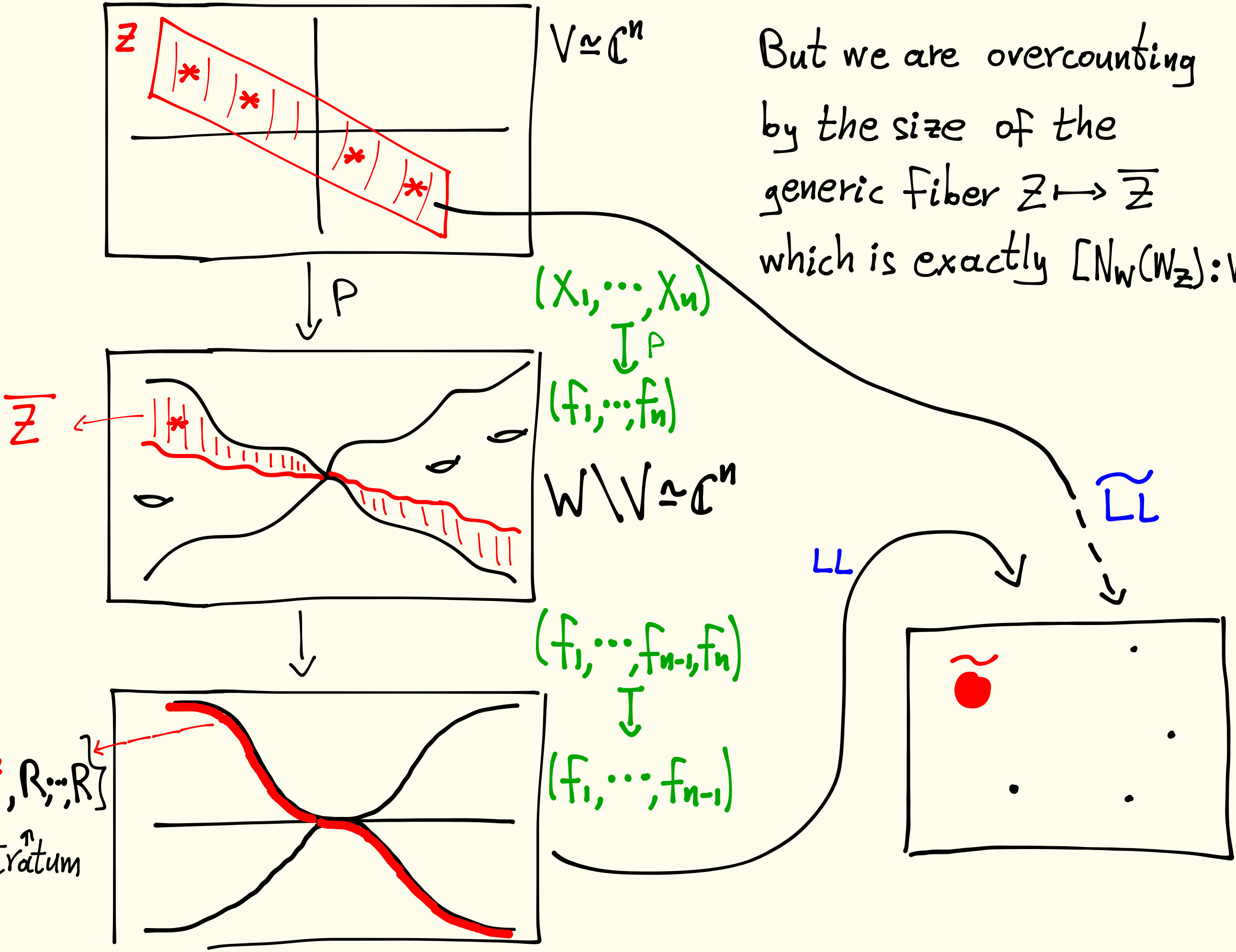
(# of shortest refl. fact. of c)

Q: Can we prove the above equality without resorting to computation?

$\Rightarrow$ Then numerology & transitivity of Hurwitz action on $\mathrm{Red}_R(c)$ would come as natural corollaries.

$\Rightarrow$ True but not trivial for type A

PRIMITIVE CASE



But we are overcounting by the size of the generic fiber $Z \mapsto \overline{Z}$ which is exactly $[N_W(W_Z) : W_Z]$

Computing $\deg LL$ for the "primitive" case:

R E D U C E D C A S E

$$(f_1,\dots,f_n)=y \xmapsto{LL} \left\{ \begin{matrix} \text{roots of} \\ t^n+a_2(y)t^{n-2}+\dots+a_n(y) \end{matrix} \right\}$$ or equivalently

$$(f_1,\dots,f_n)=y \xmapsto{LL} (a_2(y),\dots,a_n(y))$$

Now $$\deg LL = \frac{\prod_{i=2}^{n} \deg a_i}{\prod_{i=1}^{n-1} \deg f_i} = \frac{h^n\cdot n!}{|W|}$$

P R I M I T I V E C A S E

The corresponding (union of) stratum (-a) will be $\{Z, R,\dots,R\}$

↳ the (one or two) strata of reflections (in $W\backslash V$)

Here we can lift LL to a finite morphism $\widetilde{LL}$ defined over Z.

$$Z \ni (x_1,\dots,x_\ell)=:x \xmapsto{\widetilde{LL}} \left(f_n(x), \left\{ \begin{matrix} \text{roots of} \\ [t-f_n\underbrace{(x)]^{n-\ell}\cdot[t^\ell+b_1}_{\text{linear relation}}(x)t^{\ell-1}+\dots+b_\ell(x)] \end{matrix} \right\}\right)$$

or $$(x_1,\dots,x_\ell) \xmapsto{\widetilde{LL}} (f_n(x), b_2(x),\dots,b_\ell(x))$$

So $$\deg \widetilde{LL} = \frac{\deg f_n\cdot\prod_{i=2}^{\ell}\deg b_i}{\prod_{i=1}^{\ell}\deg x_i} = \frac{h^{\dim X}\cdot(\dim X)!}{1}$$

Q: Does the formula $\dfrac{h^{\dim X}\cdot(\dim X)!}{|N_X : W_X|}$ generalize?

A: Reinterpreting Krattenthaler & Müller's results:

type A_{n-1}: $\mathrm{Fact}_{[X_1,\dots,X_d]} = n^{d-1}\cdot\prod_{i=1}^{d}\dfrac{(\dim X_i)!}{[N_{X_i}:W_{X_i}]}$

type B_n: $\mathrm{Fact}_{[X_1,\dots,X_j,\dots,X_d]} = (2^n\cdot n)^{d-1}\cdot\dfrac{(\dim X_j)!}{[N_{X_j}:W_{X_j}]}\cdot\prod_{i\neq j}\dfrac{(\dim X_i-1)!}{[N_{X_i}:W_{X_i}]}$

↳ unique of type B_x ...

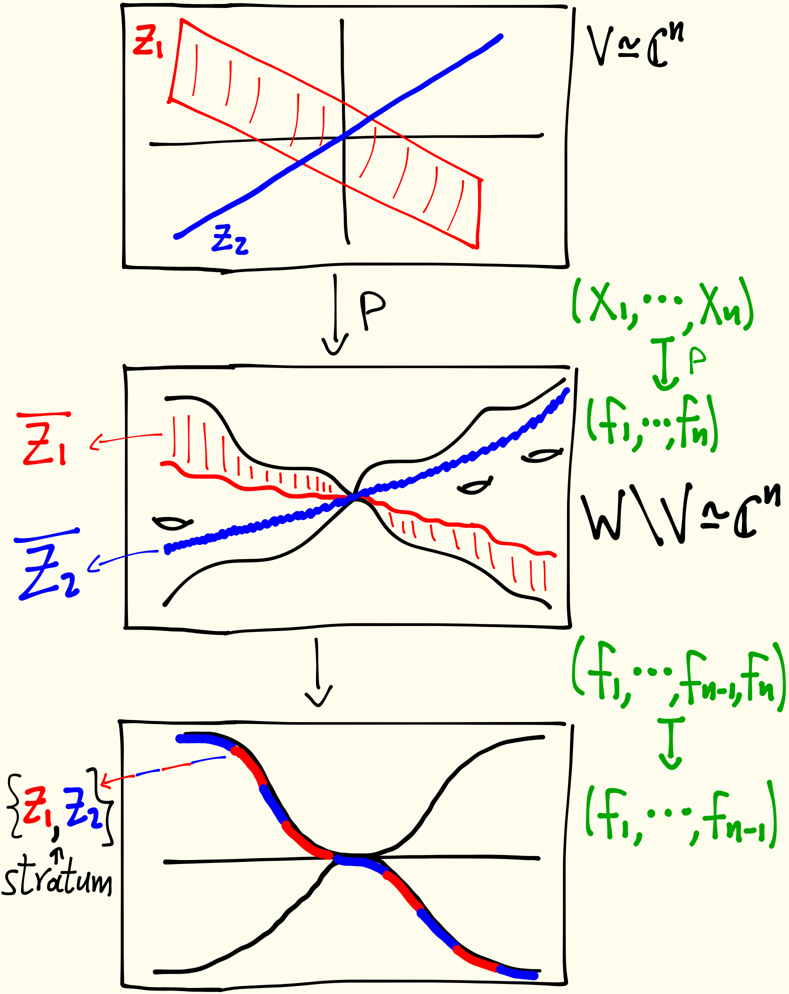
... for type D it is (as always) a bit tricky :

$$\text{type } D_n: \ \mathrm{Fact}_{[X_1,\dots,X_j,\dots X_d]} = \left[2^{n-1}\cdot(n-1)\right]^{d-1}\cdot\frac{(\dim X_j)!}{[N_{X_j}:W_{X_j}]}\cdot\prod_{i\neq j}\frac{(\dim X_i-2)!\cdot m_0^{(i)}}{[N_{X_i}:W_{X_i}]}$$

unique of type D

Many thanks to:

- The organizers for the invitation to lovely Boston and the MIT combinatorics seminar.
- Vic Reiner, David Bessis, and Vivien Ripoll for sharing this beautiful math with me
- You all for coming !!



The "shadow" stratification

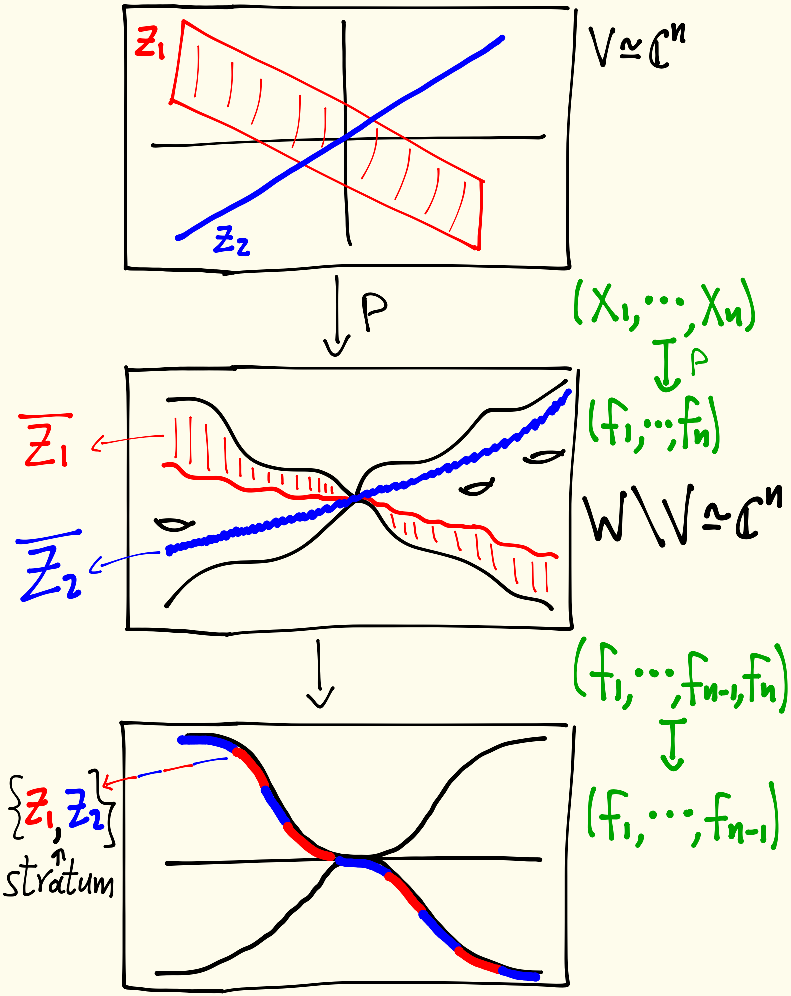
- The arrangement of the refl. hyperplanes Z_i of W stratifies the space V.
- This induces a stratification of the quotient $W \backslash V$ in strata $\overline{Z_i}$.
- This in turn induces a "shadow" stratification of the space Y, the domain of the LL-map. $\left(Y = \{(f_1, \cdots, f_{n-1})\} \simeq \mathbb{C}^{n-1} \right)$:

Given flats $Z_1, \cdots Z_k$ w/ $\sum_{i=1}^{k} \operatorname{codim}(Z_i) = n$,

define

$$\{Z_1, \cdots, Z_k\} = \left\{ \begin{array}{l} y \in Y \simeq \mathbb{C}^{n-1} \text{ s.th.} \\ \exists \text{ a perm. } \sigma \in S_k : \\ (y, x_i) \in \overline{Z_{\sigma(i)}} \ \forall i \end{array} \right\}$$

The "shadow" stratification

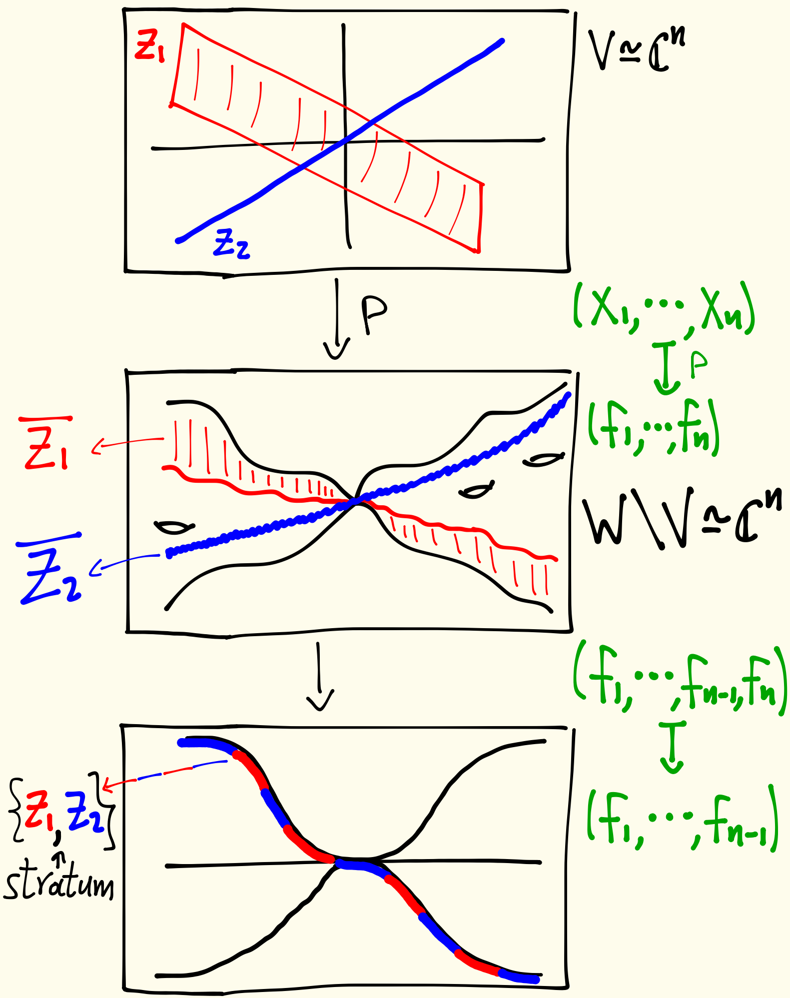


Why should I care??

- Understanding the LL-map on the strata $\{Z_1, \cdots, Z_k\}$ gives us info on block factorizations of c w/ "passport" $= [Z_1, \cdots, Z_k]$
  ↳ $w_1 \cdot \cdots \cdot w_k = c$ & $\overline{V^{w_i}} = Z_i$

In fact,

- [Ripoll, 2010]: The Hurwitz action on block factorizations of c with "passport" $[Z_1, \cdots, Z_k]$ is transitive iff the stratum $\{Z_1, \cdots, Z_k\}$ is connected.



The "shadow" stratification

Why should I care??

Evenmore,

• [via Bessis' Trivialization thm]:

If $\overline{\{Z_1, \cdots, Z_K\}}$ is the (topological) closure of the stratum, then

$\deg LL|_{\overline{\{Z_1, \cdots, Z_K\}}}$ equals the

# of block factorizations of c w/ passport $[Z_1, \cdots, Z_K]$

... generally difficult but sometimes $\overline{\{Z_1, \cdots, Z_K\}}$ is the image of an affine space.