Geometric Techniques in Coxeter-Catalan combinatorics

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Some enumeration Formulas:

- Thm [Hurwitz, 1892]

$$\#\left\{\begin{array}{l}\text{shortest factorizations of an } n\text{-cycle} \\ \quad \underset{(123\cdots n)}{c} = t_1 \cdot \cdots \cdot t_{n-1} \text{ w/ } t_i \text{ transpositions}\end{array}\right\} = n^{n-2}$$

$(12)(23) = (123)$
$(13)(12) = (123)$
$(23)(13) = (123)$

Now, given a coxeter element c in an irreducible, well-generated complex reflection group W of rank n, with $\operatorname{ord}(c) = h$:

- Thm [Bessis 2006-2016]

$$\#\{\text{shortest reflection factorizations } c = t_1 \cdots t_n\} = \frac{h^n \cdot n!}{|W|}$$

Some enumeration formulas:

- Thm [D. 2016] Given an intersection flat X,

$$\#\left\{\begin{array}{l}\text{shortest factorizations } c=\chi\cdot t_1\cdots t_\ell,\ \ell=\dim X,\\ \text{w/ } t_i \text{ reflections } \& \ V^{\chi} \text{ in the } W\text{-orbit of } X\end{array}\right\} = \frac{h^{\dim X}\cdot(\dim X)!}{[N_X : W_X]}$$

$$\Big\downarrow{\scriptstyle X=V}$$

$$\left[\text{Bessis}: \frac{h^n\cdot n!}{|W|}\right] \xRightarrow{W=A_{n-1}} \left[\text{Hurwitz}: n^{n-2}\right]$$

Coxeter-Catalan Combinatorics

- A complex reflection group W is a finite subgroup of $GL(V) \simeq GL_n(\mathbb{C})$ generated by "pseudoreflections" t_i that are of the form

$$\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ & & & & & \zeta \end{bmatrix}$$

for some root of unity ζ.

- Invariant Theory [Chevalley-Shephard-Todd]

A finite subgroup $W \leq GL(V)$ is a complex reflection group, if and only if its invariant subalgebra

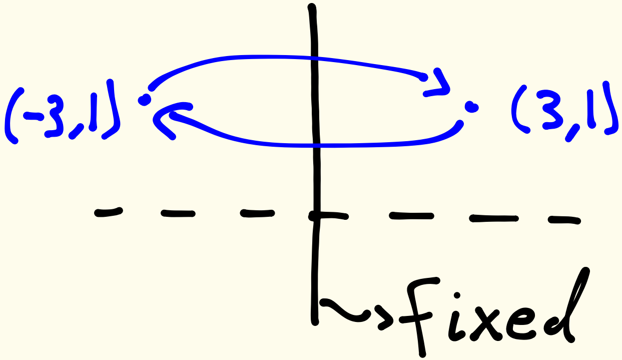
$$\mathbb{C}[V]^W := \{ f \in \mathbb{C}[V] : f(w^{-1} \cdot x) = f(x) \ \forall x \in V, \forall w \in W \}$$

is a polynomial algebra. In fact, then $W \backslash V \simeq \mathbb{C}^n$

An example and a non-example:

$C_2 = \{id, (12)\} \curvearrowright \mathbb{R}^2$ via

$(x,y) \longmapsto (-x, y)$



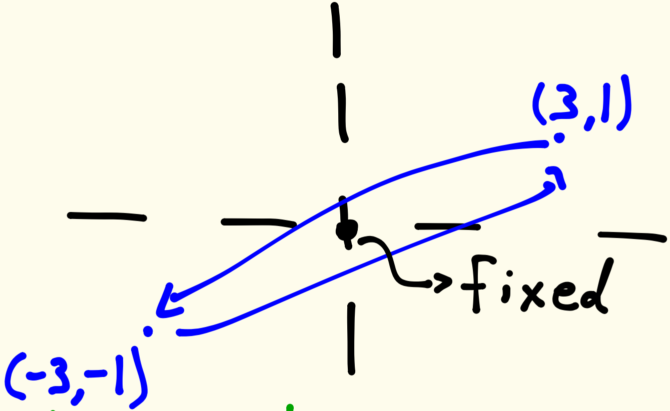
Invariant polynomials:

$f_1 = x^2, \; f_2 = y$

$* \; \deg(f_1) \cdot \deg(f_2) = |C_2|$

$C_2 = \{id, (12)\} \curvearrowright \mathbb{R}^2$ via

$(x,y) \longmapsto (-x, -y)$



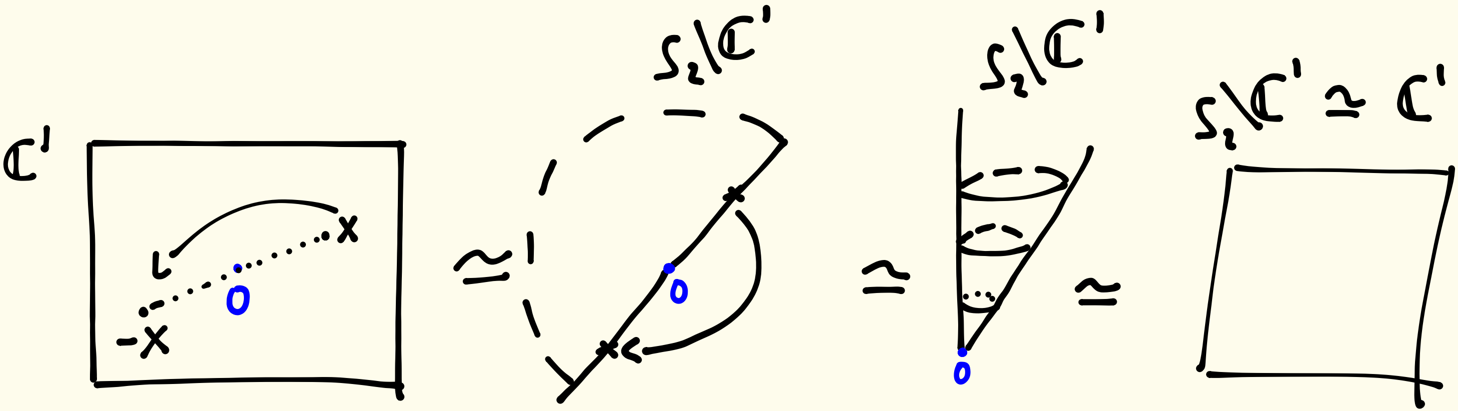
Invariant polynomials:

$f_1 = x^2, \; f_2 = y^2, \; f_3 = xy$

$\boxed{f_1 \cdot f_2 = f_3^2}$

GIT Theory: If $G \leq GL_n(\mathbb{C})$ is a complex reflection group and $\mathbb{C}[x_1,\dots,x_n]^G = \mathbb{C}[f_1,\cdots,f_n]$, then the polynomial map: $\underline{x}:(x_1,\cdots,x_n) \mapsto (f_1(\underline{x}),\cdots,f_n(\underline{x}))$ realizes $\mathbb{C}^n$ as the topological quotient $G\backslash\mathbb{C}^n$.

- The fact that $G\backslash\mathbb{C}^n \simeq \mathbb{C}^n$ is very difficult to see topologically (already non-trivial for S_2):



... Discriminant Hypersurfaces

Q: How does the GIT map $\underline{f} = (f_1, \cdots, f_n)$ act on the reflecting hyperplanes?

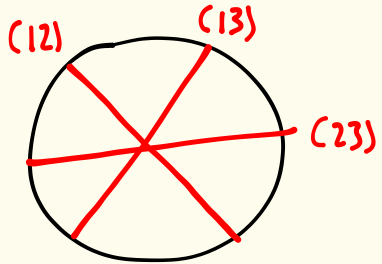
A: It "glues" them together in a hypersurface, called the discriminant.

In particular, if l_H is a linear form that cuts H and e_H the order of the associated reflection, then

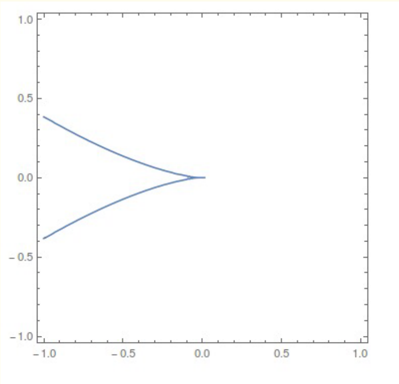
$D := \prod l_H^{e_H}$ is G-invt. (i.e. it is a polynomial in the f_i's)

... Discriminant Hypersurfaces: The Swallow's Tail

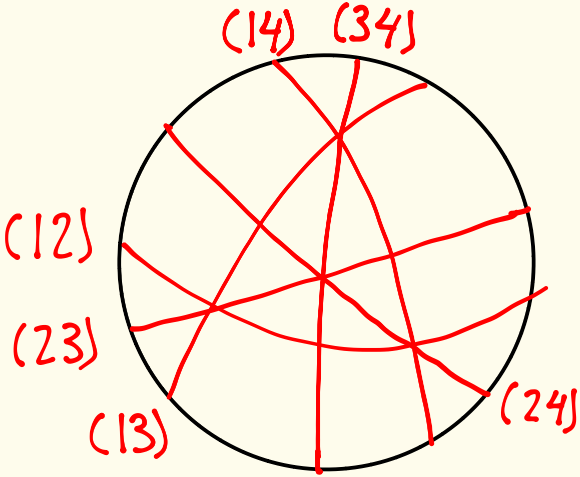
S_3:

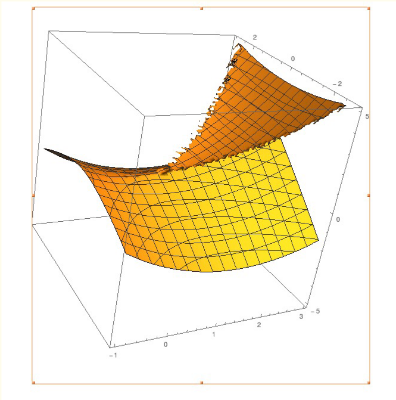


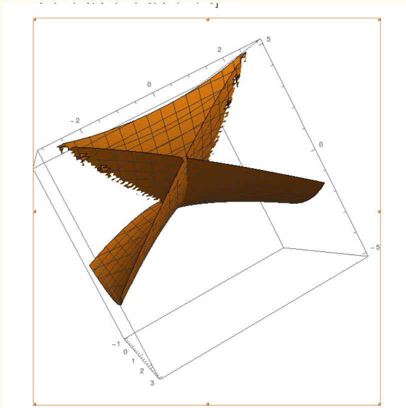
$$D(S_3) = 4c^3 + 27d^2$$
$$= \mathrm{Disc}(t^3 + bt^2 + ct + d)\big|_{b=0}$$



S_4:

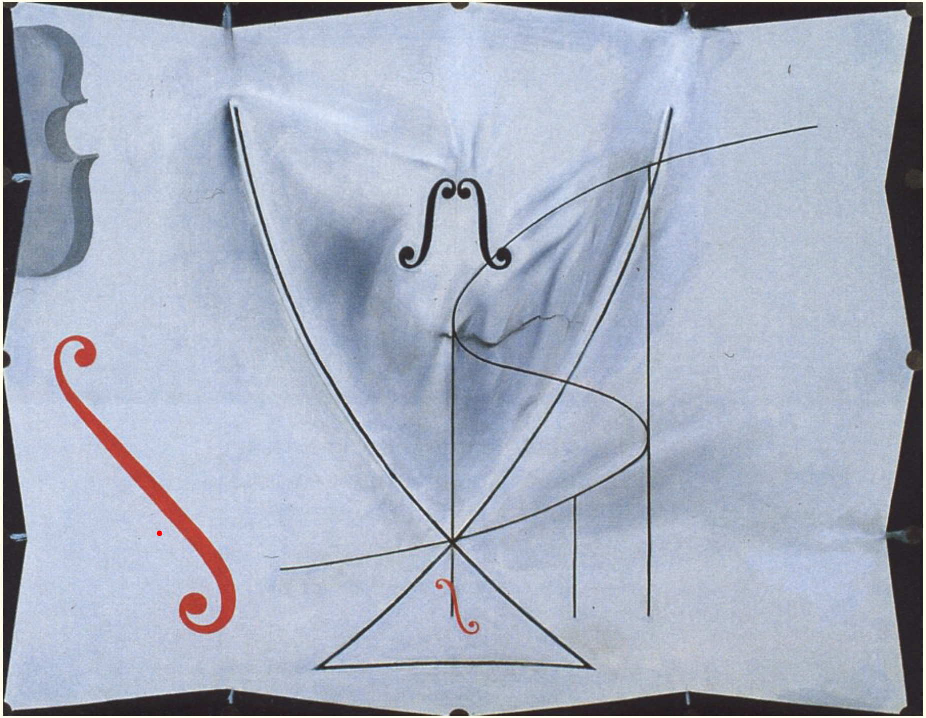






"The most beautiful aesthetic theory in the world"

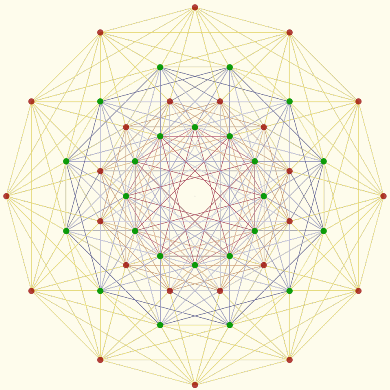
Salvador Dali, For René Thom's Catastrophe Theory



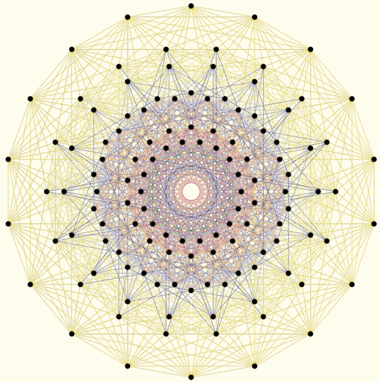
Coxeter elements via Springer

Thm Coxeter elts are characterized by having an eigenvector $\vec{v}$, which lies on no refl. hyperplane, with eigenvalue $\zeta = e^{\frac{2\pi i}{h}}$, $h = \frac{|R|+|R^*|}{n}$

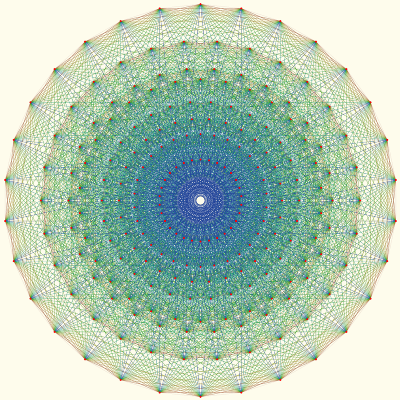
"Proof":



E_6



E_7

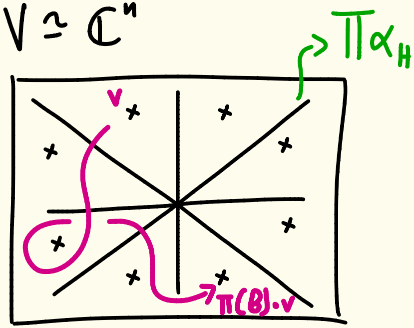


E_8

... towards a topological construction of the coxeter element...

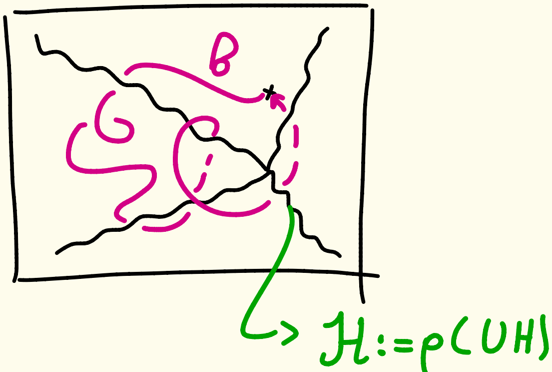
Towards a topological construction of a Coxeter element

$W \curvearrowright V \cong \mathbb{C}^n$



$$\rho \downarrow \qquad \begin{array}{c} \vec{x} := (x_1, \cdots, x_n) \\ \downarrow \\ (f_1(\vec{x}), \cdots, f_n(\vec{x})) \end{array}$$

$(W \backslash V) \cong \mathbb{C}^n$



Steinberg's Theorem:

W acts freely on $V^{reg} := V \backslash \cup H$

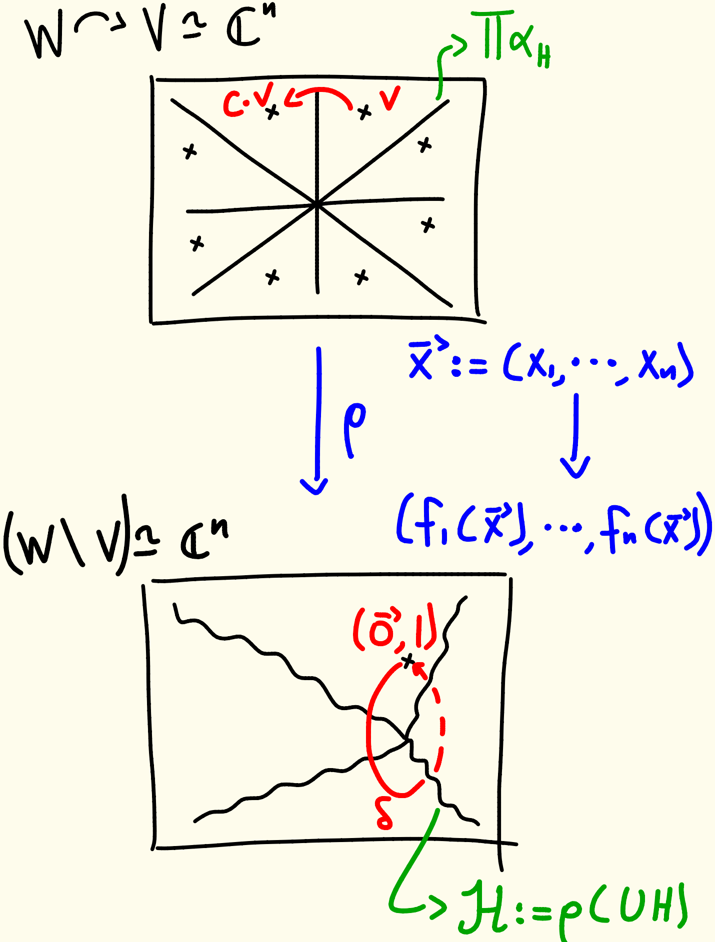
$\Rightarrow \rho : V^{reg} \to W \backslash V^{reg}$ is a covering map.

$$1 \to P(W) \hookrightarrow B(W) \xrightarrow{\pi} W \to 1$$

$$P(W) = \pi_1(V^{reg}) \qquad B(W) = \pi_1(W \backslash V^{reg}) = \pi_1(\mathbb{C}^n \backslash \mathcal{H})$$

Significance: W is realized as the group of deck-transform'ns of a covering map ρ, which is <u>explicitly</u> given via the f_i's.

Topological construction of a Coxeter element



Saito-Bessis Theorem:

W is well-gen'd $\Leftrightarrow \exists\ (f_1, \cdots, f_n)$ s.th. the discriminant $\mathcal{H}$ has equation:

$$f_n^n + a_2 f_n^{n-2} + \cdots + a_n = 0,$$

where $a_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$.

Now, pick $v \in V^{reg}$ such that:

$f_1(v) = \cdots = f_{n-1}(v) = 0,\ f_n(v) = 1$

path: $B(t) := e^{(2\pi i/h)\cdot t} \cdot v \quad t \in [0,1]$

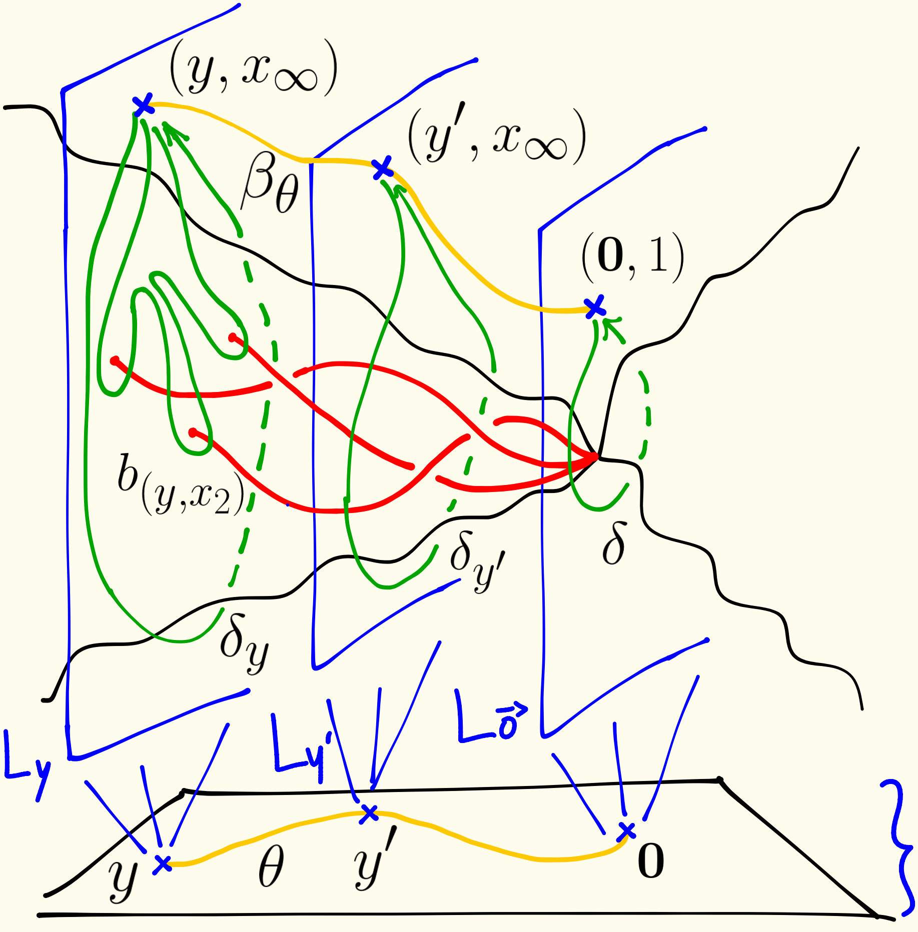
$\rho(B(t))$: $f_i(B(t)) = 0 \quad i \leq n-1$

$f_n(B(t)) = e^{(2\pi i)\cdot t}$

$\delta := \rho(B(t)) \in B(W)$

$c := \pi(\delta)$ is the Coxeter element

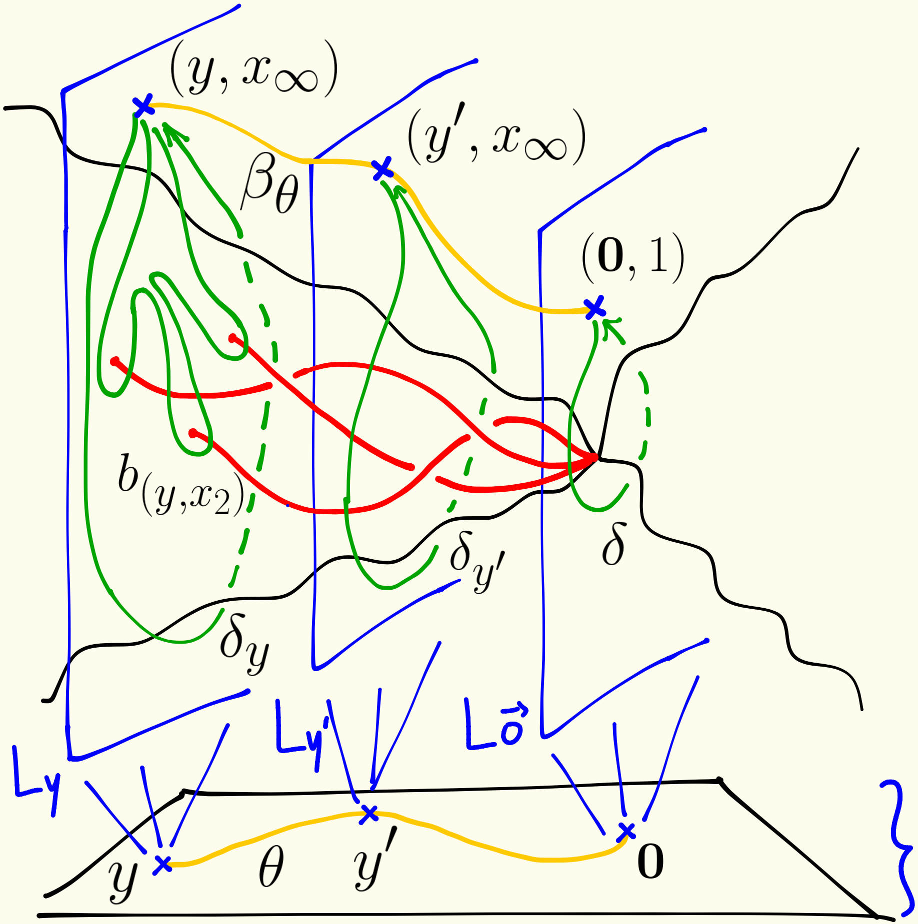
Topological Factorizations of Coxeter elements



- Pick a path $\theta : \vec{0} \to y$ in Y.
- Lift to a path β_θ in $W \backslash V$ that "stays above" $\mathcal{H}$.
- If $L_y \cap \mathcal{H} = \{x_1, \dots x_k\}$ bring little loops b_i from $x_\infty(y)$ down and around the x_i's
- Define $b_{(y,x_i)} = \beta_\theta \cdot b_i \cdot \overline{\beta_\theta}$

} "base space" $Y \simeq \mathbb{C}^{n-1} \ni (f_1, \dots, f_{n-1})$

Topological Factorizations of Coxeter elements



We define the "reduced label" map

$rlbl(y) := (c_1, \cdots, c_k)$

where $c_i := \pi(b_{(y,x_i)}) \in W$

via $1 \to P(W) \hookrightarrow B(W) \xrightarrow{\pi} W \to 1$

Notice that:

$b_{(y,x_1)} \cdots b_{(y,x_k)} \simeq \beta_\theta \cdot \delta_y \cdot \bar{\beta}_\theta \simeq \delta$

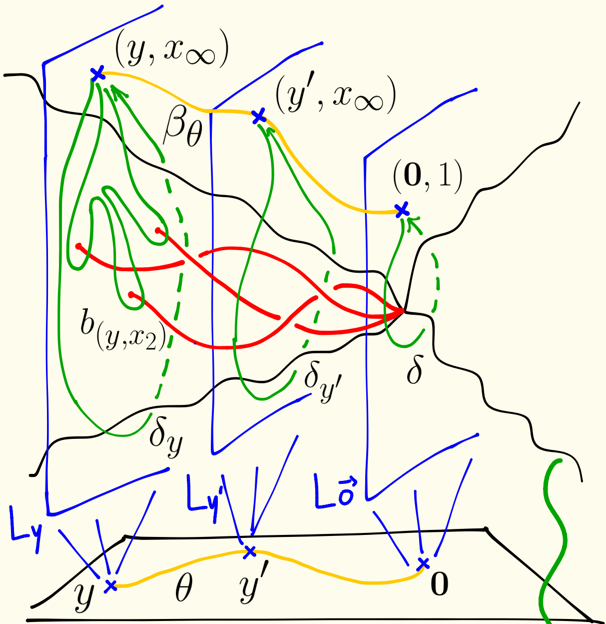
$\Longrightarrow c_1 \cdots c_k = c$

! rlbl is well-defined !

"base space"

$Y \simeq \mathbb{C}^{n-1} \ni (f_1, \cdots, f_{n-1})$

The Lyashko-Looijenga morphism



$\mathcal{H}$ is given by eqn:

$f_n^n + a_2 f_n^{n-2} + \cdots + a_n = 0$

$a_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$

We define the LL map:

$$LL: Y \longmapsto \left\{ \begin{array}{c} \text{centered configurations} \\ \text{of } n \text{ points in } \mathbb{C} \end{array} \right\} := E_n$$

$$y \longmapsto \text{multiset } L_y \cap \mathcal{H}$$

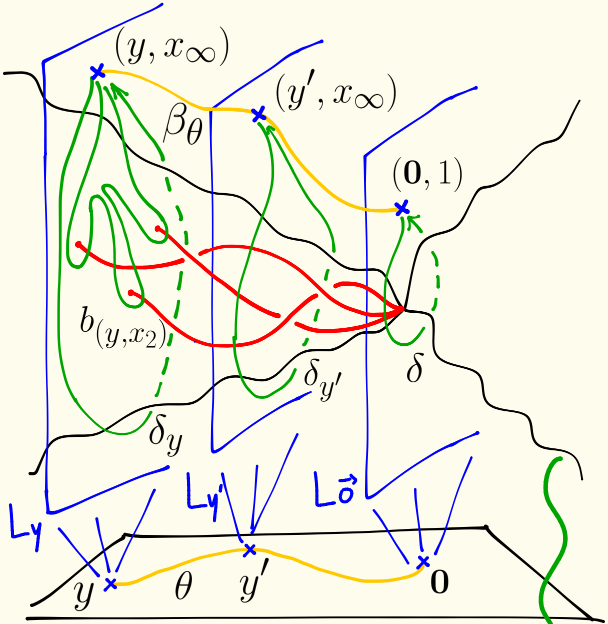
Algebraically:

$$LL: Y \simeq \mathbb{C}^{n-1} \longrightarrow E_n \simeq \mathbb{C}^{n-1}$$

$$y = (f_1, \cdots, f_{n-1}) \mapsto \left\{ \begin{array}{c} \text{roots of} \\ t^n + a_2(y)\, t^{n-2} + \cdots + a_n(y) = 0 \end{array} \right\}$$

$$(f_1, \cdots, f_{n-1}) \xmapsto{LL} (a_2(y), \cdots, a_n(y))$$

Properties of the LL & rlbl maps:



$\mathcal{H}$ is given by eqn:

$$f_n^n + \alpha_2 f_n^{n-2} + \cdots + \alpha_n = 0$$

$$\alpha_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$$

- The line L_y is transverse to $\mathcal{H}$ for all y.
- The LL map is a finite morphism. Its degree is given by

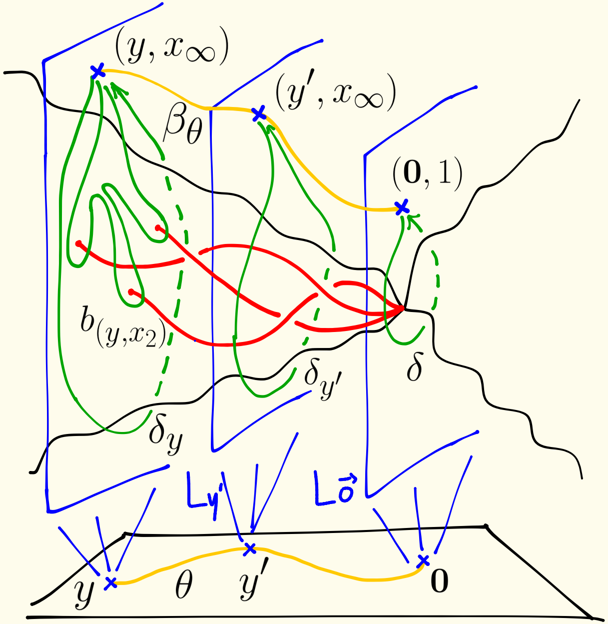
$$\deg(LL) = \frac{\prod_{i=2}^{n} \deg \alpha_i}{\prod_{i=1}^{n-1} \deg f_i} = \frac{2h \cdot \cdots \cdot nh}{d_1 \cdots d_{n-1}} = \frac{h^{n-1} \cdot n!}{\frac{|W|}{d_n}} = \frac{h^n \cdot n!}{|W|}$$

$$\left(y = (f_1, \cdots, f_{n-1}) \xmapsto{LL} (\alpha_2(y), \cdots, \alpha_n(y)) \right)$$

- LL and rlbl are compatible:

If $LL(y) = \{x_1, \cdots; x_k\}$ with $n_i := \mathrm{mult}(x_i)$ and $\mathrm{rlbl}(y) = (c_1, \cdots, c_k)$, then $\ell_R(c_i) = n_i$.

The Trivialization Theorem (Bessis)



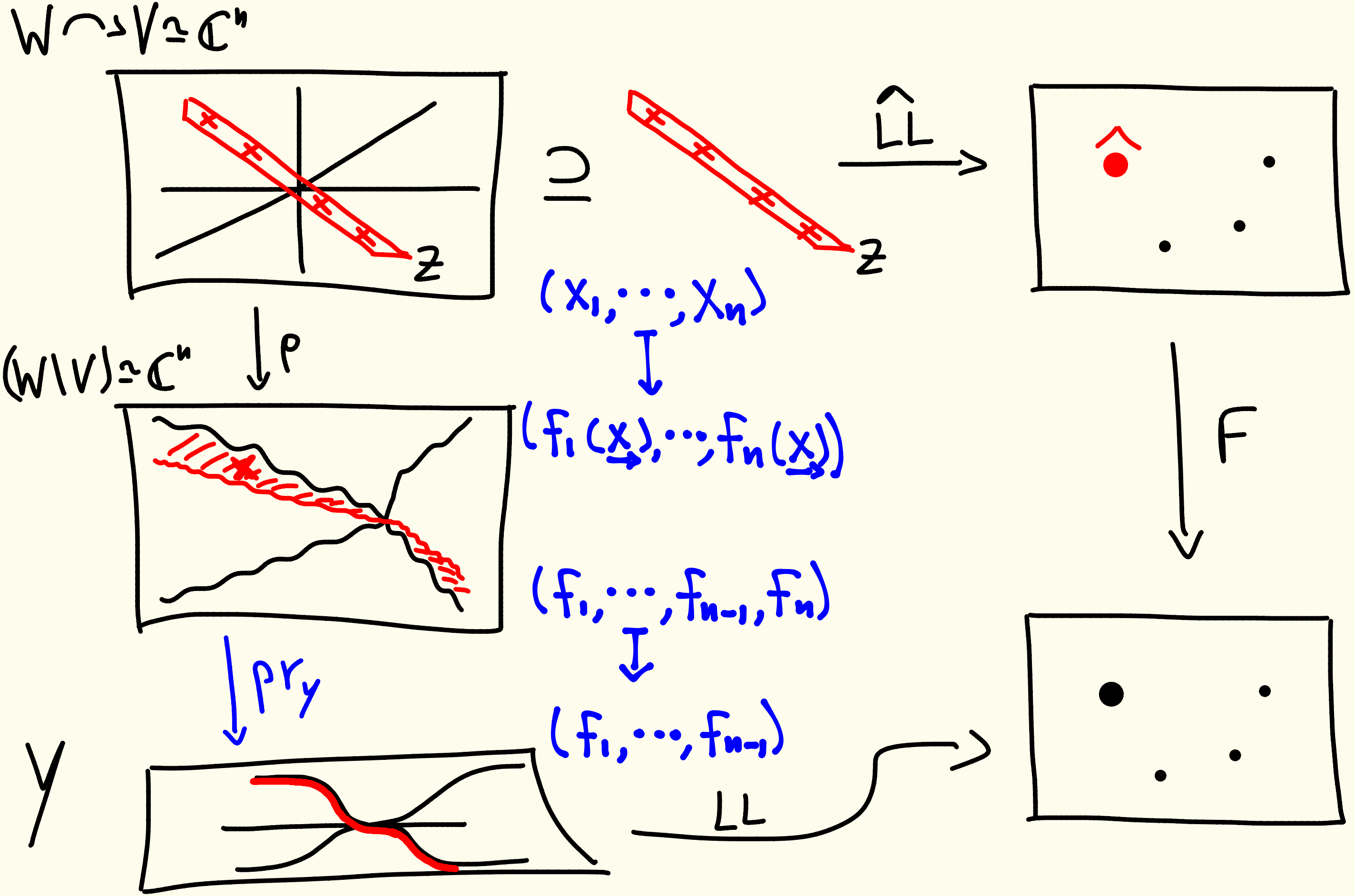
The map

$LL \times rlbl : y \mapsto \left\{ \begin{array}{l} \text{compatible pairs of} \\ (\{X_1, \cdots, X_k\}, (c_1, \cdots, c_k)) \end{array} \right\}$

is a bijection!

*! Depends on the numerological coincidence: $\deg(LL) = |\mathrm{Red}_W(c)|$

Primitive Factorizations



Primitive Factorizations

We can lift the LL map to any flat Z :

$$Z \ni (z_1, \cdots, z_k) =: \vec{z} \xmapsto{\widehat{LL}} \text{multiset } L_y \cap \mathcal{H}, \text{ decorated at } f_n(\vec{z})$$

In coordinates:

$$\widehat{LL}(\vec{z}) = \left(f_n(\vec{z}), \left\{ \underbrace{[t - f_n(\vec{z})]^{n-k} \; [t^k + b_1(\vec{z}) t^{k-1} + \cdots + b_k(\vec{z})]}_{\text{linear relation}} \right\}_{\text{roots of}} \right)$$

or:

$$(z_1, \cdots, z_k) \xmapsto{\widehat{LL}} (b_1(\vec{z}), \cdots, b_k(\vec{z}))$$

$$\text{So, } \deg \widehat{LL} = \prod_{i=1}^{k} \deg(b_i) = h \cdot 2h \cdots (kh) = h^k \cdot k! = h^{\dim Z} \cdot (\dim Z)!$$

We have overcounted factorizations by $[N_W(Z) : W_Z]$.

$$\text{So, } |\mathrm{Fact}_W(Z)| = \frac{h^{\dim Z} (\dim Z)!}{[N_W(Z) : W_Z]}$$

Towards a uniform proof of the Trivialization Theorem

Pick a configuration $e = \{x_1, \cdots, x_k\}$ with multiplicities n_i.

Compare:

$$\deg(LL) = \sum_{\substack{\sigma = (c_1, \cdots, c_k) \\ \text{compatible} \\ \text{with } e.}} \left| LL^{-1}(e) \cap rlbl^{-1}(\sigma) \right| \cdot \mathrm{mult}_{y(\sigma)}(LL)$$

and:

$$\left| Red_W(c) \right| = \sum_{\substack{\sigma = (c_1, \cdots, c_k) \\ \text{compatible} \\ \text{with } e.}} 1 \cdot \prod_{i=1}^{k} \left| Red_W(c_i) \right|$$

Some cyclic-sieving-phenomena (CSP's):

Consider the following action
on reduced reflection factorizations:

$$\Phi: (t_1, t_2, \cdots, t_n) \longmapsto ({}^{c}t_n, t_1, \cdots, t_{n-1}) \qquad ({}^{c}t_n := c \cdot t_n \cdot c^{-1})$$

Q: How many factorizations are fixed by Φ^d ?

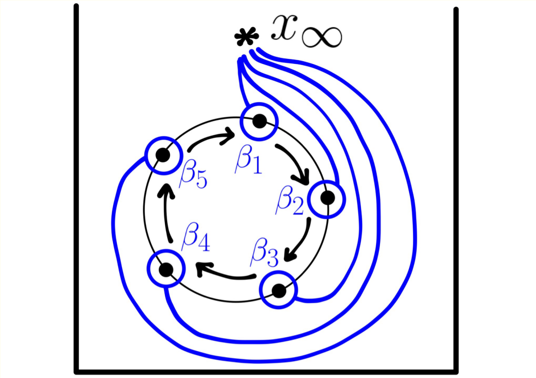
A: $$\left.\left(\underbrace{\prod_{i=1}^{n} \frac{[h \cdot i]_q}{[di]_q}}_{\mathrm{Hilb}(LL^{-1}(0),\,q)}\right)\right|_{q=\zeta^d} \qquad \text{where } \zeta = e^{2\pi i/nh}$$

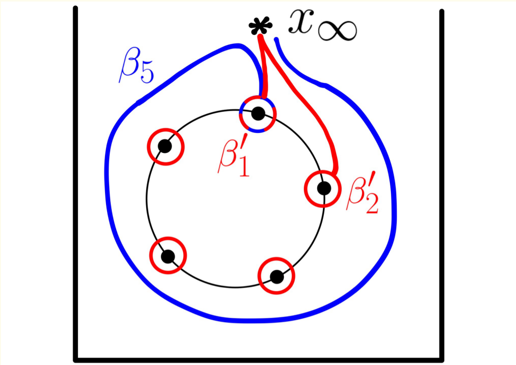
Some cyclic-sieving-phenomena (CSP's):

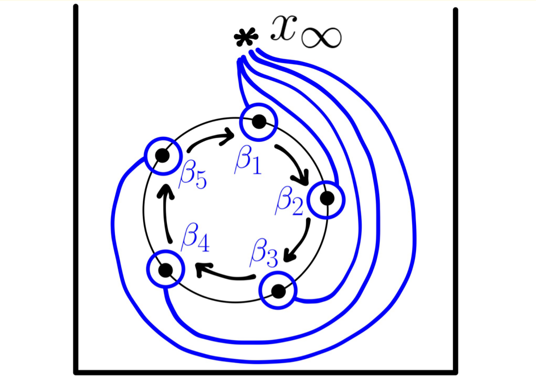
The reason is the following compatibility of the rlbl map with a scalar action:

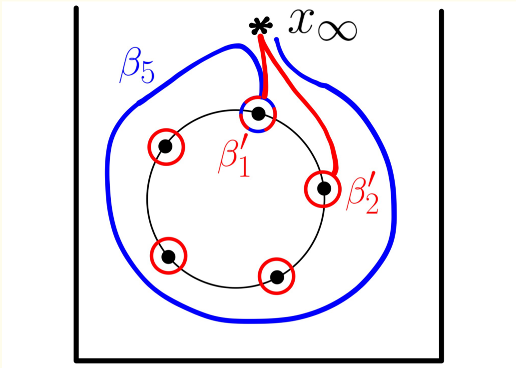
$$\mathrm{rlbl}(\xi * y) = \Phi \cdot \mathrm{rlbl}(y)$$

$\xi = e^{2\pi i/h}$ & $\xi * \underline{y} = \xi * (f_1, \cdots, f_{n-1}) = (\xi^{d_1} \cdot f_1, \cdots, \xi^{d_{n-1}} \cdot f_{n-1})$









- For particular (symmetric) point configurations $\vec{e}$, the fiber $LL^{-1}(\vec{e})$ carries a natural action of a cyclic group $C_d \leq \mathbb{C}^*$.

- On the other hand, the fiber $LL^{-1}(e)$ is always a deformation of the special fiber $LL^{-1}(\vec{0})$ and retains part of its $\mathbb{C}^*$-structure.

- The Hilbert series of the (graded) ring $K[LL^{-1}(0)]$ encodes its $\mathbb{C}^*$-structure.

Thank you!