Applications of geometric techniques in Coxeter-Catalan combinatorics

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Some enumeration formulas:

$(12)(23) = (123)$
$(13)(12) = (123)$
$(23)(13) = (123)$

- Thm [Hurwitz, 1892]

$$|\mathrm{Red}_{\mathfrak{S}_n}(c)| := \#\left\{\begin{array}{l}\text{shortest factorizations of an } n\text{-cycle} \\ \quad c = t_1 \cdot \dots \cdot t_{n-1} \text{ w/ } t_i \text{ transpositions}\end{array}\right\} = n^{n-2}$$

$c \downarrow (123\cdots n)$

Now, given a coxeter element c in an irreducible, well-generated complex reflection group W of rank n, with $\mathrm{ord}(c) = h$:

- Thm [Bessis 2006-2016]

$$|\mathrm{Red}_W(c)| := \#\{\text{shortest reflection factorizations } c = t_1 \cdots t_n\} = \frac{h^n \cdot n!}{|W|}$$

Some enumeration formulas:

- Thm[D. 2016] Given an intersection flat Z,

$$|\mathrm{Fact}_W(Z)| := \#\left\{\begin{array}{l}\text{shortest factorizations } c = w\cdot t_1\cdots t_\ell,\ \ell = \dim Z, \\ \text{w/ } t_i \text{ reflections } \& \ V^w \text{ in the } W\text{-orbit of } Z\end{array}\right\}$$

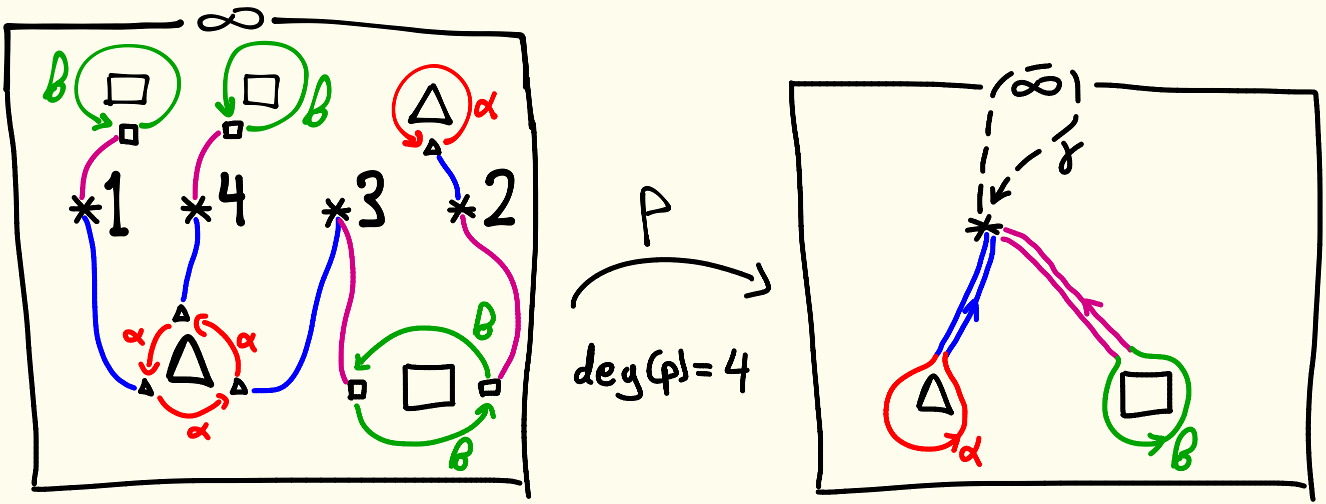
$$= \frac{h^{\dim Z}\cdot(\dim Z)!}{[N_W(Z):W_Z]}$$

$Z=V$

[Bessis: $\dfrac{h^n\cdot n!}{|W|}$] $\xrightarrow[(h=n)]{W=A_{n-1}}$ [Hurwitz: n^{n-2}]

"The simplest way to prove these theorems of [Hurwitz and] Cayley is perhaps to count the multiplicity of the quasi-homogeneous Lyashko-Looijenga map." - V. I. Arnol'd

X-ray of a polynomial map $p: \mathbb{C} \to \mathbb{C}$:



"monodromy around △" : (134)(2)

"monodromy around □" : (1)(23)(4)

$(23)\cdot(134) = (1234)$

Is there an inverse construction?

Riemann's Existence Theorem:

$$\left\{ \begin{array}{c} \text{polynomial maps } p:\mathbb{C}\to\mathbb{C} \\ \text{of the form} \\ p = z^n + a_2 z^{n-2} + \cdots + a_n \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \left(\begin{array}{c} \{w_1,\dots,w_k\} \subseteq \mathbb{C} \text{ critical values,} \\ \text{length-additive factorizations} \\ c_1 \cdots c_k = (1\,2 \cdots n) \end{array} \right) \right\}$$

$$p \longmapsto \left(\begin{array}{c} \text{critical values} \\ \text{of } p \end{array}, \text{ monodromy as before} \right)$$

So, $|\mathrm{Red}_{\mathfrak{S}_n}(c)|$ = # of polynomials of degree n, with $n-1$ fixed, distinct critical values.

Defn: The Lyashko-Looijenga (LL) morphism sends a polynomial $p = z^n + a_2 z^{n-2} + \cdots + a_n \in \mathrm{Poly}_n$ to its multiset of critical values $\{w_1, \ldots, w_k\}$.

(where $\mathrm{mult}(w_i) = \#$ of critical pts z_j (counted with multiplicity) s.th. $p(z_j) = w_i$)

Riemann's Existence Theorem $\Rightarrow$ $|\mathrm{Red}_{\mathfrak{S}_n}(c)| =$ size of generic fiber of LL

Coordinate Presentation for LL

Domain

$$p = z^n + a_2 z^{n-2} + \cdots + a_n \in \mathrm{Poly}_n$$

$$\Updownarrow$$

$$(a_2, a_3, \cdots, a_n) \in \mathbb{C}^{n-1}$$

Target

multiset of critical values

$$\{w_1, \cdots, w_{n-1}\}$$

$$\Updownarrow$$

polynomial $(t+w_1)\cdots(t+w_{n-1})$

$$= t^{n-1} + b_1 t^{n-2} + \cdots b_{n-1}$$

$$\Updownarrow$$

$$(b_1, \cdots, b_{n-1}) \in \mathbb{C}^{n-1}$$

(i.e. $b_i = e_i(w_1, \cdots, w_{n-1})$ where e_i is the i^{th} elementary symm. polynomial)

Geometry of the LL map

$$LL : \vec{a} := (a_2, \cdots, a_n) \longmapsto (b_1(\vec{a}), \cdots, b_{n-1}(\vec{a}))$$

$$LL : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}$$

Properties:

a) LL is a polynomial map

b) LL is quasi-homogeneous with weights :

$$(a_2, \cdots, a_n) \longleftrightarrow (2, 3, \cdots, n) \quad \& \quad (b_1, \cdots, b_{n-1}) \longleftrightarrow (n, 2n, \cdots, (n-1)n)$$

c) LL is a finite morphism

Geometry of the LL map

b) Domain :

Consider the scalar action on Poly_n :

$$\lambda * p = \lambda^n \cdot p \qquad (\lambda \in \mathbb{C})$$

in coordinates: $\lambda * p = \lambda^n \cdot (z^n + a_2 z^{n-2} + \cdots + a_n)$

$$= (\lambda z)^n + a_2 \lambda^2 (\lambda z)^{n-2} + \cdots + a_n \lambda^n$$

set $z' = \lambda z$: $\quad = (z')^n + a_2 \lambda^2 (z')^{n-2} + \cdots + a_n \lambda^n$

i.e.: $\lambda * (a_2, \cdots, a_n) = (a_2 \lambda^2, \cdots, a_n \lambda^n)$

Geometry of the LL map

b) ... Target:

The critical values of $\lambda * p = \lambda^n \cdot p$:

$$\{w_1, \dots, w_k\} \longrightarrow \{\lambda^n \cdot w_1, \dots, \lambda^n \cdot w_k\}$$

In coordinates:

$$b_i(\lambda * p) = e_i(\lambda^n w_1, \dots, \lambda^n w_k)$$

$$= (\lambda^n)^i \cdot e_i(w_1, \dots, w_k) = \lambda^{ni} \cdot b_i$$

So, $\lambda * (b_1, \dots, b_{n-1}) = (\lambda^n b_1, \dots, \lambda^{n(n-1)} b_{n-1})$

$\blacksquare$

Geometry of the LL map

a) Polynomiality :

w is a critical value if $p(z)-w$ has a double root. That is, if $\mathrm{Disc}_z(p(z)-w)=0$

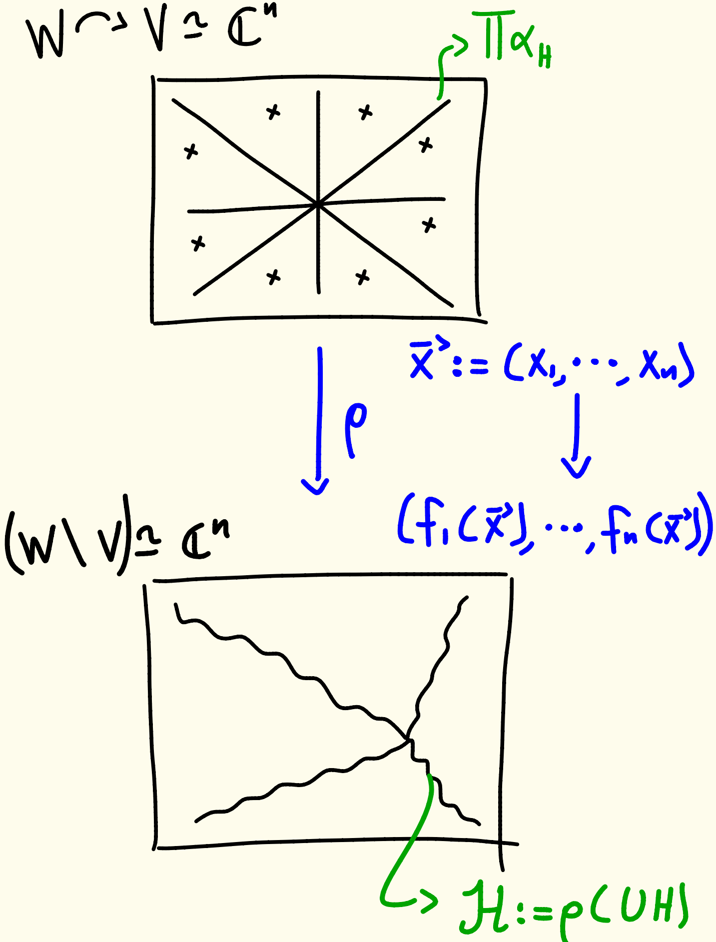
$\Longrightarrow$ w is a root of $\mathrm{Disc}_z(p(z)-t)$

c) Finiteness : $\Longleftrightarrow LL^{-1}(\vec{0})=\vec{0}$ (only z^n has all critical values equal to 0)

Consequence via Bezout's thm:

$$\deg(LL)=\frac{wt(b_1)\cdots wt(b_{n-1})}{wt(a_2)\cdots wt(a_n)}=\frac{n\cdot 2n\cdots(n-1)n}{2\cdot 3\cdot\ \cdots\cdot n}=\frac{n^{n-1}(n-1)!}{n!}=n^{n-2}$$

Basics of complex reflection groups



W is a well-generated, complex reflection group acting on $V \simeq \mathbb{C}^n$

if $W = \langle t_1, \cdots, t_n \rangle \leq GL(V)$

with $GL(V) \ni t_i \simeq \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \zeta \end{bmatrix}$

$\zeta^k = 1$ (t_i is a "pseudo-reflection")

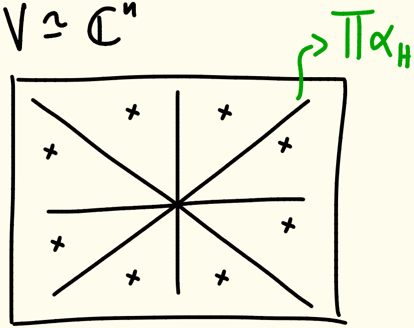
$W \curvearrowright V \Rightarrow W \curvearrowright \mathbb{C}[V]$ via

$w * f := f(w^{-1} \cdot v)$

$\mathbb{C}[V]^W := \{ f \in \mathbb{C}[V] : w * f = f \;\; \forall w \in W \}$

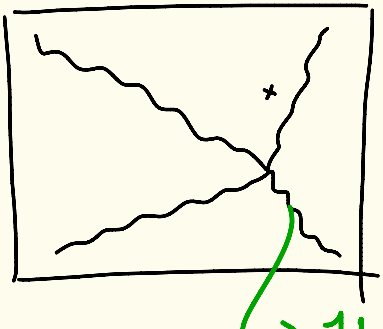
Basics of complex reflection groups

$W \curvearrowright V \cong \mathbb{C}^n$



ρ $\qquad \vec{x} := (x_1, \ldots, x_n) \mapsto (f_1(\vec{x}), \ldots, f_n(\vec{x}))$

$(W\backslash V) \cong \mathbb{C}^n$



$\mathcal{H} := \rho(\cup H)$

Shephard-Todd-Chevalley:

$$\mathbb{C}[V]^W = \mathbb{C}[\underbrace{f_1, \ldots, f_n}_{\text{"fundamental invariants"}}]$$

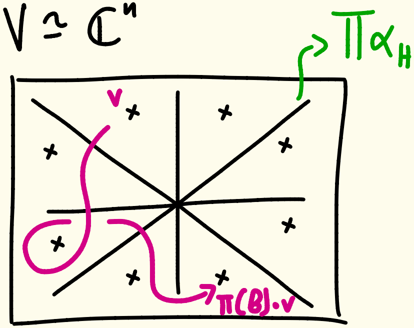
We write $d_i := \deg(f_i)$ and order them $d_1 \le d_2 \le \cdots \le d_n$

For W well-gen'd, $h := d_n$ is the Coxeter number.

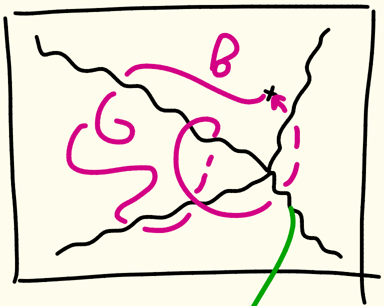
Significance: Implies that the fibers of the map
$$\rho : (x_1, \ldots, x_n) \longmapsto (f_1(\vec{x}), \ldots, f_n(\vec{x}))$$
are precisely the W-orbits.

Basics of complex reflection groups

$W \curvearrowright V \cong \mathbb{C}^n$



$$\begin{array}{ccc} V & & \vec{x} := (x_1, \cdots, x_n) \\ \big\downarrow p & & \big\downarrow \\ (W \backslash V) \cong \mathbb{C}^n & & (f_1(\vec{x}), \cdots, f_n(\vec{x})) \end{array}$$



Steinberg's Theorem:

W acts freely on $V^{reg} := V \setminus \cup H$

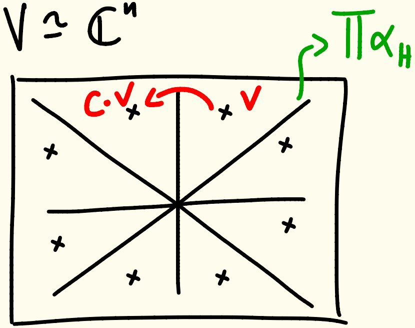
$\Rightarrow p : V^{reg} \to W \backslash V^{reg}$ is a covering map.

$$\begin{array}{ccccccccc} 1 & \to & P(W) & \hookrightarrow & B(W) & \overset{\pi}{\twoheadrightarrow} & W & \to & 1 \\ & & \parallel & & \parallel & & & & \\ & & \pi_1(V^{reg}) & & \pi_1(W \backslash V^{reg}) = \pi_1(\mathbb{C}^n \setminus \mathcal{H}) & & & & \end{array}$$

Significance: W is realized as the group of deck transformations of a covering map p, which is <u>explicitly</u> given via the f_i's.

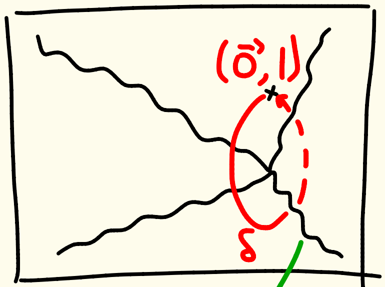
Coxeter elements and their geometric factorizations

$W \curvearrowright V \simeq \mathbb{C}^n$



ρ $\qquad \vec{x} := (x_1, \cdots, x_n) \mapsto (f_1(\vec{x}), \cdots, f_n(\vec{x}))$

$(W \backslash V) \simeq \mathbb{C}^n$



Saito-Bessis theorem:

W is well-gen'd $\iff \exists\ (f_1, \cdots, f_n)$ s.th

eqn of $\mathcal{H} =: \Delta(W; f) =$

$$= f_n^n + a_2 \cdot f_n^{n-2} + \cdots + a_n,$$

$$a_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$$

Now, pick $v \in V^{reg}$ s.th

$$f_1(v) = \cdots = f_{n-1}(v) = 0, \quad f_n(v) = 1$$

path: $B(t) := e^{(2\pi i/h)\cdot t} \cdot v \quad t \in [0,1]$

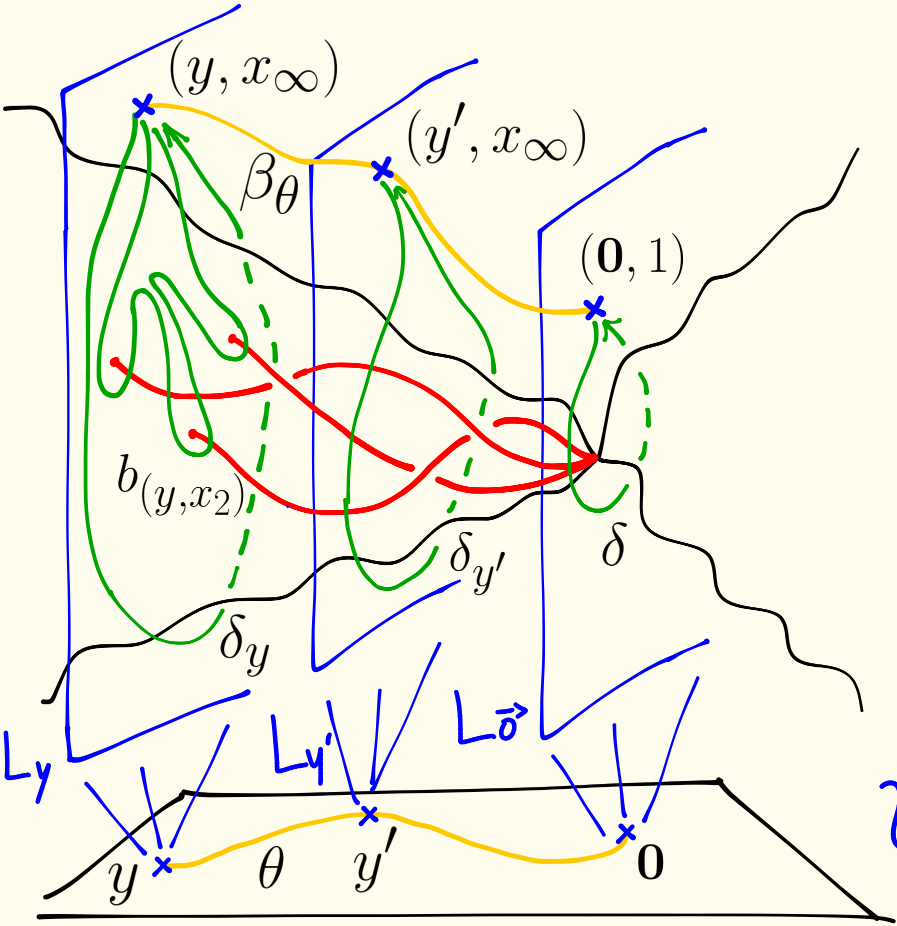
$$\rho(B(t)): \quad f_i(B(t)) = 0 \quad i \leq n-1$$

$$f_n(B(t)) = e^{(2\pi i)\cdot t}$$

$$\delta := \rho(B(t)) \in B(W)$$

$c := \pi(\delta)$ is the Coxeter element

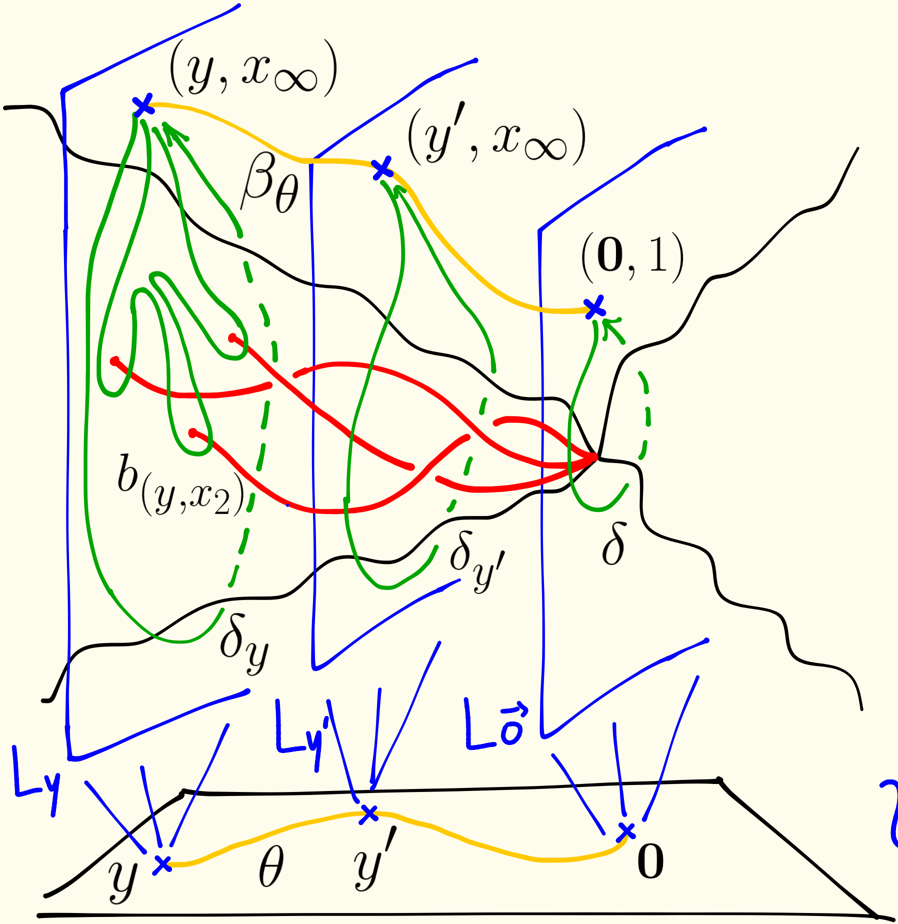
Coxeter elements and their geometric factorizations



"base space"
$Y \simeq \mathbb{C}^{n-1} \ni (f_1, \dots, f_{n-1})$

- Pick a path $\theta : \vec{0} \to y$ in Y.
- Lift to a path β_θ in $W \backslash V$ that "stays above" $\mathcal{H}$.
- If $L_y \cap \mathcal{H} = \{x_1, \cdots x_k\}$ bring little loops b_i from $x_\infty(y)$ down and around the x_i's.
- Define $b_{(y,x_i)} = \beta_\theta \cdot b_i \cdot \overline{\beta_\theta}$

Coxeter elements and their geometric factorizations



We define the "reduced label" map rlbl :

$\mathrm{rlbl}(y) := (c_1, \cdots, c_k)$

where $c_i := \pi(b_{(y,x_i)}) \in W$

(via $1 \to P(W) \hookrightarrow B(W) \xrightarrow{\pi} W \to 1$)

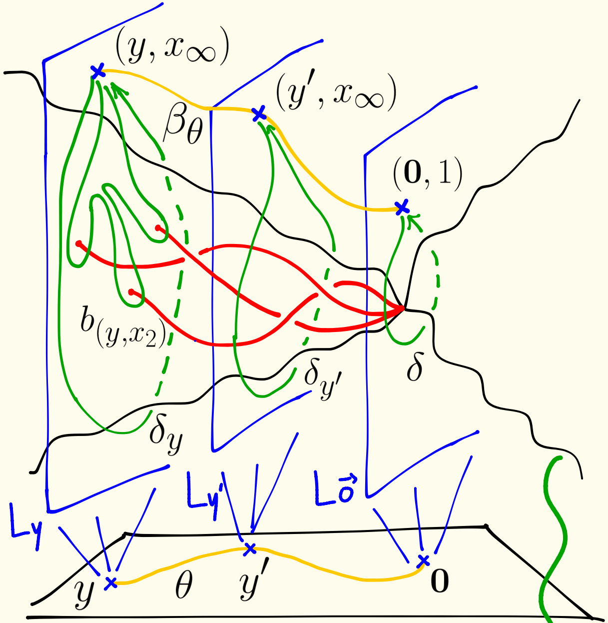
$b_{(y,x_1)} \cdots b_{(y,x_k)} \simeq \beta_\theta \cdot \delta_y \cdot \bar{\beta}_\theta \simeq \delta$

so $c_1 \cdots c_k = c$

! rlbl is well-defined !

"base space" $Y \simeq \mathbb{C}^{n-1} \ni (f_1, \cdots, f_{n-1})$

The Lyashko-Looijenga (LL) morphism



$\mathcal{H}$ is given by eqn:

$$f_n^n + a_2 f_n^{n-2} + \cdots + a_n = 0$$

$$a_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$$

We define the LL map:

$$LL : Y \longmapsto \left\{ \begin{matrix} \text{centered configurations} \\ \text{of } n \text{ points in } \mathbb{C} \end{matrix} \right\} =: E_n$$

$$y \longmapsto \text{multiset } L_y \cap \mathcal{H}$$

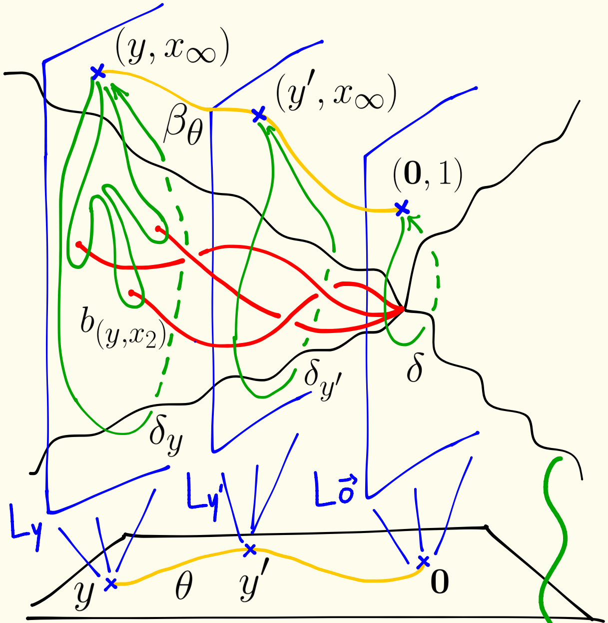
Algebraically:

$$LL : Y \simeq \mathbb{C}^{n-1} \longrightarrow E_n \simeq \mathbb{C}^{n-1}$$

$$y = (f_1, \cdots, f_{n-1}) \longmapsto \left\{ \begin{matrix} \text{roots of} \\ t^n + a_2(y) t^{n-2} + \cdots + a_n(y) = 0 \end{matrix} \right\}$$

$$y = (f_1, \cdots, f_{n-1}) \xmapsto{LL} (a_2(y), \cdots, a_n(y))$$

Properties of the LL and rlbl maps:



$\mathcal{H}$ is given by eqn:

$$f_n^n + a_2 f_n^{n-2} + \dots + a_n = 0$$

$$a_i \in \mathbb{C}[f_1, \dots, f_{n-1}]$$

- The line L_y is transverse to $\mathcal{H}$ for all y.
- The LL map is a finite morphism. Its degree is given by

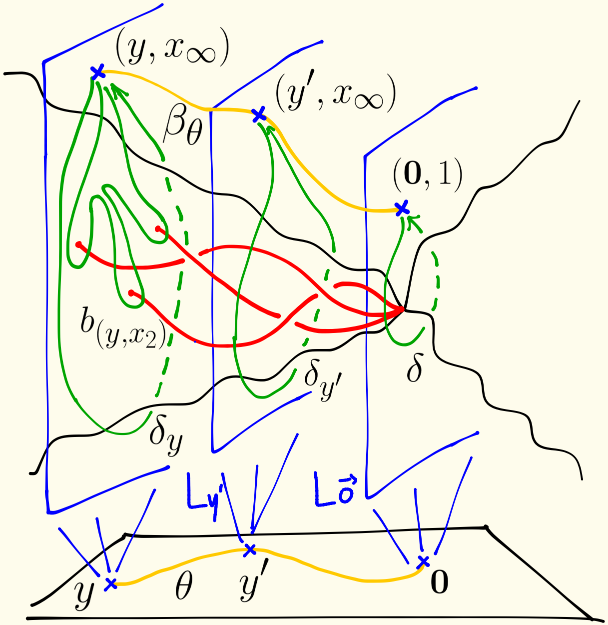
$$\deg(LL) = \frac{\prod_{i=2}^{n} \deg a_i}{\prod_{i=1}^{n-1} \deg f_i} = \frac{2h \cdot \dots \cdot nh}{d_1 \cdots d_{n-1}} = \frac{h^{n-1} \cdot n!}{\frac{|W|}{d_n}} = \frac{h^n \cdot n!}{|W|}$$

$$\left(y = (f_1, \dots, f_{n-1}) \xmapsto{LL} (a_1(y), \dots, a_n(y)) \right)$$

- LL and rlbl are compatible:

  If $LL(y) = \{x_1, \dots, x_k\}$ with $n_i := mult(x_i)$ and $rlbl(y) = (c_1, \dots, c_k)$, then $l_R(c_i) = n_i$.

The trivialization theorem. [Bessis]



· The map

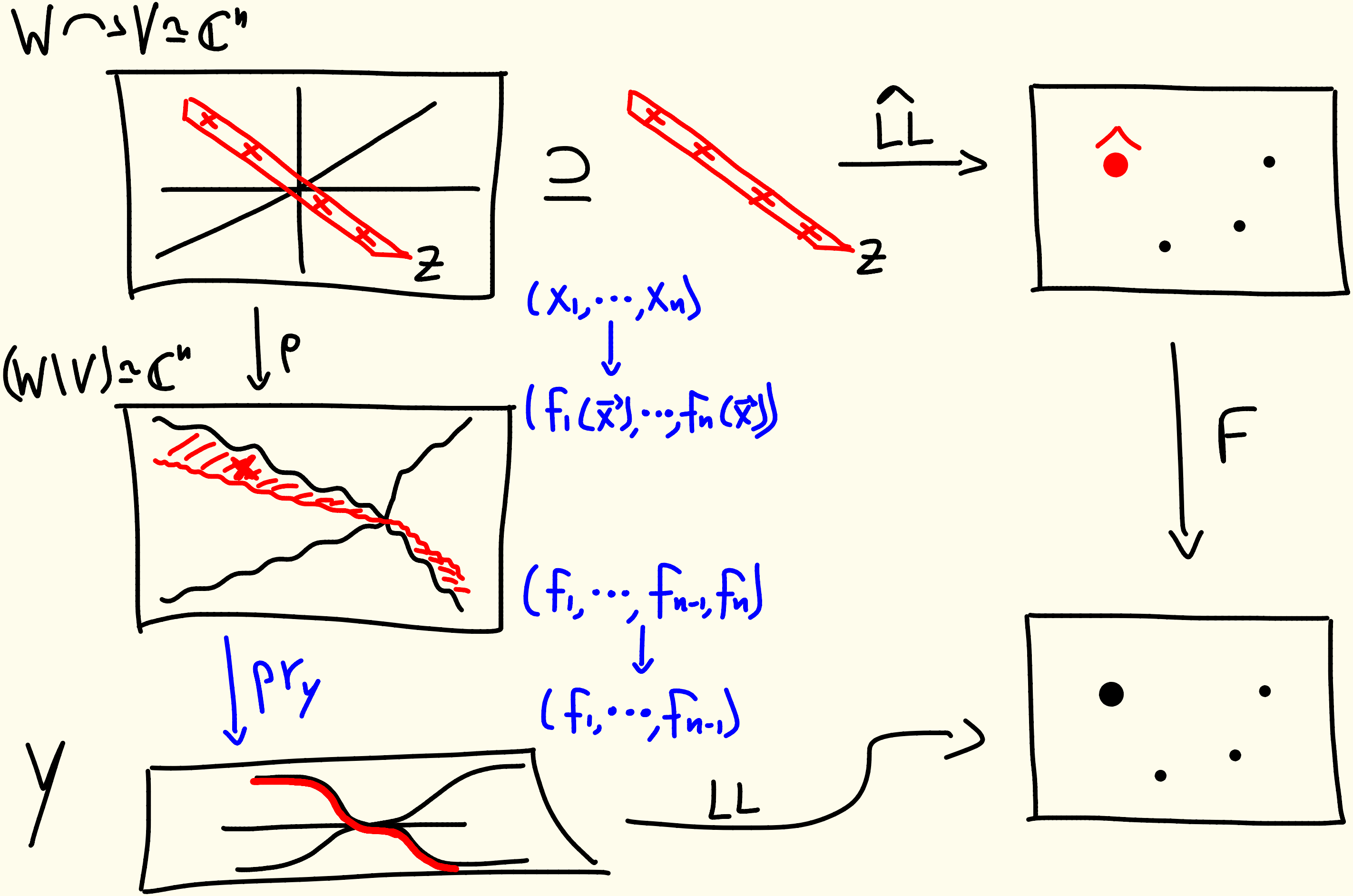
$$LL\times rlbl : y \longmapsto \left\{ \begin{array}{c} \text{compatible pairs of} \\ \left({\color{blue}\{x_1, \dots, x_k\}}, {\color{red}(c_1, \dots, c_k)} \right) \end{array} \right\}$$

is a bijection.

!! Depends on the numerological

coincidence : $\deg(LL) = |\mathrm{Red}_W(c)|$

Primitive Factorizations



Primitive Factorizations

We can lift the LL map to any flat Z :

$$Z \ni (z_1, \cdots, z_k) =: \vec{z} \xmapsto{\widehat{LL}} \text{multiset } L_y \cap \mathcal{H}, \text{ decorated at } f_n(\vec{z})$$

In coordinates :

$$\widehat{LL}(\vec{z}) = \Big(f_n(\vec{z}), \Big\{ \overset{\text{roots of}}{\underbrace{[t - f_n(\vec{z})]^{n-k} [t^k + b_1(\vec{z}) t^{k-1} + \cdots + b_k(\vec{z})]}_{\text{linear relation}}} \Big\} \Big)$$

or:

$$(z_1, \cdots, z_k) \xmapsto{\widehat{LL}} (b_1(\vec{z}), \cdots, b_k(\vec{z}))$$

So, $\deg \widehat{LL} = \prod_{i=1}^{k} \deg(b_i) = h \cdot 2h \cdots (kh) = h^k \cdot k! = h^{\dim Z} \cdot (\dim Z)!$

We have overcounted factorizations by $[N_W(Z) : W_Z]$.

So, $|\mathrm{Fact}_W(Z)| = \dfrac{h^{\dim Z} (\dim Z)!}{[N_W(Z) : W_Z]}$

Towards a uniform proof of the Trivialization Theorem

Pick a configuration $e = \{x_1, \cdots, x_k\}$ w/ multiplicities $n_i := \mathrm{mult}(x_i)$

Compare:

$$\deg(LL) = \sum_{\substack{\sigma = (c_1, \cdots, c_k) \\ \text{compatible} \\ \text{with } e}} \left| LL^{-1}(e) \cap rlbl^{-1}(\sigma) \right| \cdot \mathrm{mult}_{y(\sigma)}(LL)$$

and:

$$|\mathrm{Red}_W(c)| = \sum_{\substack{\sigma = (c_1, \cdots, c_k) \\ \text{compatible} \\ \text{with } e}} 1 \cdot \prod_{i=1}^{k} |\mathrm{Red}_W(c_i)|$$

Thank you!