Slides available at my website linked in the abstract (www.irif.fr/$\sim$douvr001/)

Coxeter factorizations and the Matrix Tree theorem with generalized Jucys-Murphy weights

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Paris VII, IRIF and CNRS, ERC CombiTop

Counting trees and counting factorizations

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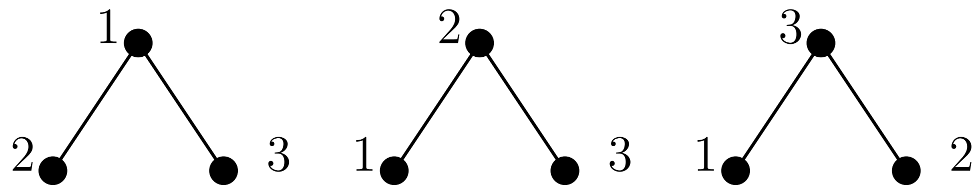
Theorem (Cayley 1889, $+$other people)**.** *There are* n^{n-2} *trees on* n *labeled vertices.*

Example, $n = 3$:
$3^{3-1} = 3$



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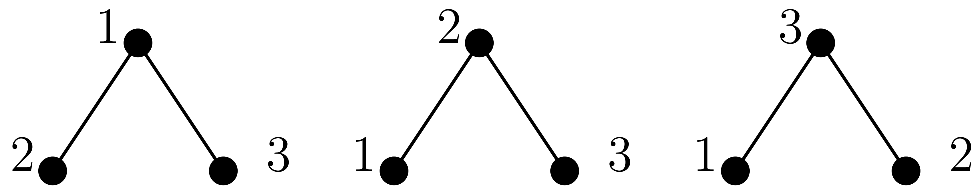
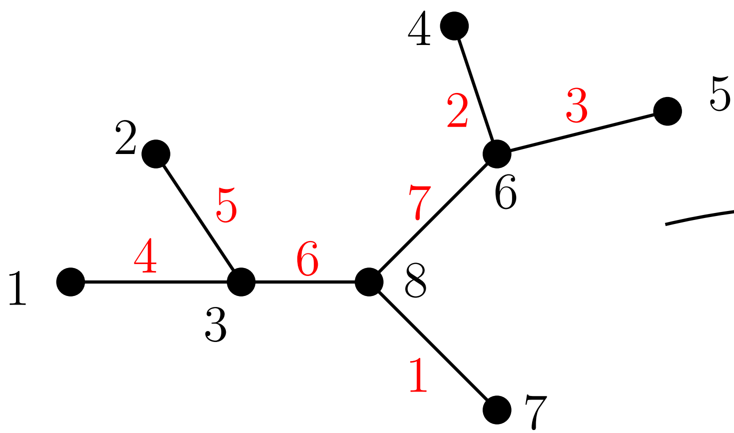


Corollary (Denes, 1959).
Call a permutation $c \in S_n$ a long cycle if it is conjugate to $(12 \cdots n)$. There are $n^{n-2} \cdot (n-1)!$ minimal length factorizations $\tau_1 \cdots \tau_{n-1} = c$ of long cycles $c \in S_n$ in transpositions τ_i.



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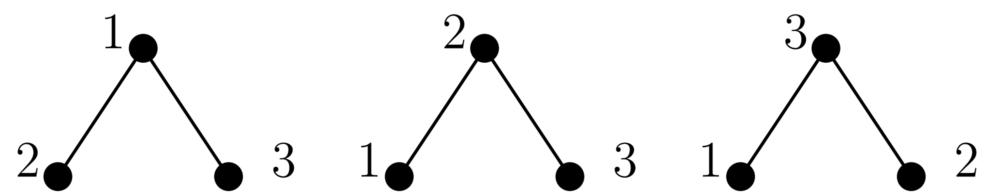

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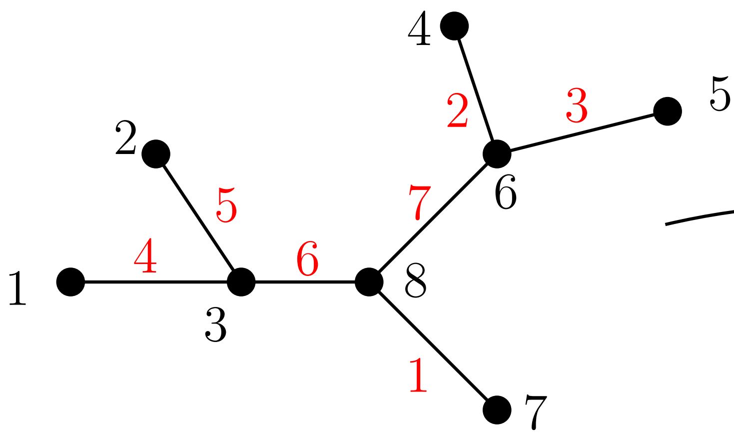

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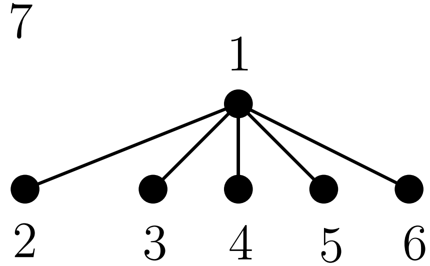
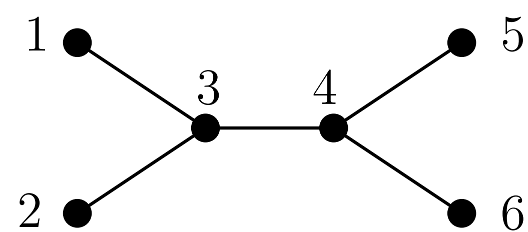


All long cycles appear here!



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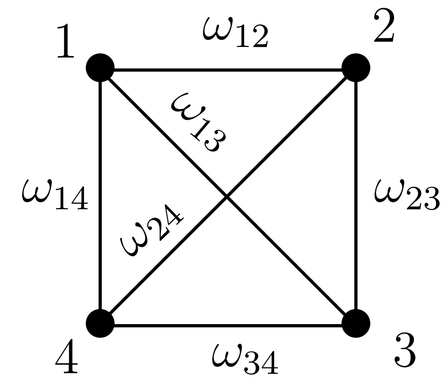

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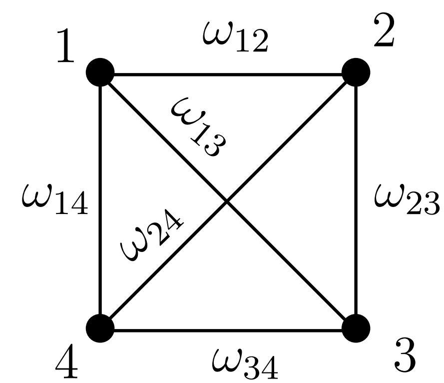

Theorem ((weighted) Matrix Tree theorem)**.**
The Laplacian of a graph G on n vertices counts the spanning trees of G via the formula

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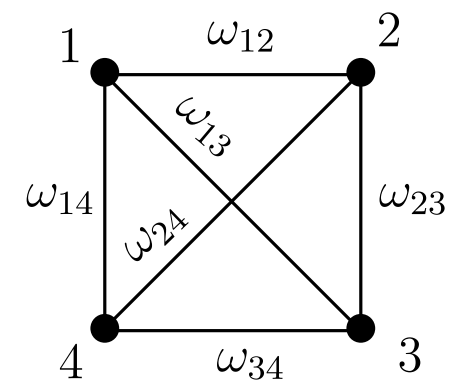

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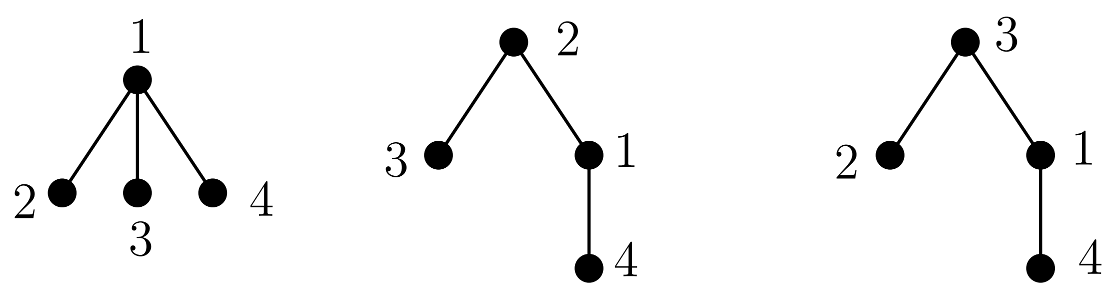
Example for $G = K_4$:

$$\operatorname{qdet}\big(L_{K_4}(\boldsymbol{\omega})\big) := \prod_{\lambda_i \neq 0} \lambda_i(\boldsymbol{\omega}) = 4 \cdot w_{12} \cdot w_{13} \cdot w_{14} + 4 \cdot w_{12} \cdot w_{23} \cdot w_{14} + 4 \cdot w_{13} \cdot w_{23} \cdot w_{14} + \cdots$$



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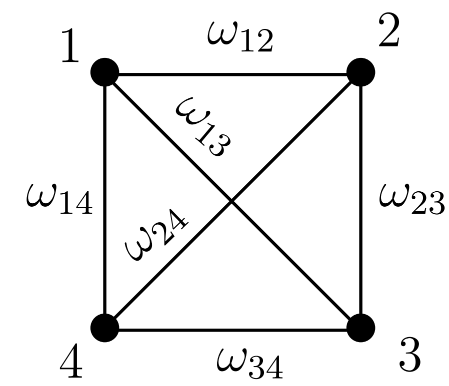

Corollary (the Denes argument)**.**
The weighted count of factorizations of long cycles $c \in S_n$ in tranpositions τ_i is given via

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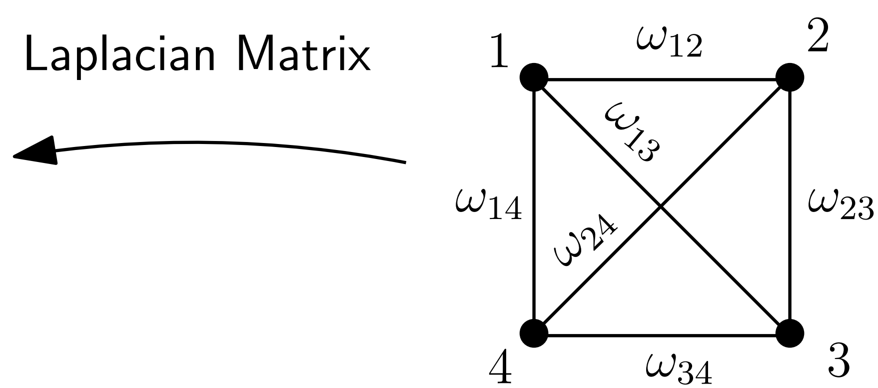

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Notice that

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exp. gen. fnc.

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We consider $\binom{n}{2}$ parameters $\boldsymbol{\omega} := (\omega_{ij})_{i<j}$ that form a weight system $\mathbf{w}\big((ij)\big) = \omega_{ij}$ for the transpositions $(ij) \in S_n$. If $\mathcal{C}$ is the class of the long cycles, define:

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which is in fact a new proof of the (weighted) Matrix Tree theorem after Denes' argument.

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A complete poset of formulas? Not yet!

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3. *Equivalently for real W, c is an element whose order satisfies $|c| \cdot n = 2|\mathcal{R}|$, and which has an invariant plane that is not orthogonal to any root and which it rotates by $2\pi i/|c|$.*

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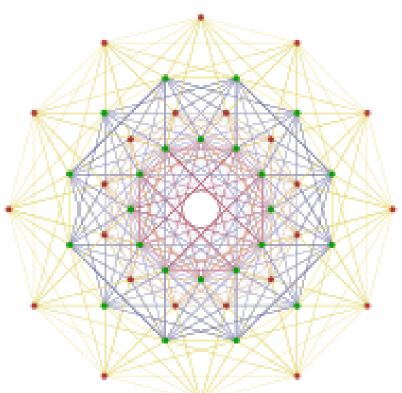
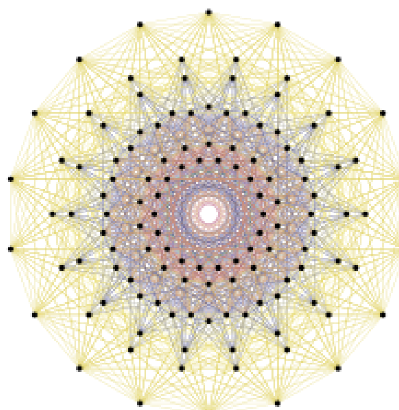
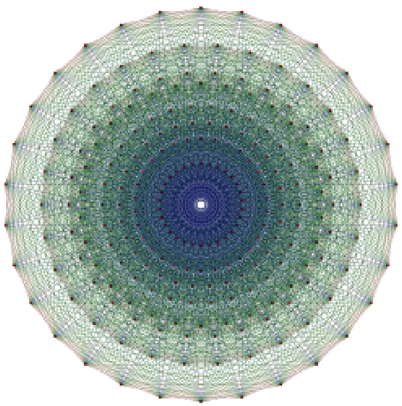


E_6



E_7



E_8

Complex reflection groups and Coxeter elements

A *finite* subgroup $W \leq \mathrm{GL}(V)$, for some $V \cong \mathbb{C}^n$ is called a complex reflection group if it is generated by pseudo-reflections. These are $\mathbb{C}$-linear maps t that fix a hyperplane. If W is generated by n reflections we say that it is well-generated.

Shephard and Todd have classified (irreducible) complex reflection groups into:

1. an infinite 3-parameter family $G(r, p, n)$ of monomial groups
2. 34 exceptional cases indexed G_4 to G_{37}.

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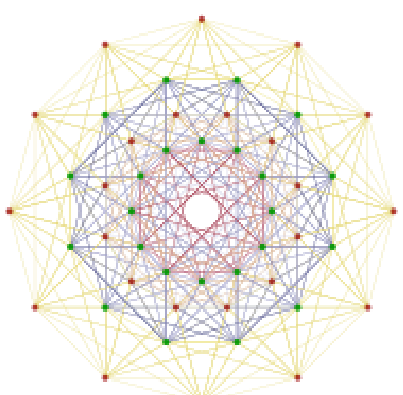

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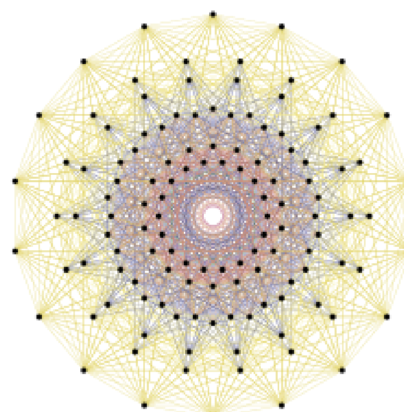

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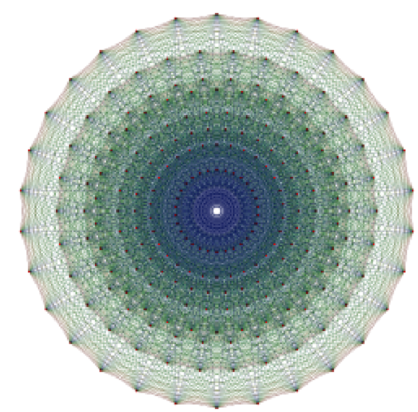

E_8

4. In the general complex case, c is a Springer $e^{2\pi i/h}$-regular element where h is the Gordon-Griffeth Coxeter number $(|\mathcal{R}| + |\mathcal{A}|)/n$.

Arbitrary length reflection factorizations of Coxeter elements c

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$$F_W(N) := \#\{(\tau_1, \cdots, \tau_N, c) \in \mathcal{R}^N \times \mathcal{C} \mid \tau_1 \cdots \tau_N = c\}.$$

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Looijenga-Deligne-Arnol'd-Chapoton-Reading-Bessis formula
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where $\{\lambda_i^T(\boldsymbol{\omega})\}$ are the eigenvaues of the W-Laplacian:

$$L_W^T(\boldsymbol{\omega}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)\cdot\big(\mathbf{1}-\rho_V(\tau)\big) \in \mathrm{GL}(V).$$

(ρ_V is the reflection representation of W)

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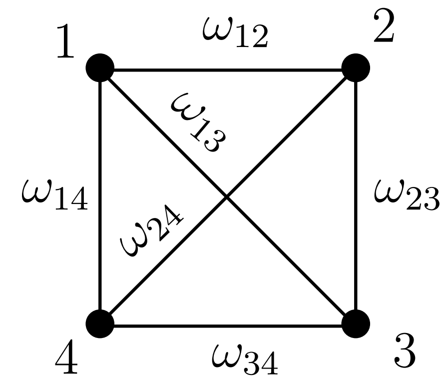
Laplacian Matrix



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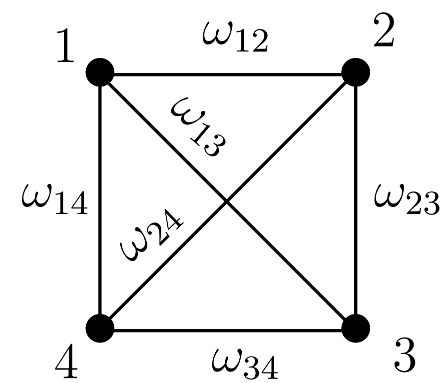

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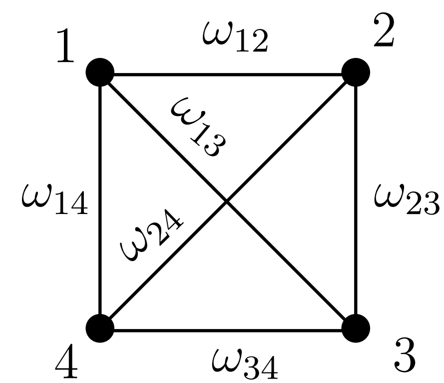

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is a direct generalization of the graph Laplacian.

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The filtration of $\mathcal{R}$ by the tower T defines natural analogs of the Jucys-Murphy elements:

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That the virtual characters agree on the identity $\mathbf{id} \in W$ and the element of the group algebra $\mathbf{R} := \sum_{i=1}^{n} J_i = \sum_{\tau \in \mathcal{R}} \tau$ is in fact equivalent with the Chapuy-Stump formula.

It has relatively difficult uniform proofs.

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Burman's theory of Lie-like elements completely determines the eigenvalues of $\boldsymbol{\mathcal{L}}_W^T(\boldsymbol{\omega})$ on $\bigwedge^k(V_{\mathrm{ref}})$. They are precisely the k-sums of the eigenvalues of the W-Laplacian $L_W^T(\boldsymbol{\omega})$.

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So now, we have

$$\mathcal{F}_W^T(t,\boldsymbol{\omega}) = \frac{e^{t\mathbf{w}(\mathcal{R})}}{h}\sum_{k=0}^{n}(-1)^k\sum_{1\leq i_1\leq\cdots\leq i_k\leq n} e^{-t\lambda_{i_1}(\boldsymbol{\omega})-\cdots-t\lambda_{i_k}(\boldsymbol{\omega})} = \frac{e^{t\mathbf{w}(\mathcal{R})}}{h}\cdot\prod_{i=1}^{n}\big(1-e^{-t\lambda_i(\boldsymbol{\omega})}\big).$$

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Their characters and parabolic subgroups are indexed by combinatorial objects and restriction to (parabolic) subgroups can be described via a variant of the Littlewood-Richardson's rules (John Stembridge's notes were very helpful).

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3) Burman's theory of Lie-like elements and our ability to experiment in Sage-Gap-Chevie were key. Also a love for the "Okounkov-Vershik approach" (thanks Vic!).

A W-matrix forest theorem!




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The whole characteristic polynomial of the Laplacian of a graph G has a combinatorial interpretation. This is usually referred to as the (weighted) Matrix-forest theorem:

$$\det\big(x + L_G(\boldsymbol{\omega})\big) = \sum_{\mathcal{F}} \mathbf{w}(\mathcal{F}) \cdot x^{c(\mathcal{F})},$$

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Corollary (Chapuy, D. '19)**.** *If we set all weights equal to 1 we get a generalization of the Deligne-Arnol'd-Bessis formula $\mathrm{Hur}(W) = \frac{h^n n!}{|W|}$:*

$$(x+h)^n = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} |W_X| \cdot \mathrm{Hur}(W_X) \cdot \frac{x^{\dim(X)}}{\big(\mathrm{codim}(X)\big)!}.$$

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For each hyperplane $H_i \in \mathcal{A}$ choose an orthogonal vector r_i of unit norm. Then

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The recursion looks very similar to Brieskorn's lemma:

$$\mathrm{Poin}(V\setminus\mathcal{A}, t) = \sum_{X\in\mathcal{L}_{\mathcal{A}}} \mathrm{rank}\big(H^{\mathrm{top}}(V\setminus\mathcal{A}_X)\big)\cdot t^{\dim(X)},$$

which in fact shows furthermore a natural decomposition of the corresponding cohomology spaces. Could the previous proposition be interpreted in a similar way?

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Write $\mathrm{Hur}(W)$ for the number of reduced reflection factorizations of a fixed Coxeter element c:

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$$\mathrm{Krew}(X) = \frac{\prod_{i=1}^{\dim(X)} (h + 1 - b_i^X)}{[N(X) : W_X]} \qquad \text{and} \qquad \mathrm{Krew}(L) = \frac{h}{[N(L) : W_L]}.$$

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Uniformely!

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$$\mathrm{Krew}(X) = \frac{\prod_{i=1}^{\dim(X)} (h + 1 - b_i^X)}{[N(X) : W_X]} \qquad \text{and} \qquad \mathrm{Krew}(L) = \frac{h}{[N(L) : W_L]}.$$

Case by case.

A uniform proof of the chain number $\frac{h^n n!}{|W|}$ of $NC(W)$.

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which after pluggin in the formula to be proven demands that

$$h^{n-1} n! = \sum_{L \in \mathcal{L}^1_{\mathcal{A}_W}/W} \frac{|W|}{|N(L)|} \cdot \Big(\prod_{i=1}^{n-1} h_i(W_L) \Big) \cdot (n-1)!,$$

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and now summing over all flats (instead of orbits of flats):

$$n \cdot h^{n-1} = \sum_{L \in \mathcal{L}^1_{\mathcal{A}_W}} \prod_{i=1}^{n-1} h_i(W_L).$$

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But in fact the recursion for the characteristic polynomial of the W-Laplacian gives us more:

$$(h + x)^n = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} \Big(\prod_{i=1}^{\mathrm{codim}(X)} h_i(X) \Big) \cdot x^{\dim(X)}.$$

The end!

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Thank you very much!

A combinatorial description of the eigenvalues of the W-Laplacian

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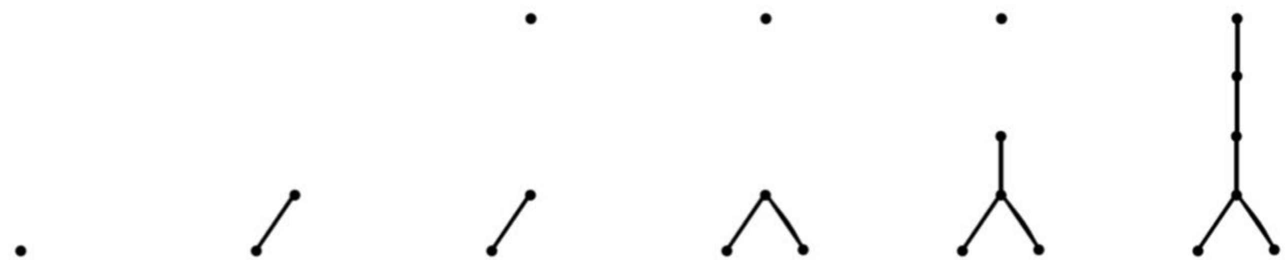

$$
\begin{array}{rc}
A = \begin{bmatrix}
2 & 1 & 0 & 1 & 2 & 4 \\
0 & 3 & 0 & 1 & 2 & 4 \\
0 & 0 & 2 & 0 & 0 & 8 \\
0 & 0 & 0 & 4 & 2 & 4 \\
0 & 0 & 0 & 0 & 6 & 4 \\
0 & 0 & 0 & 0 & 0 & 10
\end{bmatrix} & \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{array}
\end{array}
$$

$$
\{\lambda_i(\boldsymbol{\omega})\} = \left\{ \begin{array}{c}
2\omega_1 + \omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\
3\omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\
2\omega_3 + 8\omega_6, \quad 4\omega_4 + 2\omega_5 + 4\omega_6, \\
6\omega_5 + 4\omega_6, \quad 10\omega_6
\end{array} \right\}
$$