

Recursions and Proofs in Coxeter-Catalan Combinatorics: a conclusion (?)

Theo Douvropoulos

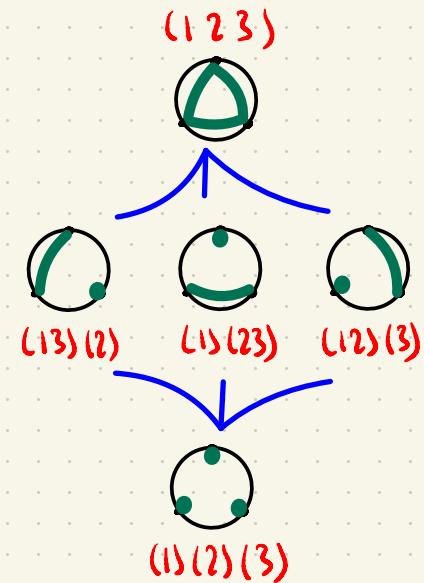
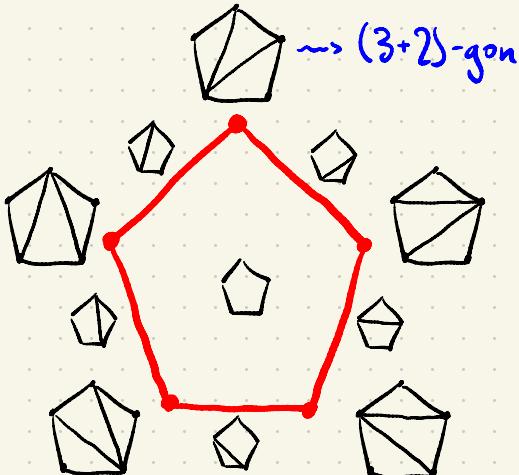
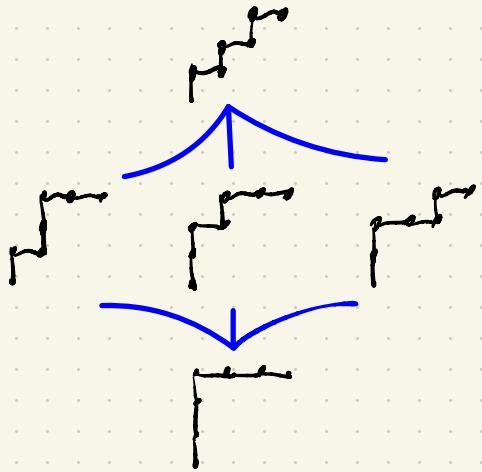
(joint w/ Matthieu Sosua-Vergé's)

UMass Amherst Discrete Math Seminar

Oct. 14th 2022

Attractions @ the Catalan Zoo

The Catalan numbers $\text{Cat}(n) := \frac{1}{n+1} \binom{2n}{n}$ count:



Dyck Paths $(0,0) \rightarrow (n,n)$

Vertices of the
associahedron

Noncrossing Partitions

$\text{Cat}(n): 1, 1, 2, 5, 14, 42, 132, 429, \dots$

AKA: the answer to the ultimate question of Life
the Universe, and Everything

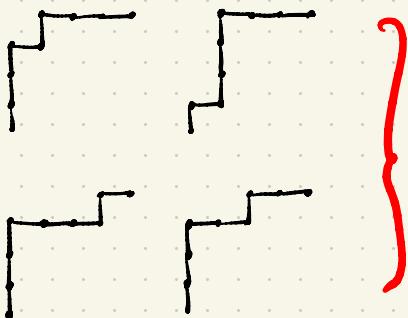
Germain Kreweras visits the Catalan Zoo

The Kreweras numbers $\text{Krew}(\beta) := \frac{n(n-1)\cdots(n-k+1)}{\text{Sym}(\beta)}$ count:

$\beta \vdash n$

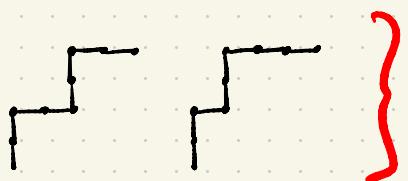
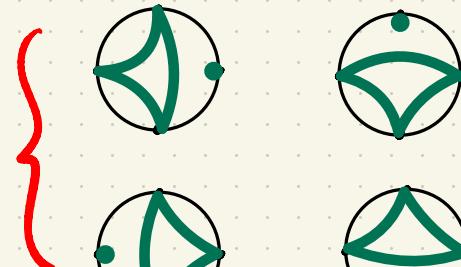
$$\beta = (\beta_1, \dots, \beta_k) = (m_1, \dots, m_{\mu_1})$$

$$\text{Sym}(\beta) := m_1! \cdots m_n!$$



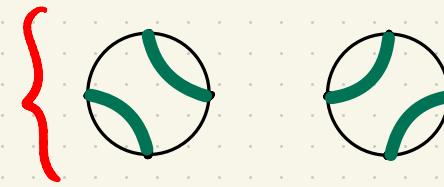
$$\beta = (3, 1)$$

$$\text{Krew}(\beta) = 4 = \frac{4}{1}$$



$$\beta = (2, 2)$$

$$\text{Krew}(\beta) = 2 = \frac{4}{2}$$



Dyck Paths whose vertical runs determine a partition β

$$\text{Krew}(\beta)$$

$$1$$

4	2
(3,1)	(2,2)

Noncrossing Partitions whose blocks determine a partition β

$$6$$

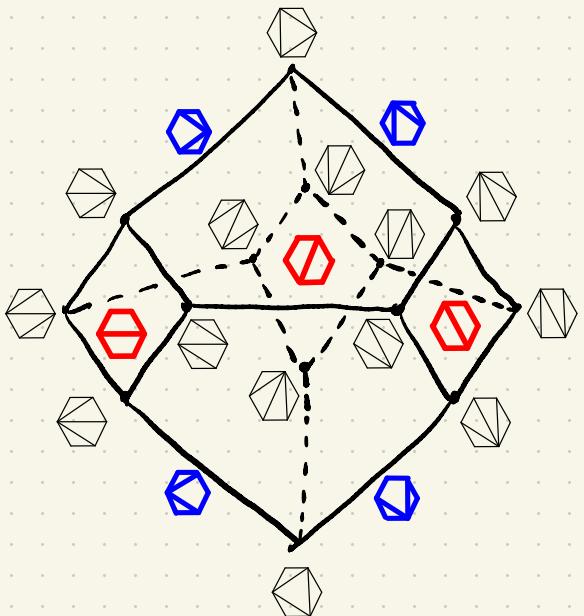
$$1$$

$$\Rightarrow 14$$

$$(2,1,1)$$

$$(1,1,1,1)$$

Jean-Louis Loday visits the Catalan Zoo



f-vector

$$f_0 = 14$$



$$f_1 = 21$$



$$f_2 = 9$$



$$f_3 = 1$$



2-diml faces

λ



(2,2)

$$\delta(\lambda) = \frac{6}{2} = 3$$

"type" =



(3,1)



"type" =

$$\delta(\lambda) = \frac{6}{1} = 6$$

The Loday numbers $\delta(\lambda)$:= $\frac{(n+2)\cdots(n+k)}{\text{Sym}(\lambda)}$ count $(n-k)$ -diml faces of the associahedron that are indexed by $(\lambda; i+2)$ -gons

Aside: Compositional Inverses

Question: Find $g(x)$ s.t. $F(g(x)) = x$ when

$$F(x) = x + a_1x^2 + a_2x^3 + \dots + a_nx^{n+1} + \dots$$

Answer:

$$g(x) = x + b_1x^2 + b_2x^3 + \dots + b_nx^{n+1} + \dots$$

$$b_1 = -a_1 \quad b_2 = 2a_1^2 - a_2$$

$$b_3 = -5a_1^3 + 5a_1a_2 - a_3 \quad b_4 = 14a_1^4 - 21a_1^2a_2 + 6a_1a_3 + 3a_2^2 - a_4$$

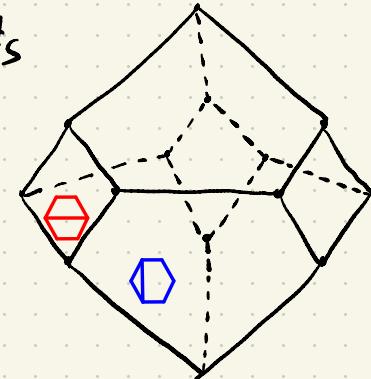
Face #s

14

21

$$6+3=9$$

1



Theorem [Loday]

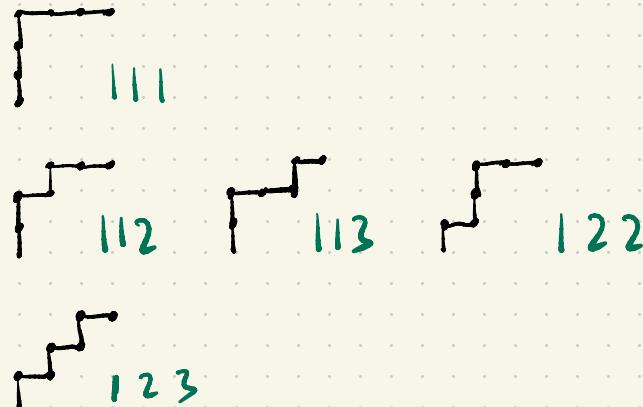
$$g(x) = \sum_{\substack{F \text{ Face of} \\ \text{Associahedron}}} (-1)^{\dim(F)} \cdot \prod_{i=2}^n$$

Parking @ the Catalan Zoo

A parking function is a word a_1, \dots, a_n such that its non-decreasing rearrangement $b_1 \leq b_2 \leq \dots \leq b_n$ satisfies $b_i \leq i$

111						
112	121	211	$\hookleftarrow S_n$			
113	131	311				
122	212	221				
123	132	213	231	312	321	

S_n acts on positions and the orbits are indexed by Dyck paths



Corollary: $\text{Park}(n) \cong \bigoplus_{S_n \models n} \text{Krew}(I) \cdot \mathbb{F}_{S_I}^{\text{triv}}$

The non-decreasing representatives prescribe the vertical runs

The algebraic Parking Space

Thm [Haiman]: The character of $\text{Park}(n)$ is given via:

Use as definition

$$S_n \ni g \longmapsto (n+1)^{\text{cyc}(g)-1}$$

$$\text{Trace}_{\text{Park}(n)}(g) = \# \left\{ \begin{array}{l} \text{Parking Functions} \\ \text{Fixed under } g \end{array} \right\}$$

Thm [Haiman-Kraft]: There exists a homogeneous system of parameters g_1, g_2, \dots, g_n such that

$$\frac{\mathbb{C}\langle V \rangle}{(g_1, g_2, \dots, g_n)} \underset{S_n}{\cong} \text{Park}(n)$$

ambient space

Corollaries: ① Natural q -versions

② Interpretation of Krew(λ) as structure coefficients

From S_n to reflection groups W

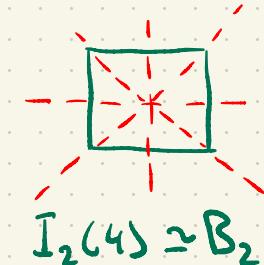
Reflection groups are...

finite subgroups $W \subseteq GL(\mathbb{R}^n)$

generated by Euclidean reflections?



$$I_2(3) \cong A_2$$



$$I_2(4) \cong B_2$$

Reflection groups are...

classified in four infinite families A_n, B_n, D_n , and $I_2(m)$

$$\begin{matrix} S_{n+1} \\ \downarrow \\ \text{hyperoctahedral} \end{matrix}$$

hyperoctahedral

$$\begin{matrix} I_2(m) \\ \downarrow \\ \text{dihedrals} \end{matrix}$$

dihedrals

and some exceptionals ($H_3, H_4, F_4, E_6, E_7, E_8$)

Reflection groups have...

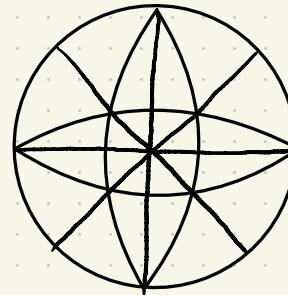
- nice presentations $W = \langle \underbrace{s_1, \dots, s_n}_{\text{simple gen's}} \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$

- Coxeter elements $c = s_1 \cdot s_2 \cdots s_n$ $\text{order}(c) = h \xrightarrow{\text{"Coxeter number"}}$

From S_n to reflection groups W : Partitions

A partition

$$\{\{1, 3, 4\}, \{2, 6, 7, 8\}, \{5\}\}$$



B_3

determines an intersection
of hyperplanes : a "flat"

$$x_1 = x_3 \cap x_3 = x_4 \cap x_2 = x_6 \cap x_6 = x_7 \cap x_7 = x_8$$

W -orbits of flats [x]
play the role of cycle types

exceptional numerology:

$\chi \rightsquigarrow$ characteristic polynomial

$\mathcal{A}_W \rightsquigarrow$ reflection arrangement

$\mathcal{A}_W^X \rightsquigarrow$ restricted arrangement

$$\chi(t_{\mathcal{A}_W}, t) = \prod_{i=1}^{\text{rank}(W)} (t - e_i)$$

in S_n : $e_i = i$

"group exponents"

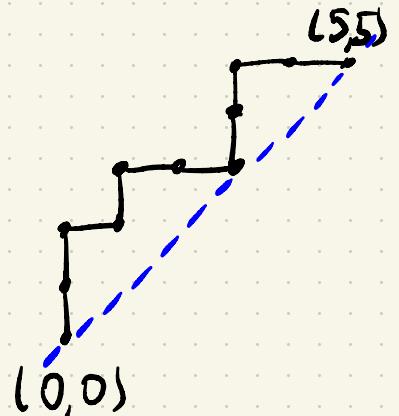
$$\chi(t_{\mathcal{A}_W^X}, t) = \prod_{i=1}^{\dim(X)} (t - b_i^X)$$

in S_n : $b_i^X = i$

"Orlik-Solomon exponents"

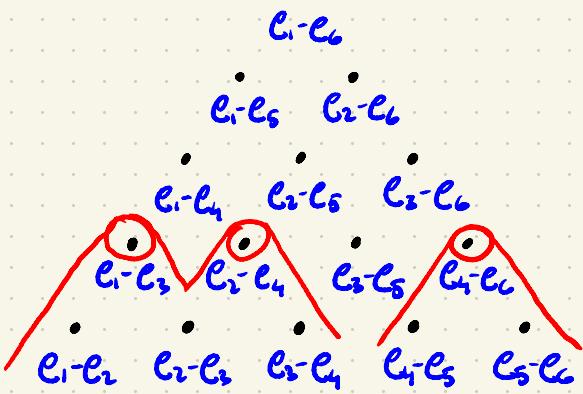
From S_n to reflection groups W : Dyck paths

Dyck paths



Peaks become top elements

Order ideals in Root Posets



$$K_{\text{rew}}(\beta) = \frac{n \cdot (n-1) \cdots (n-s+1)}{\text{Sym}(\beta)}$$

of Dyck paths of "type" β

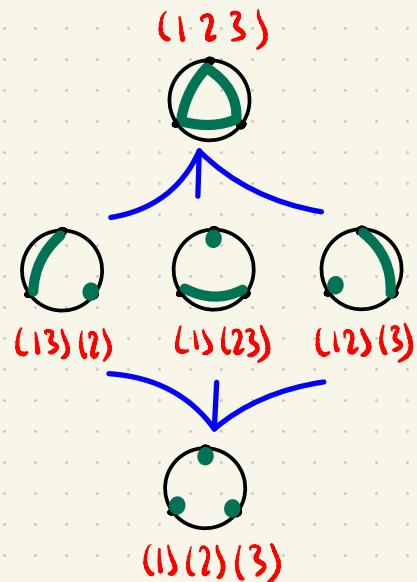
$$\xrightarrow{\sim} K_{\text{rew}}^{\text{Ord.Id.}}_W([x]) = \frac{\prod (h+1 - b_i^x)}{[N(x) : W_x]}$$

[Sommers, '03]

of order ideals of "type" $[x]$

From S_n to reflection groups W : Noncrossing Partitions

$$A_2 \cong S_3$$



The W -noncrossing partitions are defined as the interval

$$NC(W) := [1, c]_{\leq_R}$$

under the order \leq_R induced by reflection length



[Athansiadis-Reiner]

$$\text{Krew}_W([x]) = \frac{\prod (h+1 - b_i^x)}{[N(x) : W_x]}$$

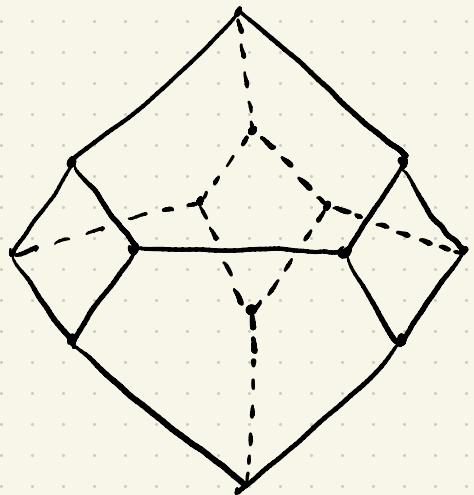
of W -noncrossing partitions of "type" $[x]$

$$\text{Krew}(r) = \frac{n \cdot (n-1) \cdots (n-k+1)}{\text{Sym}(r)}$$

of noncrossing partitions of cycle type r

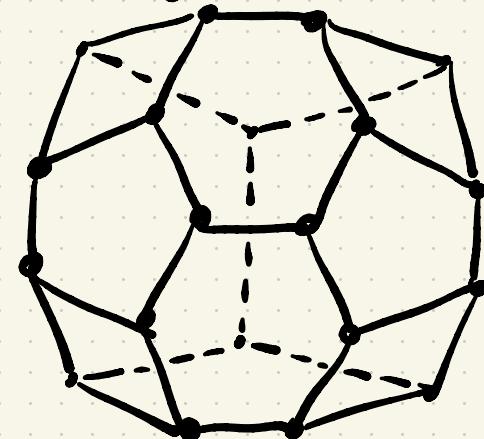
From S_n to reflection groups W : Associahedra

A_3 -Associahedron



Chapoton-Fomin-Zelevinsky
construct a simple
polytope whose normal
Fan encodes the
clusters of the
cluster algebra of W

B_3 -Associahedron
"cyclohedron"



The Loday numbers

$$\delta(\gamma) = \frac{(n+2)\cdots(n+k)}{\text{Sym}(\gamma)}$$



= # of faces of "type" γ

Thm [D., Sosua et Verges]

$$\delta_W(\{x\}) = \frac{\prod (h+1+b_i x)}{[N(x):W_x]}$$

= # of faces of "type" $\{x\}$

$$B_2: \text{hexagon} \rightsquigarrow \frac{8}{2} = 4$$

$$A_2: \text{pentagon} \rightsquigarrow \frac{8}{2} = 4$$

$$A_1 \times A_1: \text{square} \rightsquigarrow \frac{8}{2} = 4$$

From S_n to reflection groups W : Parking Spaces

$$\text{Park}(n) \simeq (\{v\}) / (\Theta)$$

$$g \mapsto (n+1)^{\text{cycles}} - 1$$

Θ : homog. degrees $> 1, n, \dots, n$
carries standard S_n -rep

Armstrong-Rhoades-Reiner
and Etingof construct
h.s.o.p.'s of suitable
degrees that generalize
those of S_n

$$\text{Park}_W^{\text{alg}} \simeq (\{v\}) / (\Theta_W)$$

$$g \mapsto (n+1)^{\dim(V^g)}$$

Θ : homog. degrees $> 1, n, \dots, n$
carries W -reflection repn.

Theorem [Sommers]
"type-independent"

$$\bigoplus_{[x] \in \mathbb{F}_w/W} K_{\text{rew}}^{\text{Ord.Id.}}_W(\{x\}) \cdot \begin{smallmatrix} \uparrow_w \\ W_x \text{ triv} \end{smallmatrix} \simeq_w \text{Park}_W^{\text{alg}}$$

Question: What about $\text{Krew}_W^{\text{NC}}(\{x\})$?

Goal: A type-independent proof of $\text{Park}_W^{\text{NC}} \simeq_w \text{Park}_W^{\text{alg}}$

$$\text{Park}_W^{\text{NC}} := \bigoplus_{\{x\} \in L_W/W} \text{Krew}_W^{\text{NC}}(\{x\}) \cdot \mathbb{P}_{Wx}^W \text{ triv}$$

\hookrightarrow W-noncrossing Partitions
 ↑↑??↑

\hookrightarrow Order ideals in Root Posets

$$\text{Park}_W^{\text{alg}} := C(V)/(\partial) \simeq \bigoplus_{\{x\} \in L_W/W} \text{Krew}_W^{\text{alg}}(\{x\}) \cdot \mathbb{P}_{Wx}^W \text{ triv}$$

$$[g \mapsto (h+1)^{\dim(V^g)}]$$

Consequence:

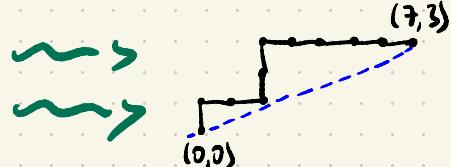
A type-independent proof that $\text{Krew}_W^{\text{NC}}(\{x\}) = \text{Krew}_W^{\text{alg}}(\{x\})$

The m-W-Fuss version : Dyck paths & Park_W^{alg}(m)

Dyck paths $(0,0) \rightarrow (n,n)$



Paths $(0,0) \rightarrow (mn+1, n)$



$$\text{Krew}(\gamma) = \frac{n \cdot (n-1) \cdots (n-k+1)}{\text{Sym}(\gamma)}$$

$$m\text{-Krew}(\gamma) = \frac{mn \cdot (mn-1) \cdots (mn-k+1)}{\text{Sym}(\gamma)}$$



$$\text{Park}_W^{\text{alg}} := C(V)/(I_W)$$

$$g \mapsto (h+1)^{\dim(Vg)}$$

$$\Theta_W \rightsquigarrow \text{h.s.o.p. degrees} = h+1$$

$$\text{Structure coefficients } \text{Krew}_W^{\text{alg}}(x) = \frac{\prod(h+1-b_i^x)}{[N(x); W_x]}$$

$$\text{Park}_{W,m}^{\text{alg}} := C(V)/(I_{W,m})$$

$$g \mapsto (mh+1)^{\dim(Vg)}$$

$$\Theta_{W,m} \rightsquigarrow \text{h.s.o.p. degrees} = mh+1$$



$$\text{Structure coefficients } m\text{-Krew}_W^{\text{alg}}(x) = \frac{\prod(mh+1-b_i^x)}{[N(x); W_x]}$$

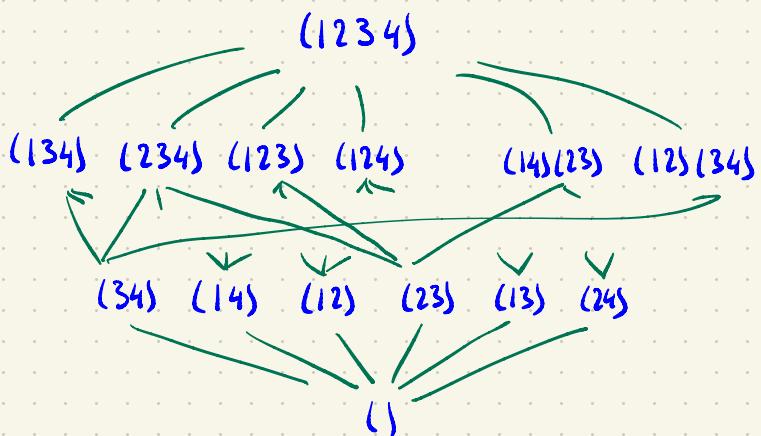
The m-W-Fuss version : Noncrossing Partitions

$m=1$ case

of elements

in $NC(W)$

of "type" $[x]$



arbitrary m case

of chains in $NC(W)$
starting at an element
of "type" $[x]$

$$K_{\text{rew}}^{NC}_W([x]) = \frac{\prod (h+1 - b_i^x)}{(N(x): W_x)}$$



$$m\text{-}K_{\text{rew}}^{NC}_W([x]) = \frac{\prod (mh+1 - b_i^x)}{(N(x): W_x)}$$

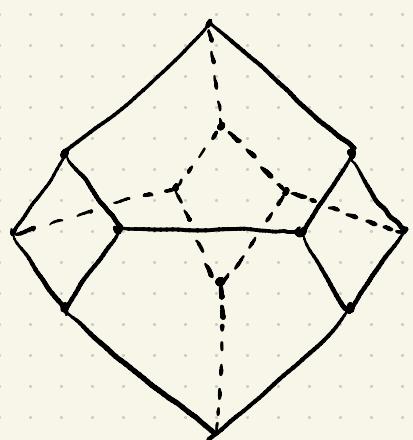
[Rhoades]

[Athansiadis-Reiner]

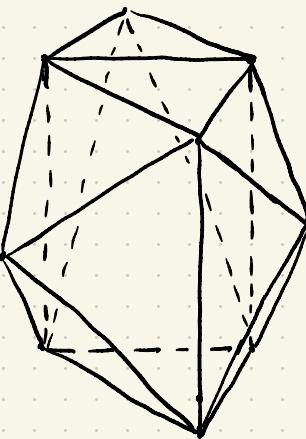
Define $\text{Park}_W^{NC}(m) := \bigoplus_{[x] \in h_w/W} m\text{-}K_{\text{rew}}^{NC}_W([x]) \cdot \text{P}_{W_x}^{\text{triv}}$

The m -W-Fuss version : W-associahedra

$m=1$ case



DUAL



W-associhedron

W-cluster complex

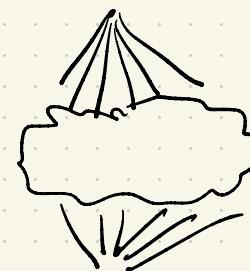
Thm [D., Sosuaat-Verges]

$$\gamma_W(\{x\}) = \frac{\prod (h+1+b_i^x)}{(N(x):W_x)}$$

= # of faces of "type" $\{x\}$



arbitrary m case



- No polytope
- No cluster algebra
- Yes complex (simplicial)

Thm [D., Sosuaat-Verges]

$$m - \gamma_W(\{x\}) = \frac{\prod (mh+1+b_i^x)}{(N(x):W_x)}$$

= # of faces of "type" $\{x\}$
in m -cluster complex

Goal: A type-independent proof of $\text{Park}_W^{NC}(m) \simeq_w \text{Park}_W^{\text{alg}}(m)$

$$\text{Park}_W^{NC}(m) := \bigoplus_{\{x\} \in L_W/W} m\text{-Krew}_W^{NC}(\{x\}) \cdot \uparrow_{W_x}^W \text{triv}$$

\hookrightarrow chains of W -noncrossing Partitions

Players:

$$\text{Park}_W^{\text{alg}}(m) := C(V)/(\Theta) \simeq \bigoplus_{\{x\} \in L_W/W} m\text{-Krew}_W^{\text{alg}}(\{x\}) \cdot \uparrow_{W_x}^W \text{triv}$$

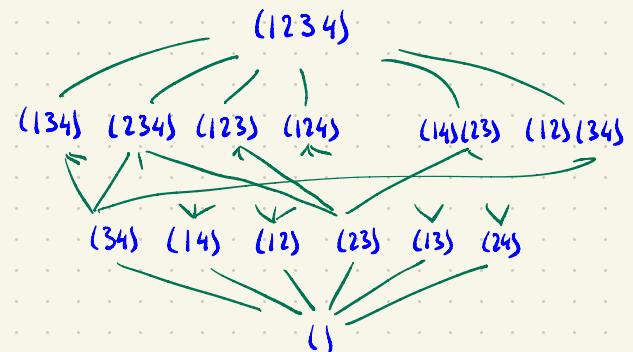
$g \mapsto (m h + 1)^{\dim(V^g)}$

$$L_s := \frac{\prod (m h + 1 - b_i)}{[N(x) : W_x]}$$

Consequence:

A type-independent proof that $m\text{-Krew}_W^{NC}(\{x\}) = \frac{\prod (m h + 1 - b_i)}{[N(x) : W_x]}$

Idea of the Proof



Whitney #s

1

$$6 = 4 + 2$$

6

1

$h-f$
transform

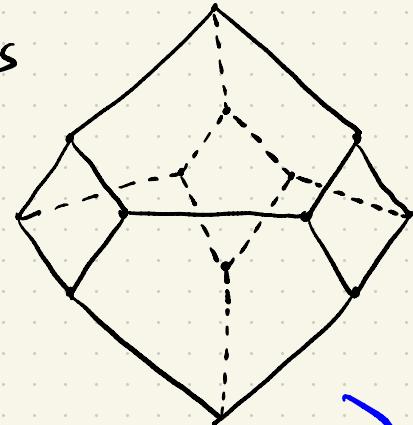
Face #s

14

21

9

1



⑥ Recursion
on chains

⑥ Recursion
on faces

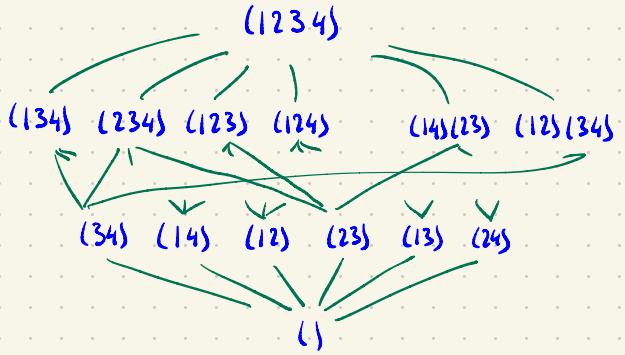
$$\frac{\prod (m_i h_i + 1 - b_i^*)}{(N(x):W_x)}$$

- ① Find combinatorial recursions that determine the chain numbers?
- ② Prove the same recursions for structure coeff's

?? Must go through cluster complex ??

Structure of the proof: Recursion on chains

A natural recursion : Count chains of length m w.r.t. their k -th element ($m = k+r$)



$$m\text{-Krew}_W^{NC}([x]) = \sum k\text{-Krew}_{W_y}^{NC}([x]) \cdot r\text{-Krew}_W^{NC}([y])$$

$$["x", c]_{\leq_R} \quad ["x", "y"]_{\leq_R} \quad ["y", c]_{\leq_R}$$

In terms of algebraic
Parking space $\text{Park}_W^{\text{alg}}(m)$

[recall $g \rightarrow (mb+1)^{\dim(V^g)}$]

$$(t + kh)^{\dim(x)} = \sum_{y \leq x} t^{\dim(y)} \cdot \prod_{i=1}^{\deg(x)-\deg(y)} kh_i(x, y)$$

\downarrow
 $\text{Park}_W^{\text{alg}}(m)$

[D. '21+] via W-Laplacians

\uparrow
 $\text{Park}_{W_y}^{\text{alg}}(k)$

Where is this coming from?

Matrix Forest Theorem (For complete graph K_n , unweighted)

$$(t+n)^{n-1} = \sum_{k=1}^n C_k \cdot t^{K-1}$$

of rooted forests on $\{n\}$ w/ k trees

Example: $(t+4)^3 = 4^3 + (\underbrace{4 \cdot 3^2}_{\text{1 tree}} + \underbrace{3 \cdot (2 \cdot 2)}_{\text{2 trees}}) \cdot t + \underbrace{6 \cdot 2}_{\text{3 trees}} \cdot t^2 + \underbrace{1}_{\dots} \cdot t^3$



↓ ↓ ↓ generalizes
to:

Laplacian Recursion Lemma [Chapuy-D.'20 & Burman]

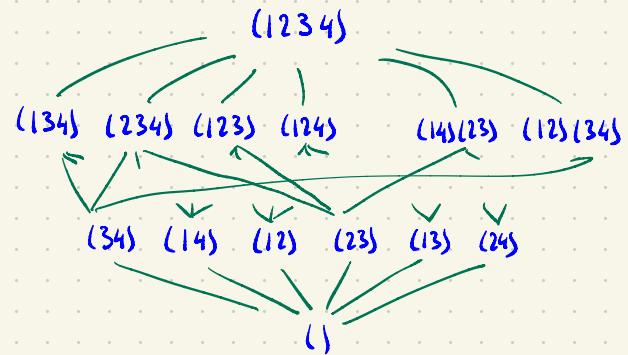
For every hyperplane arrangement A there is a Laplacian L_A and

$$\det(L_A + t) = \sum_{X \in \text{ht}(A)} p \det(L_{A_X}) \cdot t^{\dim(X)}$$

↳ product of non-zero eigenvalues

Structure of the proof: Recursion on chains Problem?

Chains may be lazy!



$$m\text{-Krew}_W^{NC}([x]) = \sum K\text{-Krew}_{W_y}^{NC}([x]) \cdot r\text{-Krew}_W^{NC}([y])$$

$\{[x], c\}_{\leq R}$ $\{[x], [y]\}_{\leq R}$ $\{[y], c\}_{\leq R}$

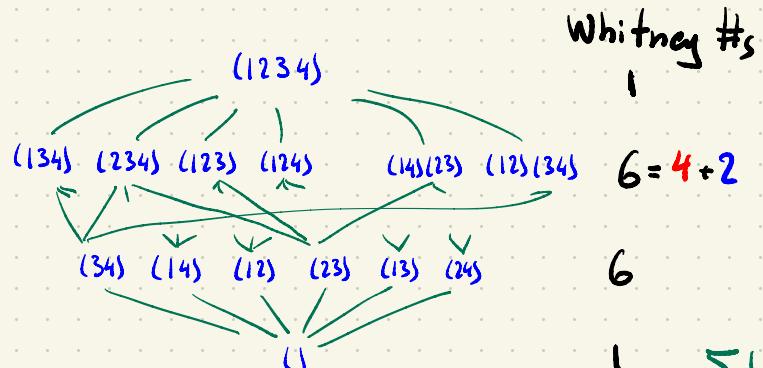
RHS contains the interval $\{[x], c\}_{\leq R}$ twice!

$$K\text{-Krew}_W^{NC}([x], m) - K\text{-Krew}_W^{NC}([x], k) - K\text{-Krew}_W^{NC}([x], r) = \sum_{y \neq x} K\text{-Krew}_{W_y}^{NC}([x], k) \cdot K\text{-Krew}_W^{NC}([y], r)$$

* Linear term is not determined!

Assumed known by induction!

Structure of the proof: Relate with m-cluster complex



Whitney #s

1

$$6 = 4 + 2$$

6

$$1 \quad \sum h_{xt} t^x = \sum f_{xt} (t-1)^x$$

$h-f$
transform



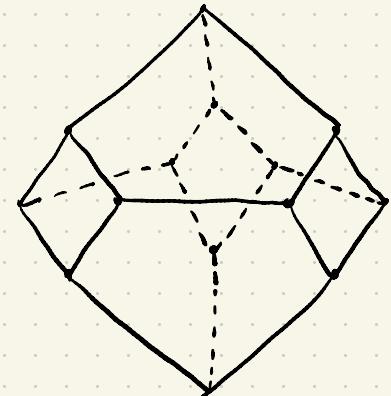
Face #s

14

21

$$9 = 6 + 3$$

1



Combinatorially: $\delta_W([X], m) = \sum_{[Y] \leq [X]} V_{X,Y} \cdot \text{Krew}_W^{NC}([Y])$ [Josuat-Vergès, D.]

$$t = mh + 1 \quad (\uparrow)$$

$$\downarrow \text{refines } \binom{n}{k} \quad (\uparrow)$$

Algebraically: $\overline{\Pi}(t + b_i x) = \sum_{Y \subset X} c(A^Y) \cdot \overline{\Pi}(t - b_i x)$ (Kung's identity)

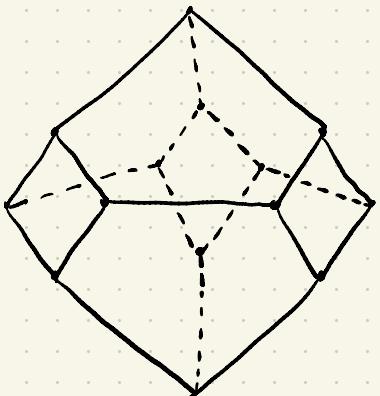
Structure of the proof: Face Recursion on cluster c_x

Face #s

14

21

$$q = \begin{matrix} 6+3 \\ \text{---} \\ 1 \end{matrix}$$



- Faces are smaller associahedra
(m-cluster c_x 's)
- Faces are simplices?

Combinatorially

$$\binom{\dim(x)}{k} \cdot f_w([x], m) = \sum_{\substack{[y] \leq [x] \\ d(y) = k}} f_w([y], m) \cdot f_{w_y}([x], m)$$

Algebraically

$$\binom{\dim(x)}{k} \cdot \prod (mh + l + b_i^{x,y}) = \sum_{\substack{Y \subseteq X \\ d(Y) = k}} \left[\prod (mh + l + b_i^{x,y}) \right] \cdot \left[\prod (mh_i(x,y) + l + b_i^{x,y}) \right]$$

Structure of the proof: Technical part

$$\textcircled{0} \quad n! \cdot \prod_{i=1}^n (mh_i + d_i) = (mh+2) \cdot \sum_{\substack{L \in h_A^{-1} \\ L \neq h_A}} \prod_{i=1}^{n-1} (mh_i (W_L) + d_i (W_L))$$

Fomin-
Reading

$$\textcircled{1} \quad n! \cdot \prod_{i=1}^n (mh_i + d_i) = \sum_{\substack{x \text{ complete} \\ \text{flags}}} \prod (mh_i (x) + 2)$$

induction on \textcircled{0}

$$\textcircled{2} \quad n! \cdot h \cdot |W| \cdot \sum_{i=1}^n \frac{1}{d_i} = 2^{n-1} \cdot \sum_{\substack{x \\ i=1 \dots n}} h_i(x)$$

extracting
linear term

$$\textcircled{3} \quad n! \cdot h \cdot |W| \cdot \sum_{i=1}^n d_i = \sum_{\substack{x \in h_A \\ x \neq 0}} \text{codim}(x)! \cdot (\dim(x)-1)! \cdot c(A_x) \cdot \sum_{z \in A^x} c(A^z) \cdot h(z, x)$$

counting
chains

$$\textcircled{4} \quad h \cdot \sum \frac{1}{d_i} = \sum_{H \in A} \frac{1}{\text{supp}(S_H)}$$

Lots of simplicial geometry

$$\textcircled{5} \quad \# \{ H : S_H \text{ is full support} \} = \frac{h_n}{2} \cdot \frac{\prod (e_i - 1)}{|W|}$$

enumerate H wrt support
of \$S_H\$

Thank You

- Counting nearest faraway flats for Coxeter chambers : 2209.06201
- The generalized cluster complex:
refined enumeration of spaces 2209.12540
(w/ M. J-V)
- Reflection Laplacians, parking spaces,
and multiderivations in Coxeter-Catalan
soon!
- Recursions and Proofs in Cataland
(w/ M. J-V) in preparation



The Algebraic Parsing Space Recursion

Theorem (D_0)

$$\text{Park}_W^{\text{alg}}(m) = \bigoplus_{[x] \in h/W} \text{Krew}_W^{\text{alg}}([x], m) \cdot q_W W_x \text{Park}_{W_x}^{\text{alg}}(r)$$

- Questions:
- Ⓐ Interpretation via shift functors in Cherednik algebras?
 - Ⓑ q -version?
 - Ⓒ Combinatorial interpretation in Order Ideals?