Recursions and Proofs in Coxeter-Catalan combinatorics

Theo Douvropoulos

(joint w/ Matthieu Josuat-Vergès)

Algebraic Combinatorics Seminar @ Waterloo

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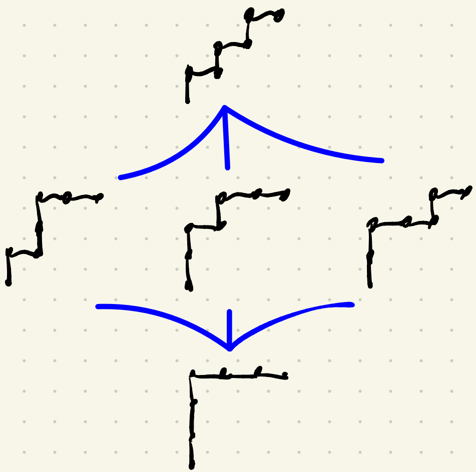
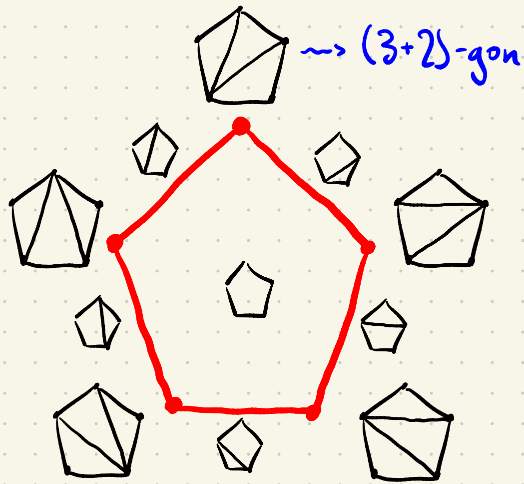
(slides @ my website)

Attractions @ the Catalan Zoo

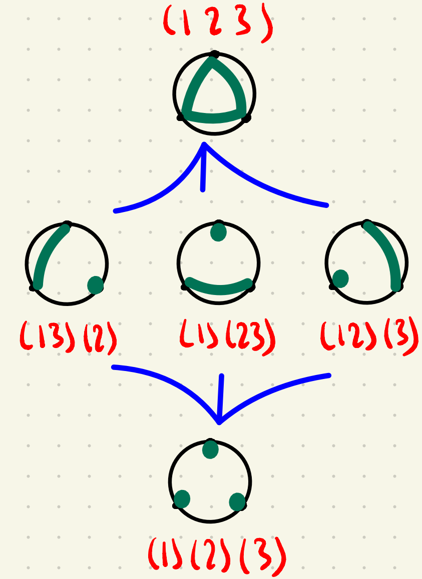
The Catalan numbers $Cat(n) := \frac{1}{n+1}\binom{2n}{n}$ count:



Dyck Paths $(0,0) \to (n,n)$



Vertices of the associahedron



Noncrossing Partitions

$Cat(n)$: 1, 1, 2, 5, 14, 42, 132, 429, ...

AKA: the answer to the ultimate question of Life the Universe, and Everything

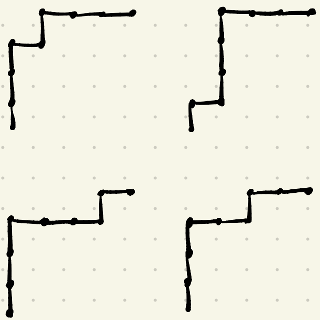
Germain Kreweras visits the Catalan Zoo

$\lambda \vdash n$

$\lambda = (\lambda_1, \dots, \lambda_k) = (1^{m_1}, \dots, n^{m_n})$

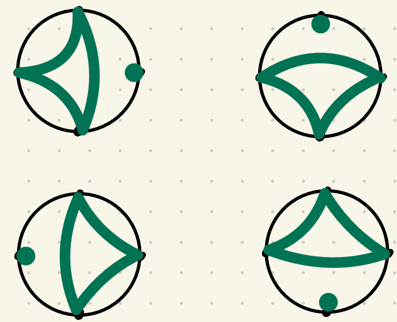
$\mathrm{Sym}(\lambda) := m_1! \cdots m_n!$

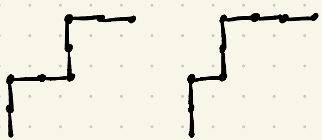
The Kreweras numbers $\mathrm{Krew}(\lambda) := \dfrac{n(n-1)\cdots(n-k+1)}{\mathrm{Sym}(\lambda)}$ count:



$\lambda = (3,1)$

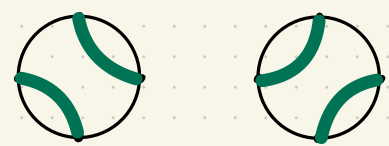
$\mathrm{Krew}(\lambda) = 4 = \dfrac{4}{1}$





$\lambda = (2,2)$

$\mathrm{Krew}(\lambda) = 2 = \dfrac{4}{2}$

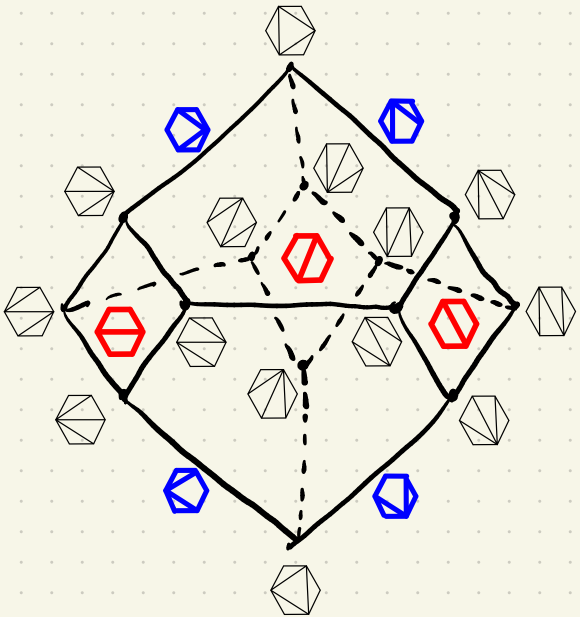


Dyck Paths whose vertical runs determine a partition λ

Noncrossing Partitions whose blocks deterine a partition λ

$\mathrm{Krew}(\lambda)$	1	$\boxed{4}$	$\boxed{2}$	6	1	$\Rightarrow$ 14
λ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$	

Jean-Louis Loday visits the Catalan Zoo



f-vector

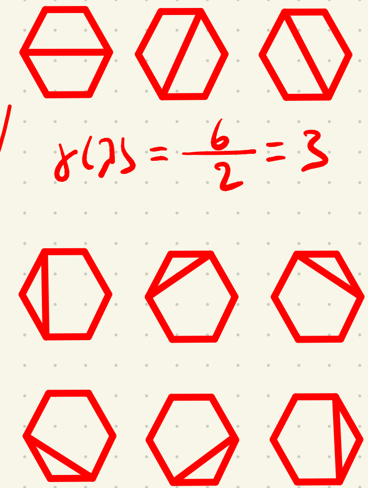
$f_0 = 14$

$f_1 = 21$

$f_2 = 9$

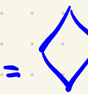
$f_3 = 1$

2-dim'l Faces λ

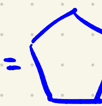


$(2,2)$

$\gamma(\lambda) = \frac{6}{2} = 3$

"type" = 

$(3,1)$

"type" = 

$\gamma(\lambda) = \frac{6}{1} = 6$

The Loday numbers $\gamma(\lambda) := \frac{(n+2)\cdots(n+k)}{\mathrm{Sym}(\lambda)}$ count $(n-k)$-dim'l faces of the associahedron that are indexed by (λ_i+2)-gons

Aside: Compositional Inverses

Question: Find $g(x)$ s.th. $F(g(x)) = x$ when

$$F(x) = x + a_1x^2 + a_2x^3 + \cdots + a_nx^{n+1} + \cdots$$

Answer:

$$g(x) = x + b_1x^2 + b_2x^3 + \cdots + b_nx^{n+1} + \cdots$$

$b_1 = -a_1 \qquad b_2 = 2a_1^2 - a_2$

$b_3 = -5a_1^3 + 5a_1a_2 - a_3 \qquad b_4 = 14a_1^4 - 21a_1^2a_2 + 6a_1a_3 + 3a_2^2 - a_4$

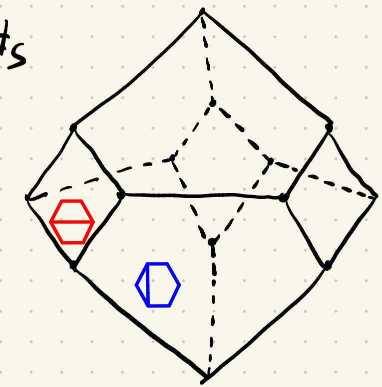
Face #s

14

21

$6 + 3 = 9$

1



Theorem [Loday] $\displaystyle g(x) = \sum_{\substack{F \text{ Face of} \\ \text{Associahedron}}} (-1)^{n-\dim(F)} \cdot \prod a_{n_i-2}$

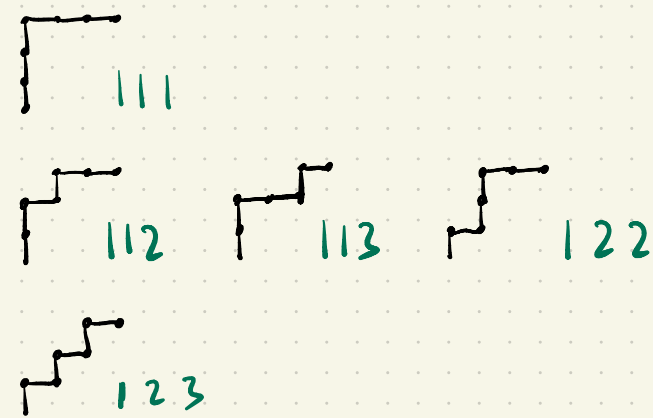
Parking @ the Catalan Zoo

A **parking function** is a word $a_1, \dots, a_n$ such that its non-decreasing rearrangement $b_1 \leq b_2 \leq \dots \leq b_n$ satisfies $b_i \leq i$

S_n acts on positions and the orbits are indexed by Dyck paths

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

S_n

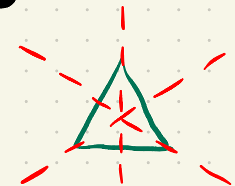


The non-decreasing representatives prescribe the vertical runs

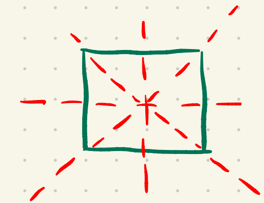
$$\boxed{\text{Corollary: } \mathrm{Park}(n) \underset{S_n}{\simeq} \bigoplus_{\lambda \vdash n} \mathrm{Krew}(\lambda) \cdot \uparrow_{S_\lambda}^{S_n} \mathrm{triv}}$$

From S_n to reflection groups W

Reflection groups are...
Finite subgroups $W \leq GL(\mathbb{R}^n)$
generated by Euclidean reflections?



$I_2(3) \simeq A_2$



$I_2(4) \simeq B_2$

Reflection groups are...
classified in four infinite families $\underset{}{\overset{S_{n+1}}{A_n}}$, B_n (hyperoctahedral), D_n, and $I_2(m)$ (dihedrals)

and some exceptionals $(H_3, H_4, F_4, E_6, E_7, E_8)$

Reflection groups have...

- nice presentations $W = \langle \underbrace{s_1, \ldots, s_n}_{\text{simple gen's}} \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$

- Coxeter elements $c = s_1 \cdot s_2 \cdots s_n$ order$(c) = h$ → "Coxeter number"

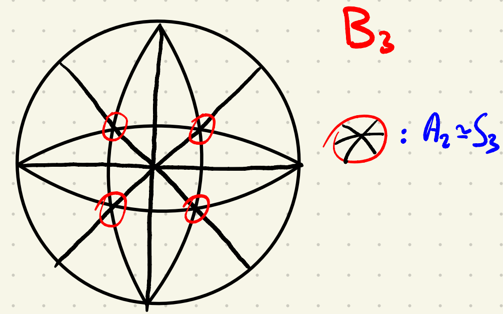
From S_n to reflection groups W: Partitions

A partition

$$\{\{1,3,4\},\{2,6,7,8\},\{5\}\}$$

determines an intersection of hyperplanes: a "Flat"

$$x_1 = x_3 \cap x_3 = x_4 \cap x_2 = x_6 \cap x_6 = x_7 \cap x_7 = x_8$$



W-orbits of Flats $[X]$ play the role of cycle types

exceptional numerology:

$\chi \rightsquigarrow$ characteristic polynomial

$\mathcal{A}_W \rightsquigarrow$ reflection arrangement

$\mathcal{A}_W^X \rightsquigarrow$ restricted arrangement

"group exponents"

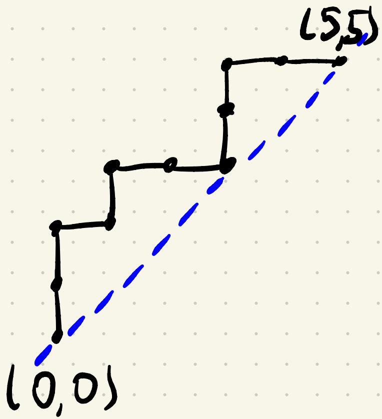
$$\chi(\mathcal{A}_W, t) = \prod_{i=1}^{\mathrm{rank}(W)} (t - e_i) \qquad \text{in } S_n:\ e_i = i$$

"Orlik-Solomon exponents"

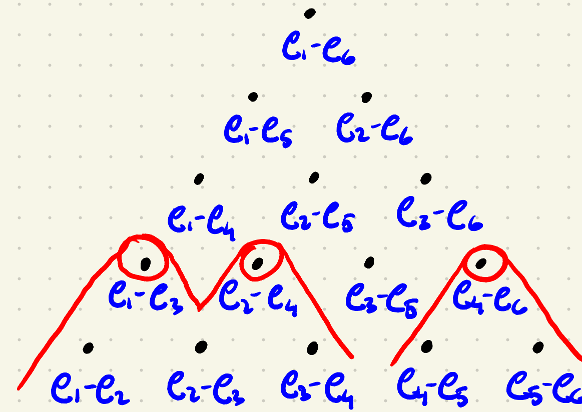
$$\chi(\mathcal{A}_W^X, t) = \prod_{i=1}^{\dim(X)} (t - b_i^X) \qquad \text{in } S_n:\ b_i^X = i$$

From S_n to reflection groups W: Dyck paths

Dyck paths



Order ideals in Root Posets



Peaks become top elements

$$\mathrm{Krew}(\lambda) = \frac{n\cdot(n-1)\cdots(n-k+1)}{\mathrm{Sym}(\lambda)} \rightsquigarrow \mathrm{Krew}_W^{\mathrm{Ord.Id.}}([X]) = \frac{\prod (h+1-b_i^X)}{[N(X):W_X]}$$

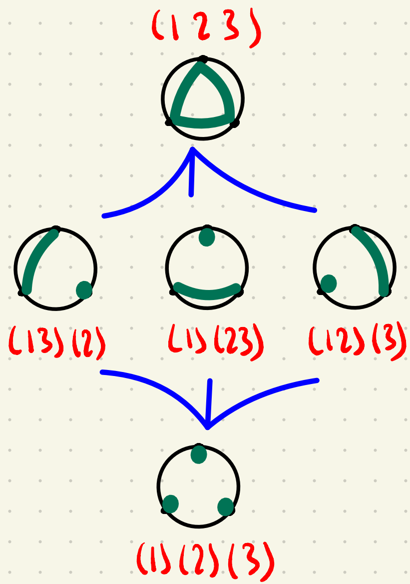
#of Dyck paths of "type" λ

[Sommers, '03, type-indep.]

# of order ideals of "type" $[X]$

From S_n to reflection groups W: Noncrossing Partitions

$A_2 \simeq S_3$

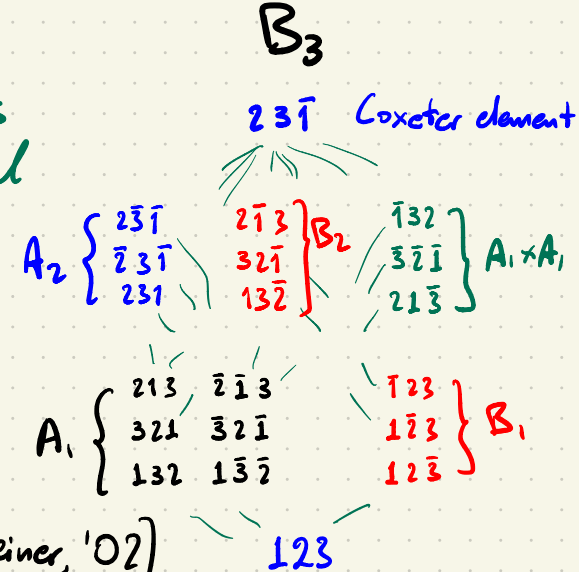


The W-noncrossing partitions are defined as the interval

$$NC(W) := [1, c]_{\leq_R}$$

under the order $\leq_R$ induced by reflection length

B_3



[Athanasiadis-Reiner, '02
case-by-case]

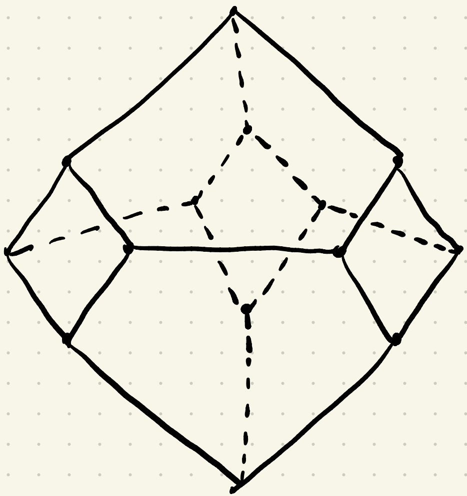
$$\mathrm{Krew}(\lambda) = \frac{n\cdot(n-1)\cdots(n-k+1)}{\mathrm{Sym}(\lambda)} \quad \rightsquigarrow \quad \mathrm{Krew}^{NC}_{W}([x]) = \frac{\prod(h+1-b_i^x)}{[N(x):W_x]}$$

# of noncrossing partitions of cycle type λ

of W-noncrossing partitions of "type" $[x]$

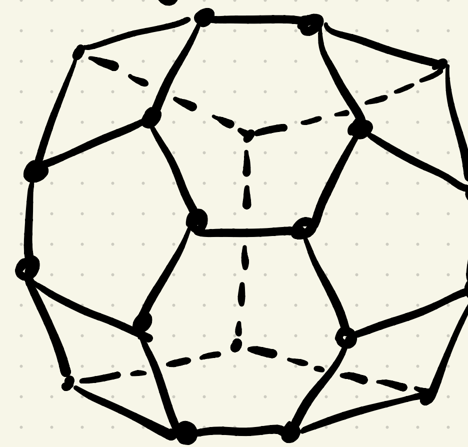
From S_n to reflection groups W: Associahedra

A_3 - Associahedron



Chapoton-Fomin-Zelevinsky construct a simple polytope whose normal fan encodes the clusters of the cluster algebra of W

B_3 - Associahedron
"cyclohedron"



The Loday numbers

$$\gamma(\lambda) = \frac{(n+2)\cdots(n+k)}{\mathrm{Sym}(\lambda)}$$

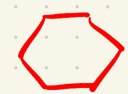
$= \#$ of faces of "type" λ

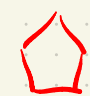
$\rightsquigarrow$

Thm [D., Josuat-Verges, '22]

$$\gamma_W([X]) = \frac{\prod (h+1+b_i^X)}{[N(X):W_X]}$$

$= \#$ of faces of "type" $[X]$

B_2:  $\rightsquigarrow \frac{8}{2} = 4$

A_2:  $\rightsquigarrow \frac{8}{2} = 4$

$A_1 \times A_1$:  $\rightsquigarrow \frac{8}{2} = 4$

Goal: A type-independent proof of the equality:

$$\mathrm{Krew}_W^{NC}([x]) = \mathrm{Krew}_W^{Ord.Id.}([x])$$

W-noncrossing Partitions $\underset{\longleftarrow}{\overset{???}{\longrightarrow}}$ Order ideals in Root Posets

Theorem [D., Josuat-Verges '22+, type-independent]

In an irreducible Coxeter group W, the number of non-crossing partitions of type [x] is given via $\mathrm{Krew}_W^{NC}([x]) = \dfrac{\prod(h+1-b_i^x)}{[N(x):W_x]}$

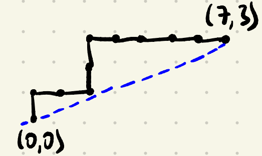
The m-W-Fuss version: Dyck paths & root ideals

Dyck paths $(0,0) \to (n,n)$

$$\mathrm{Krew}(\lambda) = \frac{n\cdot(n-1)\cdots(n-k+1)}{\mathrm{Sym}(\lambda)}$$

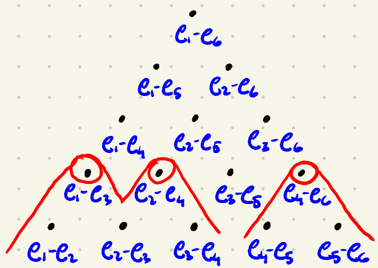
$\rightsquigarrow$

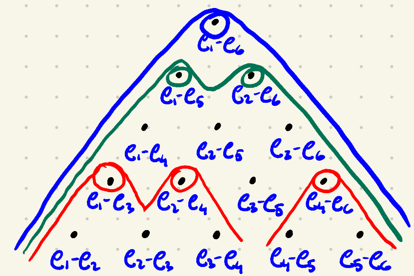
Paths $(0,0) \to (mn+1,n)$



$$m\text{-}\mathrm{Krew}(\lambda) = \frac{mn\cdot(mn-1)\cdots(mn-k+1)}{\mathrm{Sym}(\lambda)}$$

$m=1$ case
root ideals





arbitrary m case
m-chains of compatible root ideals

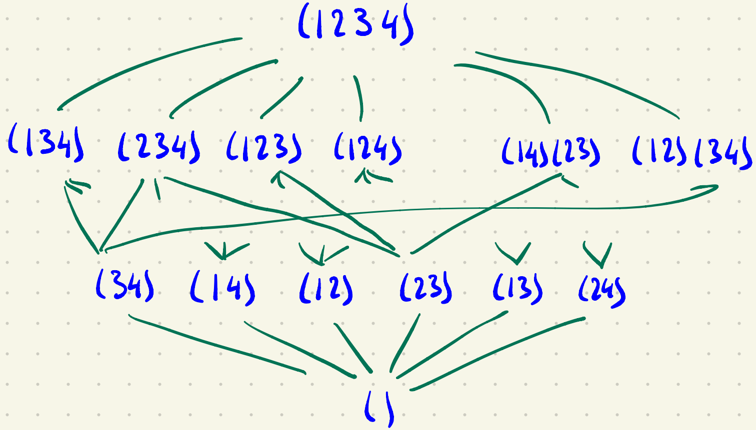
$$\mathrm{Krew}_W^{\mathrm{Ord.Id.}}([x]) = \frac{\prod (h+1-b_i^x)}{[N(x):W_x]} \rightsquigarrow m\text{-}\mathrm{Krew}_W^{\mathrm{Ord.Id.}}([x]) = \frac{\prod (mh+1-b_i^x)}{[N(x):W_x]}$$

[Sommers '03: type-independent] [Athanasiadis '04: type-independent]

The m-W-Fuss version: Noncrossing Partitions

m=1 case

# of elements in $NC(W)$ of "type" $[X]$



arbitrary m case

# of chains in $NC(W)$ starting at an element of "type" $[X]$

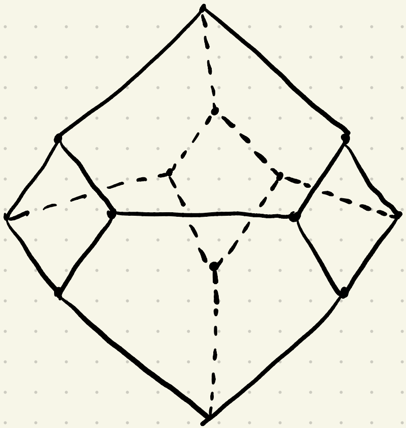
$$\mathrm{Krew}_W^{NC}([X]) = \frac{\prod(h+1-b_i^X)}{[N(X):W_X]} \rightsquigarrow m\text{-}\mathrm{Krew}_W^{NC}([X]) = \frac{\prod(mh+1-b_i^X)}{(N(X):W_X]}$$

[Athanasiadis-Reiner, '02 case-by-case]

[Rhoades, '17, case-by-case]

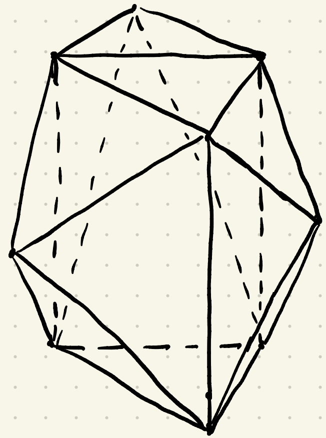
The m-W-Fuss version : W-associahedra

m=1 case



W-associahedron

DUAL



W-cluster complex

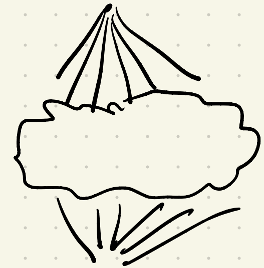
Thm [D., Josuat-Verges, '22]

$$\gamma_W([X]) = \frac{\prod (h+1+b_i^X)}{[N(X):W_X]}$$

= # of faces of "type" [X]

$\rightsquigarrow$

arbitrary m case



- No polytope
- No cluster algebra
- Yes complex (simplicial)

Thm [D., Josuat-Verges, '22]

$$m\text{-}\gamma_W([X]) = \frac{\prod (mh+1+b_i^X)}{[N(X):W_X]}$$

= # of faces of "type" [X] in m-cluster complex

Goal: A type-independent proof of the equality:

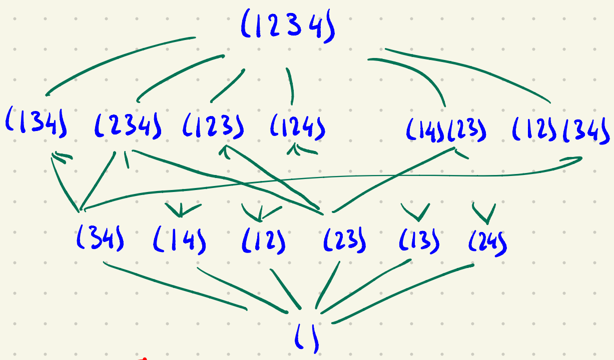
$$m\text{-}\mathrm{Krew}_W^{NC}([x]) = m\text{-}\mathrm{Krew}_W^{Ord.Id.}([x])$$

m-chains of W-noncrossing Partitions $\underset{\longleftarrow}{\overset{???}{\longrightarrow}}$ m-Chains of compatible Order ideals in Root Posets

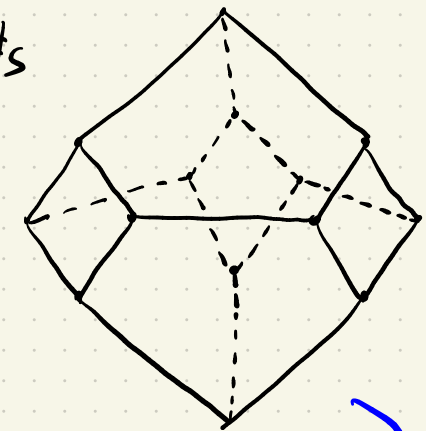
Theorem [D., Josuat-Verges '22+, type-independent]

In an irreducible Coxeter group W, the number of type [x] m-chains of W-non-crossing partitions is given via $m\text{-}\mathrm{Krew}_W^{NC}([x]) = \dfrac{\prod(mh+1-b_i^x)}{[N(x):W_x]}$

Idea of the Proof



Whitney #s		Face #s
1		14
6 = 4 + 2		21
6	h-f transform	6 + 3 = 9
1		1



Recursion on chains

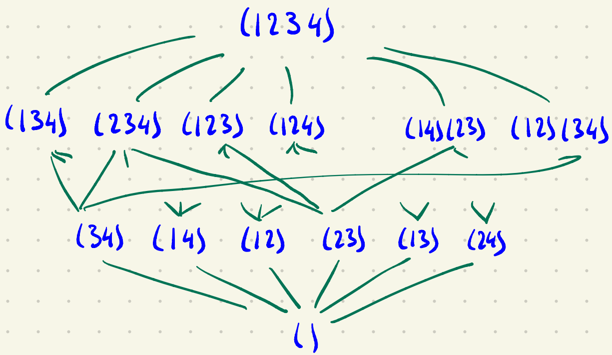
Recursion on faces

① Find combinatorial recursions that determine the chain numbers?

② Prove the same recursions for structure coeff's $\dfrac{\prod(mh+1-b_i^x)}{[N(x):W_x]}$

?? Must go through cluster complex ??

Structure of the proof: Recursion on chains



A natural recursion for Zeta polynomials: Count chains of length m w.r.t. their k-th element $(m = k+r)$

$$m\text{-Krew}^{NC}_{W}([x]) = \sum k\text{-Krew}^{NC}_{W_y}([x]) \cdot r\text{-Krew}^{NC}_{W}([y])$$

$$["x", c]_{\leq_R} \qquad ["x", "y"]_{\leq_R} \qquad ["y", c]_{\leq_R}$$

In terms of the numerology of W (after a lot of Möbius magic)

$$(t+kh)^{\dim(X)} = \sum_{Y \subset Y} t^{\dim(Y)} \cdot \prod_{i=1}^{d(x)-d(y)} k h_i(X,Y)$$

[D. '21+] via W-Laplacians

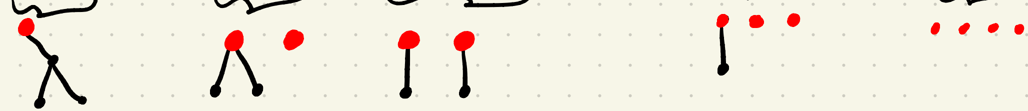
Where is this coming from?

Matrix Forest Theorem (For complete graph K_n, unweighted)

$$\boxed{(t+n)^{n-1} = \sum_{k=1}^{n} c_k \cdot t^{k-1}}$$

c_k → # of rooted forests on $[n]$ w/ k trees

Example: $(t+4)^3 = \underbrace{4^3} + (\underbrace{4\cdot 3^2} + \underbrace{3\cdot(2\cdot 2)})\cdot t + \underbrace{6\cdot 2}\cdot t^2 + \underbrace{1}\cdot t^3$



generalizes to:

Laplacian Recursion Lemma [Chapuy-D.'20 & Burman]

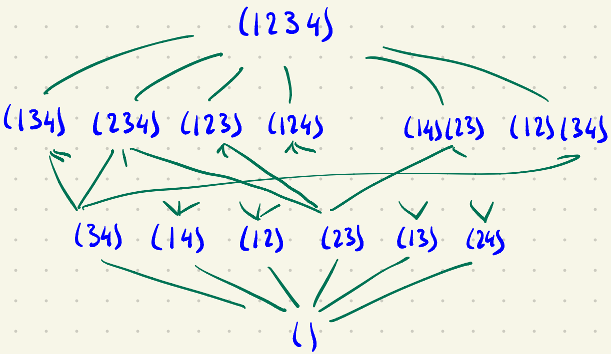
For every hyperplane arrangement $\mathcal{A}$ there is a Laplacian $L_\mathcal{A}$ and

$$\det(L_\mathcal{A} + t) = \sum_{X \in \mathcal{L}_\mathcal{A}} \mathrm{pdet}(L_{\mathcal{A}_X}) \cdot t^{\dim(X)}$$

↳ product of non-zero eigenvalues

Structure of the proof: Recursion on chains Problem!

Chains may be lazy!



$$m\text{-}Krew^{NC}_{W}([x]) = \sum k\text{-}Krew^{NC}_{W_y}([x]) \cdot r\text{-}Krew^{NC}_{W}([y])$$

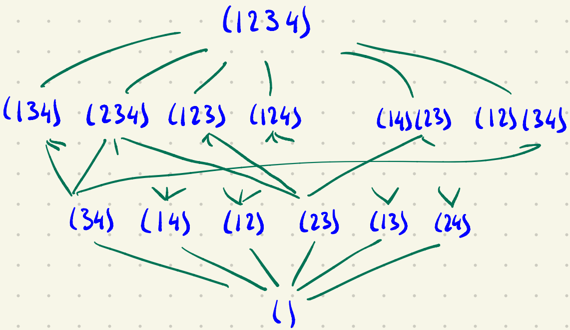
$$["x", c]_{\le_R} \qquad \underbrace{["x", "y"]_{\le_R} \qquad ["y", c]_{\le_R}}$$

RHS contains the interval $["x", c]_{\le_R}$ twice!

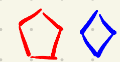
$$Krew^{NC}_{W}([x], m) - Krew^{NC}_{W}([x], k) - Krew^{NC}_{W}([x], r) = \underbrace{\sum_{y \neq x} Krew^{NC}_{W_y}([x], k) \cdot Krew^{NC}_{W}([y], r)}_{\text{Assumed known by induction!}}$$

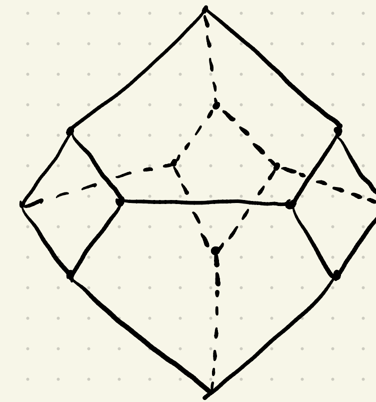
* Linear term is <u>not</u> determined!

Structure of the proof: Relate with m-cluster complex



Whitney #s		Face #s
1	h-f transform	14
6 = 4 + 2		21
6		9 = 6 + 3
1	$\sum h_k t^k = \sum f_k (t-1)^k$	1





Combinatorially: $\gamma_W([X], m) = \sum_{[Y] \leq [X]} \nu_{X,Y} \cdot \mathrm{Krew}^{NC}_W([Y])$ [Josuat-Vergès, D.]

$t = mh+1$ refines $\binom{n}{k}$

Algebraically: $\prod (t + b_i^X) = \sum_{Y \subset X} c(A^Y) \cdot \prod (t - b_i^Y)$ (Kung's identity)

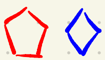
Structure of the proof: Face Recursion on cluster cx

Face #s

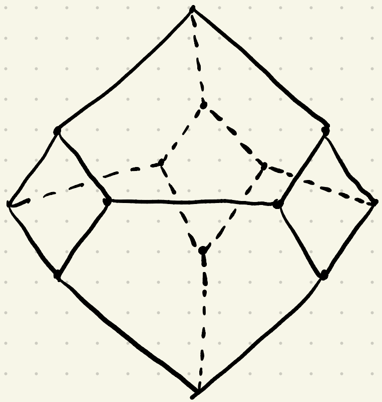
14

21

9 = 6 + 3



1



- Faces are smaller associahedra (m-cluster cx's)
- Faces are simplices?

Combinatorially $$\binom{\dim(x)}{k}\cdot\gamma_W([x],m) = \sum_{\substack{[y]\le[x]\\ d(y)=k}} \gamma_W([y],m)\cdot\gamma_{W_y}([x],m)$$

Algebraically $$\binom{\dim(x)}{k}\cdot\prod(mh+1+b_i^x) = \sum_{\substack{y\subseteq x\\ d(y)=k}} \left[\prod(mh+1+b_i^y)\right]\cdot\left[\prod(mh_i(x,y)+1+b_i^{x,y})\right]$$

We only need to show this for the linear term.

Thank You

- Counting nearest faraway flats for Coxeter chambers : 2209.06201
- The generalized cluster complex: refined enumeration of spaces 2209.12540
  (w/ M. J-V)
- Reflection Laplacians, parking spaces, and multiderivations in Coxeter-Catalan
  soon?
- Recursions and Proofs in Cataland
  (w/ M. J-V) in preparation

The algebraic Parking Space

Thm [Haiman]: The character of Park(n) is given via:

Use as definition

$$\left\{ S_n \ni g \longmapsto \underbrace{(n+1)^{cyc(g)-1}}_{\text{Trace}_{Park(n)}(g) = \#\left\{\begin{array}{c}\text{Parking Functions}\\ \text{Fixed under } g\end{array}\right\}} \right.$$

Thm [Haiman-Kraft]: There exists a homogeneous system of parameters $\underbrace{\theta_1}_{deg=1}, \underbrace{\theta_2, \ldots, \theta_n}_{deg=n+1}$ such that

$$\mathbb{C}[V] / (\theta_1, \theta_2, \ldots, \theta_n) \simeq_{S_n} Park(n)$$

ambient space

Corollaries: ⊙ Natural q-versions

⊙ Interpretation of Krew(λ) as structure coefficients

From S_n to reflection groups W: Parking Spaces

$\mathrm{Park}(n) \simeq \mathbb{C}[V]/(\Theta)$

$g \longmapsto (n+1)^{\mathrm{cyc}(g)-1}$

Θ: homog. degrees $= 1, n+1, n+1$
carries standard S_n-rep

Armstrong-Rhoades-Reiner and Etingov construct h.s.o.p.'s of suitable degrees that generalize those of S_n

$\mathrm{Park}^{\mathrm{alg}}_W \simeq \mathbb{C}[V]/(\Theta_W)$

$g \longmapsto (h+1)^{\dim(V^g)}$

Θ: homog. degrees $= h+1, .., h+1$
carries W-reflection repn.

Theorem [Sommers]
"type-independent"

$$\bigoplus_{[X]\in \mathcal{L}_W/W} \mathrm{Krew}^{\mathrm{Ord.Id.}}_W([X]) \cdot \big\uparrow_{W_X}^{W} \mathrm{triv} \simeq_W \mathrm{Park}^{\mathrm{alg}}_W$$

Question: What about $\mathrm{Krew}^{NC}_W([X])$?

The Algebraic Parking Space Recursion

Theorem [D.]

$$\mathrm{Park}^{alg}_{W}(m) = \bigoplus_{[X]\in \mathcal{L}/W} \mathrm{Krew}^{alg}_{W}([X],K)\cdot \uparrow^{W}_{W_X} \mathrm{Park}^{alg}_{W_X}(r)$$

Questions:

Ⓐ Interpretation via shift functors in Cherednik algebras?

Ⓑ q-version?

Ⓒ Combinatorial interpretation in Order Ideals?

Structure of the proof: Technical part

(0) $n \cdot \prod_{i=1}^{n} (mh + d_i) = (mh+2) \cdot \sum_{L \in \mathcal{L}_A^1} \prod_{i=1}^{n-1} (m\, h_i(W_L) + d_i(W_L))$ Fomin-Reading

(1) $n! \cdot \prod_{i=1}^{n} (mh + d_i) = \sum_{\chi \text{ complete flags}} \prod (m\, h_i(\chi) + 2)$ induction on (0)

(2) $n! \cdot h \cdot |W| \cdot \sum_{i=1}^{n} \frac{1}{d_i} = 2^{n-1} \cdot \sum_{\substack{\chi \\ i=1..n}} h_i(\chi)$ extracting linear term

(3) $n! \cdot h \cdot |W| \cdot \sum_{i=1}^{n} d_i = \sum_{\substack{X \in \mathcal{L}_A \\ X \neq 0}} \operatorname{codim}(X)! \cdot (\dim(X)-1)! \cdot c(A_X) \cdot \sum_{Z \in A^X} c(A^Z) \cdot h(Z, X)$ counting chains

(4) $h \cdot \sum \frac{1}{d_i} = \sum_{H \in A} \frac{1}{\operatorname{supp}(s_H)}$ Lots of simplicial geometry

(5) $\#\{H : s_H \text{ is full support}\} = \frac{hn}{2} \cdot \frac{\prod (e_i - 1)}{|W|}$ enumerate H wrt support of S_H