Coxeter numbers: From fake degree palindromicity to the enumeration of reflection factorizations.

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Paris VII, IRIF (ERC CombiTop)

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The number of reduced reflection factorizations of c

Theorem (Hurwitz, 1892)

There are n^{n-2} (minimal length) factorizations $t_1 \cdots t_{n-1} = (12 \cdots n) \in S_n$ where the t_i's are transpositions.

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Theorem (Deligne-Arnol'd-Bessis)

For a well-generated, complex reflection group W, with Coxeter number h, there are $\dfrac{h^n n!}{|W|}$ (minimal length) reflection factorizations $t_1 \cdots t_n = c$ of the Coxeter element c.

Arbitrary length reflection factorizations of c

If $\mathcal{R}$ denotes the set of reflections of W, we write

$$\mathrm{Fact}_{W,c}(N) := \#\{(t_1,\cdots,t_N)\in\mathcal{R}^n \mid t_1\cdots t_N = c\}.$$

Now, consider the exponential generating function:

$$\mathrm{FAC}_{S_n,c}(t) = \sum_{N\geq 0}\mathrm{Fact}_{S_n,c}(N)\frac{t^N}{N!}.$$

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If $c = (12 \cdots n) \in S_n$, then

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Notice that

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3. Lyashko-Looijenga and Bessis.
   There exist two subgroups $G_1 \leq G_2 \leq B_n$ of the braid group B_n on n strands, with *finite* indexes ν_1 and ν_2 such that:

$$\nu_1 = \frac{h^n n!}{|W|} \qquad \nu_2 = \#\{\text{reduced reflection factorizations of } c\}$$

How to count, the Frobenius way

Consider the central element $\mathfrak{R} := \sum_{t\in\mathcal{R}} t$ of the group algebra $\mathbb{C}[W]$.

$$\sum_{N\geq 0} \#\{(t_1,\cdots,t_N)\in\mathcal{R}^N \mid t_1\cdots t_N = c\} \quad \cdot\frac{t^N}{N!}$$

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Remark (Hurwitz 1901)

Exponential generating functions that enumerate factorizations of the form $a_1\cdots a_N = g$, where all a_i's belong to a set C closed under conjugation, **are finite** *(weighted)* **sums of** *(scaled)* **exponentials**.

The type-A calculation

$$\mathrm{FAC}_{S_n,c}(t) = \frac{1}{n!}\sum_{\chi\in\widehat{S_n}} \chi(1)\cdot\chi(c^{-1})\cdot\exp(t\cdot\frac{\chi(\mathfrak{R})}{\chi(1)})$$

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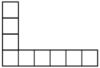
Ingredients to calculate the above sum:

1. $c = (12\cdots n)$, then $\chi(c^{-1}) \neq 0$ iff χ is a hook ().

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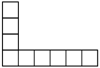
1. $c = (12 \cdots n)$, then $\chi(c^{-1}) \neq 0$ iff χ is a hook ().
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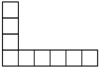
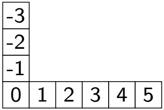
1. $c=(12\cdots n)$, then $\chi(c^{-1})\neq 0$ iff χ is a hook ().
2. If $\chi_k := \chi_{(1^k,n-k)}$, we have (using Jucys-Murphy elements on 

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Complex reflection groups and regular elements

A *finite* subgroup $G \leq GL_n(V)$ is called a *complex reflection group* if it is generated by pseudo-reflections. There are $\mathbb{C}$-linear maps t that fix a hyperplane (i.e. $\operatorname{codim}(V^t) = 1$).

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Definition

An element $g \in W$ is called ζ-regular if it has a ζ-eigenvector $\vec{v}$ that lies in no reflection hyperplane.
In particular, a *Coxeter element* is defined as a $e^{2\pi i/h}$-regular element for $h = (|\mathcal{R}| + |\mathcal{A}|)/n$.

You already know this definition of Coxeter elements

Example

1. In S_n, the regular elements are $(12\cdots n)$, $(12\cdots n-1)(n)$, and their powers. Indeed, $(\zeta^{n-1}, \zeta^{n-2}, \cdots, 1)$ with $\zeta = e^{2\pi i/n}$ is an eigenvector for $(12\cdots n)$.

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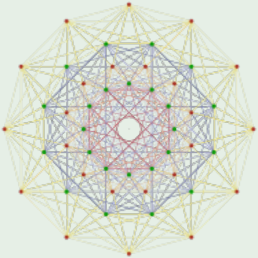
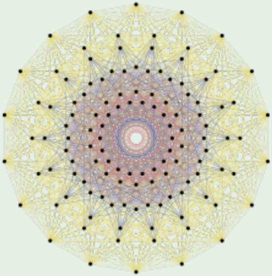
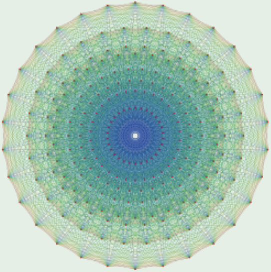
2. For real reflection groups:

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Remark

The fact that there is no uniform construction of the irreducible characters $\mathrm{Irr}(W)$ *makes it is very difficult to have a uniform proof.*

A uniform argument; the decaf version

Definition

Given a character $\chi \in \widehat{W}$, we define the Coxeter number c_χ as the normalized trace of $\sum_{t\in\mathcal{R}}(\mathbf{1}-t)$. That is,

$$c_\chi := \frac{1}{\chi(1)} \cdot \big(|\mathcal{R}|\chi(1) - \chi(\mathfrak{R})\big) = |\mathcal{R}| - \frac{\chi(\mathfrak{R})}{\chi(1)}.$$

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Lemma

*For a cpx reflection group W and a **regular** element $g\in W$, the total contribution in (1) of those characters $\chi\in\widehat{W}$ for which c_χ is not a multiple of $|g|$ is 0.*

[Just a whiff of coffee]

A uniform argument; the decaf version

Definition

Given a character $\chi \in \widehat{W}$, we define the Coxeter number c_χ as the normalized trace of $\sum_{t\in\mathcal{R}}(\mathbf{1}-t)$. That is,

$$c_\chi := \frac{1}{\chi(1)}\cdot\big(|\mathcal{R}|\chi(1)-\chi(\mathfrak{R})\big) = |\mathcal{R}| - \frac{\chi(\mathfrak{R})}{\chi(1)}.$$

The Frobenius Lemma gives then:

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\cdot\sum_{\chi\in\widehat{W}}\chi(1)\cdot\chi(g^{-1})\cdot\exp(-t\cdot c_\chi). \tag{1}$$

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[Just a whiff of coffee]There is a cyclic permutation on the characters, induced by a galois action on the corresponding Hecke characters, that cancels out the contributions in each non-singleton orbit.

A uniform argument; the decaf version

Remark

We write $l_R(g)$ for the reflection length of g, i.e. the smallest number k of (quasi-)reflections t_i needed to write $g = t_1 \cdots t_k$.

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$$\mathrm{FAC}_{W,g}(t) = 0 + 0 \cdot \frac{t}{1} + 0 \cdot \frac{t^2}{2!} + \cdots + 0 \cdot \frac{t^{l_R(g)-1}}{(l_R(g)-1)!} + (something) \cdot \frac{t^{l_R(g)}}{l_R(g)!} + \cdots .$$

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Theorem

For a complex reflection group W, and a regular element $g \in W$:

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|} \cdot \left[(1-X)^{l_R(g)} \cdot \Phi(X) \right] \Big|_{X=e^{-t|g|}}$$

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Corollary

When W is a complex reflection group and $g \in W$ a regular element, then

1. *If $|g| = d_n$ (includes Coxeter elements)* $\mathrm{FAC}_{W,g}(t) = \dfrac{e^{t|\mathcal{R}|}}{|W|} \cdot \left(1 - e^{-t|g|}\right)^{l_R(g)}$

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2. *Generally, we have that* $\mathrm{RedFact}_W(g) =$ *multiple of* $\dfrac{|g|^{l_R(g)}(l_R(g))!}{|W|}$

Can anyone guess what is happening?

Example

Below are the polynomials $\tilde{\Phi}(X)$ for $W = S_n$, $n = 4 \cdots 6$ and all regular classes

1. S_4:
    1. $(1234) \quad : (1 - X)^3$

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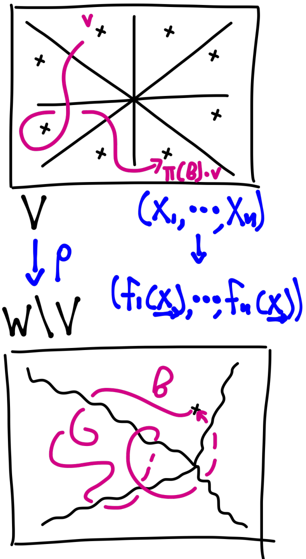
The topological braid group $B(W)$



Theorem (Steinberg)

W acts freely on the complement of the hyperplane arrangement $V^{\text{reg}} := V \setminus \bigcup H$. That is, $\rho : V^{\text{reg}} \to W \setminus V^{\text{reg}}$ is a covering map.

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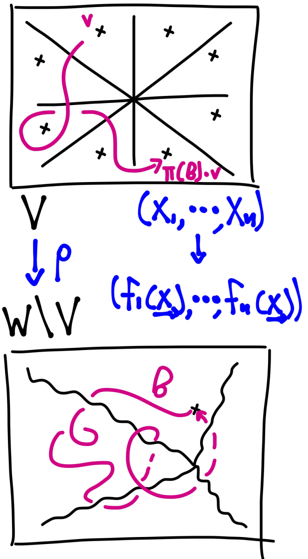

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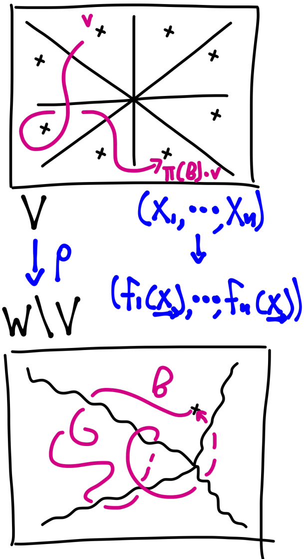
$$1 \hookrightarrow \underset{\substack{\| \\ P(W)}}{\pi_1(V^{\text{reg}})} \xrightarrow{\rho_*} \underset{\substack{\| \\ B(W)}}{\pi_1(W \backslash V^{\text{reg}})} \overset{\pi}{\twoheadrightarrow} W \to 1$$



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Theorem (Shephard-Todd-Chevalley, GIT)

W is realized as the group of deck transformations of a covering map ρ which is **explicitly** *given via the fundamental invariants f_i.*

Hecke algebras for complex reflection groups

Consider a set of parameters $\boldsymbol{u} := (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W,\ 0\leq j\leq e_{\mathcal{C}}-1)}$ where $\mathcal{A} := \bigcup H$ is the reflection arrangement, $\mathcal{C}$ an orbit of hyperplanes, and $e_{\mathcal{C}}$ the common order of the pointwise stabilizers W_H $(H\in\mathcal{C})$.

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Definition

The *generic Hecke algebra* $\mathcal{H}(W)$ associated to W is the quotient of the group ring $\mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}]B(W)$, over the ideal generated by the elements of the form

$$(\boldsymbol{s}-u_{\mathcal{C},0})(\boldsymbol{s}-u_{\mathcal{C},1})\cdots(\boldsymbol{s}-u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

which we call *deformed order relations*. Here $\boldsymbol{s}$ runs over all possible generators of the monodromy around the stratum $\mathcal{C}$ of $\mathcal{H}$.

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Theorem (Formerly known as **"The BMR-freeness conjecture"**)

The generic Hecke algebra is free over $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$ *of rank* $|W|$.

Hecke algebras , an example

Example

The generic Hecke algebra of G_{26} (over the ring $\mathbb{Z}[x_0^{\pm 1}, \cdots y_2^{\pm 1}]$) is:

$$\mathcal{H}(G_{26}) \;=\; \langle\ \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}\ |\quad \boldsymbol{stst} = \boldsymbol{tsts},\ \boldsymbol{su} = \boldsymbol{us},\ \boldsymbol{tut} = \boldsymbol{utu},$$

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After the specializations $(x_0, x_1) = (1, -1)$, $(y_0, y_1, y_2) = (1, \zeta_3, \zeta_3^2)$, we obtain the following Coxeter-like presentation of G_{26}:

$$G_{26} \ = \ \langle s, t, u \mid stst = tsts,\ su = us,\ tut = utu,\ s^2 = t^3 = u^3 = 1\ \rangle.$$

Splitting fields for Hecke algebras

Theorem (Malle)

Let K be the field of definition of W. There is a number N such that for parameters $\boldsymbol{v} := (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W,\ 0\leq j\leq e_{\mathcal{C}}-1)}$, which satisfy

$$v_{\mathcal{C},j}^N = \exp(2\pi i/e_{\mathcal{C}})u_{\mathcal{C},j}$$

the algebra $K(\boldsymbol{v},\boldsymbol{v}^{-1})\mathcal{H}(W)$ is split.

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Definition

We consider the 1-parameter specialization $u_{\mathcal{C},0}\to x$ and $u_{\mathcal{C},j}\to\exp(2\pi ij/e_{\mathcal{C}})$. Then, if y is such that $y^N = x$, $K(y)\mathcal{H}_x(W)$ is split.

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Definition (Malle's Permutation Ψ)

We write Ψ for the permutation of the irreducible modules of $\mathcal{H}_x(W)$ induced by the galois conjugation $y\to e^{2\pi i/N}\cdot y \ \in \mathrm{Gal}\big(K(y)/K(x)\big)$.

Fake degree palindromicity

Let $(f_1, \cdots, f_n)$ be *homogeneous* generators of the invariant algebra $\mathbb{C}[V]^W$ (so they satisfy $f_i(g^{-1}\boldsymbol{v}) = f_i(\boldsymbol{v})\ \forall \boldsymbol{v} \in V$).

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The fake degree $P_\chi(q) := \sum q^{e_i(\chi)}$ of a character $\chi \in \widehat{W}$ is a polynomial that records the *exponents* $e_i(\chi)$ of χ. These are the degrees of the graded components of $\mathrm{co}(W)$ that contain copies of χ.

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Theorem (Beynon-Lusztig, Malle, Opdam)

The fake degrees $P_\chi(q)$ satisfy the following palindromicity property:

$$P_\chi(q) = q^{c_\chi} P_{\Psi(\chi^*)}(q^{-1}),$$

where c_χ are the Coxeter numbers and Ψ is Malle's permutation on $\mathrm{Irr}(W)$.

The proof of the technical lemma

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|} \cdot \sum_{\chi \in \widehat{W},\ c_\chi | |g|} \chi(1) \cdot \chi(g^{-1}) \cdot \exp(-t \cdot c_\chi).$$

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5. If $k = \dfrac{d}{\gcd(c_\chi, d)} \neq 1$, we have $\displaystyle\sum_{i=1}^{k} \Psi^k(\chi)(w) = 0$.

Weighted enumeration

Definition

Consider a set of variables $\boldsymbol{w} := (w_{\mathcal{C}})_{(\mathcal{C}\in\mathcal{A}/W)}$ and the weight function

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and the exponential generating funtion of weighted reflection factorizations:

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Theorem

For a regular element $g \in W$, the weighted generating function takes the form:

$$\mathrm{FAC}_{W,g}(\boldsymbol{w}, z) = \frac{e^{z\cdot\mathrm{wt}(\mathcal{R})}}{|W|}\cdot\Big[\Phi(\boldsymbol{X})\cdot\prod_{\mathcal{C}\in\mathcal{A}/W}(1-X_{\mathcal{C}})^{n_{\mathcal{C}}}\Big]_{X_{\mathcal{C}}=e^{-zw_{\mathcal{C}}|g|}}.$$

The exponents $n_{\mathcal{C}}$ are equal to the smallest number of reflections from $\mathcal{C}$ necessary in **any** *reflection factorization of g.*

Thank you!