Hurwitz numbers for Reflection Groups

FPSAC 2021

Theo Douvropoulos
Alejandro Morales } UMass Amherst
Joel Lewis => George Washington U.

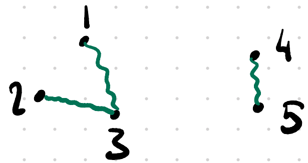
Hurwitz numbers: A brief Intro.

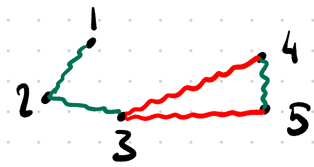
Main players: Factorizations $t_1 \cdot t_2 \cdot \cdots \cdot t_k = \sigma$ of permutations $\sigma \in S_n$ in transpositions t_i.

Property definition: A factorization is called transitive if the group $\langle t_1, \dots, t_k \rangle$ acts transitively on the set $[n] := \{1, 2, \dots, n\}$

non example: $(23)\cdot(45)\cdot(13) = (123)(45)$

yes example: $(12)\cdot(23)\cdot(34)\cdot(45)\cdot(35) = (123)(45)$





Hurwitz numbers: A brief Intro.

Theorem [Hurwitz, Goulden-Jackson, ...]
For an element $\sigma \in S_n$ of cycle-type $\lambda = (\lambda_1, \dots, \lambda_c)$ the number of minimum-length, transitive, transposition factorizations of σ is:

$$H_0(\lambda) = (n+c-2)! \cdot n^{c-3} \cdot \prod_{i=1}^{c} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}$$

Special Cases:

If $\lambda = (n)$, we have $H_0((n)) = (n-1)! \cdot n^{1-3} \cdot \frac{n^n}{(n-1)!} = n^{n-2}$

If $\lambda = (1^n)$, we have $H_0((1^n)) = (2n-2)! \cdot n^{n-3}$

Hurwitz numbers: A brief Intro.

There exist many proofs of the remarkable product formula

$$H_0(\lambda) = (n+c-2)! \cdot n^{c-3} \cdot \prod_{i=1}^{c} \frac{\lambda_i^{\lambda_i}}{(\lambda_i-1)!}$$

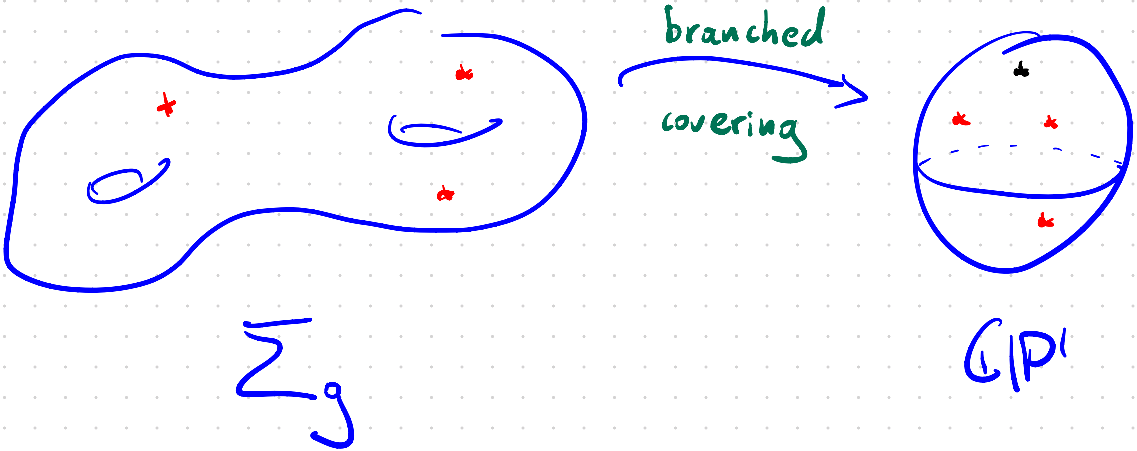
(A) Hurwitz + Strehl: Comparing a combinatorial cut-and-join recursion with the formula.

(B) Goulden-Jackson: Generating Functionology after the cut-and-join recursion and Lagrange Inversion.

(C) ELSV Formula: Via a degree calculation of an algebraic morphism (defined on a cone over $\overline{\mathcal{M}}_{0,n}$).

(D) Schaeffer-Poulalhon-Duchi: A bijective proof involving trees on the cycles of the permutation.

Why Hurwitz numbers?



- stylish bowtie
- questionable mustache
- unlucky first name

They count <u>classes</u> of branched coverings of the sphere $\mathbb{CP}^1$ by surfaces of genus g.

We are interested mostly in the $g=0$ case.

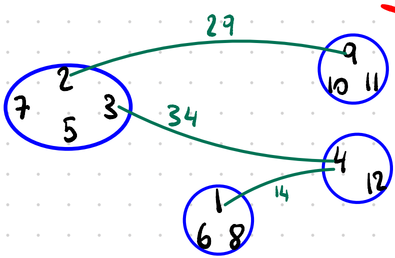
Hurwitz numbers for Reflection Groups: the setup

① Ambient group: A Weyl group W (later: well-gen'd complex refl. group)

② Property definition: $\underbrace{t_1 \cdots t_k}_{\text{reflections}} = \sigma$ is called Full if $\langle t_1, \ldots, t_k \rangle = W$

③ Class of elements: $\sigma \in W$ is called parabolic quasi-Coxeter if there exists $t_1 \cdot \cdots \cdot t_k = \sigma$ s.th.
- t_i are reflections and k is minimum
- $\langle t_1, \ldots, t_k \rangle$ is a parabolic subgroup.

④ Supporting Combinatorial Object: The Family $RGS(W, \sigma)$ of Relative Generating Sets contains all sets of reflections $\{t_{k+1}, \ldots, t_n\}$ s.th.



$\sigma = (168)(2538)(4\,12)(9,10,11)$

$\{(14), (29), (34)\} \in RGS(G_{12}, \sigma)$

$$\langle \underbrace{t_1, \ldots, t_k}_{\text{any shortest Fac'n of } \sigma}, t_{k+1}, \ldots, t_n \rangle = W$$

Hurwitz numbers for Reflection Groups: Main Theorem

$F^{full}(\sigma) :=$ # of shortest-length, full reflection factorizations of σ.

$F^{red}(\sigma) :=$ # of shortest-length reflection factorizations of σ.

[D.-Lewis-Morales '20] For a parabolic quasi-Coxeter $\sigma \in W$, with generalized cycle-decomposition $\sigma = \sigma_1 \cdot \cdots \cdot \sigma_c$, we have:

$$F^{full}(\sigma) = \ell^{full}(\sigma)! \cdot |RGS(W,\sigma)| \cdot \frac{I(W_\sigma)}{I(W)} \cdot \prod_{i=1}^{c} \frac{F^{red}(\sigma_i)}{\ell^{red}(\sigma_i)!}$$

(W_σ is the parabolic closure of σ and $I(W)$ the connection index of W.)

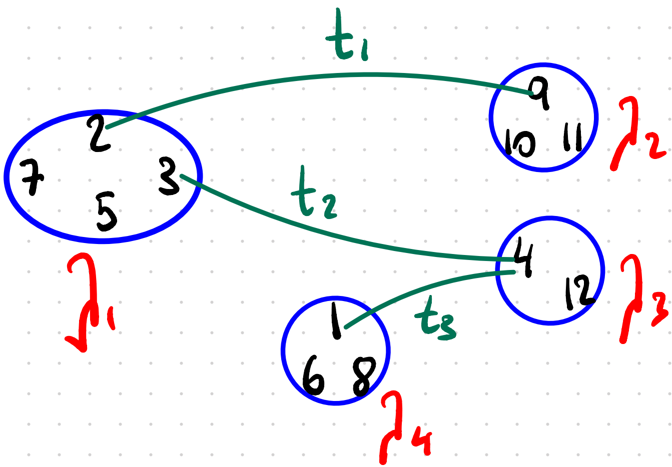
This is a full generalization of the Hurwitz formula.

$$H_0(\lambda) = (n+c-2)! \cdot n^{c-3} \cdot \prod_{i=1}^{c} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}$$

Concordance of the Hurwitz Formulas for $\mathfrak{S}_n$.

Comparing the two formulas one needs to show for $\lambda=(\lambda_1,\dots,\lambda_c)$ that

$$|RGS(S_n,\lambda)| = n^{c-2}\cdot\prod_{i=1}^{c}\lambda_i$$



$$|RGS(S_n,\lambda)| = \sum_{\substack{T \text{ is a tree} \\ \text{on } [c]}} \prod_{i=1}^{c}\lambda_i^{\deg(T_i)}$$

$= \lambda_1\cdot\dots\cdot\lambda_c\cdot(\lambda_1+\dots+\lambda_c)^{c-2}$ by Cayley's theorem.

$= \lambda_1\cdots\lambda_c\cdot n^{c-2}$

trees on $[c] := \{1,2,\dots,c\}$

Hurwitz numbers for Reflection Groups: The Proof

$$\boxed{F^{\text{full}}(\sigma) = \ell^{\text{full}}(\sigma)! \cdot |RGS(W,\sigma)| \cdot \frac{I(W_\sigma)}{I(W)} \cdot \prod_{i=1}^{c} \frac{F^{\text{red}}(\sigma_i)}{\ell^{\text{red}}(\sigma_i)!}}$$

Our proof is case-by-case (help us?) and by separately calculating the two sides:

Ⓐ Combinatorial Families A_n, B_n, D_n
- → LHS: Projection to $\mathfrak{S}_n$ and we rely on $H_0(?)$ and $H_1(?)$.
- → RHS: (relative) tree counting

Ⓑ Exceptional groups
- → LHS: Representation theory and SAGE
- → RHS: SAGE

Heuristics (Don't worry, they don't work!)

$$\boxed{F^{full}(\sigma) = \ell^{full}(\sigma)! \cdot |RGS(W,\sigma)| \cdot \frac{I(W_\sigma)}{I(W)} \cdot \underbrace{\prod_{i=1}^{c} \frac{F^{red}(\sigma_i)}{\ell^{red}(\sigma_i)!}}_{\hookrightarrow \text{ Almost } F^{red}(\sigma)}}$$

- Combining $F^{red}(\sigma)$ and $RGS(W,\sigma)$ we produce Full Factorizations.

$$\underbrace{t_1 \cdots\cdots t_k}_{\text{in } Red_W(\sigma)} \cdot \underbrace{t_{k+1} \cdot t_{k+1} \cdot \cdots \cdot t_n \cdot t_n}_{\{t_{k+1}, \ldots, t_n\} \in RGS(W,\sigma)} = \sigma \quad \text{is Full?}$$

- The existence of such Factorizations characterizes par. quasi-Coxeter elements.
- No simple way to produce remaining Factorizations but they are all in the same "Hurwitz orbit".

Numerology and Structure for quasi-Coxeter elements

The quantities $F^{red}(\sigma_i)$ always factor nicely!

- σ_i a cycle of length λ_i: $F^{red}(\sigma_i) = \lambda_i^{\lambda_i - 2}$
- σ_i a Coxeter element: $F^{red}(\sigma_i) = \dfrac{h^n \cdot n!}{|W|}$ (h = Coxeter # of W)
- In general: $F^{red}(\sigma_i)$ is conjecturally the degree of a Lyashko-Looijenga (branching) morphism of a related Frobenius manifold.

g	$F_W^{\mathrm{red}}(g)$	g	$F_W^{\mathrm{red}}(g)$	g	$F_W^{\mathrm{red}}(g)$	g	$F_W^{\mathrm{red}}(g)$
A_n	$(n+1)^{n-1}$	$E_7(a_3)$	$2 \cdot 3^4 \cdot 5^6$	$E_8(a_7)$	$2^{13} \cdot 3^6 \cdot 5 \cdot 7$	$H_4(2)$	$3^4 \cdot 5^2$
B_n	n^n	$E_7(a_4)$	$2^4 \cdot 3^8 \cdot 5 \cdot 7$	$E_8(a_8)$	$2^7 \cdot 3^9 \cdot 5^2 \cdot 7$	$H_4(3)$	$2^6 \cdot 3^3$
$I_2(m)$	m	E_8	$2 \cdot 3^5 \cdot 5^7$	F_4	$2^4 \cdot 3^3$	$H_4(4)$	$2^3 \cdot 3 \cdot 5^3$
E_6	$2^9 \cdot 3^4$	$E_8(a_1)$	$2^{18} \cdot 3^5$	$F_4(a_1)$	$2^3 \cdot 3^4$	$H_4(5)$	$2 \cdot 3^2 \cdot 5^3$
$E_6(a_1)$	3^{10}	$E_8(a_2)$	$2^{10} \cdot 5^7$	H_3	$2 \cdot 5^2$	$H_4(6)$	$3^4 \cdot 5^2$
$E_6(a_2)$	$2^6 \cdot 3^5 \cdot 5$	$E_8(a_3)$	$2^{12} \cdot 3^6 \cdot 5 \cdot 7$	$H_3(1)$	$2 \cdot 3^3$	$H_4(7)$	$2^6 \cdot 5^2$
E_7	$2 \cdot 3^{12}$	$E_8(a_4)$	$2 \cdot 3^{13} \cdot 5 \cdot 7$	$H_3(2)$	$2 \cdot 5^2$	$H_4(8)$	$2^3 \cdot 3^4 \cdot 5$
$E_7(a_1)$	$2 \cdot 7^7$	$E_8(a_5)$	$3^5 \cdot 5^7 \cdot 7$	H_4	$2 \cdot 3^3 \cdot 5^2$	$H_4(9)$	$2 \cdot 3^3 \cdot 5^2$
$E_7(a_2)$	$2^9 \cdot 3^6 \cdot 5$	$E_8(a_6)$	$2^3 \cdot 3^2 \cdot 5^8 \cdot 7$	$H_4(1)$	$2^6 \cdot 5^2$	$H_4(10)$	$2^3 \cdot 3 \cdot 5^3$

$$F_{D_n}^{\mathrm{red}}(D_n(a,b)) = 2 \cdot (n-1) \cdot \binom{n-2}{a-1,b-1} \cdot a^a \cdot b^b \text{ with } a+b=n$$

TABLE 1. The counts $F_W^{\mathrm{red}}(g)$ for quasi-Coxeter elements g of real reflection groups W.

This also comes with a dual-braid theory

Baumeister-Neaime-Rees

The case of well-generated complex reflection groups

$$\boxed{F^{full}(\sigma) = \ell^{full}(\sigma)! \cdot |RGS(W,\sigma)| \cdot \frac{I(W_\sigma)}{I(W)} \cdot \prod_{i=1}^{c} \frac{F^{red}(\sigma_i)}{\ell^{red}(\sigma_i)!}}$$

$$\boxed{F^{full}(\sigma) = \ell^{full}(\sigma)! \cdot \left(\prod_{i=1}^{c} \frac{F^{red}(\sigma_i)}{\ell^{red}(\sigma_i)!}\right) \cdot \sum_{\mathbf{t} \in RGS(W,\sigma)} \frac{\operatorname{GramDet}(\mathbf{t}_\sigma)}{\operatorname{GramDet}(\mathbf{t} \cup \mathbf{t}_\sigma)}}$$

Broue-Corran-Michel define root systems for complex reflection groups where such Grammians behave like the connection index.

roots in a shortest length reflection factorization of σ

The higher genus case

$$\mathcal{F}^{\text{Full}}_W(\sigma; z) := \sum_{N\geq 0} \#\left\{(t_1,\dots,t_N)\in \underset{\text{reflections of } W}{\underbrace{R^N}} \text{ s.th } t_1\cdots t_N=\sigma \text{ and } \langle t_1,\dots,t_N\rangle = W\right\}\cdot \frac{z^N}{N!}$$

Theorem (structural). The generating function $\mathcal{F}^{\text{Full}}_W(\sigma; z)$ is always a finite sum of exponentials. In fact,

$$\mathcal{F}^{\text{Full}}_W(\sigma; \log z) = \frac{1}{|W|}\cdot \Phi_W(\sigma; z)\cdot (z-1)^{\ell^{\text{Full}}(\sigma)}\cdot \frac{1}{z^{|\mathcal{A}_W|}}$$

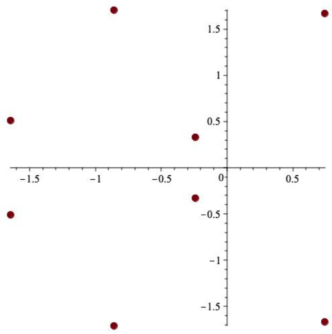
where $\Phi_W(\sigma; z)$ is a monic polynomial in z of degree $h\cdot n - \ell^{\text{Full}}(\sigma)$.

ex: $\Phi_{\widetilde{G_4}}(id; z) = z^6 + 6z^5 + 21z^4 + 40z^3 + 21z^2 + 6z + 1$

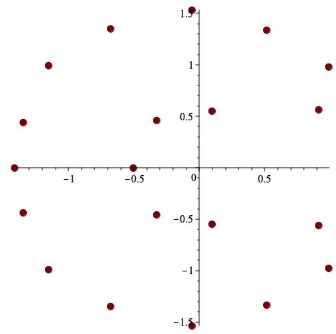
but $\Phi_{G(6,1,1)}(id; z) = z^4 + 2z^3 + 3z^2 + 2z - 2$

A.2. **Roots of the polynomials $\Phi_W(\mathrm{id}; X)$ for all exceptional complex reflection groups.** We give below the plot of roots of the polynomials $\Phi_W(\mathrm{id}; X)$ in the complex plane. For the polynomials themselves, see the data file attached as a supplementary file to this arXiv submission.

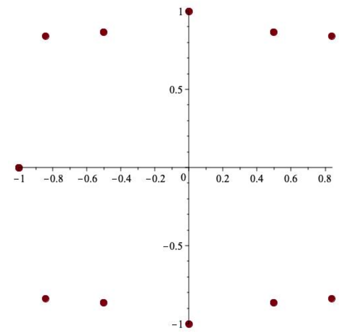
Rank 2.



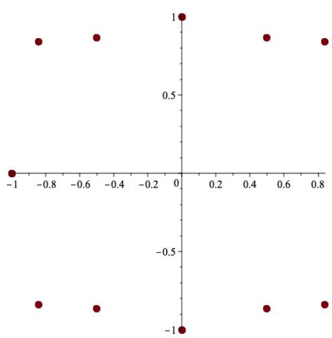
G_4



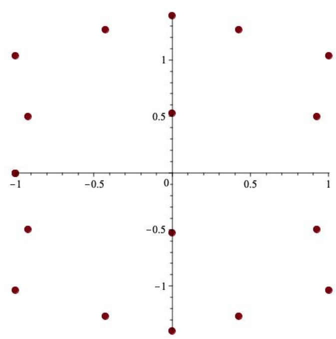
G_5



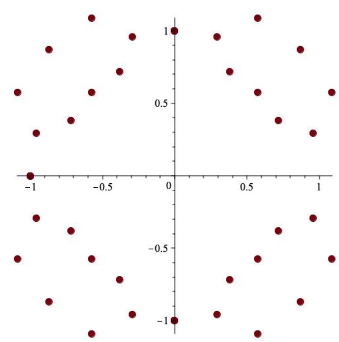
G_6



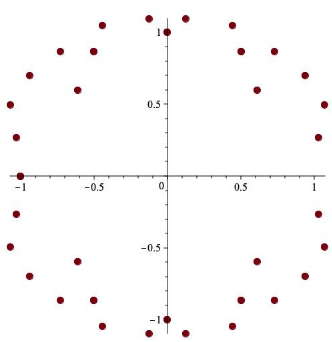
G_7



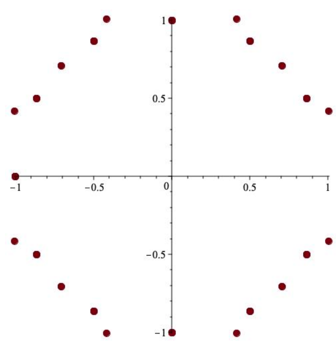
G_8



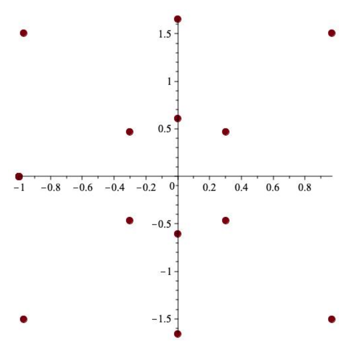
G_9



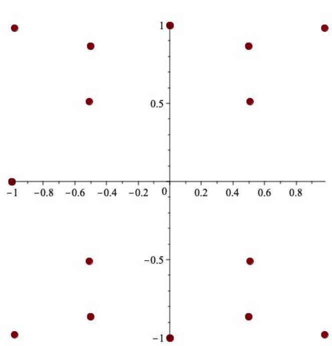
G_{10}



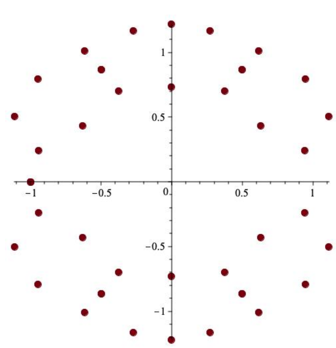
G_{11}



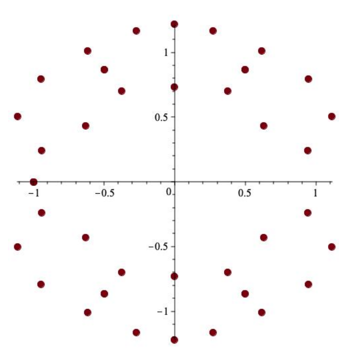
G_{12}



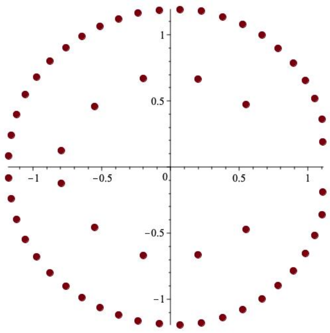
G_{13}



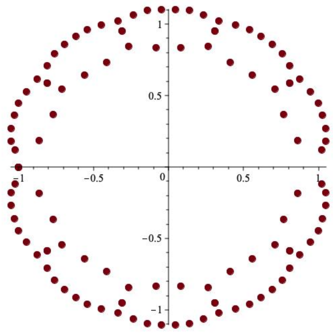
G_{14}



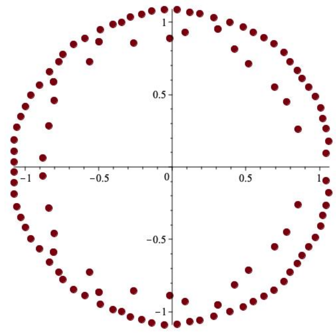
G_{15}



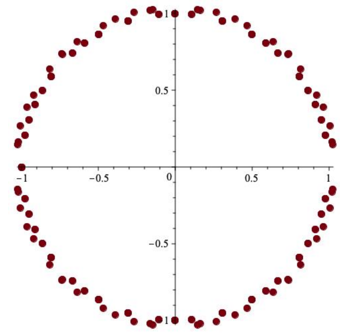
G_{16}



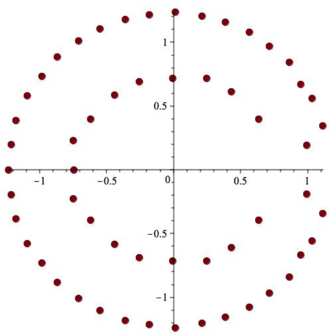
G_{17}



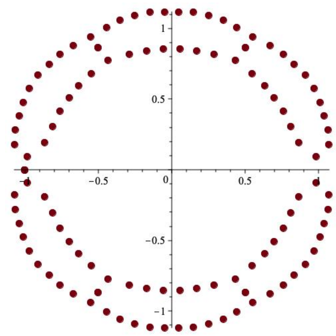
G_{18}



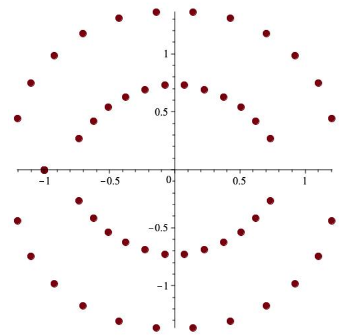
G_{19}



G_{20}

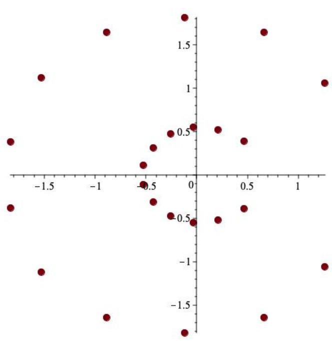


G_{21}

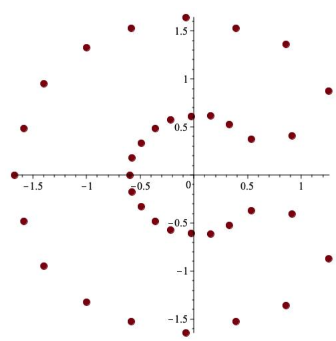


G_{22}

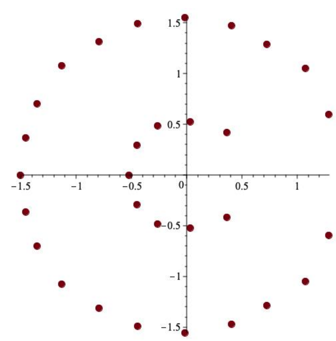
Rank 3.



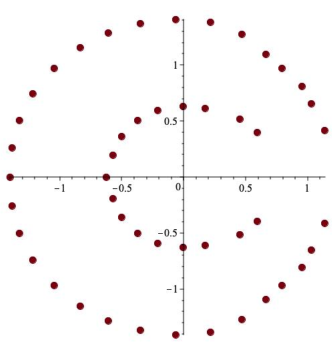
$G_{23} = H_3$



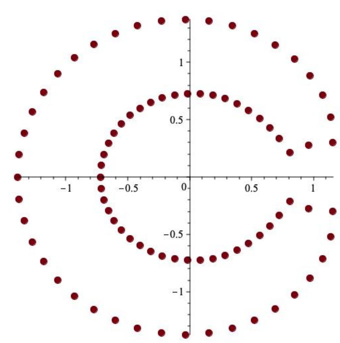
G_{24}



G_{25}

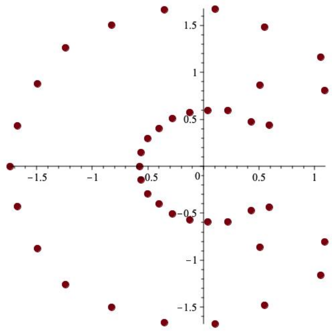


G_{26}

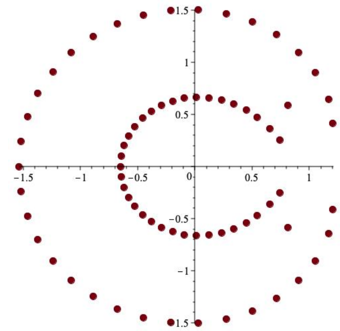


G_{27}

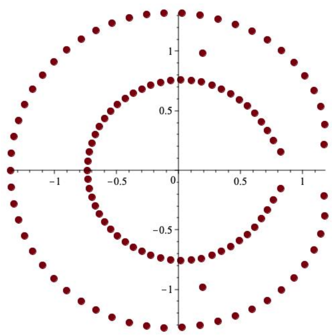
Rank 4.



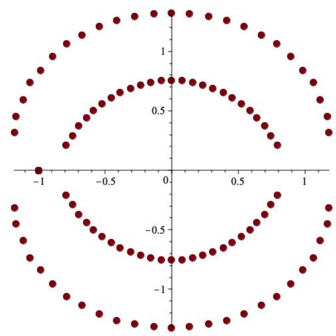
$G_{28} = F_4$



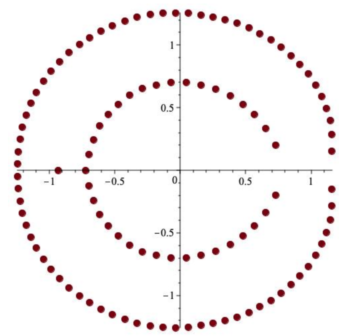
G_{29}



$G_{30} = H_4$

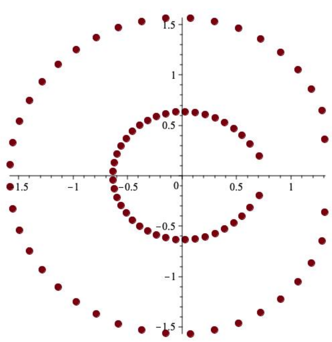


G_{31}

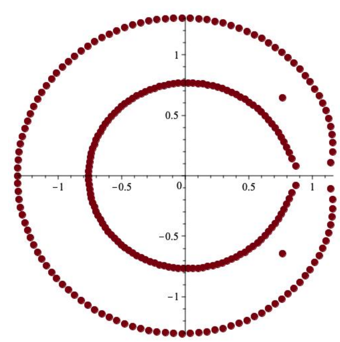


G_{32}

Ranks 5 and 6.

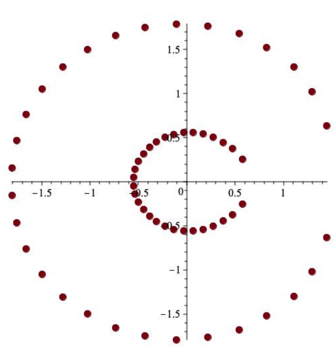


G_{33}

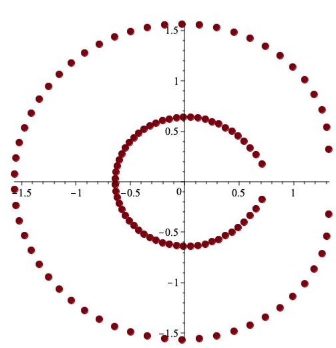


G_{34}

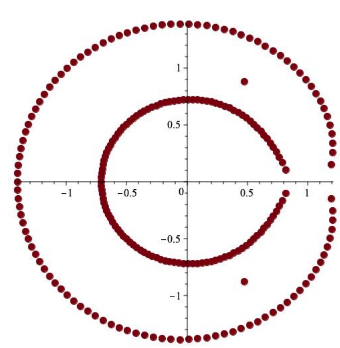
E-series.



$G_{35} = E_6$



$G_{36} = E_7$



$G_{37} = E_8$

The higher genus case

$$\mathcal{F}^{\mathrm{Full}}_{W}(\sigma; z) := \sum_{N\geq 0} \#\Big\{(t_1,\dots,t_N)\in \underset{\substack{\downarrow \\ \text{reflections of } W}}{R^N} \text{ s.th } t_1\cdots t_N=\sigma \text{ and } \langle t_1,\dots,t_N\rangle = W\Big\}\cdot \frac{z^N}{N!}$$

Theorem (enumerative for $G(m,p,n)$). For an element $\sigma \in G(m,p,n)$ with k cycles of colors $a_1,\dots,a_k$ and with $d=\gcd(a_1,\dots,a_k,p)$ we have:

$$\mathcal{F}^{\mathrm{Full}}_{m,p,n}(\sigma; z) = \underbrace{\frac{1}{m^{n-1}}}\cdot \underbrace{\mathcal{F}^{\mathrm{Full}}_{m,p,1}\big(\zeta_m^{\mathrm{col}(g)}; n\cdot z\big)}_{\substack{G(m,p,1)\\ \cong \\ \text{cyclic group of order } m/p}}\cdot\left[\sum_{r\mid d} \underset{\substack{\downarrow\\ \text{number-theoretic}\\ \text{Möbius Function}}}{\mu(r)}\cdot r^{n+k-2}\cdot \mathcal{F}^{\mathrm{Full}}_{\tilde{\mathfrak{S}}_n}\Big(\underset{\substack{\downarrow\\ \text{a projection } G(m,p,n) \to \tilde{\mathfrak{S}}_n}}{\Pi_{m/1}(\sigma)}; \frac{m}{r}\cdot z\Big)\right]$$

Thank You and a happy 2022*

With Joel B. Lewis and
Alejandro H. Morales:

Hurwitz numbers for Reflection groups:

Part I: Generating Functionology arxiv: 2112.03427

Part II: Parabolic quasi-Coxeter elements

Part III: Uniform Formulae

$\Big\}$ soon?

* I promise: There are only nine greek letters left?