Cyclic Sieving for reduced reflection factorizations of the Coxeter element

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The number of reduced reflection factorizations of c

Theorem (Hurwitz, 1892)

There are n^{n-2} (minimal length) factorizations $t_1 \cdots t_{n-1} = (12 \cdots n) \in S_n$ where the t_i's are transpositions.

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$$(12)(23)=(123)\qquad (13)(12)=(123)\qquad (23)(13)=(123).$$

Theorem (Deligne-Arnol'd-Bessis)

For a well-generated, complex reflection group W, with Coxeter number h, there are $\dfrac{h^n n!}{|W|}$ (minimal length) reflection factorizations $t_1\cdots t_n=c$ of the Coxeter element c.

What is.. a Cyclic Sieving Phenomenon (CSP)?

Ingredients:

1. A set X.
2. A polynomial $X(q)$.
3. A cyclic group $C = \langle c \rangle$ of some order n, acting on X.

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We say that the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon if for all d,

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where ζ is a(ny) primitive n^{th} root of unity.

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The polynomial $X(q)$ is sometimes a statistic on a combinatorial object, a Poincare polynomial, a Hilbert Series, a formal character.. (for more, have a look at What is... Cyclic Sieving?).

A CSP for reduced reflection factorizations of c

W is a well-generated complex reflection group of rank n, c is a Coxeter element of W and the t_i's are reflections.

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2. $$X(q) = \prod_{i=1}^{n} \frac{[hi]_q}{[d_i]_q} \qquad (\text{for } W = S_n,\ X(q) = \prod_{i=2}^{n-1} [n]_{q^i}),$$
   where the d_i's are the invariant degrees of W and $[n]_q := \frac{1-q^n}{1-q}$.

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   3. $\mathfrak{Twist} : (t_1, \cdots, t_n) \to \left({}^{(t_1 \cdots t_{n-1})} t_n, {}^{(t_1 \cdots t_{n-2})} t_{n-1}, \cdots, {}^{t_1} t_2, t_1\right)$, $2h$,
   4. ...

   where ${}^w t := wtw^{-1}$ stands for conjugation.

A CSP for reduced reflection factorizations of c

Theorem (D. '17, Conjectured in Williams' Cataland)

For an irreducible, well-generated complex reflection group W, with degrees $d_1, \cdots, d_n$, Coxeter element c and Coxeter number $h = d_n$, the triple

$$\Big(\mathrm{Red}_W(c), \prod_{i=1}^{n} \frac{[hi]_q}{[d_i]_q}, C \Big),$$

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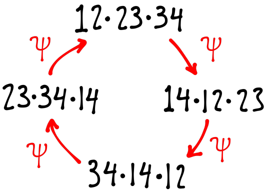
For $W = S_4$, we have:

$$X(q) = [4]_{q^2} \cdot [4]_{q^3} = \frac{1-q^8}{1-q^2} \cdot \frac{1-q^{12}}{1-q^3}$$

$$\zeta = e^{2\pi i/12}, \ \zeta^4 = e^{2\pi i/3}$$

$$X(\zeta^4) = 1 \cdot (1 + \zeta^{12} + \zeta^{24} + \zeta^{32}) = 4.$$

$$X(\zeta) = X(\zeta^2) = X(\zeta^3) = X(\zeta^6) = 0.$$



The other orbit is free and contains $12 \cdot 34 \cdot 24$.

Coxeter elements via Springer

Theorem (For us definition)

Coxeter elements are characterized by having an eigenvector $\vec{v}$, which lies on no reflection hyperplane, with eigenvalue $\zeta = e^{2\pi i/h}$, where $h = \dfrac{|R| + |R^*|}{n}$.

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For the symmetric group S_n, the Coxeter element is the (any) long cycle $(12\cdots n)$; its eigenvectors are of the form $(\zeta^{n-1}, \zeta^{n-2}, \cdots, \zeta, 1)$.

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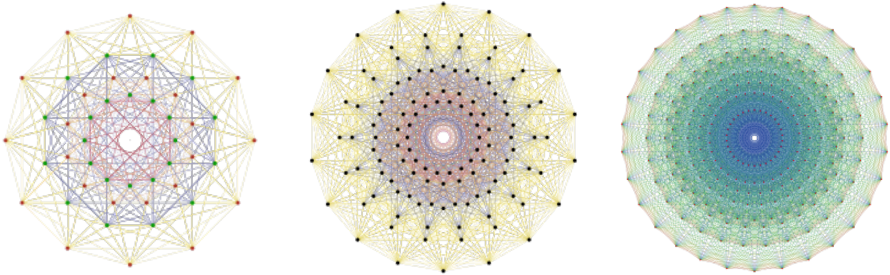
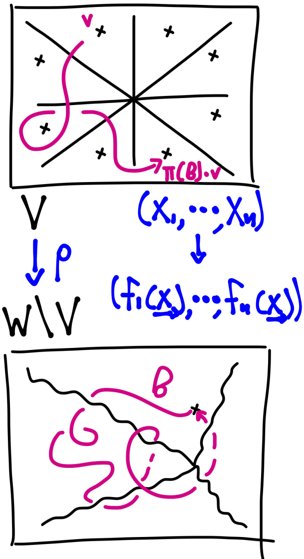
For real reflection groups:



Towards a geometric construction of the Coxeter element



Theorem (Steinberg)

W acts freely on the complement of the hyperplane arrangement $V^{\text{reg}} := V \setminus \bigcup H$. That is, $\rho : V^{\text{reg}} \to W \setminus V^{\text{reg}}$ is a covering map.

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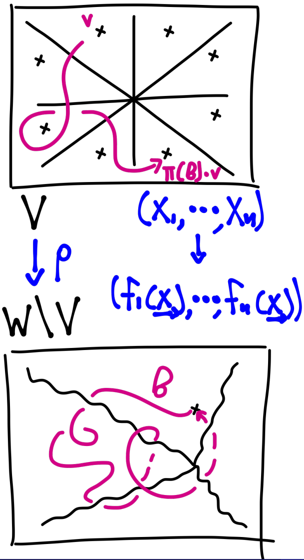

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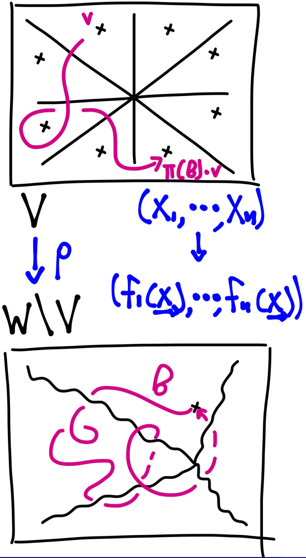

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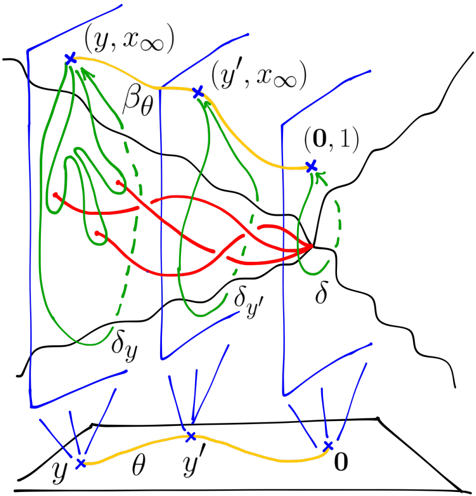
Theorem (Shephard-Todd-Chevalley, GIT)

W is realized as the group of deck transformations of a covering map ρ which is **explicitly** *given via the fundamental invariants f_i.*

Topological factorizations of Coxeter elements



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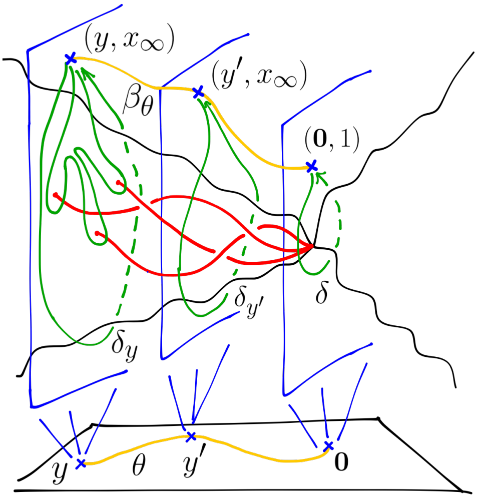

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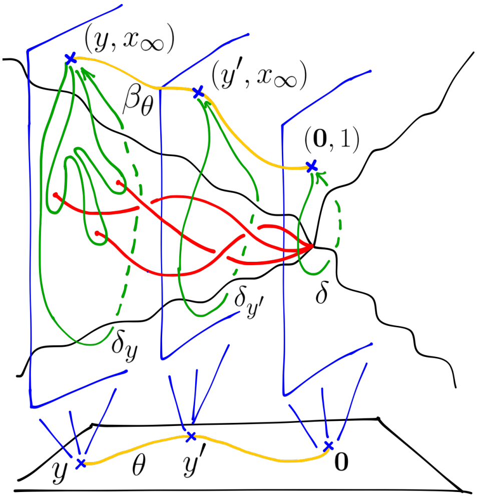

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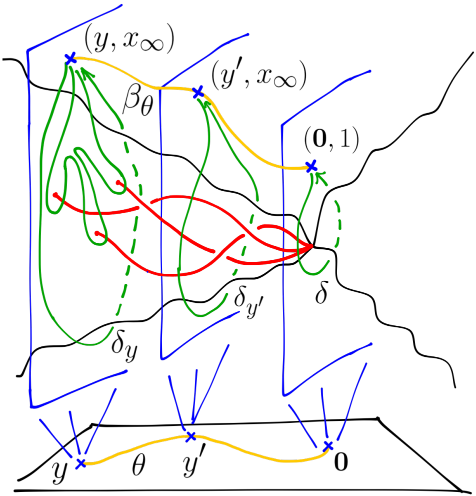

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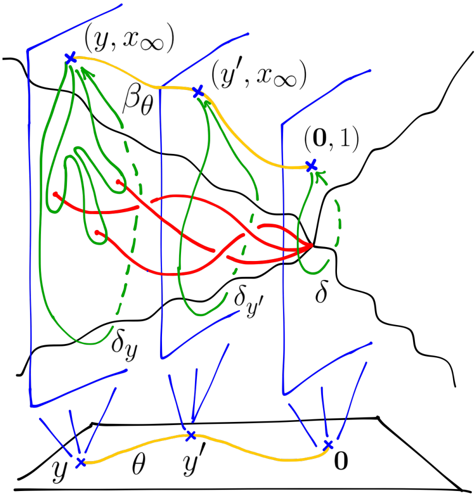
We call this construction a *labeling map* and we write

$$\mathrm{rlbl}\left(y, \quad\right) = (c_1, \cdots, c_k),$$

to indicate the dependence on the base star.

The Lyashko-Looijenga (LL) morphism



We define the LL map:

$$LL : Y \to \begin{Bmatrix} \text{centered configurations} \\ \text{of } n \text{ points in } \mathbb{C} \end{Bmatrix}$$

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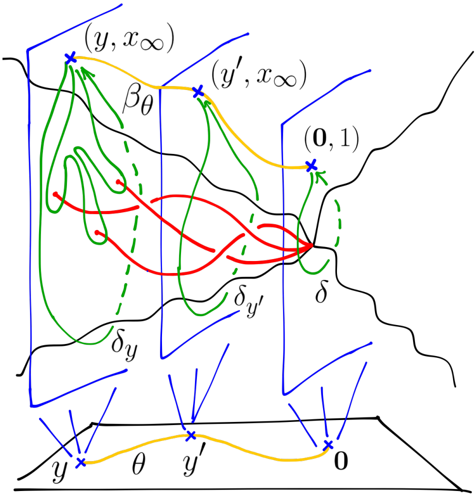

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Algebraically, it is given as:

$$(f_1, \cdots, f_{n-1}) \xrightarrow{LL} (\alpha_2(\boldsymbol{f}), \cdots, \alpha_n(\boldsymbol{f})),$$

where α_i is weighted homogeneous of degree hi.

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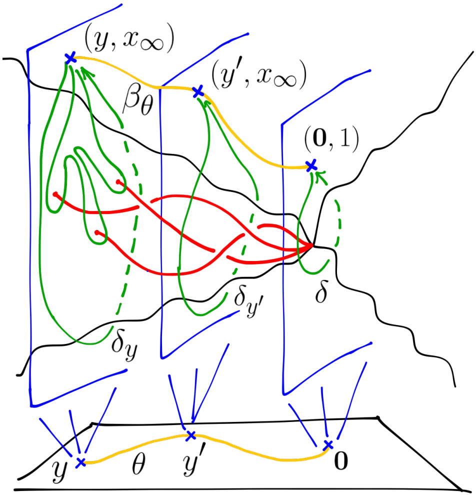

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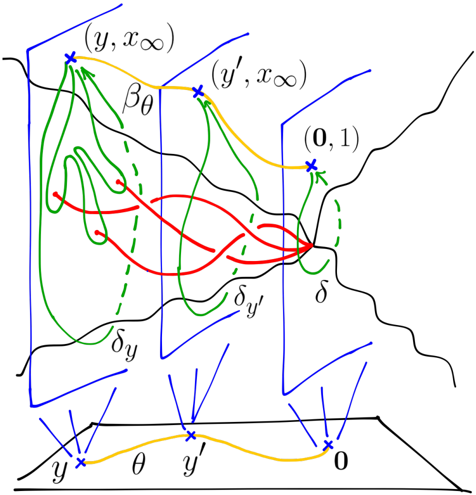

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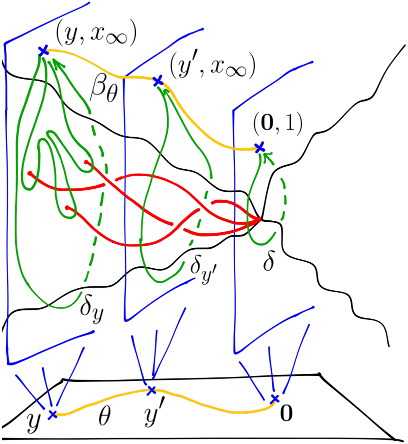
Our polynomial $X(q)$ is precisely the Hilbert series:

$$\operatorname{Hilb}\left(LL^{-1}(\mathbf{0}), q\right) = \prod_{i=1}^{n} \frac{[hi]_q}{[d_i]_q}.$$

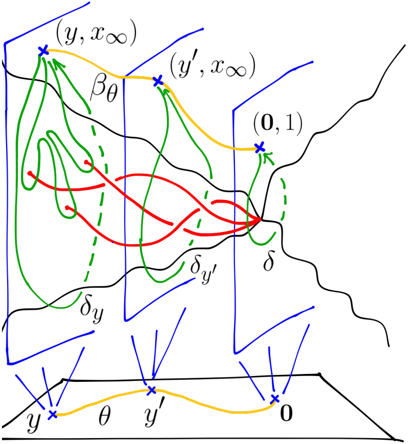
Bessis' trivialization theorem



Theorem (Bessis)

The points in a generic fiber $LL^{-1}(\boldsymbol{e})$ of the LL map are in bijection with reduced reflection factorizations of the Coxeter element c. The bijection is given by the labeling map and depends non-trivially on a choice of base-star for the configuration $\boldsymbol{e}$.

$$LL^{-1}(\boldsymbol{e}) \ni y \to \operatorname{rlbl}\left(y, \quad\right) \in \operatorname{Red}_W(c).$$



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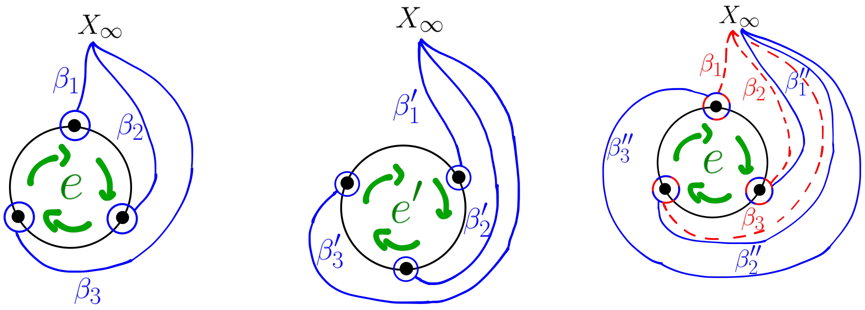
This remarkably relies on the *numerological coincidence* $\deg(LL) = |\mathrm{Red}_W(c)|$.

The cyclic actions $\Psi, \Phi, \mathfrak{Twist}, \cdots$ via the rlbl map



rlbl $\Big(y'',$  $\Big) =$ rlbl $\Big(y,$  $\Big) = (b_1, b_2, b_3)$ rlbl $\Big(y'',$  $\Big) = (b_1, b_2, b_3)$.

$(b_1, b_2, b_3) = (b_2, b_3, {}^{b_1 b_2 b_3} b_1)$ i.e. $\text{rlbl}(y) = \Psi^{-1} \cdot \text{rlbl}(y'')$

Proof of the CSP (sketch)

1. $X(q) = \mathrm{Hilb}\left(LL^{-1}(\mathbf{0}), q\right) = \displaystyle\prod_{i=1}^{n} \frac{[hi]_q}{[d_i]_q}$

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5. This allows Hilbert series $\mathrm{Hilb}\left(LL^{-1}(\mathbf{0}), q\right)$ to give a CSP for the C_{kh} (scalar) action on $LL^{-1}(\boldsymbol{e})$, and hence the combinatorial actions $\Psi, \cdots$ as well.

CSP's through finite quasi-homogeneous morphisms

Lemma (See much more generally in Broer-Reiner-Smith-Webb)

Let $\boldsymbol{f} : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial morphism, finite and quasi-homogeneous. Consider further a point ϵ, such that the fiber $\boldsymbol{f}^{-1}(\epsilon)$ is stable under **weighted** *multiplication by $c := e^{-2\pi i/N}$ for some number N. Then, if $C_N = \langle c \rangle$, we have the isomorphism of C_N-modules:*

$$\boldsymbol{f}^{-1}(\epsilon) \cong_{C_N} \boldsymbol{f}^{-1}(\mathbf{0}).$$

In particular, the triple $\left(\boldsymbol{f}^{-1}(\epsilon), \mathrm{Hilb}\left(\boldsymbol{f}^{-1}(\mathbf{0}), q\right), C_N\right)$ exhibits the CSP.

CSP's through finite quasi-homogeneous morphisms

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Let $\boldsymbol{f} : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial morphism, finite and quasi-homogeneous. Consider further a point ϵ, such that the fiber $\boldsymbol{f}^{-1}(\epsilon)$ is stable under **weighted** *multiplication by $c := e^{-2\pi i/N}$ for some number N. Then, if $C_N = \langle c \rangle$, we have the isomorphism of C_N-modules:*

$$\boldsymbol{f}^{-1}(\epsilon) \cong_{C_N} \boldsymbol{f}^{-1}(\mathbf{0}).$$

In particular, the triple $\left(\boldsymbol{f}^{-1}(\epsilon), \mathrm{Hilb}\left(\boldsymbol{f}^{-1}(\mathbf{0}), q\right), C_N\right)$ exhibits the CSP.

1. Springer's theorem $\mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_n) \cong_{W \times C} \mathbb{C}[W]$, and in particular the CSP for the q-binomial $\binom{n}{k}_q$.

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3. (Possibly..) many of the factorization enumeration formulas in S_n that have geometric interpretation. In particular, the Goulden-Jackson formula:

$$\mathrm{Fact}_{[\lambda_1,\cdots,\lambda_m]}((n))=n^{m-1}\prod_{i=1}^{m}\frac{(l(\lambda_i)-1)!}{\mathrm{Aut}(\lambda_i)}.$$

Thank you!