Recursions and Proofs in Coxeter-Catalan combinatorics

The CAGE (Combinatorics, Algebra, and Geometry) Seminar @ Drexel & UPenn

by Theo Douvropoulos (UMass Amherst)

Parking Functions and Parking Spaces

Classical

words $a_1, a_2, \ldots, a_n$ such that
non-decreasing rearrangement
$b_1 \le b_2 \le \cdots \le b_n$ has $b_i \le i$

111 — $[S_3 : S_3] \leftarrow$ 1

112 121 211 — 3

113 131 311 — $[S_3 : S_2 \times S_1]$ 3

122 212 221 — 3

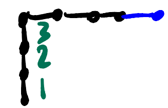
123 132 213 231 312 321 — 6 $\rightarrow [S_3 : Id]$

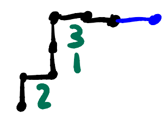
$\# = 5 = Cat(3)$

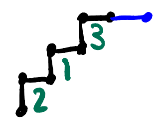
$Sum = 16 = (3+1)^{3-1}$

Dyck path

Dyck paths $(0,0) \rightarrow (n+1, n)$
with labels i on columns a_i
increasing on vertical runs





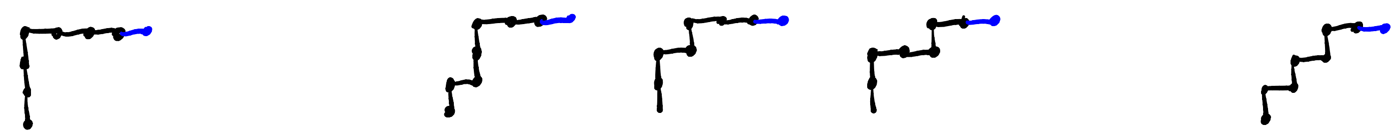


Parking Functions and Parking Spaces

Kreweras numbers:

There are $\mathrm{Krew}(\lambda) := \frac{1}{n+1}\binom{n+1}{\underbrace{\mu(\lambda)}_{\text{part-mult's}}}$ Dyck paths w/ vertical runs $\lambda = (\lambda_1, \ldots, \lambda_s)$

$\lambda = (3)$ $\qquad\qquad$ $\lambda = (2,1)$ $\qquad\qquad$ $\lambda = (1,1,1)$



$$\mathrm{Krew}((2,1)) = \frac{1}{4}\binom{4}{1,1} = \frac{1}{4}\cdot\frac{4!}{1!\cdot 1!\cdot 2!} = 3$$

S_n-action: The symmetric group S_n acts on positions with orbit representatives the non-decreasing parking functions

Parking Functions and Parking Spaces

Definition: $\underbrace{\mathrm{Park}(n)}_{S_n\text{-module}} := \left\{ \begin{array}{c}\text{parking functions} \\ \text{on } [n]\end{array} \right\}$ with natural S_n action

Character calculation: $\chi_{\mathrm{Park}(n)}(\pi) = (n+1)^{\#\mathrm{cyc}(\pi)-1}$ $\qquad (\mathrm{Dim} = (n+1)^{n-1})$

Orbit decomposition: $\mathrm{Park}(n) = \bigoplus_{\lambda \vdash n} \mathrm{Krew}(\lambda) \cdot \uparrow_{S_\lambda}^{S_n} \mathrm{triv}$

# of Dyck paths
$(0,0) \to (n+1,n)$ w/ vertical
runs given by $\lambda = (\lambda_1, .., \lambda_s)$

Possible labelings
of each Dyck path.

Rational Parking Functions and Parking Spaces

Classical m-Fuss

Words $a_1, a_2, \dots, a_n$ such that
non-decreasing rearrangement
$b_1 \le b_2 \le \dots \le b_n$ has $b_i \le 1 + m(i-1)$

111 $\longrightarrow$ 1

112 113
114 115
122 133

$6 \cdot 3 = 18$

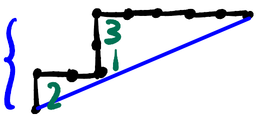
123
124 134
125 135

$5 \cdot 6 = 30$

$\text{Sum} = 49 = (2 \cdot 3 + 1)^{3-1}$

Dyck path

Dyck paths $(0,0) \to (mn+1, n)$
with labels i on columns a_i
increasing on vertical runs

313


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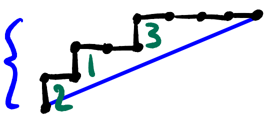

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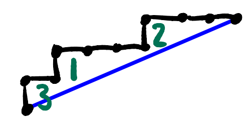


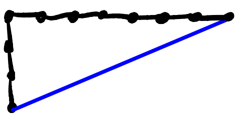
Rational Parking Functions and Parking Spaces

Kreweras numbers:

There are $\mathrm{Krew}^{\langle m\rangle}(\lambda) := \frac{1}{mn+1}\binom{mn+1}{\underbrace{\mu(\lambda)}_{\text{part-mult's}}}$ Dyck paths from $(0,0)$ to $(m\cdot n+1, n)$ w/ vertical runs $\lambda = (\lambda_1, \ldots, \lambda_s)$

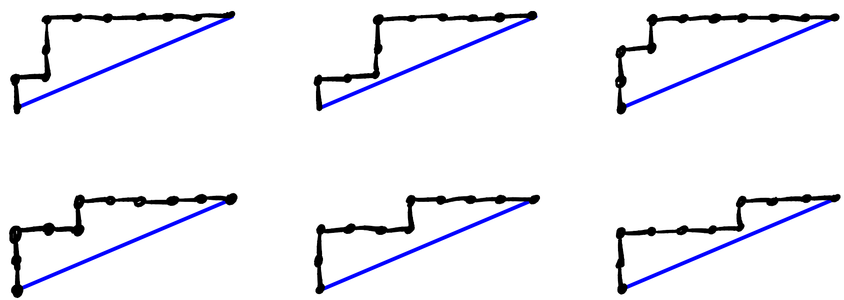
$m=2,\ n=3$

$\lambda = (3)$



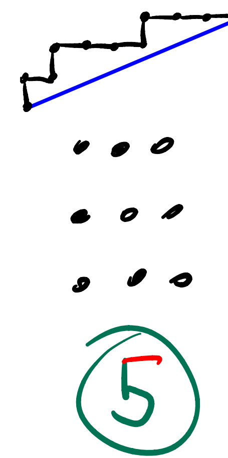
1

$\lambda = (2,1)$



$\mathrm{Krew}^{\langle 2\rangle}((2,1)) = \frac{1}{7}\binom{7}{1,1} = \frac{1}{7}\cdot\frac{7!}{1\cdot 1\cdot 5!} = 6$

$\lambda = (1,1,1)$



Rational Parking Functions and Parking Spaces

Definition: $\underbrace{\mathrm{Park}^{\langle m \rangle}(n)}_{S_n\text{-module}} := \left\{ \begin{array}{c} m\text{-Fuss} \\ \text{parking functions} \\ \text{on } [n] \end{array} \right\}$ with natural S_n action

Character calculation: $\chi_{\mathrm{Park}^{\langle m \rangle}(n)}(\pi) = (mn+1)^{\#\mathrm{cyc}(\pi)-1}$ $\qquad$ $(\mathrm{Dim} = (mn+1)^{n-1})$

Orbit decomposition: $\mathrm{Park}^{\langle m \rangle}(n) = \bigoplus_{\lambda \vdash n} \mathrm{Krew}^{\langle m \rangle}(\lambda) \cdot \uparrow_{S_\lambda}^{S_n} \mathrm{triv}$

# of Dyck paths
$(0,0) \to (mn+1, n)$ w/ vertical
runs given by $\lambda = (\lambda_1, \dots, \lambda_s)$

$\hookrightarrow$ Possible labelings
of each Dyck path.

Main Theorem for Symmetric group S_n

Orbit decomposition: $\mathrm{Park}^{\langle m\rangle}(n) = \bigoplus_{\lambda \vdash n} \mathrm{Krew}^{\langle m\rangle}(\lambda) \cdot \uparrow_{S_\lambda}^{S_n} \mathrm{triv}$

generalizes to

For all $m, k, r \in \mathbb{Z}_{\geq 0}$ with $m = k + r$

$$\mathrm{Park}^{\langle m\rangle}(n) = \bigoplus_{\lambda \vdash n} \mathrm{Krew}^{\langle k\rangle}(\lambda) \cdot \uparrow_{S_\lambda}^{S_n} \underbrace{\mathrm{Park}^{\langle r\rangle}(S_\lambda)}_{= \mathrm{Park}^{\langle r\rangle}(\lambda_1) \otimes \cdots \otimes \mathrm{Park}^{\langle r\rangle}(\lambda_s)}$$

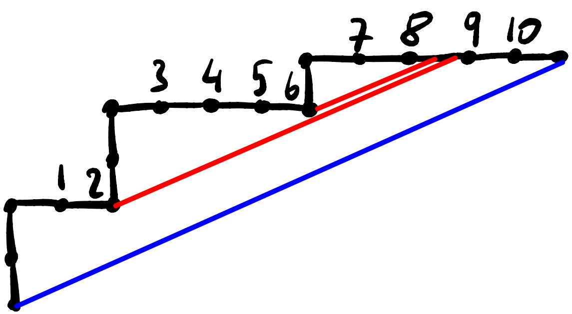
Question 1 (*): Give combinatorial interpretation.

Question 2: Give q,t-version.

Interpretation via non crossing Partitions

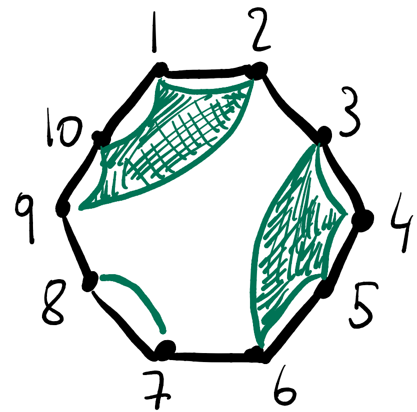
(Armstrong-Williams-...)

k-Fuss Parking Functions



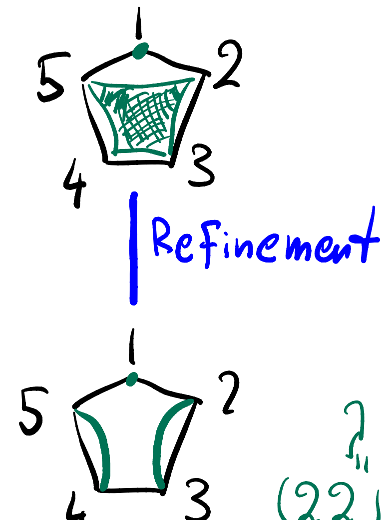
$\lambda = (2,2,1)$

k-divisible non-crossing partitions of $[kn]$



$\lambda = (2,2,1) \times 2$

k-chains of non-crossing partitions of $[n]$

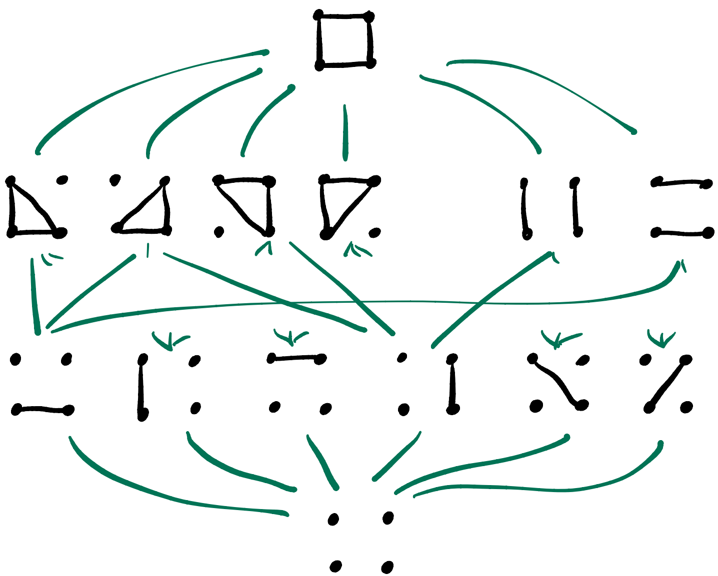


$\lambda = (2,2,1)$

Towards a combinatorial proof of Main Thm in S_n

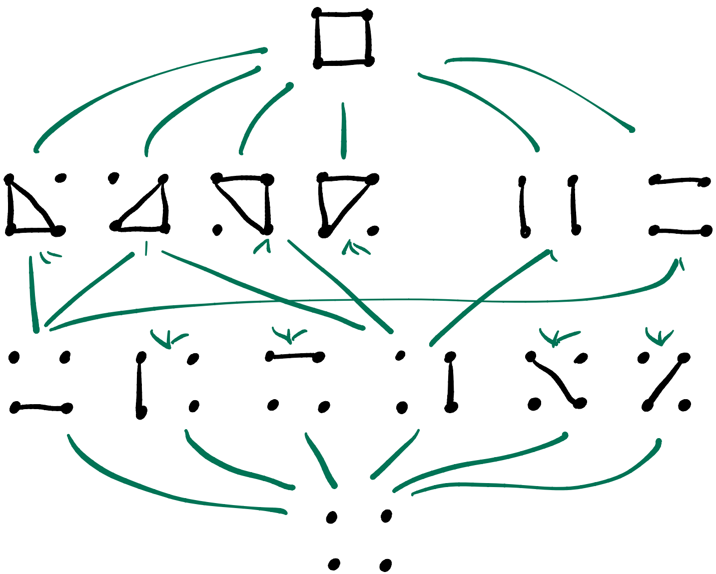
For all $m, k, r \in \mathbb{Z}_{>0}$ with $m = k + r$

$$\mathrm{Park}^{\langle m \rangle}(n) = \bigoplus_{\lambda \vdash n} \mathrm{Krew}^{\langle k \rangle}(\lambda) \cdot \uparrow_{S_\lambda}^{S_n} \underbrace{\mathrm{Park}^{\langle r \rangle}(S_\lambda)}_{\substack{\shortparallel \\ \mathrm{Park}^{\langle r \rangle}(\lambda_1) \otimes \cdots \otimes \mathrm{Park}^{\langle r \rangle}(\lambda_s)}}$$



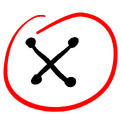
Corresponds to enumerating length-m chains wrt their k-th element

The Noncrossing partition lattice and Famous names



Catalan #s	Narayana #s	Kreweras #s
	1	1
14	6	4 + 2
		$\lambda=(3,1)$ · $\lambda=(2,2)$
	6	6
	1	1
$\frac{1}{n+1}\binom{2n}{n}$	$\frac{1}{k}\cdot\binom{n}{k}\cdot\binom{n}{k-1}$	$\frac{1}{n+1}\binom{n+1}{\mu(\lambda)}$

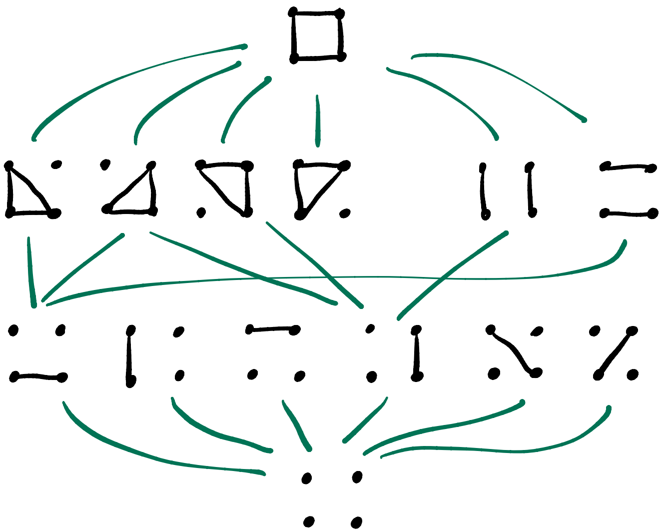
$\mu(\lambda)$: records multiplicity of parts

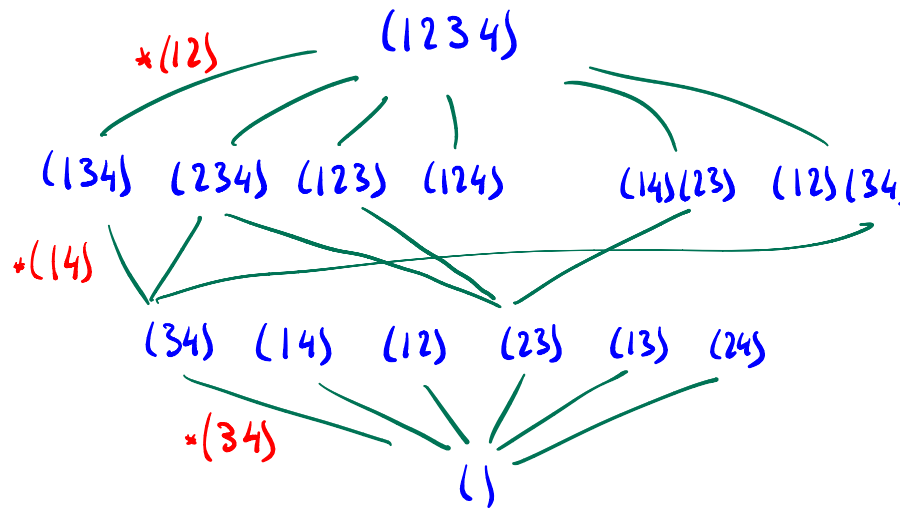


-> Not me! I'm crossing!

The noncrossing lattice as an interval in S_n

* cycle notation





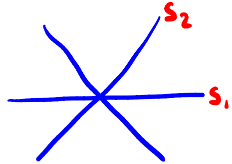
Theorem [Biane '96] Maximal chains correspond to minimum length transposition factorizations of the long cycle. There are n^{n-2} many!

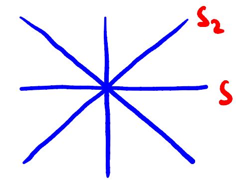
$$(34)\cdot(14)\cdot(12)=(1234)$$

Coxeter Combinatorics

Motto: If you like it in the symmetric group, you will love it in...

Finite Coxeter groups: These are the finite subgroups of $GL_n(\mathbb{R})$ generated by Euclidean reflections

$S_3 \cong A_2$: 

B_2: 

$\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 \rangle$ $\quad\quad$ $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 \rangle$

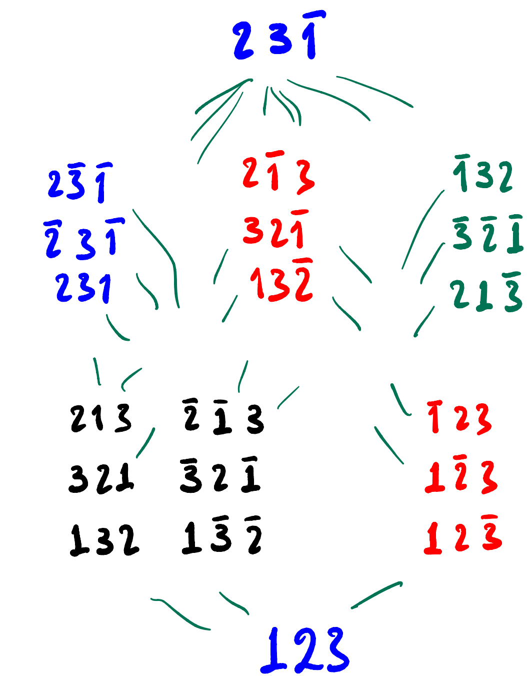
- The product $s_1 \cdots s_n$ of the simple generators and any conjugate element are called Coxeter elements. Their common order is h (the Coxeter #)
- The set of all reflections determines a length function $\ell_R(g)$ and an order $u \leq_R v \iff \ell_R(u) + \ell_R(u^{-1} \cdot v) = \ell_R(v)$

Coxeter-Catalan combinatorics

W a Coxeter group. Then $NC(W) := [e,c]_{\leq_R}$ $\left(= \{ g \in W \text{ s.th. } g \leq_R c \} \right)$ Fixed Cox. elt.

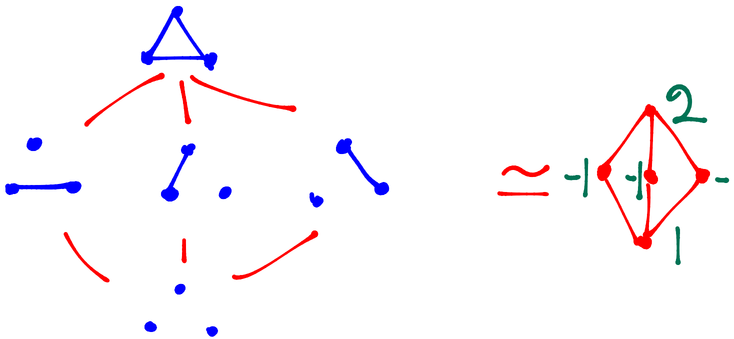
"noncrossing partition lattice of W"

* one-line notation



B_3-Catalan #s	B_3-Narayana #s	B_3-Kreweras #s
	1	1
	9	3 + 3 + 3
20		
	9	6 + 3
	1	1

Finite Coxeter groups: Numerology



$$\chi(\mathcal{A}_{S_3},t)=\sum_{X\leq V}\mu(V,X)\cdot t^{\dim(X)}$$
$$=t^2+3\cdot(-1)\cdot t+2$$
$$=(t-1)(t-2)$$

- Group exponents: The (integer) roots of $\chi(\mathcal{A}_W,t) \dashrightarrow =\prod(t-e_i)$

- Orlik-Solomon exponents: The (integer) roots of the characteristic polynomials of restricted arrangements $\chi(\mathcal{A}^X,t)=\prod(t-b_i^X)$

- In the symmetric group: $\{e_i\}=\{1,2,\cdots,n-1\}$

$$\{b_i^X\}=\{1,2,\ldots,\dim(X)\}$$

Coxeter-Catalan combinatorics

Symmetric group S_n case

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$$

$$\mathrm{Nar}(n,k) = \frac{1}{k}\cdot\binom{n}{k}\cdot\binom{n}{k-1}$$

$$\mathrm{Krew}(\lambda) = \frac{1}{n+1}\binom{n+1}{\mu(\lambda)}$$

Finite Coxeter group W

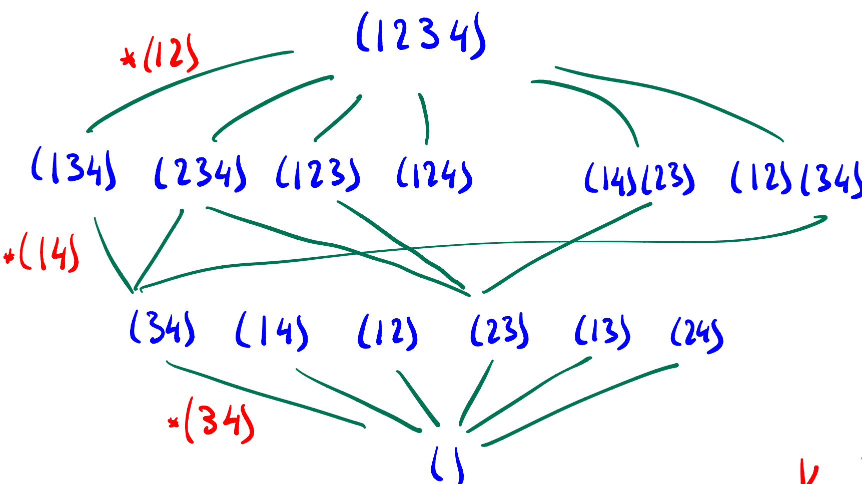
$$\mathrm{Cat}(W) = \frac{\prod_{i=1}^{n}(h+e_i+1)}{|W|}$$

$$\mathrm{Nar}(W,k) = \sum_{X\in\mathcal{L}_W^k}\mathrm{Krew}(W,X)$$

$$\mathrm{Krew}(W,X) = \frac{\prod_{i=1}^{\dim(X)}(h+1-b_i^X)}{[N(X):W_X]}$$

The (a) Coxeter-Catalan problem: Understand these formulas without relying on the Coxeter classification

m-Fuss-Catalan combinatorics



Chains of length-m ...
starting at "cycle type" X.

Fuss-versions of Famous names?

$$\mathrm{Cat}^{\langle m\rangle}(W) := \#\{ 1 \leq_R w_1 \leq_R \cdots \leq_R w_m \leq_R c \}$$

$$\mathrm{Cat}^{\langle m\rangle}(W) = \frac{\prod (mh+1+e_i)}{|W|}$$

"Chains of length m in the noncrossing lattice"

$$\mathrm{Krew}^{\langle m\rangle}(W,X) := \#\{ 1 \leq_R w_1 \leq_R \leq \cdots \leq_R c \text{ s.th. } V^{w_1} \sim X \}$$

↳ "of type X"

$$\mathrm{Krew}^{\langle m\rangle}(W,X) = \frac{\prod (mh+1-b_i^X)}{[N(X):W_X]}$$

m-Fuss-Catalan combinatorics

Symmetric group S_n case

$$\mathrm{Cat}^{\langle m\rangle}(n) = \frac{1}{mn+1}\binom{(m+1)n}{n}$$

$$\mathrm{Nar}^{\langle m\rangle}(n,k) = \frac{1}{n}\cdot\binom{n}{k}\cdot\binom{mn}{k-1}$$

$$\mathrm{Krew}^{\langle m\rangle}(\lambda) = \frac{1}{mn+1}\binom{mn+1}{\mu(\lambda)}$$

Finite Coxeter group W

$$\mathrm{Cat}^{\langle m\rangle}(W) = \frac{\prod_{i=1}^{n}(mh+e_i+1)}{|W|}$$

$$\mathrm{Nar}^{\langle m\rangle}(W,k) = \sum_{X\in\mathcal{L}_W^k}\mathrm{Krew}^{\langle m\rangle}(W,X)$$

$$\mathrm{Krew}^{\langle m\rangle}(W,X) = \frac{\prod_{i=1}^{\dim(X)}(mh+1-b_i^X)}{[N(X):W_X]}$$

The (a) Coxeter-Catalan problem: Understand these formulas without relying on the Coxeter classification

Combinatorial Recursions: Zeta Polynomials

A natural recursion for Zeta polynomials:

$$\mathrm{Krew}^{\langle m\rangle}(W,X) = \sum \mathrm{Krew}^{\langle k\rangle}(W_Y,X)\cdot\mathrm{Krew}^{\langle r\rangle}(W,Y)$$

$m=k+r$ $\qquad [``X", c]_{\leq R} \qquad [``X", ``Y"]_{\leq R} \qquad [``Y", c]_{\leq R}$

In terms of Fuss-W-parking space

$$\mathrm{Park}^{\langle m\rangle}(W) := \bigoplus_{[X]\in\mathcal{L}_W/W} \mathrm{Krew}^{\langle m\rangle}(W,X)\cdot\uparrow_{W_X}^{W}\mathrm{triv}$$

this becomes... Main theorem via combinatorics

For all $m,k,r\in\mathbb{Z}_{>0}$ with $m=k+r$

$$\boxed{\mathrm{Park}^{\langle m\rangle}(W) = \bigoplus_{[X]\in\mathcal{L}_W/W} \mathrm{Krew}^{\langle k\rangle}(W,X)\cdot\uparrow_{W_X}^{W}\underbrace{\mathrm{Park}^{\langle r\rangle}(W_X)}_{\mathrm{Park}^{\langle r\rangle}(W_1)\otimes\cdots\otimes\mathrm{Park}^{\langle r\rangle}(W_s)}}$$

Parabolic Recursions: Numerology

$$\mathrm{Park}^{\langle m\rangle}(W) = \bigoplus_{[X]\in \mathcal{L}_W/W} \mathrm{Krew}^{\langle k\rangle}(W,X)\cdot \uparrow_{W_X}^{W} \mathrm{Park}^{\langle r\rangle}(W_X)$$

For the character calculation this becomes

$$(rh+t)^{\dim(Z)} = \sum_{Y\subset Z} \chi(\mathcal{A}^Y, t)\cdot \prod_{i=1}^{\dim(Z)-\dim(Y)} (rh_i(Y,Z)+1)$$

$$t \to kh+1$$

or equivalently: (Möbius trick)

$$\circledast\left[\ (h+t)^{\dim(Z)} = \sum_{Y\subset Z} \prod_{i=1}^{\dim(Z)-\dim(Y)} h_i(Y,Z)\cdot t^{\dim(Y)}\ \right]$$

$\to$ Theorem [D. '21]

Recursions and Proofs

$$\mathrm{Krew}^{\langle m\rangle}(W,X) = \sum \mathrm{Krew}^{\langle k\rangle}(W_Y, X)\cdot \mathrm{Krew}^{\langle r\rangle}(W,Y)$$

$["X", c]_{\le R}$ $["X", "Y"]_{\le R}$ $["Y", c]_{\le R}$

Rewrite:

$$\underbrace{\mathrm{Krew}(W,X,m) - \mathrm{Krew}(W,X,k) - \mathrm{Krew}(W,X,r)}_{\text{* Linear term is not determined?}} = \underbrace{\sum_{Y\neq X} \mathrm{Krew}(W_Y,X,k)\cdot \mathrm{Krew}(W,Y,r)}_{\text{Assumed known by induction?}}$$

Help??

Still: All m-Fuss formulas reduced to $m=1$ case.

Also: Sufficient to recover formulas for "coincidental" types where $\{b_1^X,\ldots,b_{\dim(X)}^X\}$ form arithmetic progression

COROLLARIES

D. '21

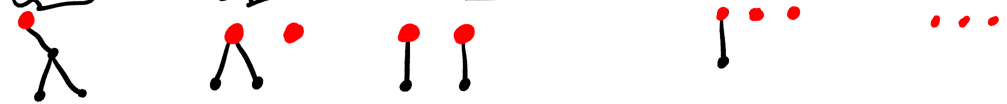
Where is this coming from?

Matrix Forest Theorem (For complete graph K_n, unweighted)

$$\boxed{(t+n)^{n-1} = \sum_{k=1}^{n} c_k \cdot t^{k-1}}$$

c_k = # of rooted forests on $[n]$ w/ k trees

Example: $(t+4)^3 = \underbrace{4^3} + (\underbrace{4\cdot 3^2} + \underbrace{3\cdot(2\cdot 2)})\cdot t + \underbrace{6\cdot 2}\cdot t^2 + \underbrace{1}\cdot t^3$



generalizes to:

Laplacian Recursion Lemma [Chapuy-D. '20 & Burman]

For every hyperplane arrangement $\mathcal{A}$ there is a Laplacian $L_{\mathcal{A}}$ and

$$\det(L_{\mathcal{A}} + t) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \operatorname{pdet}(L_{\mathcal{A}_X}) \cdot t^{\dim(X)}$$

↳ product of non-zero eigenvalues

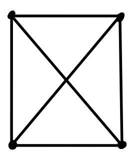
The W-Laplacian and its determinant

W is a Coxeter group acting on $V \simeq \mathbb{R}^n$. It has set of reflections $\mathcal{R}$, root system Φ, and reflection representation ρ_V. Its W-Laplacian L_W is:

Defn 1: $GL(V) \ni L_W := \sum_{\tau \in \mathcal{R}} \left(I_n - \rho_V(\tau) \right)$ Defn 2: $L_W(v) := \sum_{\sigma \in \Phi^+} \langle v, \hat{\sigma} \rangle \cdot \sigma$

↳ the $n \times n$ identity matrix

TYPE-A PICTURE



K_4 graph ⇝ ⇝ Laplacian

$$\underbrace{\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}}_{L(K_4)} = \sum_{i<j} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sum_{i<j} \left(I_4 - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

by construction and W-invariance: $\det(L_W) = h^n$ (h is the Coxeter # of W)

$$\det(L_{W_X}) = \prod_{i=1}^{\operatorname{codim}(X)} h_i(W_X)$$

Recursions and Proofs in Coxeter-Catalan Combinatorics

Thank You "The CAGE Seminar"

Part 1: arxiv:2109.04341 → joint w/ Guillaume Chapuy

Part 2: In preparation

Real coincidental reflection groups

Corollary ① was (for $X = \{0\}$)

$$\frac{\prod_{i=1}^{n}(mh+e_i+t)}{|W|} = \sum_{[X]} \frac{\chi(\mathcal{A}^X, t)}{[N(X):W_X]} \cdot \mathrm{Cat}^{\mathrm{alg},(m)}(W_X)$$

If we set $\boxed{t = e_2}$ we have

$$\frac{\prod_{i=1}^{n}(mh+e_i+e_2)}{|W|} = \mathrm{Cat}^{\mathrm{alg},(m)}(W) + \sum_{\substack{[L]\in\mathcal{L}_W/W \\ \dim(L)=1}} \frac{e_2-e_1}{[N(L):W_L]} \cdot \mathrm{Cat}^{\mathrm{alg},(m)}(W_L)$$

☆ This is easily equivalent w/ Fomin-Reading recursion

Corollary 3: For real coincidental W $\mathrm{Cat}^{\mathrm{alg},(m)}(W) = \mathrm{Cat}^{\mathrm{comb},(m)}(W)$

Additional combinatorics of Proj w/ Biane, Josuat-Verges should further give $\mathrm{Park}^{\mathrm{alg},(m)}(W) \simeq_W \mathrm{Park}^{\mathrm{comb},(m)}(W)$ (for real coincidental W...)