RESEARCH STATEMENT

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1. Introduction

My research area and classical training lies in Algebraic Combinatorics, and within that I am particularly fascinated by complex reflection groups. These groups W appear at the intersection of mathematical disciplines, and the resulting viewpoints suggest a number of natural lines of research. A central *motivating* theme in our work has been the study of structural and enumerative properties of the lattice $NC(W)$ of noncrossing partitions of W. We pursue this through two main avenues: the representation theory of W (§3), and the algebraic (§2) and differential (§4) geometry of its space of orbits. We give below, for key mathematical areas, some of their objects, techniques, or problems, that play an important role in our research:

- *In combinatorics:*
  - W-analogs of the Matrix-Tree theorem and Jucys-Murphy elements (§3.3), of cacti formulas (§2.2).
  - Chain enumeration in $NC(W)$ (§2.2, §3.1, Problem 1); cyclic sieving phenomena (§2.3, Problem 2).
  - Hyperplane arrangements (§3.4); polytopal combinatorics (§3.2, Problems 4,5).
- *In representation theory:*
  - Enumeration of factorizations via the Frobenius lemma (§3); Hecke algebras and Coxeter numbers (§3.1).
  - The exterior powers of the reflection representation of W (§3.3, Problem 6).
- *In singularity theory:*
  - Braid monodromy of algebraic functions (§2); simple (A-D-E) singularities (§4.1, Problem 9).
- *In geometric group theory:*
  - Geometric construction of cell complexes for generalized braid groups (§2.4, Problem 3).
- *In differential geometry:*
  - The Saito flat coordinates and Frobenius structure on the quotient variety V/W (§4).
  - Frobenius manifolds and quasi-Coxeter elements; Algebraic solutions of WDVV equations (Problems 7,8).

Although many results mentioned below appear enumerative in nature, our techniques come from different areas and may thus reveal non-trivial phenomena or connections between them. In one main project (§3.3) we give a formula for the weighted enumeration of certain factorizations of Coxeter elements (Thm. 3.5) . Our proof however also produces a new theorem (Thm. 3.6) on the (well-studied) exterior powers of the reflection representations of groups W. A uniform proof of the Chapuy-Stump formula (§3.1) brings up a connection between Hurwitz numbers and transportation polytopes (§3.2) while our interpretation of the numerology associated with quasi-Coxeter elements leads to new algebraic solutions of the WDVV equations (§4).

An important aspect in the theory of complex reflection groups is their Shephard-Todd classification. This has propelled the evolution of the subject with many results first proven via case-by-case arguments while a uniform explanation is pursued by the community. The following statement is a characteristic example.

Theorem 1.1 (Deligne-Arnol'd-Bessis via the classification, [Mic16] for Weyl groups, [Dou18c] in general)**.** *In a (duality) reflection group W of rank n, the set $\mathrm{Red}_W(c)$ of minimal length factorizations $\tau_1\cdots\tau_n = c$ of a Coxeter element c in reflections τ_i has size given by the Hurwitz number $\mathrm{Hur}(W) := h^n n!/|W|$, where $h := |c|$.*

This theorem and its many interpretations lie in the core of our research. In the symmetric group S_n, the Hurwitz number $\mathrm{Hur}(S_n) = n^{n-2}$ counts vertex-labeled trees and may be computed by the Matrix-Tree theorem. We give an analog of the Laplacian and prove a weighted Matrix-Forest theorem for (duality) reflection groups (§3.3). An important ingredient is a new general formula for hyperplane arrangements (§3.4) that in fact leads to a second uniform proof of Thm. 1.1 and has further applications on multi-reflection arrangements [CD19a].

A different exegesis of the Hurwitz number, popularized by Arnol'd and further developed by Bessis, is as the degree of the quasi-homogeneous Lyashko-Looijenga morphism (§2). Building on this, we produce a parabolic refinement (Thm. 2.2) of Thm. 1.1, prove a cyclic sieving phenomenon for it conjectured in Williams' thesis (Thm. 2.3), and via the theory of Frobenius manifolds we propose a version of it for quasi-Coxeter elements (§4).

We also present in what follows some open problems suggested by our projects which are often amenable to division in partial goals that could be suitable even for advanced undergraduate students.

2. Braid monodromy of the discriminant hypersurface

A cornerstone for much of the study of real reflection groups W is the chamber decomposition of the ambient space V induced by the arrangement of reflection hyperplanes $\mathcal{A}_W := \bigcup H$. Over the complex field, where such a decomposition cannot exist, a similar role is played by the quotient variety $\mathcal{H} := W \setminus \bigcup H$ which is known as the *discriminant hypersurface* of W. In the seminal work [Bes15] Bessis exploits the *braid monodromy* of $\mathcal{H}$ (albeit in the guise of the following "Trivialization Theorem") to prove a long-standing conjecture: the complement $V \setminus \bigcup H$ is a $K(\pi, 1)$ space (see also §2.4).

The braid monodromy of an algebraic function g is a refinement of its usual monodromy group: it keeps track of *how* the function values move around each other, when we vary the coefficients of g, as opposed to just recording their final permutation. To define it one usually chooses a generic direction z, for which $g = z^n + a_1(\boldsymbol{y}) \cdot z^{n-1} + \cdots + a_n(\boldsymbol{y})$ and treats the variety $V(g)$ as a branched cover over $Y := \operatorname{Spec}(\mathbb{C}[\boldsymbol{y}])$. If $\mathcal{K}$ is the branch locus, the coefficient map $\boldsymbol{a}(\boldsymbol{y})$ determines a representation of $\pi_1(Y \setminus \mathcal{K})$ into the usual braid group of n strands B_n which we call the braid monodromy of g as in [Han89; CS97].

For complex reflection groups W, the Shephard-Todd-Chevalley theorem identifies the quotient space V/W as the affine complex space $\mathbb{C}^n$ whose coordinates are given by the fundamental invariants $\boldsymbol{f} := (f_i)_{i=1\cdots n}$ of W. In the subclass of duality groups (which possess Coxeter elements and include all real reflection groups) the highest degree invariant f_n plays a special role; in particular, the equation for the discriminant hypersurface $\mathcal{H}$ is monic *and of degree* n with respect to f_n. Central in Bessis' work, the *Lyashko-Looijenga* map $LL(\boldsymbol{y})$ is essentially the coefficient map for the braid monodromy of $\mathcal{H}$ along the f_n direction (with parameter $\boldsymbol{y} \in Y := \operatorname{Spec}\big(\mathbb{C}[f_1, \cdots, f_{n-1}]\big)$).

2.1. **The noncrossing lattice and the trivialization theorem.**

A geometric interpretation of the LL map (and any coefficient map) is that it records the intersections of complex lines $L_{\boldsymbol{y}} := \boldsymbol{y} \times \mathbb{C}$, parallel to the direction of f_n, with the discriminant hypersurface $\mathcal{H}$. Bessis considers loops that surround $\mathcal{H}$ only inside these lines $L_{\boldsymbol{y}}$ and constructs in this way well-defined elements of the generalized braid group $B(W) := \pi_1(V/W - \mathcal{H})$. Taking advantage of the canonical short exact sequence $1 \to P(W) \to B(W) \to W \to 1$, he extends this to a labeling map $\operatorname{lbl}(\boldsymbol{y})$ that sends $\boldsymbol{y} \in Y$ to a tuple of elements of W.

The topological construction of the labeling map heavily restricts the resulting tuples. The *noncrossing lattice* $NC(W)$ is defined as the set of all $w \in W$ that satisfy $l_R(w) + l_R(w^{-1}c) = l_R(c)$ for a given Coxeter element c, where the reflection length $l_R(w)$ is the smallest number k of (any) reflections τ_i needed to write $w = \tau_1 \cdots \tau_k$. Then if $\boldsymbol{e} := LL(\boldsymbol{y}) = L_{\boldsymbol{y}} \cap \mathcal{H}$ is an image of the LL map, which is by definition a collection of points in $\mathbb{C}$ with total multiplicity n, Bessis proves this remarkable *Trivialization Theorem*:

Theorem 2.1. *[Bes15] The points in a fiber $LL^{-1}(\boldsymbol{e})$ are in a natural bijection via the labeling map with chains in the noncrossing lattice whose rank jumps are given by the multiplicities in $\boldsymbol{e}$.*

2.2. **Refined chain enumeration by parabolic type.**

Chains in the noncrossing lattice $NC(W)$ correspond to length-additive factorizations of a Coxeter element c, so that the trivialization theorem suggests a geometric way to enumerate such collections. In particular, maximal chains correspond precisely to reduced reflection factorizations and thus the Hurwitz number of Thm. 1.1 should agree with the degree of the LL map (this is in fact needed to prove Thm. 2.1, see [Dou17, Ch. 7]).

To produce refined enumerative results, one must study the restriction of the LL map on the branch locus $\mathcal{K} \subset Y$. The discriminant hypersurface $\mathcal{H}$ is stratified by orbits of flats $[X] \in \mathcal{L}_{\mathcal{A}_W}/W$ and their projections $[X]_Y$ on the base space Y completely cover $\mathcal{K}$. Define the *parabolic type* of an element $w \in W$ as the orbit $[V^w]$ of its fixed space in the intersection lattice; a statistic that generalizes the cycle type of permutations. By studying the local behavior of the LL and lbl maps on these constructible sets $[X]_Y$, we refine Thm. 1.1:

Theorem 2.2. *[Dou18b] The number of length-additive factorizations of a Coxeter element $c \in W$ of the form $w \cdot \tau_1 \cdots \tau_k = c$, with τ_i's reflections and w of parabolic type $[X]$, is given by the formula $h^k k!/[N_W(X) : W_X]$.*

Our techniques are in the same spirit as methods initiated by Arnol'd [Arn96] and used extensively by singularity theorists thereafter (even to some extent in the celebrated ELSV formula). One tries to lift the restriction of the map to an affine space, where it becomes quasi-homogeneous and hence its degree can be calculated via Bezout's theorem. The term $[N(X) : W_X]$ that appears in our formula is exactly the degree of such a lift.

Now, for any length additive factorization $\sigma := (w_1 \cdots w_k = c)$, we define its passport $\big(\boldsymbol{Z}\big) := \big([Z_1, \cdots, Z_k]\big)$ as the tuple of parabolic types $[Z_i]$ of the w_i. An ambitious task would then be to compute the number $\operatorname{Fact}_W[(\boldsymbol{Z})]$ of such factorizations σ with given passport $(\boldsymbol{Z})$. Lando and Zvonkine [ZL99] derive the Goulden-Jackson formula

$$\operatorname{Fact}_{S_n}\big[(\boldsymbol{Z})\big] = n^{l-1} \cdot \prod_{i=1}^{l} \frac{k_i!}{[N(Z_i) : W_{Z_i}]}, \tag{1}$$

via a geometric analysis of the LL map on the space of monic degree n polynomials (which realizes V/W when W is the symmetric group). For other reflection groups, (case-by-case) formulas of Krattenthaler and Müller [KM10] suggest a similar structure for certain passports. We describe in [Dou18b, Sec. 7] a complete stratification of Y by constructible sets $Y_{\{\boldsymbol{Z}\}}$ indexed by passports, which are often precisely the intersections of the strata $[Z_i]_Y$ we used for Thm. 2.2. We relate the enumeration problem with the local geometry of the LL map on those and ask:

Problem 1. *Find a uniform geometric extension, for suitable* $(\boldsymbol{Z})$*, of formula* (1) *to other reflection groups.*

2.3. **A cyclic sieving phenomenon.**

The cyclic sieving phenomenon (CSP) [RSW04] occurs when a polynomial $X(q)$ carries orbital information about the action of a cyclic group C on a space X. More precisely, and if C is generated by an element c of order n, we say that the triple $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon* if for all integers d, the number of elements of X fixed by c^d equals the evaluation $X(\zeta^d)$, where $\zeta = e^{2\pi i/n}$.

The set $\mathrm{Red}_W(c)$ of Thm. 1.1 supports many natural cyclic actions. The operation $\mathfrak{Pro}$ below may be realized as the Hurwitz action (5) of a particular root of the full twist in the (ordinary) braid group B_n and has order hn:

$$\mathfrak{Pro} : (\tau_1, \cdots, \tau_n) \to (c\tau_n c^{-1}, \tau_1, \cdots, \tau_{n-1}).$$

Williams conjectured the following CSP for $\mathfrak{Pro}$ which we proved by exploiting the geometry of the trivialization theorem. Via the labeling map lbl, we interpret $\mathfrak{Pro}$ as a scalar action on fibers $LL^{-1}(\boldsymbol{e})$ for certain symmetric point configurations $\boldsymbol{e}$. The polynomial $X(q)$ arises then as the Hilbert series of the special fiber $LL^{-1}(\boldsymbol{0})$:

Theorem 2.3. *[Dou18a] For a (duality) reflection group W, with invariant degrees $d_1, \cdots, d_n$ and* $\mathrm{Red}_W(c)$ *as in Thm. 1.1, the triple* $\Big(\mathrm{Red}_W(c), \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}, \langle \mathfrak{Pro} \rangle\Big)$*, where* $[m]_q := \dfrac{1-q^m}{1-q}$*, exhibits the cyclic sieving phenomenon.*

Some CSP's are proven by direct calculation of the orbit sizes and perhaps lack a satisfying explanation for the appearance of the polynomial $X(q)$. In our case, the geometry of the LL map not only resolves this but it also provides an example where the same polynomial $X(q)$ encodes CSP's for different cyclic actions on X. By choosing configurations $\boldsymbol{e}$ with different cyclic symmetries, we obtain for example a CSP with C of order $h(n-1)$.

For some passports, the enumeration of factorizations is given by the degree of a quasi-homogeneous morphism. In those cases too this method will work although there are fewer candidates for symmetric fibers. We describe in [Dou18b, § 5.3.1] what happens for the factorizations of Thm. 2.2 and ask in general:

Problem 2. *Extend Thm. 2.3 over sets of block factorizations with prescribed passports (see § 2.2).*

2.4. **The Brady complex after Bessis.**

In his proof of the $K(\pi, 1)$ conjecture Bessis uses the noncrossing lattice $NC(W)$ as a combinatorial recipe for building the universal covering space of the discriminant complement $V/W - \mathcal{H}$. The procedure is quite complicated and Bessis recently proposed a simplification [Bes16]. The idea is to construct a cell model for $V/W - \mathcal{H}$, via the trivialization theorem, and hope that its combinatorics leads to a cleaner proof of the $K(\pi, 1)$ property.

Bessis' model involves first a retraction that pushes the configurations of points inside a fixed circle and then proceeds by lifting the natural cell structure there to $V/W - \mathcal{H}$ via the LL map. On the other hand, there is already a combinatorial $K(\pi, 1)$ model for the braid group $B(W)$ defined by Brady (but which is not a priori homeomorphic to $V/W - \mathcal{H}$). It is the quotient of the order complex of $NC(W)$ where we identify the chains $(w_1, \cdots, w_k)$ and $(e, w_1^{-1}w_2, \cdots, w_1^{-1}w_k)$. The labeling map lbl is compatible with Bessis' retraction in a way that suggests:

Problem 3. *Bessis' cell complex for the discriminant complement $V/W - \mathcal{H}$ is isomorphic to the Brady complex.*

3. Representation theoretic techniques in enumeration

In the case of the symmetric group S_n, Thm. 1.1 was first proven by Hurwitz who came to it after identifying length-additive factorizations in S_n with genus-0 (branched) coverings of the sphere. In this setting, it is natural to consider factorizations with arbitrary many terms as they correspond to higher genus coverings. Moreover, it is actually easier to study the whole exponential generating function (with respect to genus or, equivalently, number of terms). Returning to reflection groups W with set of reflections $\mathcal{R}$ and for an arbitrary element $g \in W$, we wish to understand the function

$$\mathrm{FAC}_{W,g}(t) := \sum_{\ell \geq 0} \#\big\{(\tau_1, \cdots, \tau_\ell) \in \mathcal{R}^\ell : \tau_1 \cdots \tau_\ell = g\big\} \cdot \frac{t^\ell}{\ell!}. \tag{2}$$

Hurwitz also observed that a Lemma of Frobenius, from newly introduced representation theory, could be used to turn expressions like (2) into a finite sum of character evaluations (see Prop. 3.2). This idea was rediscovered and popularized in the 80's by Stanley, Jackson, and others, who exploited it in the context of S_n. Recently it proved effective for all reflection groups in this beautiful generalization, due to Chapuy and Stump, of Thm. 1.1:

Theorem 3.1. *[CS14] For a (duality) rank n reflection group W and a Coxeter element $c \in W$ of order h,*

$$\mathrm{FAC}_{W,c}(t) = \frac{e^{t|\mathcal{R}|}}{|W|} \cdot (1 - e^{-th})^n.$$

3.1. **A uniform proof and generalization of the Chapuy-Stump formula.** The original proof of Thm. 3.1 proceeded based on the Frobenius lemma but had to rely on the classification of complex reflection groups and their characters. Both because it implies Thm. 1.1 and due to its intrinsic elegance, there was an effort in the community to produce a case-free proof with a first success only for Weyl groups [Mic16].

Proposition 3.2. *[CS14, Frobenius Lemma] The function $FAC_{W,g}(t)$ of (2) is given as the finite sum*

$$\mathrm{FAC}_{W,g}(t) = \frac{1}{|W|} \cdot \sum_{\chi \in \widehat{W}} \chi(1) \cdot \chi(g^{-1}) \cdot \exp\big(t \cdot \widetilde{\chi}(\mathcal{R})\big),$$

where $\widehat{W}$ denotes the set of irreducible characters of W and $\widetilde{\chi}(\mathcal{R})$ is the normalized trace $\sum_{\tau \in \mathcal{R}} \chi(\tau)/\chi(1)$.

The difficulty to apply this lemma uniformly stems from the case-by-case construction of the irreducible characters of W. To circumvent this we group the characters with respect to an integer invariant c_χ, related to Lusztig's c-function, called the *Coxeter number* of χ. Then we prove a theorem in the Hecke algebra that allows us to discard from the summation in Prop. 3.2 those $\chi \in \widehat{W}$ for which c_χ is not a multiple of $h := |c|$.

Our argument relies only on the fact that the Coxeter element c lifts to a root of the full twist in the braid group $B(W)$ and hence can be applied to all *regular* elements $g \in W$. The previous construction in conjuction with combinatorial restrictions on the leading term of $\mathrm{FAC}_{W,c}(t)$ allows us to prove the following structural result which recovers and extends the Chapuy-Stump formula (Thm. 3.1) and with little more effort [Dou18c, § 5] also gives a uniform proof for the weighted case studied in [dHR18].

Theorem 3.3. *[Dou18c] For a complex reflection group W and any regular element $g \in W$, one has*

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|} \cdot \Big[(1-X)^{l_R(g)} \cdot \Phi_g(X)\Big]\Big|_{X=e^{-t|g|}} ,$$

where $\Phi_g(X)$ is a polynomial in X of degree $\frac{|\mathcal{R}|+|\mathcal{A}|}{|g|} - l_R(g)$ and constant term equal to 1.

In the case of a Coxeter element c, the polynomials $\Phi_c(X)$ are forced to have degree 0 by *combinatorial* considerations. This holds further whenever $|g| = d_n$, which produces explicit formulas that do not appear in [CS14] or [dHR18]. In general it seems difficult to control the $\Phi_g(X)$ but we have some success with S_n in the next section.

3.2. **Higher genus Hurwitz formulas for transitive factorizations in S_n.**
Hurwitz studied in particular minimal *transitive* factorizations of elements $g \in S_n$ in transpositions; that is, factorizations whose terms generate the whole group S_n. If we write $\mathrm{TR\text{-}FAC}_{S_n,g}(t)$ for their generating function when allowing arbitrary length then, because regular elements $g \in S_n$ remain regular in all Young subgroups that contain them, the following is a direct corollary of [Dou18c]:

Proposition 3.4 (Douvropoulos)**.** *The transitive factorizations of a regular element $g \in S_n$ are counted by*

$$\mathrm{TR\text{-}FAC}_{S_n,g}(t) = \frac{e^{t\binom{n}{2}}}{n!} \cdot \Big[(1-X)^{l_{\mathrm{Tr}}(g)} \cdot \Phi_g^{\mathrm{Tr}}(X)\Big]\Big|_{X=e^{-t|g|}},$$

where $l_{\mathrm{Tr}}(g)$ is the minimum length of such a factorization for g and $\Phi_g^{\mathrm{Tr}}(X)$ a suitable polynomial.

Computer experiments have suggested the following remarkable conjecture for the polynomials $\Phi_g^{\mathrm{Tr}}(X)$. The central transportation polytope [DK14] denoted $T(p,q)$ is the set of all real $p \times q$ matrices with non-negative entries, all row sums equal to q, and all column sums equal to p; it is simple when $(p,q) = 1$. The conjecture is proven by direct calculation when $k = 2$ (and for arbitrary d) and for the general case we find the interpretation in [KM16] of such h-polynomials as plethystic coefficients to be promising.

Problem 4. *For a regular element $g \in S_{kd}$ of cycle type $(d)^k$, the polynomial $\Phi_g^{\mathrm{Tr}}(X)$ agrees with the h-polynomial of the dual of the (central) transportation polytope $T(k, kd-1)$.*

This same enumerative question is reduced via the ELSV formula to computing (highly non-trivial) integrals over the spaces $\overline{\mathcal{M}_{g,n}}$; we hope that our interpretation may lead to more explicit formulas. Moreover, even though Thm. 3.3 does not apply for non-regular classes, experiments suggest that the following is worth pursuing in S_n:

Problem 5. *Extend this interpretation of higher genus Hurwitz formulas to arbitrary classes $\lambda \subset S_n$.*

3.3. **Weighted factorizations with generalized Jucys-Murphy weights.**
The derivation of the type-A Hurwitz number $H(S_n) = n^{n-2}$ of Thm. 1.1 by calculating the Laplacian of the complete graph K_n can be extended to allow assigning weights ω_{ij} on the transpositions (ij). Burman and Zvonkine [BZ10] proved a striking higher-genus analog of this by providing a product formula for the weighted generating function that involved the *eigenvalues* of the (weighted) Laplacian.

With Chapuy we extend their work to all (duality) reflection groups W with Thm. 3.5. Unfortunately, it turns out that arbitrary weight assignments do not lead to product formulas; we consider instead special weight functions $\mathbf{w}_T : \mathcal{R} \to \boldsymbol{\omega} := (\omega_i)_{i=1}^n$ indexed by towers of parabolic subgroups $T := \big(\{\mathbf{1}\} = W_0 < W_1 < \cdots < W_n = W\big)$.

These $\mathbf{w}_T$ are defined by the filtration of $\mathcal{R}$ by T; that is, for a reflection $\tau \in \mathcal{R}$ we have $\mathbf{w}_T(\tau) = \omega_i$ if and only if $\tau \in W_i \setminus W_{i-1}$. We are interested in the exponential generating function $\mathrm{FAC}_W^T(t,\boldsymbol{\omega})$ of weighted reflection factorizations of *any* element c of the Coxeter class $\mathcal{C}$, an analog of (2):

$$\mathrm{FAC}_W^T(t,\boldsymbol{\omega}) := \sum_{\ell\geq 0} \frac{t^\ell}{\ell!} \cdot \Big(\sum_{\substack{(\tau_1,\cdots,\tau_\ell,c)\in\mathcal{R}^\ell\times\mathcal{C} \\ \tau_1\cdots\tau_\ell=c}} \mathbf{w}_T(\tau_1)\cdots\mathbf{w}_T(\tau_\ell)\Big). \tag{3}$$

Thm. 3.5 gives a product formula for (3) which generalizes the Chapuy-Stump formula of Thm. 3.1. If we write ρ_V for the reflection representation of W, the matrix $L_W^T(\boldsymbol{\omega}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)\big(\mathrm{id} - \rho_V(\tau)\big)$ is a W-analog of the Laplacian of K_n and its eigenvalues weighted analogs of the Coxeter number h. In this sense, the equality of the two leading terms below may be considered as a (weighted) Matrix-Tree theorem for W:

Theorem 3.5. *[CD19b] For a (duality) reflection group W the weighted enumeration* (3) *is given by*

$$\mathrm{FAC}_W^T(t,\boldsymbol{\omega}) = \frac{e^{t\mathbf{w}_T(\mathcal{R})}}{h} \cdot \prod_{i=1}^n \big(1 - e^{-t\lambda_i^T(\boldsymbol{\omega})}\big),$$

where $\mathbf{w}_T(\mathcal{R}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)$, and the $\lambda_i^T(\boldsymbol{\omega})$ are the eigenvalues of the W-Laplacian $L_W^T(\boldsymbol{\omega})$.

In the process of proving Thm. 3.5 we produce a generalization of the Frobenius Lemma (Prop. 3.2) for any group G where the elements of a generating conjugacy class $\mathcal{G}$ are weighted via an arbitrary tower of subgroups. Heavily influenced by the work of Okounkov and Vershik [OV96], we consider in the group algebra $\mathbb{C}[W]$ generalized Jucys-Murphy elements $J_i := \sum_{\tau\in\mathcal{R}\cap W_i\setminus W_{i-1}} \tau$. For any parabolic tower T, they generate a commutative subalgebra $\mathbb{C}[\boldsymbol{J}^T]$ and the weighted enumeration is given in terms of its spectrum.

The product structure of the formula comes down to a connection with the exterior powers of the reflection representation V_{ref}. We say that two virtual characters χ and ψ are *tower-equivalent* if they agree on the subalgebras $\mathbb{C}[\boldsymbol{J}^T]$ for any choice of parabolic tower T. Then Thm. 3.5 is equivalent with the following:

Theorem 3.6. *[CD19b] The virtual characters $\sum_{\chi\in\widehat{W}} \chi(c^{-1})\cdot\chi$ and $\sum_{k=0}^n (-1)^k \bigwedge^k(V_{\mathrm{ref}})$ are tower-equivalent.*

We prove this theorem by computer calculation for the exceptional types and an inductive argument, which involves working out some non-trivial Littlewood-Richardson coefficients, for the infinite families. In the work of Michel [Mic16] for Weyl groups, and in a much weaker sense, it is the unipotent characters U_χ indexed by $\chi = \bigwedge^k(V_{\mathrm{ref}})$ that are related to the virtual sum $\sum_{\chi\in\widehat{W}} \chi(c^{-1})\cdot\chi$. Either by building on this or otherwise, we ask:

Problem 6. *Give a uniform proof of Thm. 3.6.*

3.4. **A new formula for hyperplane arrangements and the Deligne-Reading recursion.**
In [CD19b] we in fact give a W-Matrix-*Forest* theorem for the whole characteristic polynomial of $L_W^T(\boldsymbol{\omega})$. This is done by combining Thm. 3.5 with the following general formula for hyperplane arrangements $\mathcal{A}$, where the $\mathcal{A}$-Laplacian is the sum of rank 1 operators $L_{\mathcal{A}}(\boldsymbol{\omega}) := \sum_{H\in\mathcal{A}} \omega_H(\mathrm{Id} - s_H)$ with weights $\boldsymbol{\omega} = (\omega_H)_{H\in\mathcal{A}}$.

Theorem 3.7. *[CD19b] The characteristic polynomial of the $\mathcal{A}$-Laplacian $L_{\mathcal{A}}(\boldsymbol{\omega})$ is given by*

$$\det\big(t\cdot\mathrm{Id} + L_{\mathcal{A}}(\boldsymbol{\omega})\big) = \sum_{X\in\mathcal{L}_{\mathcal{A}}} \mathrm{qdet}\big(L_{\mathcal{A}_X}(\boldsymbol{\omega}_X)\big)\cdot t^{\dim(X)}.$$

This formula produces some very interesting numerology for reflection arrangements $\mathcal{A}_W$. For instance, identifying all weights to 1, it implies the following relation for (the multiset $\{h_i(W_X)\}$ of) Coxeter numbers of parabolics:

$$(t+h)^n = \sum_{X\in\mathcal{L}_W} t^{\dim(X)} \cdot \prod_{i=1}^{\mathrm{rk}(W_X)} h_i(W_X). \tag{4}$$

Comparing the coefficients of t^{n-1} in the two sides of this equation, and with some known results on noncrossing lines $L \in \mathrm{NC}^{n-1}(W)$, this gives [CD19a] a uniform derivation of Thm. 1.1 from the Deligne-Reading recursion $\mathrm{Hur}(W) = \sum_{L\in\mathrm{NC}^{n-1}(W)} \mathrm{Hur}(W_L)$ (which so far had only led to case-by-case proofs).

4. Frobenius manifolds and Quasi-Coxeter elements

The theory of Frobenius manifolds was developed by Dubrovin to give a coordinate-free formulation of the WDVV equations from 2D topological field theory. In it, a Frobenius algebra structure is specified on any tangent plane T_xM of a manifold M and its structure coefficients encode the WDVV associativity equations for a prepotential F.

The quotient varieties V/W for real reflection groups W form an important class of Frobenius manifolds. For them the algebra structure is defined via a special choice of fundamental invariants, known as *Saito flat coordinates* that provide an Euclidean metric for the orbit space V/W. Dubrovin conjectured [Dub99] and Hertling later proved that, in fact, these are the only examples of Frobenius manifolds with associated *polynomial* prepotentials.

In his classification [Dub99, Lect. 4] of massive Frobenius manifolds Dubrovin encodes the local algebra structure in a Stokes matrix or equivalently a tuple of euclidean reflections $\boldsymbol{\tau} := (\tau_1, \cdots, \tau_n)$, while he describes its analytic continuation via the Hurwitz action of the Braid group B_n on $\boldsymbol{\tau}$:

$$B_n \ni \sigma_i * (\tau_1, \cdots, \tau_n) = (\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}^{-1}\tau_i\tau_{i+1}, \cdots, \tau_n). \tag{5}$$

Then *algebraic* prepotentials correspond to tuples $\boldsymbol{\tau}$ with finite Hurwitz orbits and Dubrovin asks for the construction of the corresponding Frobenius manifolds. After work of Michel [Mic06] however, finite Hurwitz orbits occur if and only if $\boldsymbol{\tau}$ generates a reflection group, so that the problem of algebraic Frobenius manifolds in some sense lives entirely in the world of finite Coxeter groups.

From a different viewpoint [Bau+17] studies the Hurwitz action of B_n on the set $\mathrm{Red}_W(g)$ of reduced reflection factorizations of an element $g \in W$ and shows that when g is quasi-Coxeter, i.e. when there is no proper reflection subgroup $W' \lneq W$ that contains it, then the action is transitive. Reduced tuples $\boldsymbol{\tau}$ always determine a quasi-Coxeter element $g := \prod_{i=1}^n \tau_i$ of the group $W' = \langle \boldsymbol{\tau} \rangle$ so that we can index the possible corresponding Frobenius manifold by g (and write F_g as opposed to $F_{\boldsymbol{\tau}}$).

Stump calculated the sizes of the (single orbit) sets $\mathrm{Red}_W(g)$ for quasi-Coxeter elemenets g and discovered that they always factor in small primes. He asked if there is an explanation for this or even a generalization of Thm. 1.1. In fact, a lot of the relevant geometric objects of §2 appear in the theory of Frobenius manifolds and in particular the LL map, which relates two natural coordinate systems of F_g. It sends the flat coordinates, on which the prepotential is given, to (the elementary symmetric polynomials of) the canonical coordinates, which are the eigenvalues in the algebra structure of the multiplication by the Euler field. Given the prepotential, it is easy to calculate the degree of the LL map; this and Dubrovin's construction described previously suggest the following:

Problem 7. *For a quasi-Coxeter element g, assuming F_g exists, the degree of the map $LL(F_g)$ equals $|\mathrm{Red}_W(g)|$.*

In the case of Weyl groups, there are deep reasons [Pav00; Din13] that the weights of the flat coordinates of F_g should be given by $(e_i(g)+1)/|g|$, where the *exponents* $e_i(g)$ determine the eigenvalues $e^{2\pi i e_i(g)/|g|}$ of g. Because the LL map is weighted-homogeneous, this would give its degree as the right hand side of (6) where d_g is the *algebraicity degree* of the Frobenius prepotential.

Observation 4.1 (Douvropoulos)**.** For a regular quasi-Coxeter element g in a crystallographic W we have that

$$|\mathrm{Red}_W(g)| = \frac{|g|^n n!}{\prod_{i=1}^n (e_i(g)+1)} \cdot d_g, \tag{6}$$

where the exponents $e_i(g)$ are defined as above and d_g is a small integer given (using Carter's notation) by:

$g \in W$	$D_{2n}(n-1)$	$F_4(1)$	$E_6(1)$	$E_6(2)$	$E_7(1)$	$E_7(4)$	$E_8(1)$	$E_8(2)$	$E_8(3)$	$E_8(5)$	$E_8(6)$	$E_8(8)$
d_g	n	3	2	5	2	$2 \cdot 3^2$	2	3	2^3	7	$2^2 \cdot 5$	$3^3 \cdot 5$

Indeed, in all the cases that algebraic Frobenius manifolds have been constructed our interpretation of the numbers d_g is confirmed. Applied in the opposite direction, this enumerative data can be exploited to guess solutions to the WDVV equations. Sekiguchi [Sek19] has been succesful in doing so in small dimensions using the information from our calculations with Stump. We state here Dubrovin's refinement of his original conjecture:

Problem 8 (Dubrovin)**.** *Construct an algebraic Frobenius manifold F_g for any quasi-Coxeter element g.*

4.1. **Applications on the trivialization theorem.** For the symmetric group S_n, the trivialization theorem (Thm. 2.1) is equivalent to Riemann's existence theorem while more generally for types A-D-E the LL map may be interpreted as the morphism that sends a deformation of a simple singularity to its set of critical values. Currently the proof of Thm. 2.1 relies on the numerological coincidence between the degree of the LL map and the Hurwitz number $\mathrm{Hur}(W)$ [Dou17, Ch. 7]. However, Hertling and Roucairol [HR18] prove an equivalent version for simple singularities by exploiting the Frobenius structure. We ask to extend their approach to (duality) reflection groups:

Problem 9. *Give a case-free conceptual proof of Thm. 2.1 that does not rely on the numerological coincidence.*

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