PARKING SPACE CONJECTURES

THEO DOUVROPOULOS

UNIVERSITY OF MINNESOTA

ABSTRACT. A prominent line of research in Coxeter combinatorics has been for a better understanding of the noncrossing lattice $NC(W)$, associated to a reflection group W. In [ARR15], Armstrong, Reiner and Rhoades, defined two new Parking Spaces, an isomorphism between which would give uniform proofs and understanding to many a combinatorial formulae. The purpose of this report is to describe a rephrasing of their Main Conjecture, due to Gordon and Ripoll [GR12], in terms of the geometric framework for $NC(W)$, introduced by Bessis in [Bes15].

1. Introduction

It might be true that there are people who dislike the Catalan numbers. For the rest of us, apart from featuring 42 as its fifth element, the Catalan sequence $\mathrm{Cat}(n) = \dfrac{1}{n+1}\dbinom{2n}{n}$ has the uncanny ability to present itself, again and again, as the answer to many combinatorial riddles and can be a source of endless imagination.

Of the many attempts to explain the prolificity of Catalan numbers we mention here the humanistic one: The generating function for $\mathrm{Cat}(n)$ is merely the simplest non-trivial function one could think of; us therefore, as humans, often come up with examples that being barely non-trivial, happen to satisfy the Catalan recursion.[1]

Some of the most far-reaching realizations of the Catalan sequence regard objects that are associated to the symmetric group $\mathfrak{S}_n$. It often happens that such objects and phenomena can be generalised to other reflection groups as well. This is the world of Coxeter-Catalan combinatorics.

A distinguished resident of this world is the noncrossing lattice $NC(W)$ associated to a reflection group W (see Defn. 2.1). When $W = \mathfrak{S}_n$, its elements are Catalan objects and correspond, among other things, to $\mathfrak{S}_n$-orbits of parking functions:

A classical *parking function* is a map $f : [n] := \{1, 2, \cdots, n\} \to \mathbb{N}$ such that the increasing rearrangement $(b_1, b_2, \cdots, b_n)$ of the sequence $(f(1), f(2), \cdots, f(n))$ satisfies $b_i \leq i$. Let Park_n be the set of parking functions on $[n]$; below is the set Park_3:

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

The group $\mathfrak{S}_n$ acts on Park_n by permuting *positions*. Each row of the table corresponds to an orbit and the increasing orbit representatives are on the left. The permutation action turns Park_n into an $\mathfrak{S}_n$-module which we call the *standard parking space* (see Section 2.3). The cardinality of Park_n is $(n+1)^{n-1}$.

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[1]Overheard at FPSAC Chicago, attributed to Doron Zeilberger

The world of Coxeter-Catalan combinatorics is plagued by a terrible misfortune. Many things are *known* about the numerology, structure and properties of its residents, but few are *understood*. The classification of complex reflection groups by Shephard and Todd [ST54] made it easy for many proofs to be carried out case by case. To this day, there is no uniform proof even for the formula that describes the size of the non-crossing lattice:

$$|NC(W)| = \prod_{i=1}^{n} \frac{h+d_i}{d_i} =: \mathrm{Cat}(W)$$

where d_i's are the fundamental degrees of W (see Defn. 4.1), h the Coxeter number and where we call $\mathrm{Cat}(W)$ the W-Catalan number.

In [ARR15], Armstrong, Reiner and Rhoades state a conjecture about two new parking spaces that sheds new light on the situation. A uniform proof of that conjecture would give a better understanding for the structure of $NC(W)$, its relation to the nonnesting partitions $NN(W)$ (see Defn. 2.3) and a lot of their remarkable numerology (as described for instance in [Arm09]). A significant tool towards its proof might be the new geometric understanding of $NC(W)$ introduced by Bessis, [Bes15] as he tackled the $K(\pi,1)$ conjecture for complex reflection groups.

The rest of this report is structured as follows: In Section 2, we recall necessary definitions and give a historical exposition of the ideas that led to [ARR15]. In Section 3, we build up to their Main Conjecture and its consequences (see Figure 6). In Sections 4 and 5, we describe the Bessis interpretation of the noncrossing lattice and give a rephrasing of the Main Conjecture in this new context. Along the way we do a case study for the dihedral group $I_2(m)$.

2. A short story of nearly everything - we'll need

It is difficult to give a linear narration of the ideas that led to the introduction of Parking Spaces. Many of the core concepts were developed at the same time and deep connections between them arose in a parallel fashion, rather than sequentially. In this presentation, we will try to follow three strands of thought that have been unfolding the last 25 years, but surely we'll be unable to observe most of the very beautiful scenery of Coxeter-Catalan Combinatorics, to which they belong. Before that, we remind the reader of some basic reflection group terminology. For references consult [Kan01] or [Hum90].

Let W be a real reflection group acting irreducibly on $V \cong \mathbb{R}^n$ and T its set of reflections. It has an associated root system $\Phi = \{\pm\alpha\}$ comprised of the normal vectors to its reflecting hyperplanes $\{H_\alpha\}_{\alpha\in\Phi}$, which in turn form the *Coxeter Arrangement* $\mathcal{A}_W$. The complement $V \setminus \mathcal{A}_W$ is a disjoint union of connected components called *chambers*, each of which induces a Coxeter System structure (W,S) for W.

Choose now a fundamental chamber C_0 and let

$$\Delta \subset \Phi^+ \subset \Phi \subset Q \subset V$$

be the simple roots, positive roots, root system and root lattice (only when W is a Weyl group) respectively. Call $S = \{s_1, \cdots s_n\}$, the set of simple reflections associated to C_0 and fix for the rest of this paper a Coxeter element $c = s_1 \cdots s_n$ by ordering S. Let h be the multiplicative order of c and call it the *Coxeter number* of W.

Finally, let $\mathcal{L}$ be the intersection lattice of $\mathcal{A}_W$ and recall Steinberg's theorem (e.g. at [Bro10, Theorem 4.7] that any element X of $\mathcal{L}$ is equal to the fixed space V^g for some $g \in W$. Therefore, the action of W on V induces an action on the elements of $\mathcal{L}$ (called *flats*) which can be expressed as:

$$w \cdot X = w \cdot V^g = V^{wgw^{-1}}$$

We are ready now to kickstart our narration of how Parking Spaces came to be, by introducing a classical Catalan object:

2.1. **The set of non-crossing partitions** of $[n] := \{1, 2, \ldots, n\}$, denoted $NC(n)$, was studied by Kreweras in [Kre72]. It contains those set partitions, for which no two blocks contain elements a, c and b, d respectively, such that $a < b < c < d$. It forms a self-dual lattice and has rich enumerative properties. The term *non-crossing* comes from the following geometric depiction of a partition p of $[n]$ (Figure 1).

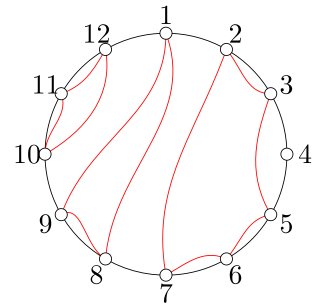


FIGURE 1. A non-crossing partition with blocks $\{1, 8, 9\}$, $\{2, 3, 5, 6, 7\}$, $\{4\}$, and $\{10, 11, 12\}$

In the mid 90's, Reiner was the first to extend the pictorial definition of non-crossing partitions and study their properties for the reflection groups B_n and D_n (see [Rei97]); he observed that they are enumerated by a generalization of $\text{Cat}(n)$ which he called the W-Catalan numbers. At that paper he asked for a natural definition of the lattice of non-crossing partitions for all finite Coxeter groups.

Quoting [AR04] a few years later, "the main idea for this may be described as folklore, but only fairly recently, and in particular after the work of Bessis [Bes03] and Brady and Watt [BW02a], it became apparent that such a definition is both available and useful." Recall first the following about the absolute order of a real reflection group:

For any $w \in W$, call $l_T(w)$ the least number k of reflections $t_i \in T$ needed to write $w = t_1 \cdots t_k$. This is the *reflection length* on W and it defines the *absolute order* $\leq_T$ on W by setting

$$u \leq_T v \iff l_T(v) = l_T(u) + l_T(u^{-1}v)$$

Then Brady and Watt [BW02a] and Bessis [Bes03] introduce the following notion of non-crossing partitions (which we copy from [ARR15]):

Definition 2.1. *Define the poset $NC(W)$ of W-noncrossing partitions as the interval $[1, c]_T$ in absolute order. Brady and Watt [BW02b] showed that this poset embeds into the intersection lattice*

$$NC(W) \hookrightarrow \mathcal{L}$$
$$w \mapsto V^w$$

We will sometimes identify $NC(W)$ with its image under this embedding, and refer to the elements of $NC(W)$ as the **noncrossing flats** *$X \in \mathcal{L}$.*

Note that conjugation by $w \in W$ is a poset isomorphism $[1, c]_T \cong [1, wcw^{-1}]_T$ and since all Coxeter elements are conjugate,[2] $NC(W)$ is well defined up to isomorphism. Furthermore, the subgroup $C = \langle c \rangle$ acts on $NC(W)$, and one has an action of elements g in $C \leq W$ on a fixed space V^w defined by $g(V^w) = V^{g^{-1}wg}$ so that the embedding described above is C-equivariant.

The poset $NC(W)$ was also very important in Bessis's work on complex reflection arrangements [Bes15]. There he observed that $NC(W)$ can actually be defined for well generated complex reflection groups W, it is still self-dual, a lattice, and its cardinality is given by the W-Catalan number. Sadly, there is no uniform proof of the last two statements (but see [BW08] or [Rea11] for the lattice property in the real case).

Example 2.2. We'll start here to develop our running example, which will be the dihedral group $I_2(m)$, the group of symmetries of a regular m-sided polygon. We can represent it as

$$I_2(m) = \{r, s | r^n = s^2 = 1, srs = r^{-1}\}$$

where s acts by reflecting over the y-axis and r is a counter-clockwise rotation by $\theta = 2\pi/m$:

$$s : (x, y) \mapsto (-x, y)$$
$$r : (x, y) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

The hyperplanes $\{x = 0\}$ and $\{x = -y \tan(\theta/2)\}$ bound a chamber and their corresponding reflections are s and rs. The two different products $rss = r$ and $srs = r^{-1}$ form a conjugacy class, so are the only possible Coxeter elements for $I_2(m)$. We pick r to define $NC(I_2(m))$.

[2] For a generalization of the Coxeter elements that still gives isomorphic non-crossing lattices, see [RRS14]

Since it is not a reflection itself, $l_T(r) = 2$. Now, for any reflection $r^k s$ of $I_2(m)$, we have $r^k s \cdot r^{k-1}s = r^k s \cdot sr^{1-k} = r$. That is, all reflections of $I_2(m)$ (and since $l_T(r) = 2$, by necessity no other elements) lie below r in $NC(I_2(m))$. (see Figure 2)

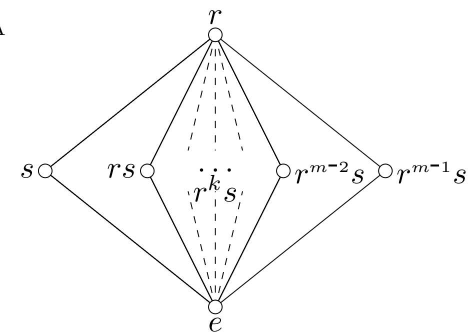


Figure 2. The non-crossing lattice for $I_2(m)$.

Notice that the $|NC(I_2(m))| = m + 2$, which agrees with Bessis's formula: The Coxeter number h for $I_2(m)$ is m (the order of r) and the fundamental degrees are 2 and m (see Example 4.11), so

$$\mathrm{Cat}(I_2(m)) = \prod_{i=1}^{n} \frac{h + d_i}{d_i} = \frac{m+2}{2} \cdot \frac{m+m}{m} = m + 2$$

2.2. **The set of Nonnesting Partitions**, denoted $NN(n)$, is the second part of our narration and is somewhat more recent. They are defined by the restriction that whenever $a < b < c < d$ and a, d are *consecutive* elements of a block B, then b, c cannot belong to the same block B'.

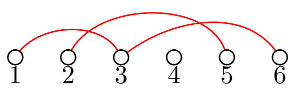


Figure 3. A non-nesting partition with blocks $\{1, 3, 6\}$, $\{2, 5\}$, and $\{4\}$.

They owe their name to a (different) depiction of partitions of $[n]$ (see Figure 3). Interestingly, they are not only equinumerous to $NC(n)$ but they also have the same distribution according to number of blocks.

In the early 90's, Postnikov (see [Rei97, Remark 2]), gave a definition of nonnesting partitions that works for *all* Weyl groups. Again, we copy it from [ARR15]:

Definition 2.3. *Let W be a Weyl group, so that there exists a root poset $(\Phi^+, \leq)$, defined by setting $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{N}\Phi^+$. The set $NN(W)$ of W-nonnesting partitions is the collection of antichains (sets of pairwise-incomparable elements) in $(\Phi^+, \leq)$.*

Postnikov observed a bijection (that had also appeared in Shi's study [Shi97,Shi87] of the sign types of affine Weyl groups) between $NN(W)$ and the set of regions into which the fundamental chamber C_0 is dissected by Cat_W, a certain deformation of the Coxeter arrangement $\mathcal{A}_W$.

Shortly after the introduction of noncrossing partitions, Athanasiades (ref) studied the aforementioned regions, using the finite field method, and found them, for types A,B,C,D, to be enumerated by the same W-Catalan numbers. Moreover he defined a notion of *block type* that extended the block size statistic in type A, and refining results of Reiner, showed that $NN(W)$ and $NC(W)$, for W of types A, B, C, are equidistributed with respect to type.

Finally, in 2004, Athanasiades and Reiner [AR04], also answering a question of Bessis, completed the proof of the equidistribution property for all Weyl groups. More importantly, they gave an embedding $NN(W) \hookrightarrow \mathcal{L}$, and showed they could replace the case-special *block type* statistic with the W-orbit of the corresponding flat X:

Theorem 2.4. *[AR04, Thm 6.3] Let W, be a Weyl group and $NN(W)$ its nonnesting partitions. There is an embedding*

$$NN(W) \hookrightarrow \mathcal{L}$$
$$A \mapsto \cap_{\alpha \in A} H_\alpha$$

which defines a partial order on $NN(W)$. We will sometimes identify $NN(W)$ with its image under this embedding and speak about **nonnesting flats**. *Then, $NC(W)$ and $NN(W)$ are equidistributed with respect to W-orbits.*

That is, every W-orbit in the intersection lattice $\mathcal{L}$ contains the same number of noncrossing and nonnesting flats.

Sadly, the proof was done case by case for the infinite families and via computer for the remaining ones. It seems that nonnesting flats don't enjoy as nice properties as the noncrossing ones. In particular, even for Weyl groups, $NN(W)$ is not a lattice under the induced partial order from $\mathcal{L}$.

Example 2.5. Despite the fact that nonnesting partitions *can* be defined for arbitrary root systems, the resulting structures only have nice properties in the crystallographic case. That's why we will only present the root system $I_2(4)$ here (which is indeed crystallographic and for that matter isomorphic to B_2).

(A) A choice of positive roots for $I_2(4)$.

(B) The root poset for $I_2(4)$.

FIGURE 4. $I_2(4)$ is one of the three crystallographic dihedral groups.

We can see a choice of positive roots for the root system of $I_2(4)$ in Figure 4a; $\alpha, \beta, \gamma, \delta$ satisfy relations $\gamma - \beta = \delta$ and $\beta - \alpha = \delta$. These imply $\gamma \geq \beta$ and $\beta \geq \alpha$, $\beta \geq \delta$. This explains Figure 4b which depicts the root poset $(\Phi^+, \leq)$ of $I_2(4)$ and from which we can tell that it has exactly 6 antichains: $\big\{\emptyset, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{\alpha, \delta\}\big\}$. This agrees with the W-Catalan number $Cat(I_2(4)) = 4 + 2$.

Notice finally, that $NN(I_2(4)) \subset \mathcal{L}$ is made up of the 4 reflecting hyperplanes, $\{0\}$ and all of $V \cong \mathbb{R}^2$ (in this case precisely equal to $NC(W) \subset \mathcal{L}$).

2.3. **Haiman's work** is the third part of the story. In [Hai94], he considered the diagonal action of the symmetric group $\mathfrak{S}_n$ on the doubly graded ring $\mathbb{Q}[x_1, \cdots, x_n, y_1, \cdots, y_n]$. Haiman studied the quotient $R_n = \mathbb{Q}[\underline{\mathrm{X}}, \underline{\mathrm{Y}}]/I$ where I is the homogeneous ideal generated by all $\mathfrak{S}_n$-invariant polynomials without constant term and made various conjectures on its character, and Hilbert and Frobenius series.

One of them in particular, was that $R_n \cong \epsilon \otimes V$ as an $\mathfrak{S}_n$-module, where ϵ is the sign representation and $V = \mathbb{Z}^n_{n+1}/\mathbb{Z}_{n+1}$ with the natural $\mathfrak{S}_n$-action on coordinates. Haiman (who credits Gessel for the suggestion) found that the permutation action on the parking functions (that we described in the introduction) is isomorphic to the $\mathfrak{S}_n$-action on V. He called V the *parking space module*.

Attempting to generalize the above, he considered the diagonal module $\mathbb{Q}[U \oplus U]$ where U is the reflection representation of a Weyl group W. He defined R_W to be the quotient by a suitably chosen ideal and conjectured, again, that $R_W \cong \epsilon \otimes Y$ where T is the permutation representation on the finite torus $Q/(h+1)Q$ (where Q is the root lattice of W and h its Coxeter number).

He proved that the total number of W-orbits in $Q/(h+1)Q$ is given by the W-Catalan number (although the term was not introduced until [Rei97]). We call $Q/(h+1)Q$ the *standard parking space*.

Another significant contribution by Haiman, was the use of homogeneous systems of parameters (*h.s.o.p.'s* see Defn ref). Many of his conjectures involved quotients of Schur functions that need not be polynomials (or have positive coefficients). Haiman used h.s.o.p.'s to interpret these quotients as Hilbert functions of certain rings.

Example 2.6. We test here the formula for the number of orbits of $Q/(h+1)Q$. We will only work with $I_2(4)$ again. Figure 5 describes the orbit structure of the action of $I_2(4)$ on $Q/(h+1)Q$, where Q is the root lattice generated by the simple roots α and δ of Figure 4a and $h = 4$. Different shapes correspond to different orbits; there are 4 orbits of size 4, corresponding to the four reflections, one orbit of size 8 corresponding to the Coxeter element r and one orbit of size 1 corresponding to the identity e.

Furthermore, their stabilizers are conjugate (actually the same in this case) to the stabilizers W_X for the corresponding nonnesting flats X (see discussion above (3.1)).

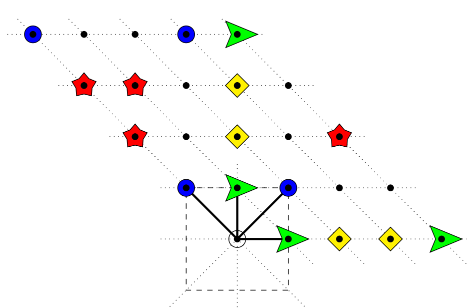


FIGURE 5. The standard parking space of $I_2(4)$.

3. Parking Spaces

In [ARR15] Armstrong, Reiner and Rhoades describe two generalizations of the standard W-parking space, called the **noncrossing parking space** and the **algebraic parking space**. These are defined for the larger set of real reflection groups and they carry not just W-actions but $W\times C$-actions, where C is the cyclic subgroup of W generated by a Coxeter element c.

3.1. Noncrossing and nonnesting parking functions.

We follow their presentation:

Define an equivalence relation on the set of ordered pairs

$$W\times\mathcal{L}=\{(w,X):w\in W,X\in\mathcal{L}\}$$

by setting $(w,X)\sim(w',X')$ when one has both

(1) $X=X'$, that is, the flats are equal, and
(2) $wW_X=w'W_{X'}$ where W_X is the pointwise W-stabilizer of the flat X.

Let $[w,X]$ denote the equivalence class of (w,X), and note that the left-regular action of W on itself in the first coordinate descends to a W-action on equivalence classes: $v\cdot[w,X]:=[vw,X]$.

Definition 3.1. *Define the* W-**nonnesting** *and* W-**noncrossing parking functions** *as the following* W*-stable subsets of* $(W\times\mathcal{L})/\sim$*:*

$$\mathrm{Park}_W^{NN}:=\{[w,X]:w\in W and X\in NN(W)\}$$

$$\mathrm{Park}_W^{NC}:=\{[w,X]:w\in W and X\in NC(W)\}$$

Both of these subsets inherit the W*-action* $v\cdot[w,X]=[vw,X]$*, but the second set* Park_W^{NC} *also has a* $W\times C$ *action, defined by letting the cyclic group* $C=<c>$ *(c being the Coxeter element we fixed) act on the right:*

$$(v,c^d)\cdot[w,Z]:=[vwc^{-d},c^d(X)]$$

As $\mathbb{C}[W]$-modules, the new parking spaces are by construction, sums of coset representations indexed by noncrossing and nonnesting flats. Indeed, let

$$\mathbb{C}[W/W_X]\cong\mathrm{Ind}_{W_X}^W\mathbf{1_{W_X}}$$

denote the action of W by left-translation on left cosets $\{wW_x\}$ (equivalently the W-action on the orbit $W\cdot X$) and notice that

$$\mathrm{Park}_W^{NN}\cong\bigoplus_{X\in NN(W)}\mathbb{C}[W/W_X]$$

$$\mathrm{Park}_W^{NC}\cong\bigoplus_{X\in NC(W)}\mathbb{C}[W/W_X]$$

This models Haiman's observation for the original parking space module, where the $\mathfrak{S}_n$-orbits were indexed by Catalan objects.

The Nonnesting parking space is actually isomorphic to Haiman's standard parking space. Cellini and Papi [CP00] and separately Shi [Shi97], established a bijection between antichains $NN(W)$ and W-orbits on $Q/(h+1)Q$. Athanasiades [Ath05, Lemma 4.1, Theorem 4.2] showed that this bijection viewed now from the nonnesting flats $NN(W)\subset\mathcal{L}$ to the finite torus, respects stabilizers (up to conjugacy). Furthermore, our previous discussion on the equidistribution of nonnesting and noncrossing flats with respect to W-orbits, implies that the newly defined spaces are also isomorphic to each other. Put together, we have:

$$Q/(h+1)Q\cong_W\mathrm{Park}_W^{NN}\cong_W\mathrm{Park}_W^{NC}\tag{3.1}$$

3.2. **The algebraic W-parking space.** As we mentioned earlier, one of the important contributions of Haiman's work was the use of homogeneous systems of parameters (hsop's) to interpret various formulae that appeared in his conjectures. The same object was used to provide conceptual proofs of cyclic sieving phenomena in the seminal paper [RSW04] by Reiner,Stanton and White. We recall some definitions:

Definition 3.2. *[BR11, Defn 4.1] [ARR15, Section 2.5] Let W be a real reflection group acting on $V = \mathbb{C}^n$, and $\mathbb{C}[V] = \mathrm{Sym}(V^*)$ the algebra of polynomial functions on V. We say that a collection $\Theta = \{\theta_1, \cdots, \theta_n\}$ of n homogeneous elements, of degree $h+1$ in $\mathbb{C}[V]$ forms a homogeneous system of parameters (hsop) carrying the dual reflection representation of W if*

1. *they are a system of parameters for $\mathbb{C}[V]$, meaning that they are algebraically independent and the quotient $\mathbb{C}[V]/(\Theta)$ is finite dimensional over $\mathbb{C}$, and*
2. *the $\mathbb{C}$-linear isomorphism defined by*

$$\begin{aligned} V^* &\to \mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n \\ x_i &\mapsto \theta_i \end{aligned} \tag{3.2}$$

is W-equivariant. In particular, the linear span $\mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n$ carries a copy of the dual[3] *reflection representation V^*.*

The existence of such systems of parameters is by no means trivial, but nonetheless guaranteed by the representation theory of **rational Cherednik algebras** (see [BEG03]). For the infinite families B/C and D, the naive choice $(\theta_1, \cdots, \theta_n) = (x_1^{h+1}, \cdots, x_n^{h+1})$ actually works but even for type A, although Haiman gave a method, his construction is inductive[4] and does not provide us with closed formulas, see [Hai94, Prop. 2.5.4]. For that matter, there is no known simple construction of hsop's for the exceptional real reflection groups.

Similarly to the coinvariant algebra in classical Springer theory (see Prop 4.2), once we have the homogeneous ideal (Θ), it is natural to consider the quotient ring $\mathbb{C}[V]/(\Theta)$. This arises for instance in the rational Cherednik theory, and in the crystallographic case it is known to be isomorphic to the representation $\mathbb{C}[Q/(h+1)Q] \cong_{\mathbb{C}[W]} \mathbb{C}[\mathrm{Park}_W^{NN}]$. As an *affine scheme* therefore, $\mathbb{C}[V]/(\Theta)$ encodes all the information of the standard parking space. In order to get a better understanding of its geometry we consider a slight deformation:

Definition 3.3. *[ARR15, Defn 2.10] Let W be a real reflection group, with $\Theta = (\theta_1, \cdots, \theta_n)$ and $(x_1, \cdots, x_n)$ chosen as in 3.2. Consider the ideal*

$$(\Theta - \boldsymbol{x}) := (\theta_1 - x_1, \cdots, \theta_n - x_n)$$

and define the **algebraic W-parking space** *as the quotient ring*

$$\mathrm{Park}_W^{alg} := \mathbb{C}[V]/(\Theta - \boldsymbol{x})$$

This quotient has the structure of a $W \times C$ representation since the ideal $(\Theta - \boldsymbol{x})$ is stable under the two commuting actions on $\mathbb{C}[V] = \mathbb{C}[x_1, \cdots, x_n]$:

1. the action of W by linear substitutions, and
2. the action of $C =< c >$ by scalar substitutions $c^d(x_i) = \omega^{-d} x_i$, with $\omega := e^{\frac{2\pi i}{h}}$.

We will only be interested in the $\mathbb{C}[W \times C]$-structure of Park_W^{alg} and in that context the choice of Θ is irrelevant:

Proposition 3.4. *[ARR15, Prop. 2.11] For every irreducible real reflection group W, and for* **any** *choice of Θ satisfying 3.2, one has an isomorphism of $W \times C$ representations*

$$\mathrm{Park}_W^{alg} := \mathbb{C}[V]/(\Theta - \boldsymbol{x}) \cong_{\mathbb{C}[W \times C]} \mathbb{C}[V]/(\Theta)$$

[3] In the case of real reflection groups, this is the same as V since we always have a $\mathbb{C}[W]$-module isomorphism $V \cong_{\mathbb{C}[W]} V^*$.

[4] For a non-inductive construction, see [Dun98].

Example 3.5. We describe here the set of possible hsop's (Θ) for $I_2(m)$ that satisfy 3.2.

After we extend scalars to $\mathbb{C}$ and make a change of basis, $I_2(m)$ can be shown to act on $V=\mathbb{C}^2$ via

$$r:\quad (x,y)\to(\zeta x,\frac{y}{\zeta}) \qquad\qquad \text{in other words, } s=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad r=\begin{bmatrix}\zeta&0\\0&\zeta^{-1}\end{bmatrix}$$
$$s:\quad (x,y)\to(y,x)$$

where $\zeta=e^{\frac{2\pi i}{m}}$.

Two arbitrary homogeneous polynomials of degree $h+1=m+1$ are given by

$$\theta_1=a_{m+1}x^{m+1}+a_mx^my+\cdots+a_1xy^m+a_0y^{m+1} \qquad \text{and}$$
$$\theta_2=b_{m+1}y^{m+1}+b_my^mx+\cdots+b_1yx^m+b_0x^{m+1}$$

For the $\mathbb{C}$-linear isomorpshism $x_i\to\theta_i$ (from V^* to the linear span of the θ_i's) to be W-equivariant, we need $r:\quad(\theta_1,\theta_2)\to(\zeta\theta_1,\theta_2\zeta^{-1})$ and $s:\quad(\theta_1,\theta_2)\to(\theta_2,\theta_1)$.

The action of r forces $a_i=a_i\zeta^i\zeta^{i-m}$ which can only be true if $a_i=0$ or $2i-m\equiv 0 \bmod m$. This leaves us with two cases for m:

$$m\ \text{ odd:}\qquad \theta_1=ax^{m+1}+bxy^m \tag{3.3}$$
$$\theta_2=ay^{m+1}+byx^m$$
$$m=2k:\qquad \theta_1=ax^{m+1}+bx^{k+1}y^k+cxy^m \tag{3.4}$$
$$\theta_2=ay^{m+1}+by^{k+1}x^k+cyx^m$$

Not all of these (Θ) are, however, systems of parameters for $\mathbb{C}[x,y]$. An equivalent condition is (ref? see also next proof) that the only common root of the θ_i's is 0. We work out the case m is odd:

In the above equation, $\theta_1=0$ implies $x=0$ or $ax+by=0$. Similarly for θ_2, we get $ay+bx=0$. This has non-trivial solutions exactly when $a=b$ and $a=-b$.

3.3. **The Parking space conjectures.** In Figure 6 we can see a chain of $\mathbb{C}[W]$-module isomorphisms, a summary of the ideas that we have described, that connect Park_W^{alg} and Park_W^{NC}. The chain breaks outside the crystallographic case but even where it works, most proofs are either case by case or not illuminating. The main conjecture in [ARR15] is an attempt to provide a geometric interpretation, a missing link to complete that chain.

In [BR11, Question 5.3], Bessis and Reiner, attempting to do exactly that, had asked for a finite set P, that carries a suitable $W\times C$ action. It should consist of $(h+1)^n$ points, its W-orbits should be naturally indexed by $NC(W)$ and it should be isomorphic to $\mathbb{C}[V]/(\Theta)$ as a $\mathbb{C}[W\times C]$-module. The role of P in [ARR15] is played by nothing else than the zero-set of the ideal $(\Theta-\boldsymbol{x})$ itself; we call the resulting 0-dimensional variety V^Θ. The notation is meant to suggest that one views V^Θ as the fixed points of the polynomial map $\Theta:V\to V$ that sends an element $\boldsymbol{x}$ with coordinates $(x_1,\cdots,x_n)\in V$ to the element $\Theta(\boldsymbol{x})=\big(\theta_1(\boldsymbol{x}),\cdots,\theta_n(\boldsymbol{x})\big)$.

Notice that the definition of V^Θ as the zero set of $(\Theta-\boldsymbol{x})$ implies that it carries a natural $W\times C$ action as well. Furthermore, Proposition 3.4 indicates that the variety V^Θ has at most $(h+1)^n$ *distinct* points (and exactly that many when counted with multiplicity).

Main Conjecture. *[ARR15] Let W be an irreducible real reflection group.*

> *(weak version) The spaces* $\mathbb{C}[\mathrm{Park}_W^{alg}]$ *and* $\mathbb{C}[\mathrm{Park}_W^{NC}]$ *are isomorphic as* $\mathbb{C}[W\times C]$*-modules.*
> *(intermediate version)* **There exists a** *choice of* Θ *as in* (3.2) *such that...*
> *(strong version)* **For all choices** *of* Θ *as in* (3.2)*, one has that...*

.. the subvariety V^Θ inside V consists of $(h+1)^n$ distinct points, that have a $W\times C$-equivariant bijection to the set Park_W^{NC}*, that is,* $V\supset V^\Theta\cong_{W\times C}\mathrm{Park}_W^{NC}$.

3.4. **Known cases.** There is quite a lot of evidence for the Main Conjecture. We present here a few results.

Define the dimension of a point $p \in V^\Theta$ to be the dimension of the flat $X \in \mathcal{L}$ that contains p. Then we can decompose the variety V^Θ as

$$V^\Theta = V^\Theta(0) \uplus V^\Theta(1) \uplus \cdots \uplus V^\Theta(n)$$

where $V^\Theta(i)$ is the set of points $p \in V^\Theta$ of dimension i and $\uplus$ denotes disjoint union.

Proposition 3.6. *[ARR15, Prop. 2.13] For W an irreducible real reflection group of rank n, and for all choices of Θ as in (3.2), one has the following.*

1. *The set $V^\Theta(0) = \{0\}$ of 0-dimensional points in V^Θ is the unique $W \times C$-orbit in V^Θ carrying the trivial $W \times C$-representation.*
2. *There exists a $W \times C$-equivariant injection $V^\Theta(1) \hookrightarrow \mathrm{Park}_W^{NC}$ whose image is precisely the set of noncrossing parking functions of the form $[w, X]$ with $\dim(X) = 1$.*
3. *The set $V^\Theta(n)$ of n-dimensional points in the unique W-regular orbit of points in V^Θ.*

Furthermore, every point in the subsets $V^\Theta(0)$, $V^\Theta(1)$, and $V^\Theta(n)$ is reduced in V^Θ, that is, it is cut out by the ideal $(\Theta - \boldsymbol{x})$ with multiplicity 1.

This in particular implies:

Corollary 3.7. *The strong version of the Main Conjecture holds in rank ≤ 2.*

Much of [ARR15] is devoted to proving different versions of the Main Conjecture for the infinite families $A, B/C, D$. The current status is summarized in the following table [ARR15, Table 1, p.662]:

Reflection group W	Strongest version of the Main Conjecture proven for W
rank ≤ 2	Strong; Corollary 3.7
type A_3	Strong; Brendon Rhoades writeup in progress
type A_{n-1}	Intermediate; Brendon Rhoades writeup in progress
type D_n	Intermediate; [ARR15, Section 7]
type $H_3,\ H_4,\ F_4,\ E_6$	Weak; computer verification
type $E_7,\ E_8$	Open

3.5. **Applications of the Main Conjecture.**

Even the weak form of the Main Conjecture has remarkable applications for Coxeter-Catalan combinatorics:

3.5.1. *The W-action gives* $\mathrm{Park}_W^{NN} \cong_W \mathrm{Park}_W^{NC}$.

The W-set isomorphism between the noncrossing and nonnesting parking spaces is the weakest link in Figure 6; that is, it is the only part that still has no uniform proof.[5] Even the weak form of the main conjecture provides a $\mathbb{C}[W]$-module isomorphism between the two spaces. In this particular case, this is enough to give a W-set isomorphism because of the following proposition.

Proposition 3.8. *[ARR15, Prop. 3.1] [Mil, Thm. 1] For real reflection groups W and finite sets A_1, A_2 whose W -orbits are all W-equivariant to W-orbits $\{W \cdot X\}$ of flats X in $\mathcal{L}$, one has $\mathbb{C}[A_1] \cong_{\mathbb{C}[W]} \mathbb{C}[A_2]$ if and only if $A_1 \cong_W A_2$.*

3.5.2. *The C-action is a cyclic sieving phenomenon.*

We recall the definition of the cyclic sieving phenomenon, introduced in [RSW04]:

Definition 3.9. [RSW14] Let C be a cyclic group generated by an element c of order n acting on a finite set X. Given a polynomial $X(q)$ with integer coefficients in a variable q, we say that the triple $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon (CSP)* if for all integers d, the number of elements fixed by c^d equals the evaluation $X(\zeta^d)$ where $\zeta = \exp(2\pi i/n)$.

[5] For an attempt to a uniform bijection between $NC(W)$ and $NN(W)$ that sadly does not respect block type (or W-orbits), see [AST13]

In [BR11], Bessis and Reiner showed that the triple $\big(NC(W), \mathrm{Cat}(W,q), C\big)$ exhibits the cyclic sieving phenomenon where $\mathrm{Cat}(W,q)$ was defined to be the following q-Catalan number for W

$$\mathrm{Cat}(W,q) = \prod_{i=1}^{n} \frac{1-q^{h+d_i}}{1-q^{d_i}}$$

The weak conjecture provides a conceptual proof for this CSP, as $\mathrm{Cat}(W,q)$ may be interpreted as the Hilbert series in q for $(\mathbb{C}[V]/(\Theta))^W$ (and then the theorem follows by classical techniques in the style of [RSW04]). The original proof of [BR11, Theorem 1.1] relied on certain facts proved case by case and a counting argument. If nothing else, this would be a suggestion that $\mathrm{Cat}(W,q)$ is the *correct* q-version of the W-Catalan numbers.

3.5.3. *Kirkman and Narayana numbers for W.*

The $\mathbb{C}[W]$-module isomorphism between the noncrossing and nonnesting parking spaces (which only has a case by case proof as we described before (3.1) gives an interesting interpretation to the Kirkman and Narayana numbers. These can be defined by:

$$\mathrm{Nar}_W(t) := \sum_{X\in NC(W)} t^{\dim_{\mathbb{C}}(X)} = \sum_{w\in[1,c]_T} t^{\dim V^w}$$

$$\mathrm{Kirk}_W(t) := \mathrm{Nar}_W(t+1)$$

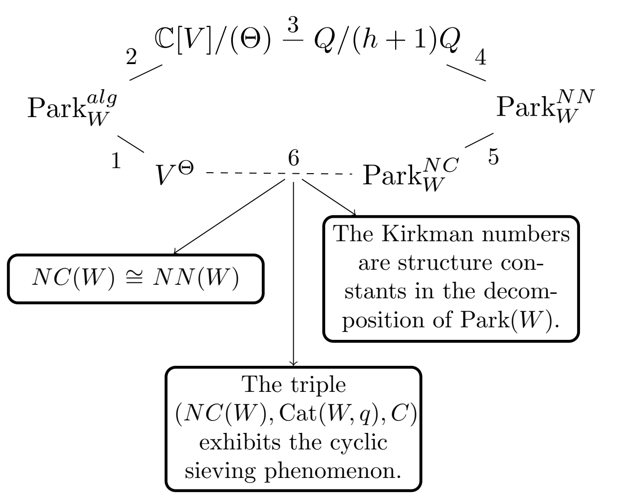
where the Kirkman numbers are the generating functions for the face numbers of the cluster complexes of finite type, or the Cambrian fan associated to W.

Let V be the reflection representation of W. Recall that the exterior powers $\bigwedge^k V$ for $k=0,1,\cdots,n$ are irreducible and pairwise inequivalent. In the case of the symmetric group, they correspond to hook-shaped partitions $(m-k,1^k)\vdash n$. If we denote by $\mathrm{Park}(W)$ either of the isomorphic $\mathbb{C}[W]$-modules $\mathbb{C}[\mathrm{Park}_W^{NC}] \cong_{\mathbb{C}[W]} \mathbb{C}[\mathrm{Park}_W^{alg}]$, we have the following interpretation for $\mathrm{kirk}_w(t)$:

Corollary 3.10. *[ARR15, Cor. 3.3] The Kirkman numbers are the multiplicities of the exterior powers $\bigwedge^k V$ in the irreducible decomposition of the parking space* $\mathrm{Park}(W)$*. That is, for W irreducible, one has*

$$\mathrm{Kirk}_W(t) = \sum_{k=0}^{n} \langle \chi_{\bigwedge^k V}, \chi_{\mathrm{Park}(W)}\rangle_W \cdot t^k$$

We close this section with a depiction that summarizes the relations between the various players so far.



1. This is just the Spec operator if $\mathbb{C}[V]/(\Theta-\boldsymbol{x})$ is reduced.
2. Recall that $\mathrm{park}_W^{alg} := \mathbb{C}[V]/(\Theta-\boldsymbol{x})$. This is then, Proposition 2.11 in [ARR15].
3. This was just a character calculation, known to Gordon and Haiman, see [BR11, Prop. 5.4], uniform.
4. Athanasiadis [Ath05], see discussion before (3.1), uniform.
5. Athanasiadis and Reiner, [AR04, Thm. 6.3] in a case by case fashion; see discussion before Theorem 2.3 .
6. The Parking Space Conjecture.

FIGURE 6. A chain of $\mathbb{C}[W]$-isomorphisms between various parking spaces, the Main Conjecture and its applications.

4. Braid groups, discriminants and the Lyashko-Looijenga morphism.

4.1. **Invariant theory of reflection groups.** Let G be a finite group acting on the vector space $V = \mathbb{C}^n$ via linear transformations. This induces an action of G on the coordinate ring $\mathbb{C}[V] := \mathbb{C}[x_1, \cdots, x_n]$ of V; call $\mathbb{C}[V]^G$ the ring of invariants.

It is a famous result by Noether that $\mathbb{C}[V]^G$ is generated by *finitely many* polynomials, that is, it is the coordinate ring of some affine variety X. Moreover, each point of X corresponds to an orbit in V in a way that makes X homeomorphic to the quotient $G\backslash V$ and the projection map $V \mapsto G\backslash V$ a morphism of affine varieties.

The case of a reflection group W acting via its reflection representation is particularly nice. The Chevalley-Shephard-Todd theorem states that this is exactly the case when the ring of invariants is polynomial. Now the extension $\mathbb{C}[V]^W \subset \mathbb{C}[V]$ is finite so $\mathbb{C}[V]^W$ will also have dimension n:

Definition 4.1. We call a set $\boldsymbol{f} = \{f_1, \cdots, f_n\}$ of *homogeneous* generators for $\mathbb{C}[V]^W$, a *system of basic invariants* for W. We order it so that $\deg f_i \leq \deg f_{i+1}$.

A system of basic invariants $\boldsymbol{f}$ encodes quite a lot of information about the group W and the Coxeter arrangement $\mathcal{A}_W$. The degrees $d_i := \deg(f_i)$ are independent of $\boldsymbol{f}$ and their product equals the size of the group $|W|$. In fact (see [ST54] and [Sol63]) we have the following stronger result, known as the Shephard-Todd formula:

$$\sum_{w \in W} q^{\dim(V^w)} = \prod_{i=1}^{n} (q + (d_i - 1))$$

where $V^w = \{v \in V = \mathbb{C}^n : w(v) = v\}$. Also, the highest degree d_n is the Coxeter number h for W.

In the context of classical Springer theory it is natural to consider the quotient $\mathbb{C}[V]/(\boldsymbol{f})$:

Proposition 4.2. *[ST54] [Che55] [Spr74] The coinvariant algebra $\mathbb{C}[V]/(\boldsymbol{f})$ carries the regular representation of W. Let $C = \langle c \rangle$ be generated by a Coxeter element c and ζ one of the eigenvalues of c. Then we actually have the stronger isomorphism of $\mathbb{C}[W \times C]$-modules*

$$S/(\boldsymbol{f}) \cong_{\mathbb{C}[W \times C]} \mathbb{C}[W]$$

A somewhat dual version of basic invariants exists:

The space $\mathbb{C}[V] \otimes V$ corresponds to the vector fields of V; its arbitrary element ϕ can be written as $\phi = \sum_{k=1}^{l} g_k \, \partial/\partial x_i$, where $g_k \in \mathbb{C}[V]$. Such an element ϕ is called homogeneous of degree p if all g_k have the same degree p. This gives a grading for $\mathbb{C}[V] \otimes V$.

Theorem 4.3. *[OT92, Lemma 6.48] The space $\big(\mathbb{C}[V] \otimes V\big)^W$ is a free, homogeneous $\mathbb{C}[f_1, f_2, \cdots, f_n]$-module of rank n.*

Definition 4.4. We call a homogeneous basis $\{\boldsymbol{\xi}\} = (\xi_1, \xi_2, \cdots, \xi_n)$ of $\big(\mathbb{C}[V] \otimes V\big)^W$ a *system of basic derivations* for W.

Basic derivations correspond to W-invariant vector fields of V. For $j \in \{1, 2, \cdots, n\}$ the vector field ξ_i defines a vector field $\overline{\xi_i}$ of the quotient variety W/V. The degrees d_i^* of the ξ_i's are also independent of the choice of $\boldsymbol{\xi}$ and are called the *co-degrees* of W. We order them in decreasing order.

Degrees and codegrees satisfy a sort of duality in the case of real reflection groups.[6] For all $i \in \{1, 2, \cdots, n\}$ we have $d_i + d_i^* = d_n = h$, see [Bes15, Thm 2.4].

[6] Actually more generally, for well-generated complex reflection groups, that is, those generated by $\dim(V)$ reflections.

4.2. **Braid Groups and the Discriminant Hypersurface.** Choosing a system of basic invariants for W is tantamount to choosing a *graded* isomorphism between $\mathbb{C}[V]^W$ and $\mathbb{C}[y_1,\cdots,y_n]$. Geometrically, this means choosing a way to identify $W\backslash V$ with the affine space $\mathbb{C}^n$. The induced morphism

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{C}^n \cong W\backslash V \\ \boldsymbol{x}=(x_1,\cdots,x_n) \mapsto & & \big(f_1(\boldsymbol{x}),\cdots,f_n(\boldsymbol{x})\big) \end{array}$$

realizes (the affine space) $\mathbb{C}^n$ as the quotient $W\backslash V$. That is, the preimage of the point $\big(f_1(\boldsymbol{x}),\cdots,f_n(\boldsymbol{x})\big)$ is exactly the orbit $W\cdot\boldsymbol{x}$.

Now, let $\mathcal{A}_W$ be the hyperplane arrangement associated to W and set

$$V^{reg} := V - \bigcup_{H\in\mathcal{A}_W} H$$

to be its complement. It is a classical theorem due to Steinberg that for any point $x\in V$, the subgroup W_x of W that fixes x is generated by the reflections that fix x. This means that the action of W on V^{reg} is free.

In particular, because the map $\big(\boldsymbol{x}\mapsto\boldsymbol{f}(x)\big)$ is continuous, the action $W \curvearrowright V^{reg}$ is a *covering space action*. This means that if $p: V \twoheadrightarrow W\backslash V$ is the quotient map and we choose a basepoint $v_0\in V^{reg}$, the following sequence is exact:

$$1\to p_*\big(\pi_1(V^{reg},v_0)\big)\to\pi_1\big(W\backslash V^{reg},p(v_0)\big)\to W\to 1$$

Definition 4.5. We call $B(W):=\pi_1\big(W\backslash V^{reg},p(v_0)\big)$ the *braid group* of W and $P(W):=p_*\big(\pi_1(V^{reg},v_0)\big)$ the *pure braid group.*

Later, we will update the definition of $B(W)$, substituting v_0 with a *fat basepoint* (see Section **??**).

The surjection $B(W)\twoheadrightarrow W$ will be crucial to Bessis' new, geometric interpretation of $NC(W)$. In order to understand it better, we return to the hyperplane arrangement $\mathcal{A}_W$.

For each $H\in\mathcal{A}_W$, let $\alpha_H\in V^*$ be a linear form with kernel H. The product $\prod_{H\in\mathcal{A}_W}\alpha_H$ is a homogeneous polynomial whose zero-set is $\mathcal{A}_W$. It is however not W-invariant; whenever $\boldsymbol{x}'=s_H\cdot\boldsymbol{x}$ (so $\boldsymbol{x}'$ is in the same orbit as $\boldsymbol{x}$), it is not very difficult to see that

$$\prod_{H\in\mathcal{A}_W}\alpha_H(\boldsymbol{x}) = -\prod_{H\in\mathcal{A}_W}\alpha_H(\boldsymbol{x}')$$

This leads us to consider its square:

Definition 4.6. Given a basic system of invariants $\boldsymbol{f}$, we can express $\prod_{H\in\mathcal{A}_W}\alpha_H(\boldsymbol{x})^2$ as a polynomial in the f_i's. We write that polynomial as $\Delta(W,\boldsymbol{f})$ and call it the *discriminant* of W with respect to $\boldsymbol{f}$. Notice that because it is homogeneous on the x_i's, as a polynomial on the f_i's, $\Delta(W,\boldsymbol{f})$ is *weighted homogeneous* for weights $\deg(f_i)=d_i$.

Now, the zero set $\Delta(W,\boldsymbol{f})=0$ in $\operatorname{Spec}\mathbb{C}[f_1,f_2,\cdots,f_n]\cong W\backslash V$ is called the *discriminant hypersurface* $\mathcal{H}$ and is precisely the image of the Coxeter arrangement $\mathcal{A}_W$ under the quotient map $p: V\to W\backslash V$.

The braid group $B(W)$ is therefore realised as the fundamental group of the complement of an algebraic variety in $\mathbb{C}^n$. Such fundamental groups are generated by particular elements called *generators-of-the-monodromy* or *meridians* [**?**, Section 2]; those correspond to small loops wrapping once around smooth points of the variety.

For a braid group $B(W)$, all meridians come from small *paths* around the hyperplanes H in V that connect a point x close to H with the image $s_H\cdot x$. These paths map to loops (meridians) in the quotient space $W\backslash V$ that we call *braid reflections.* For future reference, we record this as:

Proposition 4.7. *[Bro10, Thm. 4.15] The braid group $B(W)$ is generated by the meridians of $\mathcal{H}$ which we call braid reflections.*

4.3. **Discriminant stratification.** The intersection lattice $\mathcal{L}$ of $\mathcal{A}_W$ describes a stratification $\mathcal{S}$ of the space V via its flats. The action of W on $\mathcal{L}$ defines a quotient stratification $\overline{\mathcal{S}}$ on $W\backslash V$ which we call the *discriminant stratification.*

The discriminant stratification has been studied extensively (e.g. [Orl89]) and is intimately related to the invariant theory of W. We assemble here a few results we will need:

Proposition 4.8. *[OT92, Corol. 6.114] Let $v \in V$ with image $\overline{v} \in W\backslash V$. The tangent space to the stratum of $\mathcal{S}$ containing v is spanned by the vectors $\xi_1(v), \cdots, \xi_n(v)$ (where $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is a system of basic derivations for W). The tangent space to the stratum of $\overline{\mathcal{S}}$ containing $\overline{v}$ is spanned by the vectors $\overline{\xi_1}(v), \cdots, \overline{\xi_n}(v)$.*

Proposition 4.9. *[Bes15, Lem 5.4] Let v be as above and set*

$$V_v := \bigcap_{H \in \mathcal{A}_W,\ v \in H} H$$

The multiplicity of the discriminant hypersurface $\mathcal{H}$ at $\overline{v}$ is $\dim_{\mathbb{C}} V/V_v$.

Steinberg's theorem [Bro10, Theorem 4.6] gives a natural bijection between the set of flats in V and the set of parabolic subgroups of W. Each flat L is mapped to its fixator which is always a parabolic subgroup of W generated by those reflections that fix L. Taking the quotient by W (which acts on the parabolic subgroups by conjugation), this gives yet another decription of the stratification $\overline{\mathcal{S}}$:

Proposition 4.10. *[Rip12, Prop. 3.4] The set $\overline{\mathcal{S}}$ is in canonical bijection with:*

1. *the set of conjugacy classes of parabolic subgroups of W;*
2. *the set of conjugacy classes of parabolic Coxeter elements (i.e. Coxeter elements of parabolic subgroups);*
3. *the set of conjugacy classes of elements of $NC(W)$*

Through these bijections, the codimension of a stratum $\overline{L}$ corresponds to the rank of the associated parabolic subgroup and to the reflection length of the parabolic coxeter element.

Example 4.11. Assume $I_2(m)$ acts on $V = \mathbb{C}^2$ as in Example 3.5.

A choice of a system of basic invariants for $I_2(m)$ is given by the polynomials $f_1 = xy$ and $f_2 = x^m + y^m$. Therefore, $W\backslash V \cong \mathbb{C}^2$, with parameters (f_1, f_2). The hyperplane corresponding to the reflection $r^k s$ (for $k = 1, 2, \cdots, m$) is the zero-set of the form $x - \zeta^k y$. The discriminant is then given by

$$\Delta(I_2(m), \boldsymbol{f}) = \prod_{k=1}^{m} (x - \zeta^k y)^2 = (x^m - y^m)^2 = (x^m + y^m)^2 - 4(xy)^m = f_2^2 - 4f_1^m$$

The following Figure 7 shows the discriminant hypersurfaces $\Delta(I_2(m), \boldsymbol{f}) = 0$ for $m = 3, 4$. Both images depict only the real coordinate of f_2, that is, they live in

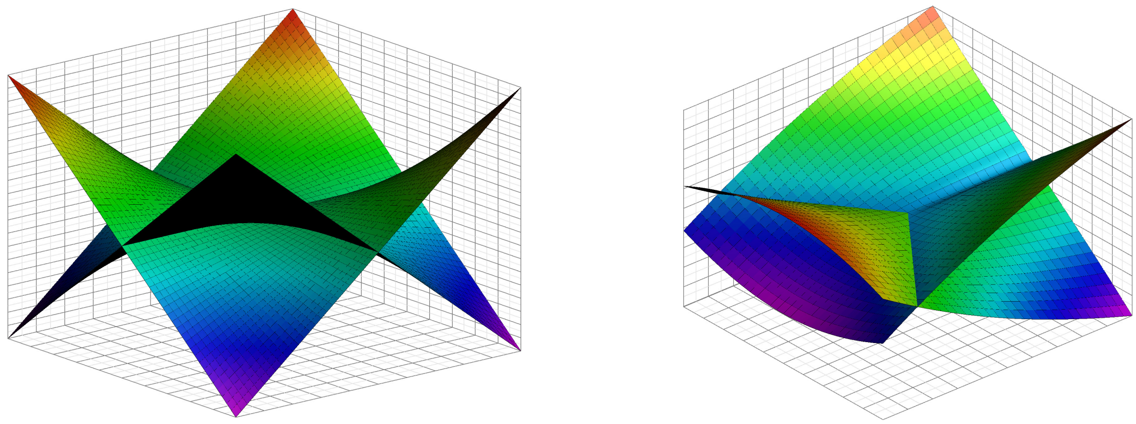
$$[0,1]^3 = [0,1]^2 \times [0,1] \subset \mathbb{C} \times \mathrm{Re}(\mathbb{C}) \subset \mathbb{C} \times \mathbb{C} \quad (= \{(f_1, f_2)\}).$$

The lines of self-intersection disappear in $\mathbb{C}^2$; the only singular point of both surfaces is the origin. Notice that, as expected by the formula $f_2^2 = 4f_1^m$, the surfaces wrap around the origin with rotational speed $m/2$.

The visual information is enough to reconstruct the stratification. In Figure 7a, removing the origin leaves two disconnected components; $I_2(4)$ has two different strata of (co-)dimension 1, because its reflecting hyperplanes form two orbits under W. This is not the case in Figure 7b, where there is a single 1-dimensional stratum.

4.4. **The generalised Lyashko-Looijenga covering.** It is known [Bes15, Thm 2.4 part (v)] that when W is a real reflection group,[7] the discriminant $\Delta(W, \boldsymbol{f})$, viewed as a polynomial in f_n is monic

[7]This is actually true for all *well-generated* complex reflection groups W.



(A) The reflections in $I_2(4)$ form two separate orbits. (B) All reflections in $I_2(3)$ are conjugate.

FIGURE 7. The discriminant hypersurfaces for $I_2(3)$ and $I_2(4)$.

of degree n. Making an extra substitution, we can eliminate the coefficient of f_n^{n-1} resulting in an equation:

$$\Delta(W, \boldsymbol{f}) = f_n^n + \alpha_2 f_n^{n-2} + \alpha_3 f_n^{n-3} + \cdots + \alpha_n$$

with $\alpha_i \in \mathbb{C}[f_1, \cdots, f_{n-2}]$. Notice that since $\Delta(W, \boldsymbol{f})$ is weighted homogeneous, so are the α_i's, of weighted degree $nd_n - (n-i)d_n = ih$. The fact that the discriminant is monic with respect to f_n implies that fixing $y = (f_1, f_2, \cdots, f_{n-1}) \in \mathbb{C}^{n-1}$, the equation $\Delta(W, \boldsymbol{f}) = 0$ has exactly n solutions for the unknown f_n, counted with multiplicity.

Recall that we have identified $W\backslash V$ with $\operatorname{Spec}\mathbb{C}[f_1, f_2, \cdots, f_n]$. Let us now define $Y := \operatorname{Spec}\mathbb{C}[f_1, f_2, \cdots, f_{n-1}]$, so that $W\backslash V \cong Y \times \mathbb{C}$, with coordinates written (y, t), or sometimes (y, z). Then, for a fixed $y = (f_1, f_2, \cdots, f_{n-1})$, the solutions of $\Delta(W, \boldsymbol{f}) = 0$ correspond to the intersections of the discriminant hypersurface $\mathcal{H}$ and the line $\{(y, t) : t \in \mathbb{C}\}$. This allows us, following [Rip12], to give the subsequent definition:

Definition 4.12. We denote by E_n the set of centered[8] configurations of n unordered, not necessarily distinct points in $\mathbb{C}$, i.e.,

$$E_n := \mathfrak{S}_n \backslash H_0, \text{ where } H_0 = \left\{(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \middle| \sum_{i=1}^n x_i = 0\right\} \cong \mathbb{C}^{n-1}$$

The Lyashko-Looijenga map of type W is defined by

$$\begin{array}{ccc} Y & \xrightarrow{LL} & E_n \\ y = (f_1, \cdots, f_{n-1}) & \to & \text{multiset of roots of } \big(\Delta(W, \boldsymbol{f}); f_n\big) = 0 \end{array}$$

Notice there is a simple description of LL as an algebraic morphism. Indeed the coordinate ring of E_n is polynomial and generated by the $(n-1)$ elementary symmetric functions $\{e_2, e_3, \cdots, e_n\}$.[9] Now, the elementary symmetric polynomials evaluated at the roots of $\big(\Delta(W, \boldsymbol{f}); f_n\big) = 0$ will give, according to Vieta's formulas the coefficients α_i (up to sign). Therefore, we can express LL as the map

$$\begin{array}{ccc} Y \cong \mathbb{C}^{n-1} & \xrightarrow{LL} & E_n \cong \mathbb{C}^{n-1} \\ y = (f_1, \cdots, f_{n-1}) & \longrightarrow & \big((-1)^2 \alpha_2(f_1, \cdots, f_{n-1}), \cdots, (-1)^n \alpha_n(f_1, \cdots, f_{n-1})\big) \end{array}$$

[8] The configurations will be centered, exactly because we eliminated the coefficient of f_n^{n-1} in the discriminant equation $\big(\Delta(W, \boldsymbol{f}); f_n\big) = 0$

[9] This is also a special case of our previous discussion for geometric invariants, since E_n is the quotient of V under the action of the *complex reflection group* $\mathfrak{S}_n$.

The generic point $(f_1, f_2, \cdots, f_{n-1}) \in Y$ is mapped via LL to a configuration of n *distinct* points. We call $E_n^{reg} \subset E_n$ the space of such configurations and define the *bifurcation locus* to be the inverse image $\mathcal{K} := LL^{-1}(E_n - E_n^{reg})$. Notice that $\mathcal{K}$ is the zero-set of the LL-discriminant:

$$\mathrm{Disc}(\Delta(W, \boldsymbol{f}); f_n) = 0$$

The Lyashko-Looijenga morphism behaves particularly nicely outside $\mathcal{K}$:

Proposition 4.13. *[Bes15, Thm 5.3] The restriction of $LL : Y - \mathcal{K} \twoheadrightarrow E_n^{reg}$ is a topological covering of degree*

$$\frac{n!h^n}{|W|}$$

Proof. The morphism LL is quasi-homogeneous between the spaces $Y \cong E_n \cong \mathbb{C}^{n-1}$; algebraically, it describes a graded extension $\mathbb{C}[\alpha_2, \cdots, \alpha_n] \subset \mathbb{C}[f_1, \cdots, f_{n-1}]$. Under this setting, the extension is finite if and only if $LL^{-1}(0) = \{0\}$ (where 0 is the origin $(0, 0, \cdots, 0)$). To see that,[10] notice that

$$LL^{-1}(0) = 0 \iff \sqrt{(\alpha_1, \alpha_2, \cdots, \alpha_n)} = (f_1, f_2, \cdots, f_{n-2})$$

$$\iff (f_1, f_2, \cdots, f_{n-1})^N \subset (\alpha_2, \alpha_3, \cdots, \alpha_n)$$

for a suitably large N which is precisely the definition for $(\alpha_2, \alpha_3, \cdots, \alpha_n)$ being a homogeneous system of parameters. This in turn implies that the extension is finite; in particular $(\alpha_2, \alpha_3, \cdots, \alpha_n)$ have to be algebraically independent (so that the two rings have the same dimension).

Now, $\mathbb{C}[f_1, f_2, \cdots, f_{n-1}]$ is Cohen-Macaulay, therefore its extension over the *polynomial* ring $\mathbb{C}[\alpha_2, \alpha_3, \cdots, \alpha_n]$, has to be free. The rank can be computed by a Hilbert series calculation. Since $\deg f_i = d_i$, the Hilbert series of the first ring is given by $\prod_{i=1}^{n-1} \frac{1}{1-t^{d_i}}$; since $\deg \alpha_i = ih$, the second ring has Hilbert series $\prod_{i=2}^{n} \frac{1}{1-t^{ih}}$. The rank of the free extension is the limit as $t \to 1$ of the quotient of these series, equal to

$$\lim_{t\to 1} \prod_{i=1}^{n-1} \frac{1 - t^{(i+1)h}}{1 - t^{d_i}} = \prod_{i=1}^{n-1} \frac{(i+1)h}{d_i} = \frac{n!(h^{n-1}}{d_1 \cdots d_{n-1}} = \frac{n!h^n}{|W|}$$

The following two lemmas complete the proof:

Lemma 4.14. *[Bes15, Lemma 5.6] $LL^{-1}(0) = \{0\}$*

sketch. The proof is non-trivial; it comes down to comparing the multiplicity $m_{(y,0)}(\mathcal{H})$ of $\mathcal{H}$ at an element $(y, 0) \in LL^{-1}(0)$ with the intersection multiplicity $i\big((y, 0), L_y \cdot \mathcal{H}; \mathbb{C}^n\big)$ of $\mathcal{H}$ and L_y at $(y, 0)$ The latter is by definition equal to n, the order of 0 as a root of $(\Delta(W, \boldsymbol{f}); f_n) = 0$. A refined Bezout theorem gives

$$i\big((y, 0), L_y \cdot \mathcal{H}; \mathbb{C}^n\big) = m_{(y,0)}(\mathcal{H} \tag{4.1}$$

that is, $m_{y,0}(\mathcal{H})$ should be n but this can only happen at the origin 0 (by Prop. 4.9). ◻

Lemma 4.15. *LL is unramified on $Y - \mathcal{K}$.*

sketch. The proof uses systems of basic derivations to describe tangent hyperplanes to the discriminant hypersurface $\mathcal{H}$. Now, a result of Looijenga [Loo74] implies it is enough to check that the hyperplanes are in general position. The calculation is done in [Bes15, Lemma 5.7] and in more detail in [Rip12, Section 4.1]. ◻

This completes the proof of Proposition 4.13; the first important property of the morphism LL.

◻

[10] For a different approach using Bezout's theorem see [LZ04, Thm 5.1.5]

4.5. **The stratification compatibility.** The configuration $LL(y)$ defines a partition of n via the multiplicities of the elements $x_i \in LL(y)$. Each of the points (y, x_i) has its own multiplicity as an element of the variety $\mathcal{H}$. To be complete, we mention that these are compatible:

Proposition 4.16. *[Bes15, Corol. 5.9] Let $v \in V$. Denote by $(y, z) \in Y \times \mathbb{C}$ (identified with $W \backslash V$) the image of v. The following integers coincide:*

(1) *the multiplicity of z in $LL(y)$;*
(2) *the intersection multiplicity of L_y with $\mathcal{H}$ at (y, z);*
(3) *the multiplicity of $\mathcal{H}$ at (y, z);*
(4) *the rank $\dim_{\mathbb{C}} V/V_v$ of the parabolic subgroup W_v.*

In fact, the equivalence (ii)$\leftrightarrow$(iii) is nontrivial, given by a refined Bezout theorem as in (4.1).

5. Geometric factorizations

Let W be an irreducible real reflection group acting on the complexified space $V = \mathbb{C}^n$ and $B(W) = \pi_1\big(W/V - \mathcal{H}\big) = \pi_1\big(V^{reg}\big)$ its associated braid group. As we mentioned earlier, Bessis [Bes15] used the surjection $\pi : B(W) \twoheadrightarrow W$ to build a geometric framework for $NC(W)$.

In order to make use of the Lyashko-Looijenga morphism, we will parametrize the quotient W/V as $Y \times \mathbb{C} = \bigcup_{y \in Y} L_y$ where Y is the subspace of the first $n-1$ coordinates $(f_1, \cdots, f_{n-1})$ and L_y is the vertical complex line $(y, t) : t \in \mathbb{C}$. Notice that the image of the LL morphism is given by

$$LL(y) = L_y \cap \mathcal{H}$$

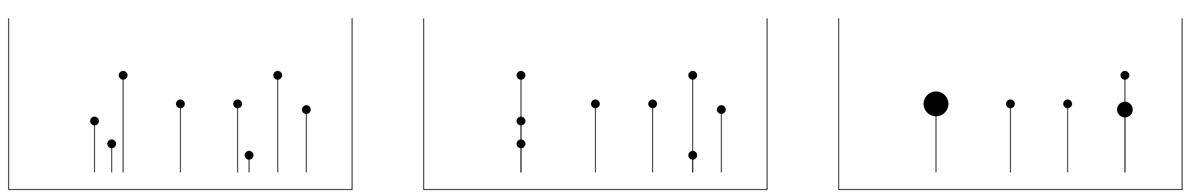
5.0.1. *The fat basepoint trick.* In general, there is no *canonical* projection $\pi : B(W) \twoheadrightarrow W$. Every such map depends on the choices of a basepoint in W/V^{reg}, and one of its $|W|$-many preimages in V^{reg}. In order to simplify notations, Bessis introduced the following contractible subset $\mathcal{U} \in W/V^{reg}$:

Definition 5.1. [Bes15, Defn. 6.2] For each $y \in Y$, let U_y be the complement in L_y of the vertical imaginary half-lines below the points of $LL(y)$ (see Figure 8). Then, the *fat basepoint* of W/V^{reg} is the subset $\mathcal{U}$ defined by

$$\mathcal{U} := \bigcup_{y \in Y} U_y$$

or equivalently by

$$\mathcal{U} := \{(y, z) \in Y \times \mathbb{C} \mid \forall x \in LL(y), \mathrm{re}(z) = \mathrm{re}(x) \Rightarrow \mathrm{im}(z) > \mathrm{im}(x)\}$$



(A) The points in $LL(y)$ might all be distinct...

(B) ...or some of them might have the same real part

(C) ...or they may appear with multiplicity > 1.

Figure 8. The complement U_y for various cases of $LL(y)$.

The proof that $\mathcal{U}$ is open and contractible and therefore can be used 'as if' it was an actual basepoint is given in [Bes15, Lemma 6.3]. Notice now, that $\mathcal{U}$ lifts to $|W|$-many disjoint *fat points* in V^{reg}. Make, once and for all, a choice of one of the preimages of $\mathcal{U}$ and consider the unigue surjection $B(W) \twoheadrightarrow W$ it defines.

5.1. **Tunnels and the Hurwitz rule.** Bessis describes a canonical way to construct factorizations of Coxeter elements, geometrically from the discriminant hypersurface $\mathcal{H}$. The starting point is the following map (following [Rip12] and [Bes15, Section 6])

$$\begin{aligned} \rho: \quad \mathcal{H} &\to W \\ (y,x) &\mapsto c_{y,x} \end{aligned} \tag{5.1}$$

that is constructed thus:

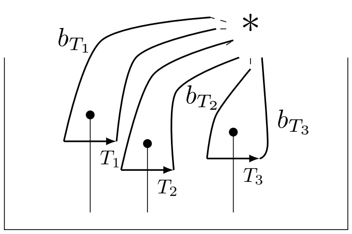
(1) Consider a small loop in $W/V-\mathcal{H}$, which always stays in the line L_y and which turns once around x (but not any other x' in $LL(y)$).
(2) This loop determines an element $b_{y,x}$ of the braid group $B(W)=\pi\big(W/V-\mathcal{H}\big)$ of W.
(3) Send $b_{y,x}$ to an element $c_{y,x}$ in W via the fixed surjection $\pi: B(W)\twoheadrightarrow W$.

In view of the fat basepoint $\mathcal{U}$, these loops can be encoded via the following semi-algebraic object:

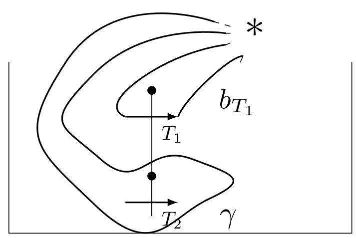
Definition 5.2. A *tunnel* is a triple $T=(y,z,L)\in Y\times\mathbb{C}\times\mathbb{R}_{\geq 0}$ such that (y,z) and $(y,z+L)$ are both in $\mathcal{U}$ and the affine segment $[(y,z),(y,z+L)]$ lies completely in W/V^{reg}. Each tunnel T represents an element $b_T\in\pi_1\big(W/V^{reg},\mathcal{U}\big)$

Figure 9 shows how we can identify various kinds of loops with products of tunnels. Notice that we will picture the fat basepoint as lying somewhere in the $i\cdot\infty$ direction, in *all* lines L_y.

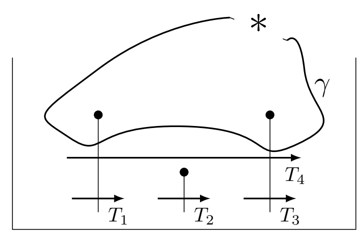
The concept of tunnels allows us to be more specific about the loop b_{y,x_i} described above. That is, if (y,x_i) has real part different than the other elements of $LL(y)$, b_{y,x_i} is just b_{T_i} where T_i is a tunnel crossing the interval below x_i but no other. If not, then say (y,x_{i+1}) is the point in $LL(y)$ immediately above (y,x_i); then b_{y,x_i} is exactly $b_{T_i}\cdot b_{T_{i+1}}^{-1}$ (as in Figure 9b).



(A) Each tunnel T_i represents a loop b_{t_i} in $B(W)$.



(B) Here the loop γ is represented by $b_{T_2}\cdot b_{T_1}^{-1}$.



(C) A tunnel can be equal to a product of tunnels: $\gamma=b_{T_4}=b_{T_1}\cdot b_{T_3}$.

FIGURE 9. Tunnels and loops.

Definition 5.3. [Bes15, Defn 6.7] An element $b\in B(W)$ is *simple* if $b=b_T$ for some tunnel T. The set of simple elements is denoted by S.

Remark 5.4. The set S will be shown to be finite and eventually will biject to the noncrossing partition lattice (see §5.2.1) This is the geometric interpretation of $NC(W)$ that we mentioned in the introduction.

The next definition associates a tuple of simple elements of $B(W)$ to $y\in Y$. We will show that this is always a factorization of a Coxeter element c.

Definition 5.5. If $(x_1,\cdots,x_p)$ is the ordered support of $LL(y)$ (for the lexicographical order on $\mathbb{C}\cong\mathbb{R}^2$), we call the sequence $\mathrm{rlbl}(y):=(b_{y,x_1},\cdots,b_{y,x_p})$, the *reduced label* of y.[11]

Remark 5.6. When y is generic, there are precisely n points in $LL(y)$, all smooth in $\mathcal{H}$, therefore $\mathrm{rlbl}\,y$ is an n-tuple of braid reflections (see Prop. 4.7).

The loops and tunnels we consider always lie in a single complex line L_y. However, loops in different lines L_y and L_y' can definitely be homotopic. We have the following rule:

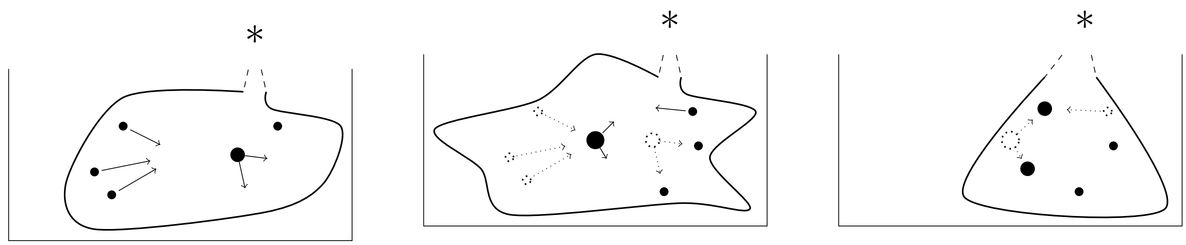
[11] We have kept the terminology from [Bes15] where Bessis initially used a slightly different assignment $\mathrm{lbl}(y)$ based more on tunnels.

Proposition 5.7. *[Bes15, Lemma 6.14 (The Hurwitz rule)] A continuous family of loops $\{\gamma_t\}$ that each lie in a single line L_{y_t}, and whose basepoints all lie in $\mathcal{U}$*

$$\gamma_t = (y_t, x_t) : [0,1] \to W/V - \mathcal{H} \subset Y \times \mathbb{C}$$
$$s \mapsto (y_t, x_t(s))$$

is a homotopy if the configurations $LL(y_t)$ never intersect the loop $x_t : [0,1] \to \mathbb{C}$.

In particular, given a starting point y_0, a continuous deformation of the central configuration $LL(y)$ lifts to a path in Y (that is unique if we stay in E_n^{reg}). This is a consequence of the finiteness of LL (Prop. 4.13) and gives us an easy way to describe homotopies (see Figure 10).[12]



(A) As the points in $LL(y)$ move, ...

(B) ...they are allowed to merge and unmerge

(C) ...and the loop might change shape as well.

FIGURE 10. A pictorial description of a homotopy

5.1.1. *The geometric interpretation for the Coxeter element.*

Definition 5.8. [Bes15, Defn 6.11] We denote by δ the simple element such that $\text{rlbl}(\mathbf{0}) = \delta$. Here $\mathbf{0} = (0, 0, \cdots, 0)$ is the origin in $Y = \mathbb{C}^{n-1}$; recall also that the Lyashko-Looijenga map $LL(\mathbf{0})$ is 0 in $L_0 \cong \mathbb{C}$ with multiplicity n (as in Lemma 4.14).

Given a point $\overline{v} \in L_0$ (the complex line above 0), we can parametrize δ as the loop in W/V^{reg}:

$$[0,1] \to W/V^{reg}$$
$$t \mapsto \overline{v} \exp(2\pi i t)$$

Its galois action on V^{reg} rotates any element $v \in p^{-1}(\overline{v})$ to $v \cdot \zeta_h$ where $\zeta_h := \exp(2\pi i/h)$. Therefore, the image $\pi(\delta)$ in W is ζ_h-regular in the sense of Springer (see [Bes15, Lem. 6.13]). We have now:

Lemma 5.9. *[Bes15, Lem 7.3] When W is irreducible, the element $c := \pi(\delta)$ is a Coxeter element of W.*

The other Coxeter elements, which are conjugates of c, appear when considering other basepoints over $\mathcal{U}$.

The map $\text{rlbl} : Y \to B(W)^n$ gives us factorizations of δ (and therefore of $\pi(\delta) = c$ as well):

Proposition 5.10. *[Bes15, Corol. 6.18] Let $y \in Y$. Let $(b_{y,x_1}, \cdots, b_{y,x_k})$ be the reduced label of y. We have $b_{y,x_1} \cdots b_{y,x_k} = \delta$*

Proof. We need only describe a homotopy between the loops δ and $b_{y,x_1} \cdots b_{y,x_k}$. The latter one corresponds to a loop that encloses all points x_i. Since the arrangement is central, we can safely assume that the origin lies inside the loop as well. Now, according to our discussion after Prop 5.7, we can move all the points x_i, keeping them *inside* the loop until they all merge in a single point at the origin (see Figure 11). □

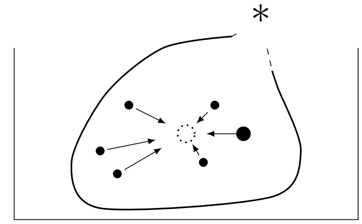


FIGURE 11. A homotopy from $b_{y,x_1} \cdots b_{y,x_k}$ to δ.

[12]Actually [Bes15, Remark 7.21] implies that if we are only allowed to merge but not unmerge points in the configuration, then the lift is still unique.

5.1.2. *The Hurwitz action.* In Prop. 5.10, we showed that to each $y \in Y$ we can associate a factorization of δ (and hence of the Coxeter element $c = \pi(\delta)$). We wish to know how that factorization may change as y moves in Y. As in Prop. 5.7, we will juxtapose movement in Y with movement in the space E_n of central configurations, via the LL mapping. First we recall the following:

Definition 5.11. [Bes15, Defn 6.19 (The Hurwitz action)] Let G be a group, and let B_n be the braid group on n strings with its usual system of generators $\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_{n-1}$. The *Hurwitz action* of B_n on G^n, denoted as a multiplication on the right, is the unique (right) group action such that

$$(g_1, \cdots, g_n) \cdot \boldsymbol{\sigma}_i = (g_1, \cdots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+1}, \cdots, g_n)$$

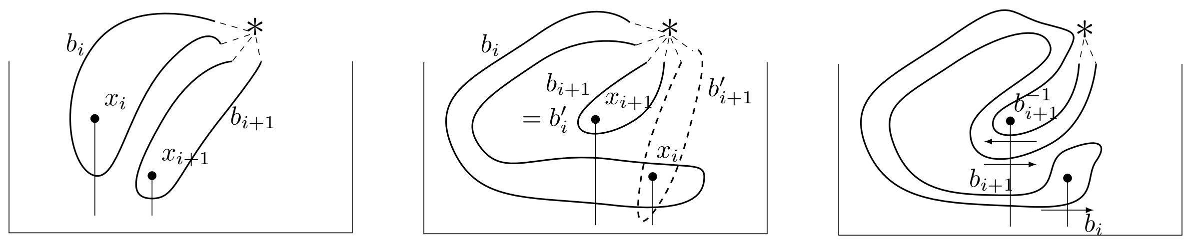
The following allows us to compare the (reduced) labels of two points y and y' in Y, given a path β in E_n that connects $LL(y)$ and $LL(y')$:

Proposition 5.12. *[Bes15, Cor. 6.20] Let $x \in E_n^{reg}$. Let $\beta \in \pi_1(E_n^{reg}, x)$,*[13] *$y \in LL^{-1}(x)$ and $y' := y \cdot \beta$ (the galois action of β). Let $(b_1, \cdots, b_n)$ be the label of y and $(b'_1, \cdots, b'_n)$ be the label of y'. Then*

$$(b'_1, \cdots, b'_n) = (b_1, \cdots, b_n) \cdot \beta$$

where β acts by right Hurwitz action.

Proof. It is enough to show it for a path β that interchanges x_i and x_{i+1} in the lexicographic order in $\mathbb{C}$ (this would be exactly the Galois action of $\boldsymbol{\sigma}_i$). Figures 12a and 12b describe the path β from y to y' and a homotopy between the corresponding loops. However, since x'_i and x'_{i+1} have exchanged positions, the map rlbl will first assign a label to x'_{i+1} and then to x'_i. Figure 12c describes why $b'_{i+1} = b_{i+1}^{-1} b_i b_{i+1}$.



(A) As we move x_i and x_{i+1} around each other...

(B) ...the loop b_i streches to avoid b_{i+1}.

(C) The loop $b_{i+1}^{-1} b_i b_{i+1}$ is homotopic to b'_{i+1}.

FIGURE 12. The Hurwitz action.

□

5.2. **The Gordon-Ripoll interpretation of the Main Conjecture.** One of the most important consequences of the Hurwitz rule, is that it identifies the set of simple elements S with the non-crossing lattice $NC(W)$. We will only need the following to state the Main Conjecture:

Proposition 5.13. *Let s be a simple element. There exists $y \in Y^{reg}$ and $i \in \{1, \cdots, n\}$ such that $s = s_1 \cdots s_i$, where $(s_1, \cdots, s_n) := \mathrm{rlbl}(y)$.*

Proof. Let T be a tunnel representing s (that is, $b_T = s$) in the line L_{y_0}. We can always deform the configuration $LL(y_0)$ slightly so that we get one with n distinct points. Now, as shown in figures a and b we can also move the points around, always in a central arrangement, so that in the end, the points enclosed in the loop $b_{T'}$, homotopic to b_T, are first in lexicographical order in the set $LL(y')$. □

Corollary 5.14. *For every simple element s, the projection $\pi(s) \in W$ is an element of the noncrossing lattice $NC(W)$.*

[13] Since E_n^{reg} is just W/V^{reg} for $W = \mathfrak{S}_n$, we can consider the fat basepoint $\mathcal{U}$ here as well, and β need not be a closed loop, rather, just a path in E_n^{reg} between points in $\mathcal{U}$

Proof. Pick a $y \in Y^{reg}$ as in Prop. 5.13 and call $w = \pi(s_1 \cdots s_i)$ and $u = \pi(s_{i+1} \cdots s_n)$. Now, Prop. 5.10 implies that $w \cdot u = c$, the coxeter element. Evenmore, because all the x_i's in $LL(y)$ are of multiplicity 1, the points (y, x_i) are of multiplicity 1 in $\mathcal{H}$ (by Prop. 4.16), which means the loops s_i are meridians. The discussion before Prop. 4.7 implies that these map to reflections in W.

Therefore $l_R(w) \leq i$ and $l_R(u) \leq n - i$, but since $l_R(c) = n$ both equalities hold, so $w \leq_R c$ and w is an element of $[1, c]_T = NC(W)$. □

We return to the context of the Main Conjecture. Let Θ be a hsop as in 3.2 and V^Θ the (finite) variety cut out by $(\Theta - \boldsymbol{x})$. Since V^Θ carries a $W \times C$ action, we can consider its quotient W/V^Θ in $W/V \cong Y \times \mathbb{C}$. Prop. 3.6, part(i) implies that only one point of W/V^Θ lies outside $\mathcal{H}$; call it $(y, x)^{\Theta, reg}$.

Recall now from (5.1) that each of the points $(y, x) \in V^\Theta \cap \mathcal{H}$ lives in the configuration $\{y\} \times LL(y)$ and is therefore assigned a simple element $b_{y,x}$ by the map ρ. We can extend this definition and naturally set $\text{Label}\big((y, x)^{\Theta, reg}\big) = e$, the identity element.[14]

We can finally ask for:

Main Conjecture. *[GR12] The map $\rho : W/V^\Theta \to NC(W)$ is a bijection.*

5.2.1. *Elaboration on the simple elements.* In fact, quite more is known about the structure of the set S of simple elements. It has an order $\leq$ that is given by a natural length function defined in terms of Braid reflections (Thm. 4.7) that makes it *isomorphic* to $NC(W)$:

Proposition 5.15. *[Bes15, Prop. 8.5]. The map $\pi : B(W) \twoheadrightarrow W$ restricts to an isomorphism $(S, \leq) \xrightarrow{\sim} ([1, c], \leq_T)$.*

The proof of this fact depends mainly on the Hurwitz action, Propositions 4.10 and 4.9 and the following enumerative result on the size of the set $\text{Red}_R(c)$, of reduced decompositions of c.

Proposition 5.16. *[Bes15, Prop. 7.6] Let W be a real (actually, well-generated complex) reflection group. Let c be a Coxeter element in W. The Hurwitz action is transitive on $\text{Red}_R(c)$. When W is irreducible, one has $|\text{Red}_R(c)| = n!h^n/|W|$.*

Indeed, recall that the map rlbl between the Galois orbit $y \cdot B(W)$ and the Hurwitz orbit $\text{rlbl}(y) \cdot B(W)$ is $B(W)$-equivariant (Prop. 5.12). Now surjectivity is implied by the transitivity of the Hurwitz action on $\text{Red}_R(c)$ and injectivity by the fact that $|\text{Red}_R(c)|$ is "big enough"; as big as the generic fiber of the LL-covering (Prop. 4.13).

Sadly, the proof of Prop. 5.16 involves quite a lot of case-by-case computations. The transitivity part is however uniform, see [Bes03, Prop. 1.6.1].[15]

6. Future Work

6.0.1. *The trivialization theorem.* The Hurwitz action and Prop. 5.16 have yet another fundamental consequence.

Definition 6.1. *[Bes15, Defn. 7.10] Let k be a positive integer. We set*

$$D_k(c) := \{(w_1, w_2, \cdots, w_k) \in W^k | c = w_1 \cdots w_k \text{ and } l_T(c) = \sum_i l_T(w_i)\},$$

$$D_\bullet(c) := (D_k(c))_{k \in \mathbb{Z}_{\geq 0}}$$

and call $D_\bullet(c)$ the set of block-factorizations of the Coxeter element c.

Theorem 6.2. *[Bes15, Thm 7.20 (trivialization of Y] The map $LL \times \text{rlbl}$ induces a bijection*

$$LL \times \text{rlbl} : Y \xrightarrow{\sim} E_n \boxtimes D_\bullet(c)$$

[14]This is compatible with our notion of "small loop around (y, x)" since $(y, x)^{\Theta, reg}$ is outside $\mathcal{H}$, so a small loop around it will be contractible, hence the identity element in $B(W)$.

[15]Recently we have a uniform proof [Mic14] for the size of $\text{Red}_R(c)$ in the Weyl case.

where $\boxtimes$ indicates a compatibility between the two partitions of n defined by the multiplicities in E_n, and the length function l_T in $D_\bullet(c)$.

The trivialization theorem allows us to consider an interesting stratification of $\mathcal{H}$, where each stratum has a constant image under the map $\rho : \mathcal{H} \to W$. To prove the Main Conjecture, we would have to show that there exists exactly one point of V^Θ in each stratum. This approach can be used to give a simpler proof of the (difficult) part (ii) of Prop. 3.6; in particular this proves the Gordon-Ripoll conjecture for $I_2(m)$.

6.0.2. *Chapoton's formula.* Chapoton observed the following formula for counting multichains in the noncrossing lattice:

Proposition 6.3. *[Rip12, Chapoton's formula] Let W be an irreducible, well-generated complex reflection group of rank n. Then, for any $p \in \mathbb{N}$, the number of multichains $w_1 \leq \cdots \leq w_p$ in the poset $NC(W)$ is equal to*

$$\mathrm{Cat}^{(p)}(W) = \prod_{i=1}^{n} \frac{d_i + ph}{d_i}$$

Again, our only understanding of this formula is case by case computation. Ripoll in [Rip12] used the Bessis-framework for $NC(W)$ to give a more geometric explanation to some special cases of Chapoton's formula. His techniques do not seem to generalize though and he mentions that "a more promising approach would be to understand globally Chapoton's formula as some ramification formula for the morphism LL".

6.0.3. *The Fuss case.* Recently [Rho14], Rhoades generalised the Main conjecture for the *Fuss-Catalan* case. He defined $W \times \mathbb{Z}_{kh}$ actions on the sets of chains of length k in $NC(W)$ and the corresponding k-**algebraic parking space** and k-**noncrossing parking space**. A proof of Rhoades' Fuss version will imply the Main Conjecture (with all its applications) and Chapoton's formula.

However, a Gordon-Ripoll style interpretation of the Main Conjecture in the Fuss case seems more complicated at this point. In particular it seems to be strongly connected with the "cyclic" $\mathbb{C}^\times$ action on V (as in [Bes15, Section 11]) that encompasses (via the Hurwitz action) all $\mathbb{Z}_{kh}$ actions on the Fuss-parking spaces $\mathrm{Park}(k)$.

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Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: `douvr001@umn.edu`