RESEARCH STATEMENT

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1. Introduction

My research area and classical training lies in Algebraic Combinatorics, and within that I am particularly fascinated by complex reflection groups. These groups W appear at the intersection of mathematical disciplines, and the resulting viewpoints suggest a number of natural lines of research. A central *motivating* theme in my work has been the study of structural and enumerative properties of the lattice $NC(W)$ of noncrossing partitions of W. This is pursued through several avenues: the combinatorics of hyperplane arrangements (§2), the representation theory of W (§4), and the algebraic (§5) and differential (§6) geometry of its space of orbits. Listed below, for key mathematical areas, are some of the objects, techniques, or problems, that play an important role in the research presented in this statement.

- *In combinatorics:*
  - Parking spaces and their W-module structure; enumeration of chains in $NC(W)$ (§2; Thm.2.1; Problems 1-3).
  - Hurwitz numbers and transportation polytopes, and W-analogs (§3; Thm.3.2; Conjectures 4-6; Problems 7,8).
  - Matrix-Tree theorem and Jucys-Murphy elements (§4.2; Thm.4.3) and Laplacians (§2) for reflection groups.
  - W-analogs of cacti formulas (§5.1; Problem 10); cyclic sieving phenomena (§5.2; Thm.5.3,5.4; Problem 11).
- *In representation theory:*
  - Enumeration of factorizations via the Frobenius lemma (§4); Hecke algebras and Coxeter numbers (§4.1).
  - The exterior powers of the reflection representation of W; the exotic Fourier transform (§4.2; Problem 9).
- *In algebraic geometry:*
  - Braid monodromy of algebraic functions (§5); enumeration via degree counting (Thm.5.2; Problem 10).
- *In differential geometry:*
  - Free multiplicities for hyperplane arrangements and the local-to-global formulas (§6.1; Conjecture 13).
  - Frobenius manifolds, quasi-Coxeter elements, algebraic solutions of WDVV equations (§6.2; Conj.14,Prob.15).
- *In geometric group theory:*
  - Geometric construction of cell complexes for generalized braid groups (§5.3; Problem 12).

An important aspect in the theory of complex reflection groups is their Shephard-Todd classification (subsuming Coxeter's classification in the real case). This has propelled the evolution of the subject with many results first proven via case-by-case arguments while a case-free explanation is pursued by the community. In that direction, we have given type-independent proofs and generalizations of many enumerative results, including the Chapuy-Stump (§4.1), Deligne-Arnold-Bessis (§2), and Chapoton (§2.2) formulas. Our most important contribution however is on the W-module structure of parking spaces, which we discuss below.

The module of parking functions of length n under a natural S_n-action, which has orbits indexed by Dyck paths, has been a central object in Algebraic Combinatorics since the work of Haiman more than 30 years ago. A vast line of research has spawned around it with the aim of extending the theory to reflection groups, leading to the introduction of the space of W-noncrossing parking functions as $\mathrm{Park}_W^{NC} := \oplus_{g\in NC(W)} \uparrow_{W_g}^{W} \mathbf{1}$ where the set of Dyck paths was replaced by $NC(W)$. As in the S_n-case, an *algebraic* parking space has also been defined as the quotient ring $\mathrm{Park}_W^{\mathrm{alg}} := \mathbb{C}[V]/(\mathbf{\Theta})$, where V is the ambient space of W and $\mathbf{\Theta}$ is an appropriate homogeneous system of parameters. One of the central open problems in the area, since the early 2000's, has been to give a *type-independent* proof that these two spaces are isomorphic and our main contribution is such a proof. We achieve this (in fact we prove the Fuss generalization) by combining a variety of new techniques described in §2.

Theorem (see §2)**.** *For any real reflection group W, the algebraic and combinatorial parking spaces are isomorphic W-modules; that is,* $\mathrm{Park}_W^{NC} \cong_W \mathrm{Park}_W^{\mathrm{alg}}$.

Another central line of research in our work involves the study of the Hurwitz numbers $H_g(\lambda)$. They count many equivalent objects, including (classes of) branched Riemann surfaces of genus g, and *transitive* factorizations of permutations with cycle type $\lambda \vdash n$ into transpositions t_i (meaning that the group $\langle t_i\rangle$ acts transitively on $[n] := \{1,\dots,n\}$). The genus-0 Hurwitz numbers for a partition $\lambda := (\lambda_1,\dots,\lambda_k)$ of n are given by the remarkable product formula

$$H_0(\lambda) = (n+k-2)!\cdot n^{k-3}\cdot\prod_{i=1}^{k}\frac{\lambda_i^{\lambda_i}}{(\lambda_i-1)!}. \tag{1}$$

We have two main contributions in this area. First, in a large project described in §3.2, we recast the notion of transitivity of factorizations in a way that makes sense for (complex) reflection groups and give a product formula that fully generalizes (1). Second, back in the symmetric group case, we give a formula for the generating function of the numbers $H_g\big((1^n)\big)$, where n is fixed and the genus g varies, in terms of the h-vector of a central transportation polytope and we conjecture a version for arbitrary λ which recovers (1) as a leading term (§3.1).

Although many results mentioned in the following sections appear enumerative in nature, our techniques come from different areas and may thus reveal non-trivial phenomena or connections between them. One especially telling example is the Laplacian for a hyperplane arrangement $\mathcal{A}$, which we introduced in [CD22b] generalizing the usual graph Laplacian. It is a weighted rank 1 operator acting on the ambient space V of the arrangement defined via

$$L_{\mathcal{A}}(\boldsymbol{\omega}) := \sum_{H\in\mathcal{A}} \omega_H \cdot (\mathbf{Id} - s_H), \tag{2}$$

where $\mathbf{Id}$ is the identity on V, s_H is an appropriately defined reflection across H, and $\boldsymbol{\omega} : (\omega_H)_{H\in\mathcal{A}}$ a collection of scalar weights. When $\mathcal{A}$ is a reflection arrangement and with special in each case choices of the weights $\boldsymbol{\omega}$, the eigenvalues of $L_{\mathcal{A}}(\boldsymbol{\omega})$ may encode (§4: Thm. 4.3) –reduced *and* of arbitrary length– factorizations of Coxeter elements, or (§2: Thm. 2.2, Thm. 2.3) identities between the Coxeter numbers of W and its subgroups that eventually describe the characters of parking spaces, or (§6.1, Probl. 13) the Hilbert series of logarithmic derivation modules associated to $\mathcal{A}$ and its restrictions.

Some of our work in factorization enumeration led us to prove results in the representation theory of complex reflection groups W; this includes Thm. 4.4 which is a new structural property of the (well-studied) exterior powers of the reflection representation of W. Separately, the algebraic geometric study of the Lyashko-Looijenga morphism (§5) –popularized by Arnold and further developed by Bessis, and whose degree calculation is a different exegesis of the product structure in formulas counting factorizations of Coxeter elements– helped us produce finer enumerative results (Thm. 5.2) and prove cyclic sieving phenomena (Thm. 5.3, 5.4), including one conjectured in Williams' thesis. Finally, via the theory of Frobenius manifolds we proposed a version of this degree-calculation for quasi-Coxeter elements and, in fact, our interpretation of the associated numerology led to new algebraic solutions of the WDVV equations (§6.2).

Many of the following sections end with a discussion of problems suggested by the new result; they are often amenable to division in partial goals and could form projects ranging from the level of a PhD theses to an REU.

2. Coxeter-Catalan combinatorics

Some of the most fascinating results about the symmetric group S_n are special cases of theorems that hold for all (finite) Coxeter groups W, or more generally complex reflection groups; this is the world of Coxeter combinatorics. Coxeter-Catalan combinatorics study in particular the poset of *W-noncrossing partitions* $NC(W) := [1, c]_{\leq_R}$ of elements $w \in W$ that lie below a Coxeter element c under the absolute order $\leq_R$ of W. It is a lattice, it has many applications outside of combinatorics –in particular it encodes $K(\pi, 1)$ spaces for the braid group $B(W)$, see §5– and when $W = S_n$, it is isomorphic to the lattice of noncrossing partitions due to Kreweras which are enumerated by the Catalan numbers.

The number $MC(W)$ of *maximal* chains of the noncrossing partition lattice $NC(W)$ is given by the *Deligne-Arnold-Bessis formula* $MC(W) = h^n n!/|W|$ where n is the rank and h the Coxeter number of W. In [CD22a] we gave the first Coxeter theoretic, type-inpendent proof of this formula by solving a recursion on $MC(W)$ due to Deligne and Reading, which counted maximal chains with respect to their last element. The success of this approach led us to apply this idea, of counting chains with respect to their last (or k-th) element, towards a much more general setting.

One of the main open problems in Coxeter-Catalan combinatorics (for more than twenty years [AR04, §7]) has been to explain, i.e. give type-independent proofs for, the remarkable product formula below for the enumeration of chains in $NC(W)$. Our main contribution in the area is the first such proof, for which we developed a framework of techniques in [Dou22a; DJ22; Dou], and which is finally presented at [DJ].

Theorem 2.1 (Athanasiadis-Reiner [AR04] and Rhoades [Rho17] via the classification, [DJ] uniformly)**.** *For any real reflection group W, the number $\mathrm{Krew}_{W,[X]}(m)$ of length-m chains in $NC(W)$ whose first element has parabolic type $[X]$ (given as a W-orbit of flats $[X] \in \mathcal{L}_W/W$) is given by the formula*

$$\mathrm{Krew}_{W,[X]}(m) = \frac{\prod_{i=1}^{\dim(X)}(mh + 1 - b_i^X)}{[N(X) : W_X]}, \tag{3}$$

where $N(X)$ and W_X are the setwise and pointwise normalizers of X, and b_i^X its Orlik-Solomon exponents.

There is a natural way to define parking functions associated to m-chains of $NC(W)$; they carry a natural W-action and the resulting module is called the *m-Fuss noncrossing parking space* $\mathrm{Park}_W^{NC}(m) := \oplus_{[X]} \mathrm{Krew}_{W,[X]}(m) \cdot \uparrow_{W_X}^W \mathbf{1}$. A sibling object to this space is the so called *algebraic parking space* $\mathrm{Park}_W^{\mathrm{alg}}(m) := \mathbb{C}[V]/(\boldsymbol{\Theta})$ introduced in [ARR15] as the quotient of the ambient polynomial ring over a system of parameters $\boldsymbol{\Theta} := (\theta_1, \ldots, \theta_n)$ of homogeneous degrees $\deg(\theta_i) = mh + 1$ that carry the reflection representation of W (the existence of such h.s.o.p. relies on Rouquier's shift functors for rational Cherednik algebras). The numbers given by the product formula of Theorem 2.1 are naturally structure coefficients for $\mathbb{C}[V]/(\mathbf{\Theta})$ so that the theorem can be equivalently phrased as the W-isomorphism between the two parking spaces; that is, we prove that $\mathrm{Park}_W^{NC}(m) \cong_W \mathrm{Park}_W^{\mathrm{alg}}(m)$.

2.1. **A comparison of recursions.** We prove Theorem 2.1 by expanding the ideas of [CD22a]. There is a natural recursion on the numbers $\mathrm{Krew}_{W,[X]}(m)$ if one counts length-m chains with respect to the parabolic type of their k-th element. The main ingredient of our proof is to show that the same recursion is satisfied by the right hand side of (3). Phrased in terms of the algebraic parking spaces, this becomes the following theorem, which we prove by comparing the characters of the two representations.

Theorem 2.2 ([Dou])**.** *For any natural numbers m, k, r such that $m = k + r$, we have the expansion formula*

$$\mathrm{Park}_W^{\mathrm{alg}}(m) = \bigoplus_{[X]\in\mathcal{L}_W/W} \mathrm{Krew}_{W,[X]}(k)\cdot \big\uparrow_{W_X}^{W} \mathrm{Park}_{W_X}^{\mathrm{alg}}(r). \tag{4}$$

The proof of Theorem 2.2 relies on our work on arrangement Laplacians (2) and their spectrum. We showed in [CD22b] that the characteristic polynomial of the $\mathcal{A}$-Laplacian $L_{\mathcal{A}}(\boldsymbol{\omega})$, for arbitrary weights $\boldsymbol{\omega}$, is given in terms of the Laplacians of the localizations $\mathcal{A}_Y$:

$$\det\big(t\cdot \mathrm{Id} + L_{\mathcal{A}}(\boldsymbol{\omega})\big) = \sum_{Y\in\mathcal{L}_{\mathcal{A}}} \mathrm{qdet}\big(L_{\mathcal{A}_Y}(\boldsymbol{\omega}_Y)\big)\cdot t^{\dim(Y)}. \tag{5}$$

In the setting of Theorem 2.2, we considered the restricted reflection arrangements $\mathcal{A}^X$ and a special selection of weights. For any hyperplane $Z\in\mathcal{A}^X$, the *relative Coxeter number $h(X,Z)$* is defined as the Coxeter number of the unique irreducible component of W_Z that does not belong to W_X. We prove in [Dou] that the recursion (5) for the arrangement $\mathcal{A}^X$ with weights $\omega_Z := h(X,Z)$, gives essentially the equality of characters for the two sides of (4).

From a different perspective, the equality between the structure coefficients in the two sides of (4) can be seen as a relation between Coxeter numbers and Orlik-Solomon exponents of a reflection arrangement $\mathcal{A}$ and its flats. We prove that the following positive expansion theorem is equivalent to the parking space recursion of Theorem 2.2. In §6.1 we give a conjectural interpretation for it in terms of special multi-derivation modules for the arrangements $\mathcal{A}^X$.

Theorem 2.3 ([Dou])**.** *For an irreducible real reflection arrangement $\mathcal{A}$ and a flat $X\in\mathcal{L}_{\mathcal{A}}$, we have that*

$$\prod_{i=1}^{\dim(X)}(t+mh+b_i^X) = \sum_{Y\in\mathcal{L}_{\mathcal{A}^X}} t^{\dim(Y)}\cdot \prod_{i=1}^{\dim(X)-\dim(Y)}\big(mh_i(X,Y)+b_i^{X,Y}\big),$$

where h and $h_i(X,Y)$ are Coxeter numbers and $b_i^X, b_i^{X,Y}$ Orlik-Solomon exponents for $\mathcal{A}$, $\mathcal{A}^X$, and $\mathcal{A}_Y^X$.

2.2. **The linear term.** The recursion that we described in §2.1 has a significant drawback: assuming knowledge of the chain counts $\mathrm{Krew}_{W',[X']}(m)$ for all previously considered intervals, it can determine all coefficients of the polynomial (in m) $\mathrm{Krew}_{W,[X]}(m)$ *apart from its linear term.* To remedy this we give a separate argument in which we show that the linear term is as prescribed by formula (3). This is technically difficult and combines our work in [DJ22] which relates the chain counts of Theorem 2.1 with a type-refined face enumeration in the cluster complex, with the following theorem (its formula is essentially the linear term in question) which we proved in a previous work by a double counting argument involving Crapo's beta invariant for matroids. The special case of the theorem for $X = V$ gives in fact the first type-independent proof of Chapoton's formula [Cha06] for the number of reflections of full support.

Theorem 2.4 ([Dou22a])**.** *In an irreducible real reflection group W, the set $\mathcal{G}_{\mathrm{se}}\big([X]\big)$ of parabolic subgroups of full support, that are simple extensions of some* standard *parabolic subgroup of type $[X]$, has size given by the formula*

$$\Big|\mathcal{G}_{\mathrm{se}}\big([X]\big)\Big| = \frac{2\cdot|\mathcal{A}^X|}{[N(X):W_X]}\cdot\prod_{i=2}^{\dim(X)}(b_i^X-1),$$

where $|\mathcal{A}^X|$ is the number of hyperplanes in $\mathcal{A}^X$ and b_i^X the Orlik-Solomon exponents of X.

2.3. **Future directions.** The two theorems in §2.1 are very suggestive of further research. Our proof of Theorem 2.2 did not make use of the graded module structure of the parking spaces. It is natural to ask for a q-version:

Problem 1. *Give a q-version of Theorem 2.2, for instance via Rouquier's shift functors for Cherednik algebras, or by generalizing the Lie-theoretic q-Kreweras numbers of [RS18], or via the freeness conjecture of §6.1.*

In Theorem 2.3 and when the flat X is the whole ambient space V, the left hand side agrees with the Poincare polynomial of the m-Fuss-Catalan deformation $\mathcal{A}^{[-m,m]}$ of $\mathcal{A}$ (which is non central and adds for each hyperplane $H\in\mathcal{A}$ an extra $2m$-many, parallel, equally spaced copies of it). In [Dou] we give a separate Ehrhart theoretic proof of Theorem 2.3 for $X = V$ relying on reciprocity theorems of Athanasiadis [Ath10]. The following problem would form an excellent PhD thesis while special cases of it (for instance restricting to the symmetric or hyperoctahedral groups S_n, B_n) would be great for senior or master projects. It is particularly amenable to computer experimentation.

Problem 2. *In Weyl groups W, generalize Athanasiadis' works [Ath04; Ath10] and construct deformations $\mathcal{A}^{X,\boldsymbol{m}}$ of the restricted arrangements $\mathcal{A}^X$ so that their resulting Poincare polynomials are given by the formulas*

$$P(\mathcal{A}^{X,\boldsymbol{m}},t)=\prod_{i=1}^{\dim(X)}(t+mh+b_i^X).$$

In recent work [Gal+22] a *rational* version of W-noncrossing partitions has been introduced, resolving another old open problem in the area. The authors gave type-independent proofs for the enumeration of these objects that recover the special case $X=V$ of our Theorem 2.1 (but their proof naturally produced a q-version for that case as well). The algebraic recursions we prove in Theorem 2.2 do generalize to that setting and it seems likely that the rational noncrossing partitions also satisfy combinatorial recursions analogous to the chain decomposition we described in §2.1. A natural next project is to combine the two techniques.

Problem 3. *Refine the combinatorial models for rational Catalan objects and generalize Theorem 2.1 in that setting.*

3. Hurwitz numbers in the symmetric group and in reflection groups

The Hurwitz numbers $H_g(\lambda)$ described in the introduction and enumerated by the product formula (1), have formed a very popular object of study in the last decades; the community has developed connections to representation theory, algebraic geometry, and combinatorics, but still there is much that is not understood. Our work focuses on two major questions; first in §3.1 for a generalization of the beautiful product formula (1) from genus $g=0$ to the arbitrary genus case (we answer this partially), and in §3.2 for a generalization of (1) to reflection groups (we give here, in a manner, a complete answer).

3.1. **Hurwitz numbers in the symmetric group and transportation polytopes.**

The ELSV formula [Eke+01] gives the Hurwitz numbers $H_g(\lambda)$ as integrals over the moduli spaces of stable curves $\overline{\mathcal{M}}_{g,n}$. This is a remarkable connection but still, in some sense, it fails to give a proper generalization of the product formula (1) (the ELSV integrals are computable only for small values of g). We proceed in a different direction, that views the numbers $H_g(\lambda)$ for all values of g simultaneously, by considering their generating function:

$$\mathcal{F}(\lambda;t):=\sum_{g\geq 0}H_g(\lambda)\cdot\frac{t^{n+k+2g-2}}{(n+k+2g-2)!}. \tag{6}$$

The interpretation of Hurwitz numbers as counting transitive transposition factorizations in the symmetric group, along with standard techniques in representation theory, implies that the generating function $\mathcal{F}(\lambda;t)$ above is expressible as a finite sum of exponentials e^{mt} with integer exponents m; i.e. as a Laurent polynomial on e^t. Our work in [Dou22c] and computer experimentation had suggested that these polynomials might have rigid expressions and that they exhibit a sort of unimodality on their coefficients.

With the following theorem, we explain this behavior for the cycle type $\lambda=(1^n)$ and in Problems 4, 5 we give explicit conjectures for the general case. The polynomials in question are the h-polynomials for certain simple polytopes (hence they are unimodal). The transportation polytope [DK14] denoted $T(p,q)$ is the set of all real $p\times q$ matrices with non-negative entries, all row sums equal to q, and all column sums equal to p; it is simple when $(p,q)=1$.

Theorem 3.1 ([CDL])**.** *The generating function $\mathcal{F}\big((1^n);t\big)$ of (6) for the Hurwitz numbers $H_g\big((1^n)\big)$ is given as*

$$\mathcal{F}\big((1^n);t\big)=\frac{e^{t\binom{n}{2}}}{n!}\cdot(1-e^{-t})^{2n-2}\cdot\Phi_n(e^{-t}),$$

where $\Phi_n(X)$ is the h-polynomial of the transportation polytope $T(n,n-1)$.

We proved this Theorem by comparing two recursions (on the index n) that are satisfied by $\mathcal{F}\big((1^n);t\big)$ and $\Phi_n(x)$; the first due to Okounkov and Dubrovin-Yang-Zagier [DYZ17] and the second due to Pak [Pak00]. Remarkably, extended calculations have suggested that the transportation polytope $T(n,n-1)$ in fact encodes all generating functions $\mathcal{F}(\lambda;t)$.

The relation is through the *S_n-equivariant Ehrhart theory* [Sta11] of the associated polytope $T_c^\circ(n,n-1)$, which is defined as the polar of the *centered* (at the origin) transportation polytope $T_c(n,n-1)$. While the usual Ehrhart theory studies the number $L(P,s)$ of lattice points in the s-th dilation of a polytope P, the equivariant Ehrhart theory keeps track of the permutation representation χ_{sP} induced by the action of some group on those lattice points in sP. In our case, the symmetric group S_n acts by permuting the rows of the matrices that form $T(n,n-1)$ and the equivariant Ehrhart series is formally defined as

$$\mathrm{Ehr}_{S_n,T_c^\circ(n,n-1)}(X):=\sum_{s\geq 0}\chi_s X^s, \tag{7}$$

where χ_s is the permutation representation of S_n on the lattice points of $s\cdot T_c^\circ(n,n-1)$. The coefficients of this formal power series are characters of S_n and can be evaluated on cycle types λ, so that (7) encodes in fact a different power series for each partition $\lambda\vdash n$. In the following statement we conjecture that they essentially agree with the generating functions $\mathcal{F}(\lambda;t)$ (note how the form of the denominator in (8) matches Stapledon's [Sta11] set-up of the theory).

Conjecture 4. *The generating functions $\mathcal{F}(\lambda;t)$ of (6) for the Hurwitz numbers $H_g(\lambda)$ are given via*

$$\mathrm{Ehr}_{S_n,T_c^\circ(n,n-1)}\big(\lambda;X\big)=\frac{n!\cdot X^{\binom{n}{2}}\cdot\mathcal{F}(\lambda;\log X)}{(1-X)\cdot\det\big(\mathrm{Id}-X\cdot\rho(\lambda)\big)},\tag{8}$$

where $\rho:S_n\to\mathrm{GL}\left(\mathbb{C}^{n\cdot(n-1)}\right)$ is induced by the S_n action on the ambient space of $T(n,n-1)$ described earlier.

Understanding the S_n-equivariant theory of the polytope $T_c^\circ(n,n-1)$ is equivalent to understanding the usual Ehrhart theory of its fixed subpolytopes $\big(T_c^\circ(n,n-1)\big)^g$ by elements $g\in S_n$. These are also related to transportation polytopes and often have unimodular triangulations so that their Ehrhart theory is completely encoded in their (topological) h-vectors; in particular, this is how our Theorem 3.1 becomes a special case of Conj. 4. In the equal cycles case $\lambda=(d^k)$ this gives a simpler version of the previous conjecture.

Conjecture 5. *The generating functions $\mathcal{F}\big((d^k);t\big)$ of (6) for the Hurwitz numbers $H_g\big((d^k)\big)$ are given via*

$$\mathcal{F}\big((d^k);t\big)=\frac{e^{t\binom{n}{2}}}{n!}\cdot(1-e^{-td})^{n+k-2}\cdot\Phi_{k,d}(e^{-td}),\tag{9}$$

where $\Phi_{k,d}(X)$ is the h-polynomial of the (simple, central) transportation polytope $T(k,kd-1)$.

Theorem 3.1 above and the following two conjectures generalize the Hurwitz formula (1) in a novel and meaningful way. The transportation polytopes that appear have vertices indexed by certain labeled trees and taking the leading terms of the (proven or claimed) expressions for $\mathcal{F}(\lambda;t)$ comes down to product formulas involving the number of such trees (since the h-vectors are statistics on the vertices). On the other hand, Duchi-Poulalhon-Schaeffer [DPS14] have proven the genus-0 Hurwitz formula via a bijective argument that relates the factorization counts with exactly such collections of labeled trees.

Apart from the remarkable connection they reveal between Hurwitz numbers and transportation polytopes, the polynomials $\Phi_n(X)$ of Theorem 3.1 have a very interesting root behavior. In a different project [DLM22a], we made the following conjecture for them. It can possibly be reduced to the quadratic recursion satisfied by $\Phi_n(X)$ but, even so, any conceptual justification for it would be highly desirable.

Conjecture 6 ([DLM22a])**.** *As n approaches infinity, the roots of the polynomial $\Phi_n(X)$ tend to the unit circle.*

3.2. **Hurwitz numbers for complex reflection groups.** The combinatorial interpretation of the Hurwitz numbers $H_g(\lambda)$ as counting transposition factorizations in S_n invites the problem of finding a generalization for Coxeter groups W, where transpositions are replaced by reflections. This question became particularly popular after Chapuy-Stump [CS14] who showed that the reflection factorizations of Coxeter elements $c\in W$ have a similar enumerative structure with the transitive transposition factorizations of the long cycle $(12\cdots n)\in S_n$, for any genus g.

Despite many attempts [BGJ08; PR21; LM21] there was no satisfying answer outside the combinatorial families (types $S_n=A_{n-1},B_n,D_n$ or $G(m,p,n)$ in the complex case). A main difficulty was that for an arbitrary reflection group W, there does not always exist a set on which W acts and that plays the role of $[n]=\{1,2,\dots,n\}$ in S_n, and therefore there is no natural way to define *transitive* factorizations. In [DLM22a] we resolved this issue by defining *full factorizations* as those reflection factorizations $t_1\cdots t_k=g$ in W whose terms generate the *full* group (i.e. $\langle t_i\rangle=W$); the two notions are equivalent in the symmetric group.

In a series of papers [DLM22a] and [DLM22b] and [DLM], joint with Lewis-Morales, we prove the following theorem which can be seen as an almost term by term generalization of the Hurwitz formula (1). It addresses the wide class of parabolic Coxeter elements (we also prove a version for the quasi-Coxeter case) which includes all the elements in the symmetric group S_n. One factor in the formula is the cardinality of the collection $\mathrm{RGS}(W,g)$ of *relative generating sets* of W with respect to g; these are sets of reflections that when combined with a reduced factorization of g give a system of $\mathrm{rank}(W)$-many reflections *that generate* W. The combinatorial object $\mathrm{RGS}(W,g)$ generalizes the tree-like structures that appear in the work of Duchi-Poulalhon-Schaeffer [DPS14] for the usual Hurwitz numbers $H_0(\lambda)$.

Theorem 3.2 ([DLM21])**.** *Let W be a Weyl group and $g\in W$ a parabolic Coxeter element fixing a flat $X\in\mathcal{L}_W$, and let $W_X=W_1\times\cdots\times W_k$ be the decomposition into irreducibles. If $\mathrm{RGS}(W,g)$ is the set of reflection generating sets relative to g, then the number $\mathcal{F}_W^{\mathrm{full}}(g)$ of minimum-length full reflection factorizations of g is given as*

$$\mathcal{F}_W^{\mathrm{full}}(g)=(2n-\sum_{i=1}^k n_i)!\cdot\big|\,\mathrm{RGS}(W,g)\big|\cdot\frac{\prod_{i=1}^k I(W_i)}{I(W)}\cdot\prod_{i=1}^k\frac{h_i^{n_i}n_i!}{|W_i|},\tag{10}$$

where h_i and n_i are respectively the Coxeter numbers and ranks of W_i and $I(W)$ denotes the connection index of W.

We prove this theorem in a type-by-type fashion and by separately computing the two sides of (10). For the combinatorial types, we first [DLM22a] give the left side in terms of the Hurwitz numbers of S_n and then [DLM22b] we calculate the right side by enumerating the tree-like structures that form the sets $\mathrm{RGS}(W,g)$. The exceptional types are dealt with via (very heavy) computer calculations. In [DLM] we in fact prove a version of Theorem 3.2 for (well generated) complex reflection groups. The connection indices $I(W), I(W_g)$ are replaced in the general formula by a Grammian statistic on the relative generating sets that is only constant in Weyl groups.

The main open problem arising in this work is to give a conceptual explanation of this remarkable product structure in (10). We develop in [DLM] analogs of the *cut and join* combinatorics of Hurwitz and Goulden-Jackson that could possibly be used towards a uniform proof. Moreover, in [DL22] we prove a Hurwitz transitivity result for full factorizations that might help explain formula (10) via a k-to-1 map over a special collection of factorizations that are essentially shuffles of reduced factorizations of an element g and the factors in an element of $\mathrm{RGS}(W,g)$.

Problem 7. *Give a case-free proof of Thm. 3.2 or its generalizations (to complex types and quasi-Coxeter elements).*

Another highly interesting pursuit related to this project is to generalize the setting of 3.1 to reflection groups. In [DLM22a] we listed analogs of the generating function $\mathcal{F}\big((1^n);t\big)$ for all reflection groups W (with (1^n) replaced by the identity element of W). In all cases, the expression was very similar to Theorem 3.1.

Problem 8. *Construct a geometric object analogous to $T(n,n-1)$ that generalizes Theorem 3.1 to reflection groups.*

4. Representation theory: techniques and interaction with enumeration

A standard approach in the enumeration of factorizations in groups is via applying a representation theoretic lemma that goes back to Frobenius and was in fact used already by Hurwitz. We are still working with reflection groups W, with set of reflections $\mathcal{R}$, and for an arbitrary element $g\in W$ we wish to understand the generating function

$$\mathrm{FAC}_{W,g}(t) := \sum_{\ell\geq 0} \#\big\{(\tau_1,\cdots,\tau_\ell)\in\mathcal{R}^\ell : \tau_1\cdots\tau_\ell = g\big\}\cdot\frac{t^\ell}{\ell!}. \tag{11}$$

The Lemma of Frobenius states that we can express this function as a finite sum of character evaluations:

$$\mathrm{FAC}_{W,g}(t) = \frac{1}{|W|}\cdot\sum_{\chi\in\widehat{W}}\chi(1)\cdot\chi(g^{-1})\cdot\exp\big(t\cdot\widetilde{\chi}(\mathcal{R})\big), \tag{12}$$

where $\widehat{W}$ denotes the set of irreducible characters of W and $\widetilde{\chi}(\mathcal{R})$ is the normalized trace $\sum_{\tau\in\mathcal{R}}\chi(\tau)/\chi(1)$. Recently, this technique proved effective for enumerating Coxeter factorizations in this beautiful theorem due to Chapuy-Stump.

Theorem 4.1. *[CS14] For a (duality) rank n reflection group W and a Coxeter element $c\in W$ of order h,*

$$\mathrm{FAC}_{W,c}(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\cdot(1-e^{-th})^n.$$

4.1. **A uniform proof and generalization of the Chapuy-Stump formula.** The original proof of Thm. 4.1 proceeded based on the Frobenius lemma (12) but had to rely on the classification of complex reflection groups and their characters. Both because it implies the Deligne-Arnold-Bessis formula of §2 and due to its intrinsic elegance, there was an effort in the community to produce a case-free proof with a first success only for Weyl groups [Mic16], while we do the general case in [Dou22c]. The difficulty to apply the lemma of Frobenius (12) in a type-independent manner stems from the case-by-case construction of the irreducible characters of W. To circumvent this, we use Malle's cyclic action on $\widehat{W}$ that is induced by a Galois automorphism in the Hecke algebra, to group together characters that share an integer invariant c_χ, related to Lusztig's c-function, called the *Coxeter number* of χ. This allows us to discard from the summation in (12) those $\chi\in\widehat{W}$ for which c_χ is not a multiple of $h:=|c|$.

Our argument relies only on the fact that the Coxeter element c lifts to a root of the full twist in the braid group $B(W)$ and hence can be applied to all *regular* elements $g\in W$. The previous construction in conjuction with combinatorial restrictions on the leading term of $\mathrm{FAC}_{W,c}(t)$ allows us to prove the following structural result which recovers and extends the Chapuy-Stump formula (Thm. 4.1) and with little more effort [Dou22c, § 5] also gives a uniform proof for the weighted case studied in [dHR18].

Theorem 4.2. *[Dou22c] For a complex reflection group W and any regular element $g\in W$, one has*

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|}\cdot\Big[(1-X)^{l_R(g)}\cdot\Phi_g(X)\Big]\Big|_{X=e^{-t|g|}},$$

where $\Phi_g(X)$ is a polynomial in X of degree $\frac{|\mathcal{R}|+|\mathcal{A}|}{|g|}-l_R(g)$ and constant term equal to 1.

In the case of a Coxeter element c, the polynomials $\Phi_c(X)$ are forced to have degree 0 by *combinatorial* considerations. This holds further whenever $|g| = d_n$, which produces explicit formulas that do not appear in [CS14] or [dHR18]. In general it seems difficult to control the $\Phi_g(X)$ but we have had some success with S_n, as we discussed in §3.1.

4.2. **Weighted factorizations with generalized Jucys-Murphy weights.**
There is a beautiful derivation of the type-A Hurwitz number $H_0\big((n)\big) = n^{n-2}$ –a special case of (1)– which is due to Dénes and proceeds by relating transposition factorizations of the long cycle $(12\dots n) \in S_n$ to labeled trees. Of the many ways to count trees, the approach using the Matrix-Tree theorem and the Laplacian of the complete graph K_n allows us to assign weights ω_{ij} on each edge (i,j); in the factorization side, this means assigning weights on the transpositions (ij). Burman and Zvonkine [BZ10] proved a striking higher-genus analog of this by providing a product formula for the generating function of weighted factorizations that involved the *eigenvalues* of the (weighted) Laplacian.

With Chapuy we extended their work to all (duality) reflection groups W with Thm. 4.3. Unfortunately, it turned out that arbitrary weight assignments did not lead to product formulas; we considered instead special weight functions $\mathbf{w}_T : \mathcal{R} \to \boldsymbol{\omega} := (\omega_i)_{i=1}^n$ indexed by towers of parabolic subgroups $T := \big(\{\mathbf{1}\} = W_0 < W_1 < \cdots < W_n = W\big)$.

These $\mathbf{w}_T$ are defined by the filtration of $\mathcal{R}$ by T; that is, for a reflection $\tau \in \mathcal{R}$ we have $\mathbf{w}_T(\tau) = \omega_i$ if and only if $\tau \in W_i \setminus W_{i-1}$. We are interested in the exponential generating function $\mathrm{FAC}_W^T(t, \boldsymbol{\omega})$ of weighted reflection factorizations of *any* element c of the Coxeter class $\mathcal{C}$, an analog of (11):

$$\mathrm{FAC}_W^T(t,\boldsymbol{\omega}) := \sum_{\ell\geq 0} \frac{t^\ell}{\ell!} \cdot \Big(\sum_{\substack{(\tau_1,\cdots,\tau_\ell,c)\in\mathcal{R}^\ell\times\mathcal{C} \\ \tau_1\cdots\tau_\ell=c}} \mathbf{w}_T(\tau_1)\cdots\mathbf{w}_T(\tau_\ell)\Big). \tag{13}$$

Thm. 4.3 gives a product formula for (13) which generalizes the Chapuy-Stump formula of Thm. 4.1. If $L_W^T(\boldsymbol{\omega})$ denotes the Laplacian (2) with weights $\mathbf{w}_T$, its eigenvalues are weighted analogs of the Coxeter number h. In this sense, the equality of the two leading terms below may be considered as a (weighted) version of the Deligne-Arnold-Bessis formula of §2 and thus as a Matrix-Tree theorem for W.

Theorem 4.3. *[CD22b] For a (duality) reflection group W the weighted enumeration* (13) *is given by*

$$\mathrm{FAC}_W^T(t,\boldsymbol{\omega}) = \frac{e^{t\mathbf{w}_T(\mathcal{R})}}{h} \cdot \prod_{i=1}^n \big(1 - e^{-t\lambda_i^T(\boldsymbol{\omega})}\big),$$

where $\mathbf{w}_T(\mathcal{R}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)$, and the $\lambda_i^T(\boldsymbol{\omega})$ are the eigenvalues of the Laplacian $L_W^T(\boldsymbol{\omega})$.

In the process of proving Thm. 4.3 we produce a generalization of the Frobenius Lemma (12) for any group G where the elements of a generating conjugacy class $\mathcal{G}$ are weighted via an arbitrary tower of subgroups. Heavily influenced by the work of Okounkov and Vershik [OV96], we consider in the group algebra $\mathbb{C}[W]$ generalized Jucys-Murphy elements $J_i := \sum_{\tau\in\mathcal{R}\cap W_i\setminus W_{i-1}} \tau$. For any parabolic tower T, they generate a commutative subalgebra $\mathbb{C}[\boldsymbol{J}^T]$ and the weighted enumeration is given in terms of its spectrum.

The product structure of the formula comes down to a connection with the exterior powers of the reflection representation V_{ref}. We say that two virtual characters χ and ψ are *tower-equivalent* if they agree on the subalgebras $\mathbb{C}[\boldsymbol{J}^T]$ for any choice of parabolic tower T. Then Thm. 4.3 is equivalent with the following:

Theorem 4.4. *[CD22b] The virtual characters $\sum_{\chi\in\widehat{W}} \chi(c^{-1})\cdot\chi$ and $\sum_{k=0}^n (-1)^k \bigwedge^k(V_{\mathrm{ref}})$ are tower-equivalent.*

We prove this theorem by computer calculation for the exceptional types and an inductive argument, which involves working out some non-trivial Littlewood-Richardson coefficients, for the infinite families. Michel [Mic22] later gave a remarkable proof for it for Weyl groups; he showed that the (truncated) exotic Fourier transform of any virtual character χ is *tower-equivalent* to χ and that the exterior algebra of V_{ref} is precisely the transform of $\sum_{\chi\in\widehat{W}} \chi(c^{-1})\cdot\chi$. Remarkably our notion of *Tower equivalence* seems to agree precisely with the kernel of the exotic Fourier transform in Weyl groups. Given the importance of this construct also in the recent work [Gal+22] (see Problem 3) a natural question is as follows.

Problem 9. *Further explore the relation between tower equivalence and the Fourier transform; give a type-independent proof of Thm. 4.4 for* all *duality groups W.*

In [CD22b] we in fact give a W-Matrix-*Forest* theorem for the whole characteristic polynomial of $L_W^T(\boldsymbol{\omega})$. This is done by combining Thm. 4.3 with the parabolic recursions for Laplacians discussed in (5). In this way, the eigenvalues of the Laplacian encode simultaneously arbitrary length factorizations of Coxeter elements and reduced length factorizations of parabolic Coxeter elements.

5. Braid monodromy of discriminant hypersurfaces

A cornerstone for much of the study of real reflection groups W is the chamber decomposition of the ambient space V induced by the arrangement of reflection hyperplanes $\mathcal{A}_W := \bigcup H$. Over the complex field, where such a decomposition cannot exist, a similar role is played by the quotient variety $\mathcal{H} := W \setminus \bigcup H$ which is known as the *discriminant hypersurface* of W. In the seminal work [Bes15] Bessis exploits the *braid monodromy* of $\mathcal{H}$ (albeit in the guise of the following "Trivialization Theorem") to prove a long-standing conjecture: the complement $V \setminus \bigcup H$ is a $K(\pi, 1)$ space (see also §5.3).

The braid monodromy of an algebraic function g is a refinement of its usual monodromy group: it keeps track of *how* the function values move around each other, when we vary the coefficients of g, as opposed to just recording their final permutation. To define it one usually chooses a generic direction z, for which $g = z^n + a_1(\boldsymbol{y}) \cdot z^{n-1} + \cdots + a_n(\boldsymbol{y})$ and treats the variety $V(g)$ as a branched cover over $Y := \mathrm{Spec}(\mathbb{C}[\boldsymbol{y}])$. If $\mathcal{K}$ is the branch locus, the coefficient map $\boldsymbol{a}(\boldsymbol{y})$ determines a representation of $\pi_1(Y \setminus \mathcal{K})$ into the usual braid group of n strands B_n which we call the braid monodromy of g as in [Han89; CS97].

For complex reflection groups W, the Shephard-Todd-Chevalley theorem identifies the quotient space V/W as the affine complex space $\mathbb{C}^n$ whose coordinates are given by the fundamental invariants $\boldsymbol{f} := (f_i)_{i=1\cdots n}$ of W. In the subclass of duality groups (which possess Coxeter elements and include all real reflection groups) the highest degree invariant f_n plays a special role; in particular, the equation for the discriminant hypersurface $\mathcal{H}$ is monic *and of degree* n with respect to f_n. Central in Bessis' work, the *Lyashko-Looijenga* map $LL(\boldsymbol{y})$ is essentially the coefficient map for the braid monodromy of $\mathcal{H}$ along the f_n direction (with parameter $\boldsymbol{y} \in Y := \mathrm{Spec}\big(\mathbb{C}[f_1, \cdots, f_{n-1}]\big)$).

5.1. The trivialization theorem and refined chain enumeration by parabolic type.

A geometric interpretation of the LL map (and any coefficient map) is that it records the intersections of complex lines $L_{\boldsymbol{y}} := \boldsymbol{y} \times \mathbb{C}$, parallel to the direction of f_n, with the discriminant hypersurface $\mathcal{H}$. Bessis considers loops that surround $\mathcal{H}$ only inside these lines $L_{\boldsymbol{y}}$ and constructs in this way well-defined elements of the generalized braid group $B(W) := \pi_1(V/W - \mathcal{H})$. Taking advantage of the canonical short exact sequence $1 \to P(W) \to B(W) \to W \to 1$, he extends this to a labeling map $\mathrm{lbl}(\boldsymbol{y})$ that sends $\boldsymbol{y} \in Y$ to a tuple of elements of W. The topological construction of the labeling map heavily restricts the resulting tuples and if $\boldsymbol{e} := LL(\boldsymbol{y}) = L_{\boldsymbol{y}} \cap \mathcal{H}$ is an image of the LL map, which is by definition a collection of points in $\mathbb{C}$ with total multiplicity n, Bessis proves this remarkable *Trivialization Theorem*:

Theorem 5.1. *[Bes15] The points in a fiber $LL^{-1}(\boldsymbol{e})$ are in a natural bijection via the labeling map with chains in the noncrossing lattice whose rank jumps are given by the multiplicities in $\boldsymbol{e}$.*

Chains in the noncrossing lattice $NC(W)$ correspond to length-additive factorizations of a Coxeter element c, so that the trivialization theorem suggests a geometric way to enumerate such collections. In particular, maximal chains correspond precisely to reduced reflection factorizations and thus the Deligne-Arnold-Bessis formula in §2 should agree with the degree of the LL map. To produce refined enumerative results, one must study the restriction of the LL map on the branch locus $\mathcal{K} \subset Y$. The discriminant hypersurface $\mathcal{H}$ is stratified by orbits of flats $[X] \in \mathcal{L}_{\mathcal{A}_W}/W$ and their projections $[X]_Y$ on the base space Y completely cover $\mathcal{K}$. By studying the local behavior of the LL and lbl maps on these constructible sets $[X]_Y$, we prove the following.

Theorem 5.2. *[Dou22b] The number of length-additive factorizations of a Coxeter element $c \in W$ of the form $w \cdot \tau_1 \cdots \tau_k = c$, with τ_i's reflections and w of parabolic type $[X]$, is given by the formula $h^k k!/[N_W(X) : W_X]$.*

Our techniques are in the same spirit as methods initiated by Arnold [Arn96] and used extensively by singularity theorists thereafter (even to some extent in the celebrated ELSV formula). One tries to lift the restriction of the map to an affine space, where it becomes quasi-homogeneous and hence its degree can be calculated via Bezout's theorem. The term $[N(X) : W_X]$ that appears in our formula is exactly the degree of such a lift.

Now, for any length additive factorization $\sigma := (w_1 \cdots w_k = c)$, we define its passport $\big(\boldsymbol{Z}\big) := \big([Z_1, \cdots, Z_k]\big)$ as the tuple of parabolic types $[Z_i]$ of the w_i. An ambitious task would then be to compute the number $\mathrm{Fact}_W[(\boldsymbol{Z})]$ of such factorizations σ with given passport $(\boldsymbol{Z})$. Lando and Zvonkine [ZL99] derive the Goulden-Jackson formula

$$\mathrm{Fact}_{S_n}\big[(\boldsymbol{Z})\big] = n^{l-1} \cdot \prod_{i=1}^{l} \frac{k_i!}{[N(Z_i) : W_{Z_i}]}, \tag{14}$$

via a geometric analysis of the LL map on the space of monic degree n polynomials (which realizes V/W when W is the symmetric group). For other reflection groups, (case-by-case) formulas of Krattenthaler and Müller [KM10] suggest a similar structure for certain passports. We describe in [Dou22b, Sec. 7] a complete stratification of Y by constructible sets $Y_{\{\boldsymbol{Z}\}}$ indexed by passports, which are often precisely the intersections of the strata $[Z_i]_Y$ we used for Thm. 5.2. We relate the enumeration problem with the local geometry of the LL map on those and ask:

Problem 10. *Find a uniform geometric extension, for suitable $(\boldsymbol{Z})$, of formula* (14) *to other reflection groups.*

5.2. **A cyclic sieving phenomenon.**
The cyclic sieving phenomenon (CSP) [RSW04] occurs when a polynomial $X(q)$ carries orbital information about the action of a cyclic group C on a space X. More precisely, and if C is generated by an element c of order n, we say that the triple $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon* if for all integers d, the number of elements of X fixed by c^d equals the evaluation $X(\zeta^d)$, where $\zeta = e^{2\pi i/n}$.

The set $\mathrm{Red}_W(c)$ of reduced reflection factorizations of a Coxeter element c supports many natural cyclic actions. The operation $\mathfrak{Pro}$ below may be realized as the Hurwitz action (16) of a particular braid and has order hn:

$$\mathfrak{Pro} : (\tau_1, \cdots, \tau_n) \to (c\tau_n c^{-1}, \tau_1, \cdots, \tau_{n-1}).$$

Williams conjectured the following CSP for $\mathfrak{Pro}$ which we proved by exploiting the geometry of the trivialization theorem. Via the labeling map lbl, we interpret $\mathfrak{Pro}$ as a scalar action on fibers $LL^{-1}(\boldsymbol{e})$ for certain symmetric point configurations $\boldsymbol{e}$. The polynomial $X(q)$ arises then as the Hilbert series of the special fiber $LL^{-1}(\mathbf{0})$:

Theorem 5.3. *[Dou18] For a (duality) reflection group W, with invariant degrees $d_1, \cdots, d_n$ and $\mathrm{Red}_W(c)$ as above, the triple $\Big(\mathrm{Red}_W(c), \prod_{i=1}^{n} \dfrac{[ih]_q}{[d_i]_q}, \langle \mathfrak{Pro} \rangle\Big)$, where $[m]_q := \dfrac{1-q^m}{1-q}$, exhibits the cyclic sieving phenomenon.*

Some CSP's are proven by direct calculation of the orbit sizes and perhaps lack a satisfying explanation for the appearance of the polynomial $X(q)$. In our case, the geometry of the LL map not only resolves this but it also provides an example where the same polynomial $X(q)$ encodes CSP's for different cyclic actions on X. By choosing configurations $\boldsymbol{e}$ with different cyclic symmetries, we obtain for example a CSP with C of order $h(n-1)$.

For some passports, the enumeration of factorizations is given by the degree of a quasi-homogeneous morphism. In those cases too this method will work although there are fewer candidates for symmetric fibers. We describe in [Dou22b, § 5.3.1] what happens for the factorizations of Thm. 5.2 and ask in general:

Problem 11. *Extend Thm. 5.3 over sets of block factorizations with prescribed passports (see § 5.1).*

A particularly nice sub-case of Problem 11 is when the passport consists of only cycles of the same length k; that is, when $Z(k) = (k, 1^{n-k})$ and $(\boldsymbol{Z}_k) := \big([Z(k), Z(k), \ldots, Z(k)]\big)$, with $Z(k)$ appearing $p := (n-1)/(k-1)$ times. Then, the Goulden-Jackson formula (14) states that $\mathrm{Fact}_{S_n}\big[(\boldsymbol{Z})\big] = n^{p-1}$. In an REU project, we proved with Justin Bailey (UMass undergrad, now PhD student in USC) the following cyclic sieving phenomenon.

Theorem 5.4 ([BD])**.** *For the passport $(\boldsymbol{Z}_k)$ defined above, the triple $\Big(\mathrm{Fact}_{S_n}\big[(\boldsymbol{Z}_k)\big], \prod_{i=2}^{p} [n]_{q^i}, \langle \mathfrak{Pro} \rangle\Big)$ exhibits the CSP.*

5.3. **The Brady complex after Bessis.**
In his proof of the $K(\pi,1)$ conjecture Bessis uses the noncrossing lattice $NC(W)$ as a combinatorial recipe for building the universal covering space of the discriminant complement $V/W - \mathcal{H}$. The procedure is quite complicated and Bessis recently proposed a simplification [Bes16]. The idea is to construct a cell model for $V/W - \mathcal{H}$, via the trivialization theorem, and hope that its combinatorics leads to a cleaner proof of the $K(\pi,1)$ property.

Bessis' model involves first a retraction that pushes the configurations of points inside a fixed circle and then proceeds by lifting the natural cell structure there to $V/W - \mathcal{H}$ via the LL map. On the other hand, there is already a combinatorial $K(\pi,1)$ model for the braid group $B(W)$ defined by Brady (but which is not a priori homeomorphic to $V/W - \mathcal{H}$). It is the quotient of the order complex of $NC(W)$ where we identify the chains $(w_1, \cdots, w_k)$ and $(e, w_1^{-1}w_2, \cdots, w_1^{-1}w_k)$. The labeling map lbl is compatible with Bessis' retraction in a way that suggests:

Problem 12. *Bessis' cell complex for the discriminant complement $V/W - \mathcal{H}$ is isomorphic to the Brady complex.*

6. Differential geometry of reflection groups

The high symmetry of reflection arrangements $\mathcal{A}_W$ has led to the construction of rigid geometries on their ambient spaces V and the quotients V/W. We explore here two directions, one on the module of logarithmic derivations associated to $\mathcal{A}$ and one on the Frobenius manifold structure of the quotient space V/W.

6.1. **Free multiplicities for restricted reflection arrangements.** For a hyperplane arrangement $\mathcal{A}$ in some space $V \cong \mathbb{C}^n$, its module of logarithmic derivations $D(\mathcal{A})$ is defined as the ring of polynomial vector fields that are tangent to *all* hyperplanes of $\mathcal{A}$. When $D(\mathcal{A})$ is a free module over the ambient algebra $\mathbb{C}[V]$ we say that $\mathcal{A}$ itself is *free* and we call the degrees of the generators of $D(\mathcal{A})$ the *exponents* of $\mathcal{A}$. Free arrangements include all supersolvable, all reflection, and all restricted reflections arrangements, and they have particularly nice numerological properties: their characteristic polynomials are products of linear factors, i.e. $\chi(\mathcal{A}, t) = \prod(t - d_i)$ and the d_i are the exponents of $\mathcal{A}$.

Ziegler generalized this notion by introducing *multiarrangements $(\mathcal{A}, \boldsymbol{m})$*, where each hyperplane $H \in \mathcal{A}$ comes equipped with a multiplicity $\boldsymbol{m}(H) \in \mathbb{Z}_{\geq 0}$ and the module of derivations $D\big((\mathcal{A}, \boldsymbol{m})\big)$ consists of those vector fields that have order of tangency $\boldsymbol{m}(H)$ on each hyperplane H. The notion of freeness generalizes analogously and it is an important problem to classify free multiplicities for a given arrangement, and determine the corresponding exponents.

When $\mathcal{A}_W$ is a real reflection arrangement, works of Solomon, Terao, and Yoshinaga have shown that all *constant* multiplicity functions $\boldsymbol{m} : \mathcal{A}_W \to \mathbb{Z}_{\geq 0}$ determine free multi-arrangements. Moreover, for a given number $m \in \mathbb{Z}_{\geq 0}$ the free multiplicities $\boldsymbol{m}^*(H) := 2m$ and $\boldsymbol{m}^{**}(H) := 2m+1$ give exponents $\exp\big((\mathcal{A}_W, \boldsymbol{m}^*)\big) = \{mh, \dots, mh\}$ and $\exp\big((\mathcal{A}_W, \boldsymbol{m}^{**})\big) = \{mh+e_1, \dots, mh+e_n\}$, where h is the Coxeter number of W and the e_i are given via the invariant degrees d_i of W as $e_i := d_i - 1$.

Separately, Abe-Terao-Wakefield [ATW07] have proven an analog of Brieskorn's localization lemma, which they called the *local to global formulas* that relates the exponents of a free multi-arrangement $(\mathcal{A}, \boldsymbol{m})$ with those of its (necessarily free) localizations $(\mathcal{A}_X, \boldsymbol{m}_X)$. In the case of the multi-arrangement $(\mathcal{A}_W, \boldsymbol{m}^{**})$ this implies the relation

$$\prod_{i=1}^{n}(t + mh + e_i) = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} t^{\dim(X)} \cdot \prod_{i=1}^{n-\dim(X)} \big(mh_i(W_X) + e_i(W_X)\big), \tag{15}$$

which is a special case (for $X = V$) of our Theorem 2.3. It is natural then to ask whether the full case of our Theorem 2.3 suggests the existence of free multiplicities for the *restricted* arrangements $\mathcal{A}_W^X$. The following conjecture, which encompasses all previously mentioned results would achieve exactly this; if the multiplicities given in Problem 13 are indeed free, the local to global formulas for them give precisely Theorem 2.3. Moreover this would be, to our knowledge, the first uniformly defined free multiplicities that are not constant or almost constant.

Conjecture 13 ([Dou])**.** *Let $\mathcal{A}$ be an irreducible real reflection arrangement, $X \in \mathcal{L}_{\mathcal{A}}$ one of its flats and let $m \in \mathbb{Z}_{\geq 0}$. For the restricted arrangement $\mathcal{A}^X$ define two multiplicity functions $\mathfrak{m}^*$ and $\mathfrak{m}^{**}$ on the hyperplanes $Z \in \mathcal{A}^X$ via*

$$\mathfrak{m}^*(Z) := m \cdot h(X, Z) \qquad \text{and} \qquad \mathfrak{m}^{**}(Z) := m \cdot h(X, Z) + 1,$$

where $h(X, Z)$ is as in §2.1. Then the multi-arrangements $(\mathcal{A}^X, \mathfrak{m}^*)$ and $(\mathcal{A}^X, \mathfrak{m}^{**})$ are free with exponents

$$\exp\big((\mathcal{A}^X, \mathfrak{m}^*)\big) = \{\underbrace{mh, \dots, mh}_{\dim(X)\text{-many}}\} \qquad \text{and} \qquad \exp\big((\mathcal{A}^X, \mathfrak{m}^{**})\big) = \{mh + b_i^X \mid i = 1, \dots, \dim(X)\},$$

where h is the Coxeter number of $\mathcal{A}$ and b_i^X the Orlik-Solomon exponents of X.

We know that the Conjecture of Problem 13 is true in various instances; when $X = V$ it agrees with the theorems of Solomon, Terao, Yoshinaga mentioned earlier, when $X = H \in \mathcal{A}$ we prove it in [Dou] as an easy consequence of the work of [ATW08], and when W is the symmetric group S_n, it is a rephrasing of the main result of [ANN09]. The general case can be pursued in at least two ways; either by constucting a local version of the so called *primitive derivation* extending Yoshinaga's work [Yos02], or by applying the deletion-restriction theorems of [ATW08] on divisional flags. The latter case applies especially to the hyperoctahedral groups B_n and would make a very good PhD thesis project. Moreover, because of known connections between deformations of rational arrangements and the corresponding multi-arrangements [Yos04], even a partial answer to this problem would be able to resolve Problem 2.

6.2. **Frobenius manifolds and Quasi-Coxeter elements.** The theory of Frobenius manifolds was developed by Dubrovin to give a coordinate-free formulation of the WDVV equations from 2D topological field theory. In it, a Frobenius algebra structure is specified on any tangent plane T_xM of a manifold M and its structure coefficients encode the WDVV associativity equations for a prepotential F.

The quotient varieties V/W for real reflection groups W form an important class of Frobenius manifolds. For them the algebra structure is defined via a special choice of fundamental invariants, known as *Saito flat coordinates* that provide an Euclidean metric for the orbit space V/W. Dubrovin conjectured [Dub99] and Hertling later proved that, in fact, these are the only examples of Frobenius manifolds with associated *polynomial* prepotentials.

In his classification [Dub99, Lect. 4] of massive Frobenius manifolds Dubrovin encodes the local algebra structure in a Stokes matrix or equivalently a tuple of euclidean reflections $\boldsymbol{\tau} := (\tau_1, \cdots, \tau_n)$, while he describes its analytic continuation via the Hurwitz action of the Braid group B_n on $\boldsymbol{\tau}$:

$$B_n \ni \sigma_i * (\tau_1, \cdots, \tau_n) = (\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}^{-1}\tau_i\tau_{i+1}, \cdots, \tau_n). \tag{16}$$

Then *algebraic* prepotentials correspond to tuples $\boldsymbol{\tau}$ with finite Hurwitz orbits and Dubrovin asks for the construction of the corresponding Frobenius manifolds. After work of Michel [Mic06] however, finite Hurwitz orbits occur if and only if $\boldsymbol{\tau}$ generates a reflection group, so that the problem of algebraic Frobenius manifolds in some sense lives entirely in the world of finite Coxeter groups.

From a different viewpoint [Bau+17] study the Hurwitz action of B_n on the set $\mathrm{Red}_W(g)$ of reduced reflection factorizations of an element $g \in W$ and show that when g is quasi-Coxeter, i.e. when there is no proper reflection subgroup $W' \lneq W$ that contains it, then the action is transitive. Reduced tuples $\boldsymbol{\tau}$ always determine a quasi-Coxeter element $g := \prod_{i=1}^{n} \tau_i$ of the group $W' = \langle \boldsymbol{\tau} \rangle$ and we denote the possible corresponding Frobenius manifold by $F(g)$.

Stump calculated the sizes of the (single orbit) sets $\mathrm{Red}_W(g)$ for quasi-Coxeter elemenets g and discovered that they always factor in small primes. He asked if there is an explanation for this or even a generalization of the Deligne-Arnold-Bessis formula of §2. In [DLM22b] we discuss how a lot of the relevant geometric objects of §5 appear in the theory of Frobenius manifolds as well. In particular there is a version of the LL map, which relates two natural coordinate systems of F_g; it sends the flat coordinates, on which the prepotential is given, to (the elementary symmetric polynomials of) the canonical coordinates, which are the eigenvalues in the algebra structure of the multiplication by the Euler field. Given the prepotential, it is easy to calculate the degree of the LL map; this and Dubrovin's construction described previously suggest the following:

Conjecture 14 ([DLM])**.** *For a quasi-Coxeter element g, assuming F_g exists, we have that* $\deg\big(LL(F_g)\big) = |\,\mathrm{Red}_W(g)|$.

In the case of Weyl groups W and *regular* quasi-Coxeter elements $g \in W$ Dinar [Din21] has constructed the Frobenius manifolds F_g and shown that the weights of the flat coordinates are given by $(e_i(g)+1)/|g|$, where the *exponents* $e_i(g)$ determine the eigenvalues $e^{2\pi i e_i(g)/|g|}$ of g. Because the LL map is weighted-homogeneous, this would give its degree as the right hand side of (17) where d_g would be viewed as the *algebraicity degree* of the Frobenius prepotential. In [DLM22b] we prove the following enumerative result (also a generalization of the Deligne-Arnold-Bessis formula of §2) which should be seen as significant evidence for the conjecture.

Proposition 6.1 ([DLM22b])**.** *For a regular quasi-Coxeter element g in a crystallographic group W we have that*

$$|\,\mathrm{Red}_W(g)| = \frac{|g|^n n!}{\prod_{i=1}^n (e_i(g)+1)} \cdot d_g, \tag{17}$$

where the exponents $e_i(g)$ are defined as above and d_g is a small integer given (using Carter's notation) by:

$g \in W$	$D_{2n}(n-1)$	$F_4(1)$	$E_6(1)$	$E_6(2)$	$E_7(1)$	$E_7(4)$	$E_8(1)$	$E_8(2)$	$E_8(3)$	$E_8(5)$	$E_8(6)$	$E_8(8)$
d_g	n	*3*	*2*	*5*	*2*	$2\cdot 3^2$	*2*	*3*	2^3	*7*	$2^2\cdot 5$	$3^3\cdot 5$

Indeed, in all the cases that algebraic Frobenius manifolds have been constructed our interpretation of the numbers d_g is confirmed. Applied in the opposite direction, this enumerative data can be exploited to guess solutions to the WDVV equations. Sekiguchi [Sek19] has been succesful in doing so in small dimensions using the information from our calculations with Stump. We state here Dubrovin's refinement of his original conjecture:

Problem 15 (Dubrovin)**.** *Construct an algebraic Frobenius manifold F_g for any quasi-Coxeter element g.*

6.3. **Applications on the trivialization theorem.** For the symmetric group S_n, the trivialization theorem (Thm. 5.1) is equivalent to Riemann's existence theorem while more generally for types A-D-E the LL map may be interpreted as the morphism that sends a deformation of a simple singularity to its set of critical values. Currently the proof of Thm. 5.1 relies on the numerological coincidence between the degree of the LL map and the chain number $MC(W) = h^n n!/|W|$ of §2 (we explain this in [CD22a]). However, Hertling and Roucairol [HR18] prove an equivalent version for simple singularities by exploiting the Frobenius structure. We ask to extend their approach to (duality) reflection groups:

Problem 16. *Give a case-free conceptual proof of Thm. 5.1 that does not rely on the numerological coincidence.*

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