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Recursions and Proofs in Cataland

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Abstract. We give the first type-independent proof of the Kreweras-style formulas for the enumeration of noncrossing partitions in a real reflection group W, with respect to parabolic type. This answers a central open question in Coxeter-Catalan combinatorics, originally asked by Athanasiadis-Reiner in 2003, special cases of which have been open even longer. Our proof also covers the m-Fuss version of the problem, as well as similar Loday-style formulas for the refined-by-type enumeration of faces of the m-cluster complex of W. It proceeds by developing a family of combinatorial recursions that completely determine the enumeration and proving their algebraic counterparts.

Keywords: noncrossing partitions, cluster complex, Coxeter, Catalan, Kreweras, Loday

1 Introduction

Some of the most fascinating results about the symmetric group S_n are special cases of theorems that hold for all (finite) Coxeter groups W, or more generally complex reflection groups; this is the world of Coxeter combinatorics. *Coxeter-Catalan* combinatorics in particular study the poset of *W-noncrossing partitions* $NC(W) := [1, c]_{\leq_R}$ of elements $w \in W$ that lie below a Coxeter element c under the absolute order $\leq_R$ of W. It is a lattice, it has many applications outside of combinatorics –in particular it encodes $K(\pi, 1)$ spaces for the braid group $B(W)$– and when $W = S_n$, it is isomorphic to the lattice of noncrossing partitions due to Kreweras which are enumerated by the Catalan numbers.

One of the main open problems in Coxeter-Catalan combinatorics (for more than twenty years [3, §7]) has been to explain, i.e. give case-free proofs (ones that do not rely on the classification of Coxeter groups) for the remarkable product formula (1.1) below for the enumeration of chains in $NC(W)$. In this extended abstract, we present the first such proof, for which we developed a framework of techniques in [10, 12, 9], and which is finally presented at the full version of this paper [11]. In fact, we simultaneously prove these chain-counting formulas *and* analogous product formulas (1.2) that constitute refinements of the f-vectors of cluster complexes; we had proved the latter ones in a case-by-case way in [12]. In both cases the formulas are indexed by a *parabolic type*, which is a W-orbit of flats $[X] \in \mathcal{L}_W/W$ in the intersection lattice of the reflection arrangement of W, see §2.

Theorem 1.1. *For an irreducible real reflection group W and a type $[X] \in \mathcal{L}_W/W$, the number $\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m)$ of length-m chains in $NC(W)$ whose first element has type $[X]$ is given by*

$$\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m) = \frac{\prod_{i=1}^{\dim(X)}(mh+1-b_i^X)}{[N(X):W_X]}, \tag{1.1}$$

and the number $\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m)$ of faces of type $[X]$ in the cluster complex $\Upsilon(W,m)$ is given by

$$\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m) = \frac{\prod_{i=1}^{\dim(X)}(mh+1+b_i^X)}{[N(X):W_X]}, \tag{1.2}$$

where $N(X)$ and W_X are respecively the setwise and pointwise stabilizers of X, b_i^X its Orlik-Solomon exponents, and h the Coxeter number of W.

There is a version of the theorem for reducible groups W as well, which is an immediate consequence of it, but we will avoid it here to keep the notation simple. The terminology we chose for these numbers stands for Kreweras and Loday who respectively in [15] and [16] calculated them for the symmetric group $W = S_n$ and for $m = 1$.

There is a natural way to define parking functions associated to m-chains of $NC(W)$; they carry a natural W-action and the resulting module (see [2, 18]) is called the *m-Fuss noncrossing parking space* and is given as

$$\mathrm{Park}^{NC}_W(m) := \bigoplus_{[X]\in\mathcal{L}_W/W} \mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m)\cdot \uparrow^W_{W_X} \mathbf{1}.$$

A sibling object to it is the so called *algebraic parking space* $\mathrm{Park}^{\mathrm{alg}}_W(m) := \mathbb{C}[V]/(\mathbf{\Theta})$ introduced in [2] as the quotient of the ambient polynomial ring over a system of parameters $\mathbf{\Theta} := (\theta_1,\ldots,\theta_n)$ of homogeneous degrees $\deg(\theta_i) = mh+1$ that carry the reflection representation of W (the existence of such h.s.o.p. relies on Rouquier's shift functors for rational Cherednik algebras). The numbers given by the product formula (1.1) of Theorem 1.1 are naturally structure coefficients for $\mathbb{C}[V]/(\mathbf{\Theta})$ so that the theorem can be equivalently phrased as the W-isomorphism between the two parking spaces; that is, we prove that $\mathrm{Park}^{NC}_W(m) \cong_W \mathrm{Park}^{\mathrm{alg}}_W(m)$. This is a big part of a series of open problems, listed as *Parking Space Conjectures* in [2]; our approach however cannot currently deal with the full version of them which includes an extra cyclic structure on the modules.

While this project was in preparation, Galashin et al [14] resolved another long open problem in Coxeter-Catalan combinatorics and defined *rational* noncrossing partitions for real reflection groups and proved enumerative formulas for them. Among their results is a case-free proof of the special case $X = V$ of Thm. 1.1 for *Weyl* groups. It is unclear at the moment whether their techniques can be extended to our setting; they involve certain character evaluations in the Hecke algebra that are uniformly understood when it comes to lifts of the Coxeter element (the $X = V$ case) but not (yet) in general. On the other hand, the algebraic recursions we describe in Proposition 3.4 are given essentially in the rational setting; if they are combined with a refined model of rational noncrossing partitions that keeps track of the parabolic type, they may be able to extend the *full* Theorem 1.1 to the rational case.

A comparison of recursions

We prove Theorem 1.1 by expanding the ideas of [7]. There is a natural recursion on the numbers $\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m)$ if one counts length-m chains with respect to the parabolic type of their k-th element. The main ingredient of our proof is to show that the same recursion is satisfied by the right hand side of (1.1). To facilitate this, we will formally define the *algebraic Kreweras numbers* $\mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m)$ and the *algebraic Loday numbers* $\mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m)$ as

$$\mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m) := \frac{\prod_{i=1}^{\dim(X)}(mh+1-b_i^X)}{[N(X):W_X]} \quad \text{and} \quad \mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m) := \frac{\prod_{i=1}^{\dim(X)}(mh+1+b_i^X)}{[N(X):W_X]}, \tag{1.3}$$

where again $N(X)$ and W_X are respecively the setwise and pointwise stabilizers of X, b_i^X its Orlik-Solomon exponents, and h the Coxeter number of W, see §2 for details. The name stems from the fact that the numbers $\mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m)$ are naturally structure coefficients of the algebraic parking space [2] but we will not use that here.

In the following sections we prove a series of combinatorial recursions that relate the numbers $\mathrm{Krew}^{\mathrm{NC}}_{W,[W]}(m)$ and $\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m)$ with corresponding Kreweras and Loday numbers for smaller rank groups W' or parabolic types $[Z]$ of smaller dimension ($\dim(Z) \lneq \dim(X)$). We prove a sufficient part of those recursions for the algebraic Kreweras and Loday numbers as well and we show inductively in §6 that

$$\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m) = \mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m) \quad \text{and} \quad \mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m) = \mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m), \tag{1.4}$$

for all W, $[X]$, and m, which is Theorem 1.1.

The linear term

The recursion on chains that we described above (and that we present in §3 in detail) has a significant drawback: assuming knowledge of the chain counts $\mathrm{Krew}^{\mathrm{NC}}_{W',[X']}(m)$ for all previously considered groups and parabolic types, it can determine all coefficients of the polynomial (in m) $\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m)$ *apart from its linear term*. To remedy this we build a separate recursion in which we show that the linear term is as prescribed by formula (1.1). This relies on our work in [12] which relates the (combinatorial) Kreweras numbers with the Loday numbers, which themselves have a natural combinatorial recursion

(Prop. 4.1). A technically difficult part of this project is to prove the algebraic version (Lemma 5.2) of the recursion for the Loday numbers; we only need to do it for the linear coefficient but even for only this, the proof requires many non-trivial steps.

In almost all the statements we give sketches of the proof arguments but we try to encompass all the components that go into them. The full version of this project, with complete proofs, will appear in [11].

2 Real reflection groups and their invariants

A real reflection group is a finite subgroup $W \leq \mathrm{GL}(V)$ of invertible linear transformations of some space $V \cong \mathbb{R}^n$, that is generated by Euclidean reflections. The arrangement of the fixed hyperplanes of the reflections of W is called the *reflection arrangement* $\mathcal{A}_W$ of W and the rank of W is defined as the rank of $\mathcal{A}_W$. We say that W is *irreducible* if it leaves no non-trivial subspace of V invariant. Arbitrary intersections $X := \cap_{i\in I} H_i$ of the reflection hyperplanes $H_i \in \mathcal{A}_W$ are called *flats* and they form the intersection lattice $\mathcal{L}_W$ of W; their pointwise stabilisers W_X are called *parabolic subgroups*; they act as reflection groups on their *moved spaces* $M := X^{\perp}$. The group W acts on the lattice $\mathcal{L}_W$ and its orbits $[X] \in \mathcal{L}_W/W$ form the *parabolic types* of W; they generalize the notion of *cycle type* in S_n.

The real reflection groups W are precisely the finite Coxeter groups; they come with a system of simple generators $S := \{s_1, \dots, s_n\}$ (where n is the rank of W) that correspond to the reflecting hyperplanes bounding a selected chamber of $\mathcal{A}_W$. The product of the simple generators $c := s_1 \cdots s_n$ and any element conjugate to it will be called a *Coxeter element* of W and its order $h := |c|$ the *Coxeter number* of W.

Any parabolic subgroup W_X is conjugate to a *standard parabolic subgroup* $\langle J\rangle$ generated by a subset $J \subseteq S$. The parabolic type of a subset $J \subseteq S$ is the parabolic type of $\langle J\rangle$. We will make use of the numbers $N_{X,Y}$ which count the number of subsets $I \subseteq S$ of parabolic type $[X]$ that are contained in a *fixed* subset $J \subseteq S$ of parabolic type $[Y]$.

We define the (absolute) *reflection length* $\ell_R(w)$ of an element $w \in W$ as the smallest number k for which we can write a factorization $w = t_1 \cdots t_k$ in reflections t_i. Then we have a natural order $\leq_R$ on W where $u \leq_R v$ if and only if $\ell_R(u) + \ell_R(u^{-1}v) = \ell_R(v)$; we call it the *absolute reflection order* of W. We define the *noncrossing partition lattice* $NC(W)$ as the interval $[1, c]_{\leq_R}$ in W below a fixed Coxeter element c under $\leq_R$. Every element $w \in W$ has a parabolic type defined as the W-orbit of its fixed space $[V^w] \in \mathcal{L}_W/W$; this allows the definition of the combinatorial Kreweras numbers in Theorem 1.1.

There is an associated object to $NC(W)$ called the *m-cluster complex* $\Upsilon(W, m)$; it is a simplicial complex whose vertices are indexed by m-colored almost positive roots [13] and whose faces correspond to sets of pairwise compatible vertices (under a certain relation). In [12] we define a parabolic type to each face F of $\Upsilon(W, m)$; essentially this involves taking the product of the reflections that make up F, in an appropriate order, and then taking the Kreweras complement of that product. This allows us to define the Loday numbers in Theorem 1.1.

Coxeter numbers and Orlik-Solomon exponents associated to flats

To each flat $X \in \mathcal{L}_W$ we may associate various numerical invariants that play an important role in this paper. The restricted reflection arrangements $\mathcal{A}^X$ are always free and therefore their characteristic polynomials factor with positive integer roots

$$\chi(\mathcal{A}^X, t) = \prod_{i=1}^{\dim(X)} (t - b_i^X),$$

which are known as the *Orlik-Solomon exponents* b_i^X of X. When $X = V$, the exponents are given as $b_i^V = d_i - 1$, in terms of the *invariant degrees* d_i of W.

To each pair of flats $X \supset Z$ we may assign a sequence of $\big(\dim(X) - \dim(Z)\big)$-many Coxeter numbers as follows. Assume that $W_Z = W_1 \times \cdots \times W_r$ is the decomposition into irreducibles and that h_i are the Coxeter numbers of the W_i's. Write also X_i for the intersections of X with the moved spaces M_i of the W_i (i.e. $W_{X_i} = W_i \bigcap W_X$). Then we write $\{h_i(X, Z)\}$ $i = 1, \ldots, \big(\dim(X) - \dim(Z)\big)$ for the *multiset of Coxeter numbers*:

$$\{h_i(X, Z)\} := \{ \underbrace{h_1, \ldots, h_1}_{\dim(X_1)\text{-times}}, \ldots, \underbrace{h_r, \ldots, h_r}_{\dim(X_r)\text{-times}} \}.$$

Similarly we will need to consider the Orlik-Solomon exponents $b_i^{X,Z}$ of the arrangements $\mathcal{A}_Z^X$; we make sure to order the Coxeter numbers and the OS exponents consistently. A special case will be particularly important. For any hyperplane $Z \in \mathcal{A}^X$, the *relative Coxeter number $h(X, Z)$* is defined as the Coxeter number of the unique irreducible component of W_Z that does not belong to W_X; it is equal to $h_1(X, Z)$ with the previous notation. Generalizing a result of Orlik-Solomon-Terao [17] it can also be given as $h(X, Z) = |\mathcal{A}_Z| - |\mathcal{A}_Z^H| + 1$ for any hyperplane H such that $H \cap X = Z$.

3 Recursions for Kreweras numbers

We prove in this section identical recursions for the combinatorial (Prop. 3.1) and algebraic (Prop. 3.4) Kreweras numbers.

Proposition 3.1. *For a well generated complex reflection group W, parabolic type $[X] \in \mathcal{L}_{\mathcal{A}_W}/W$, and for any positive integers m, k, r such that $m = k + r$, we have that*

$$\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m) = \sum_{\substack{[Y] \in \mathcal{L}_W/W \\ \dim(Y) \leq X}} \mathrm{Krew}^{\mathrm{NC}}_{W_Y,[X]}(k) \cdot \mathrm{Krew}^{\mathrm{NC}}_{W,[Y]}(r).$$

Sketch. This is just the result of counting length-m chains $(\mathbf{1} \leq_R w_1 \leq \ldots \leq_R w_m \leq_R c)$ in $NC(W)$ such that $V^{w_1} \sim X$ with respect to the parabolic type $[Y] := [V^{w_k}]$ of their k-th element w_k. □

Remark 3.2. In the right hand side of Prop. 3.1, the number $\mathrm{Krew}^{\mathrm{NC}}_{W_Y,[X]}$ is equal to 0 whenever $\dim(Y) \geq X$ apart from the case that $Y = X$. This is important in the proof of the main result in §6.

The proof of the following algebraic recursion (Prop. 3.4) is a core ingredient of this paper and relies on previous work on arrangement Laplacians and their spectrum. It was shown in [8] that the characteristic polynomial of the $\mathcal{A}$-Laplacian $L_{\mathcal{A}}(\boldsymbol{\omega})$, for arbitrary weights $\boldsymbol{\omega}$, is given in terms of the Laplacians of the localizations $\mathcal{A}_Y$. In the setting of Theorem 1.1, the restricted reflection arrangements $\mathcal{A}^X$ and a special selection of weights were considered in [9]. The Laplacian recursion for the restrictions $\mathcal{A}^X$ with weights $\omega_Z := h(X, Z)$ –see §2– gives essentially the following Lemma 3.3.

Lemma 3.3 ([9])**.** *For any complex reflection arrangement $\mathcal{A}$ and flat $X \in \mathcal{L}_{\mathcal{A}}$, we have that*

$$\prod_{i=1}^{\dim(X)} (t + kh + b_i^X) = \sum_{Y \subseteq X} \prod_{i=1}^{\dim(X)-\dim(Y)} \left(kh_i(X,Y) + b_i^{X,Y}\right) \cdot t^{\dim(Y)},$$

where $h, h_i(X,Y), b_i^X, b_i^{X,Y}$ are Coxeter numbers and OS-exponents associated to $\mathcal{A}^X$ and $\mathcal{A}_Y^X$.

We are now ready to prove the analogous recursion of Prop. 3.1 for the *algebraic* Kreweras numbers. Notice that here we do not require that W be real.

Proposition 3.4. *For a well generated complex reflection group W, parabolic type $[X] \in \mathcal{L}_{\mathcal{A}_W}/W$, and for any positive integers m, k, r, such that $m = k + r$, we have that*

$$\mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m) = \sum_{\substack{[Y] \in \mathcal{L}_W/W \\ \dim(Y) \leq X}} \mathrm{Krew}^{\mathrm{alg}}_{W_Y,[X]}(k) \cdot \mathrm{Krew}^{\mathrm{alg}}_{W,[Y]}(r).$$

Sketch. Rewriting the summation in terms of flats as opposed to orbits of flats, it is enough to show –compare with (1.3)– that

$$\prod_{i=1}^{\dim(X)} (mh + 1 - b_i^X) = \sum_{Y \subseteq X} \prod_{i=1}^{\dim(X)-\dim(Y)} \left(kh_i(X,Y) + 1 - b_i^{X,Y}\right) \cdot \prod_{i=1}^{\dim(Y)} (rh + 1 - b_i^Y).$$

If we now set $t = -rh - 1$ and replace k with $-k$, this becomes

$$\prod_{i=1}^{\dim(X)} (t + kh + b_i^X) = \sum_{Y \subseteq X} \prod_{i=1}^{\dim(X)-\dim(Y)} \left(kh_i(X,Y) - 1 + b_i^{X,Y}\right) \cdot \prod_{i=1}^{\dim(Y)} (t + b_i^Y), \tag{3.1}$$

which is equivalent with Lemma 3.3 after a Möbius inversion described in [9, §3]. □

4 Interdependence between Kreweras and Loday numbers

In this section we present relations between the Kreweras and Loday numbers that allow the complete determination of one family from the other. These relations hold identically in the combinatorial (Prop. 4.1) and algebraic (Prop. 4.3) setting.

Proposition 4.1. *For a real reflection group W and parabolic type $[X] \in \mathcal{L}_{\mathcal{A}_W}/W$, we have that*

$$\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m) = \sum_{\substack{[Y]\in\mathcal{L}_W/W \\ \dim(Y)\leq X}} N_{X,Y} \cdot \mathrm{Krew}^{\mathrm{NC}}_{W,[Y]}(m).$$

Sketch. This is proven in detail in [12, Prop. 8.1] which builds upon the work of [6]. We relate faces of type $[X]$ in $\Upsilon(W,m)$ with chains in $NC(W)$ that start at a *standard* parabolic element of type X but must then continue first with another standard parabolic (this explains the term $N_{X,Y}$). □

In [12, §10] we discuss how this statement should be seen as a refinement of an f-to-h transformation, perhaps due to a shelling of the cluster complex $\Upsilon(W,m)$ via the chains in $NC(W)$ generalizing the work of Athanasiadis-Tzanaki in [4].

The following Proposition 4.3 was originally a conjecture of Drew Armstrong (personal communication). It can be proven in many ways; for Weyl groups it has an interpretation via Ehrhart theory and the inside-out polytopes of Beck and Zaslavsky [5], in [12] we reduce it to a representation-theoretic description of tensoring with the sign character (see [12, Lem. 3.4, Lem. 5.3, Lem. 7.3]) but here we present a proof that is solely based in hyperplane arrangement theory. The following Lemma 4.2 is known as Kung's identity (see [1]) and is immediate after expanding the characteristic polynomials and standard manipulations of the Möbius functions.

Lemma 4.2. *For a hyperplane arrangement $\mathcal{A}$, and parameters s,t, we have*

$$\chi(\mathcal{A},st) = \sum_{Y\in\mathcal{L}_{\mathcal{A}}} \chi(\mathcal{A}_Y,s)\cdot\chi(\mathcal{A}^Y,t).$$

Proposition 4.3. *For a real reflection group W and parabolic type $[X] \in \mathcal{L}_{\mathcal{A}_W}/W$, we have that*

$$\mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m) = \sum_{\substack{[Y]\in\mathcal{L}_W/W \\ \dim(Y)\leq X}} N_{X,Y} \cdot \mathrm{Krew}^{\mathrm{alg}}_{W,[Y]}(m).$$

Sketch. Rewriting the summation in terms of flats of the reflection arrangement $\mathcal{A} := \mathcal{A}_W$ instead of orbits of flats, it is enough to show –compare with (1.3)– that

$$\prod_{i=1}^{\dim(X)} (mh+1+b_i^X) = \sum_{Y\subseteq X} c(\mathcal{A}_Y^X)\cdot \prod_{i=1}^{\dim(Y)} (mh+1-b_i^Y),$$

where $c(\mathcal{A}_Y^X)$ denotes the number of chambers in $\mathcal{A}_Y^X$. This is precisely Kung's identity (Lemma 4.2) for the arrangement $\mathcal{A}^X$ after setting $t = mh+1$ and $s=-1$. □

5 Recursions for Loday numbers

In this section, we return to Loday numbers and present identical recursions for them; combinatorial ones (Prop. 3.1) and algebraic ones (Prop. 3.4). We start with the combinatorial case which generalizes the Fomin-Reading recursion [13, Prop. 8.3].

Proposition 5.1 ([12, Prop. 11.3])**.** *For a real reflection group W, with arrangement $\mathcal{A}$ and simple generators S, and for any parabolic type $[X] \in \mathcal{L}_{\mathcal{A}}/W$ with $\dim(X) = k$, we have that*

$$\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m) = \sum_{s\in S} \frac{mh+2}{2k} \cdot \mathrm{Lod}^{\mathrm{NC}}_{W_{(s)},[X]\cap W_{(s)}}(m).$$

Sketch. This is proven in detail in [12, Prop. 11.3]; it relies on a double counting of pairs (F, v) of faces F of $\mathrm{Y}(W, m)$ and vertices $v \in F$. □

The previous Proposition can be extended to a double counting of pairs (F, K) where the face F has type $[X]$ and the face $K \subseteq F$ has fixed dimension; in that case the factor $(mh + 2)/2k$ will also be replaced by a Loday number $\mathrm{Lod}^{\mathrm{NC}}_{W,[K]}(m)$.

The next lemma is the most technically complicated part of this project. Proposition 5.1 is the only combinatorial recursion for which we cannot uniformly prove the full algebraic analog. However, we will only need the linear term case (see §6). To simplify the presentation we prove only the case $X = V$ but the same argument works in general.

Lemma 5.2. *For a real reflection group W, with arrangement $\mathcal{A}$ and simple generators S, and for any parabolic type $[X] \in \mathcal{L}_{\mathcal{A}}/W$ with $\dim(X) = k$, we have that*

$$[m^1]\left(\mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m)\right) = [m^1]\left(\frac{mh+2}{2k} \cdot \sum_{s\in S} \mathrm{Lod}^{\mathrm{alg}}_{W_{(s)},[X]\cap W_{(s)}}(m)\right).$$

Sketch for $X = V$. Again, we can rewrite the summation as one over flats and then we would have to show –compare with (1.3)– that

$$[m^1]\left(\prod_{i=1}^{n}(mh+d_i)\right) = [m^1]\left(\frac{mh+2}{n} \cdot \sum_{L\in\mathcal{L}^1_{\mathcal{A}}} \prod_{i=1}^{n-1}(mh_i(W_L) + d_i(W_L))\right),$$

where the sum is over 1-dimensional flats L. We can inductively assume the result to be true for the parabolic subgroups W_L and the above equation becomes then equivalent to

$$[m^1]\left(n! \cdot \prod_{i=1}^{n}(mh+d_i)\right) = [m^1]\left(\sum_{\mathcal{F}}\left(\prod_{i=1}^{n}(mh_i(\mathcal{F})+2)\right)\right),$$

where the summation is over all complete flags $\mathcal{F} = (\{\mathbf{0}\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = V)$, with $X_i \in \mathcal{L}_{\mathcal{A}}$ and $\dim(X_i) = i$, and where $h_i(\mathcal{F})$ is the relative Coxeter number

$h(X_i, X_{i-1})$ for the pair of flats X_i and X_{i-1} in $\mathcal{F}$. It is easy to extract the linear terms from both sides of the equation; we must now show that

$$n! \cdot h \cdot |W| \sum_{i=1}^{n} \frac{1}{d_i} = 2^{n-1} \cdot \sum_{\mathcal{F}} \sum_{i=1}^{n} h_i(\mathcal{F}), \tag{5.1}$$

where we have used that $|W| = d_1 \cdots d_n$.

Any pair of flats $Z \subseteq X$, with $\dim(X) = \dim(Z) + 1$ can be extended to a complete flag $\mathcal{F}$. Moreover, by [8, Prop. 4.3] the number of complete flags in a simplicial arrangement $\mathcal{K}$ of rank n equals $c(\mathcal{K}) \cdot n!/2^n$ where $c(\mathcal{K})$ denotes the number of chambers of $\mathcal{K}$. Since in our case both arrangements $\mathcal{A}_X$ and $\mathcal{A}^Z$ are simplicial, there will be $c(\mathcal{A}_X) \cdot \operatorname{codim}(X)!/2^{\operatorname{codim}(X)}$ (respectively $c(\mathcal{A}^Z) \cdot \dim(Z)!/2^{\dim(Z)}$) complete flags in the interval $[V, X]$ (resp. $[Z, \mathbf{0}]$) in $\mathcal{L}_{\mathcal{A}}$. We can thus rewrite the RHS of (5.1) as

$$\begin{aligned} \text{RHS}\big((5.1)\big) &= \sum_{\substack{X \in \mathcal{L}_{\mathcal{A}} \\ X \neq \mathbf{0}}} \operatorname{codim}(X)! \cdot c(\mathcal{A}_X) \cdot \sum_{Z \in \mathcal{A}^X} \dim(Z)! \cdot c(\mathcal{A}^Z) \cdot h(X, Z) \\ &= \sum_{\substack{Z \in \mathcal{L}_{\mathcal{A}} \\ Z \neq V}} \big(n - 1 - \dim(Z)\big)! \cdot \dim(Z)! \cdot c(\mathcal{A}_Z) \cdot c(\mathcal{A}^Z) \cdot |\mathcal{A}_Z|, \end{aligned} \tag{5.2}$$

where the second equality is a reordering of the summation and an application of the definition of $h(X, Z)$ in §2 and where $|\mathcal{A}_Z|$ denotes the number of hyperplanes in $\mathcal{A}_Z$.

Now, the quantity $c(\mathcal{A}_Z) \cdot c(\mathcal{A}^Z) \cdot |\mathcal{A}_Z|$ counts, by a standard double counting of faces and chambers in $\mathcal{A}$, the triples (C, H, F) of chambers $C \in \mathcal{C}(\mathcal{A})$, hyperplanes $H \in \mathcal{A}$, and faces $F \subseteq C$ such that $\operatorname{span}(F) = Z$ and $F \subseteq H$. Rewriting (5.2) as a sum over such triples, we get that (below $[C \cap H, \mathbf{0}]$ denotes an interval in the *face lattice* of $\mathcal{A}$)

$$\begin{aligned} \text{RHS}\big((5.1)\big) &= \sum_{C \in \mathcal{C}(\mathcal{A})} \sum_{H \in \mathcal{A}} \sum_{F \in [C \cap H, \mathbf{0}]} \big(n - 1 - \dim(F)\big)! \cdot \dim(F)! \\ &= |W| \cdot \sum_{H \in \mathcal{A}} \sum_{F \in [C_0 \cap H, \mathbf{0}]} \big(n - 1 - \dim(F)\big)! \cdot \dim(F)! \\ &= |W| \cdot \sum_{H \in \mathcal{A}} \sum_{J \subseteq \operatorname{cosup}(H)} \big(n - 1 - |J|\big)! \cdot |J|!. \end{aligned} \tag{5.3}$$

Here the second equality holds because in the reflection arrangement $\mathcal{A}$ the intersection patterns of hyperplanes and chambers are all the same; so we only need to address the fundamental chamber C_0. The third equality above is a consequence of [10]; any face of the fundamental chamber corresponds to a subset $J \subseteq S$ and, in particular, the face $C_0 \cap H$ corresponds to the complement of the support of H, which we denote $\operatorname{cosup}(H)$.

It is an easy consequence of Pascal's rule for binomial coefficients –in particular, the interpretation of $\operatorname{cosup}(H)$ is irrelevant– that

$$\sum_{J \subseteq \operatorname{cosup}(H)} \big(n - 1 - |J|\big)! \cdot |J|! = \frac{n!}{n - |\operatorname{cosup}(H)|}.$$

Combining this with (5.1) and(5.3) it is sufficient to show that

$$h \cdot \sum_{i=1}^{n} \frac{1}{d_i} = \sum_{H \in \mathcal{A}} \frac{1}{|\operatorname{supp}(H)|}, \tag{5.4}$$

which should be seen as a generalization of the formula $hn = 2|\mathcal{A}|$ and is itself a consequence of the main theorem of [10] and (3.1). We use [10] for its case-free proof of Chapoton's formula that counts the number of reflections of *full* support and we use (3.1) to compare the coefficient of k^1 in its two sides when $X = V$ and $t = 1$. □

6 The main result

We are now ready to give the proof of our main theorem by combining the combinatorial and algebraic recursions of the previous sections. The inductive proof relies on the following simple lemma that addresses the stucture of Propositions 3.1, 3.4.

Lemma 6.1. *Assume that for two polynomials $P(x), Q(x) \in \mathbb{C}[X]$ of the same degree n, and for all triples (m, k, r) of positive integers such that $m = k + r$, we have*

$$P(m) - P(k) - P(r) = Q(m) - Q(k) - Q(r).$$

Then, $P(x)$ and $Q(x)$ have all their coefficients equal apart from possibly the linear term.

Proof. Let us denote the coefficients of $P(x), Q(x)$ by p_i, q_i respectively, and write

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0 \quad \text{and} \quad Q(x) = q_n x^n + q_{n-1} x^{n-1} + \cdots + q_1 x + q_0.$$

Now we consider the triples $(m, m-1, 1)$ for all integers $m > 1$. Treating the expression $P(m) - P(m-1) - P(1)$ as a polynomial in m we may recover its coefficients in terms of those of P. An easy calculation gives us

$$\begin{aligned}\left[m^{n-k}\right]\Big(P(m) - P(m-1) - P(1)\Big) = (-1)^{k-1}\binom{n}{n-k} p_n + (-1)^{k-2}\binom{n-1}{n-k} p_{n-1} + \cdots \\ + (-1)\binom{n-k+2}{n-k} p_{n-k+2} + (n-k+1) p_{n-k+1},\end{aligned}$$

for all $1 \le k \le n-1$, while we get that

$$\left[m^0\right]\Big(P(m) - P(m-1) - P(1)\Big) = -p_0 - 2p_2 - 2p_4 - \cdots$$

Of course, the same is true for $Q(x)$ and the assumption of the lemma gives us a triangular list of equations (one for each $0 \le k \le n-1$) that force $p_0 = q_0$, $p_2 = q_2$, $p_i = q_i$, $i \ge 3$. This completes the proof (note that p_1 and q_1 never appear in the expressions above; indeed, the identity $p_1(m-k-r) = 0 = q_1(m-k-r)$ is empty). □

We are now ready to prove Theorem 1.1 as formulated in (1.4).

Proof of Theorem 1.1. Our proof proceeds via two nested inductions. First we assume that

$$\mathrm{Krew}^{\mathrm{NC}}_{W,[Z]} = \mathrm{Krew}^{\mathrm{alg}}_{W,[Z]} \quad \text{and} \quad \mathrm{Lod}^{\mathrm{NC}}_{W,[Z]} = \mathrm{Lod}^{\mathrm{alg}}_{W,[Z]}, \tag{6.1}$$

for all real reflection groups W of rank less than or equal to n and for all parabolic types $[Z] \in \mathcal{L}_W/W$; the base case $n = 1$ is trivially true.

Now, we pick an irreducible W such that $\mathrm{rank}(W) = n + 1$ and we run a separate induction on its flats. We assume that (6.1) is true for all types $[Z] \in \mathcal{L}_W/W$ with $\dim(Z) \leq k$ (the case $k = 0$, i.e. $Z = \mathbf{0}$, is trivially true) and we pick some parabolic type $[X]$ with $\dim(X) = k + 1$. Combining our inductive assumptions with Proposition 5.1 and Lemma 5.2 we must have that

$$[m^1]\left(\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m)\right) = [m^1]\left(\mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m)\right). \tag{6.2}$$

Now Prop. 4.1, 4.3 again in conjuction with our inductive assumptions imply that also

$$[m^1]\left(\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m)\right) = [m^1]\left(\mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m)\right). \tag{6.3}$$

Looking at the Kreweras recursions now, we rewrite the left hand side of Prop. 3.1 as

$$\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m) - \mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(k) - \mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(r),$$

and this leaves in the right hand side only expressions involving smaller rank groups or lower dimension flats; similarly for Proposition 3.4. This means that combining our inductive assumptions and these two Propositions we will have for the polynomials $P(m) := \mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m)$ and $Q(m) := \mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m)$ that

$$P(m) - P(k) - P(r) = Q(m) - Q(k) - Q(r),$$

for all positive integers (m, k, r) such that $m = k + r$. Now, we just showed in (6.3) that the coefficients of the *linear* terms of $P(m)$ and $Q(m)$ are equal and combining the above expression with Lemma 6.1 we must have that *all* coefficients are equal. In other words, we have shown that

$$\mathrm{Krew}^{\mathrm{NC}}_{W,[X]}(m) = \mathrm{Krew}^{\mathrm{alg}}_{W,[X]}(m),$$

and by applying Propositions 4.1, 4.3 again with our inductive assumptions we must also have that

$$\mathrm{Lod}^{\mathrm{NC}}_{W,[X]}(m) = \mathrm{Lod}^{\mathrm{alg}}_{W,[X]}(m).$$

This completes the inductive argument and so it completes the proof as well. $\square$

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