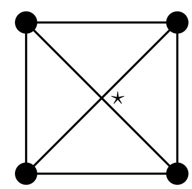
Honors Discovery Seminar: First Day Auction

You will work in small groups on the following problems for 30 minutes. At the end of the 30 minute time frame, teams will have the opportunity to auction off their solution: you will bid to present your answer, and the team with the highest bid wins the opportunity to present. After the presentation, if anyone thinks they have a better answer, the auction begins again. The team with the best answer wins! **No calculators, computers, phones, or checking the internet are allowed.**

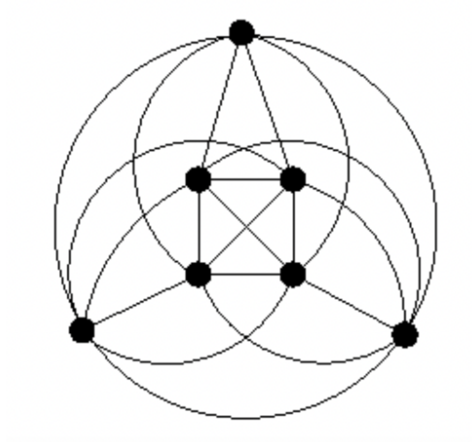
1. Draw 7 dots on a piece of paper. Draw a line or curve segment connecting every pair of dots such that the curves cross each other as *few* times as possible, and no three curves can cross each other at the same point.

   For example, if we only had 4 dots, here is a drawing with one crossing (the crossing is labeled with a star):

   This problem is asking about the 'minimal crossing number' for the complete graph with n vertices. (A complete graph is just n dots drawn on the page with a line segment connecting each dot to every other dot.) The *minimal crossing number* is the smallest number of crossings that must occur in the drawing of the graph, and we only know the minimal crossing number for $n \leq 12$. Here are the papers that proved it for $n \leq 10$ (Guy 1972) and $n = 11, 12$ (Pan, Richter 2007).

   For this question, when $n = 7$, we know the answer is 9, and here is a drawing!

2. Given any positive integer (whole number) n, create a sequence of numbers as follows: if n is even, divide it by 2. If n is odd, multiply it by 3 and add 1. Repeat on the output.

   For example, if we start with $n = 5$, the sequence is:

   $$5, 16, 8, 4, 2, 1.$$

   Find an integer n such that this process takes as *long* as possible to reach 1.

The *Collatz Conjecture* (here's the Wikipedia page) is a famous open conjecture that says: given any integer n, this process always reaches 1 is a finite number of steps. It has been checked by computer for all numbers $\leq 2^{68}$, and proven for 'almost all' numbers (a precise notion) (Tao 2022). Here is a Numberphile video on this. By a computer check, the number < 100 that takes the longest to reach 1 is 97, which does it in 118 steps, and there are similar other results.

3. Find the *largest* Fibonacci number that is also a prime number.

   As a reminder, the Fibonacci numbers are defined by the relationship $F_n = F_{n-1} + F_{n-2}$ (take the previous two numbers and add them together), so the series starts as:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

   This is a very hard and open problem. We have no idea if there are infinitely many prime Fibonacci numbers, but it is expected that the answer is yes. Here is a Numberphile Video (Krieger 2014) summarizing some of this.

   We know some things, like that every prime number appears as a factor of a Fibonacci number (Carmichael, 1913), and Krieger has several related (but much more recent) results.

4. A unit-distance coloring of the xy-plane is a way to color the plane such that any two points exactly 1 unit apart are colored with different colors. Find a unit-distance coloring of the plane using as *few* colors as possible.

   For example, here is a coloring of the plane with 16 colors (each number represents a different color; the side length of each square is $1/\sqrt{2}$; the boundary grid lines are considered as the color to the right or above, and the boundary corners are considered to be the color to the above right; and the pattern repeats indefinitely).

1	2	3	4	1
13	14	15	16	13
9	10	11	12	9
5	6	7	8	5
1	2	3	4	1

   This is known as the *chromatic number of the plane*, and the question of finding it is the Hadwinger-Nelson problem. It is known that the answer is between 5 and 7 (de Gray, 2018) but we do not know what number actually works. Here is a coloring with 7 colors (proving that the upper bound is 7):