Honors Discovery Seminar: Foundations of Mathematical Reasoning

Definition. A *proof* is an argument establishing a fact or the truth of a statement.

The following pages contain several *proofs*. For each one, we will read through it and comment on the following:

- are there any assumptions being made in the proof?
- does each step logically follow from the previous step?
- is there anything that needs to be further explained?
- are you convinced that the result is true?

1. *Statement:* If a whole number n is a multiple of 2, then n^2 is a multiple of 2.

   Proof: If n is a multiple of 2, that means $n = 2m$ for some whole number m. Then, $n^2 = (2m)^2 = 2m2m = 4m^2$. Because $4m^2$ can be written as $2(2m^2)$, $n^2 = 2(2m^2)$, so n^2 is a multiple of 2.

2. *Statement:* If $xy = 0$, then either x or y is equal to 0.

   Proof: If neither x nor y was zero, then $xy \neq 0$, so in order for $xy = 0$, we must have at least one of them equal 0.

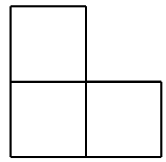
3. *Statement:* If T is a triangle, the sum of the interior angles of T equals 180 degrees.

   Proof: Draw a line through one vertex of the triangle parallel to the opposite side of the triangle. The sum of the angles on the parallel line is 180 degrees and this is equal to the sum of the interior angles of the triangle.

4. *Statement:* There are infinitely many prime numbers.

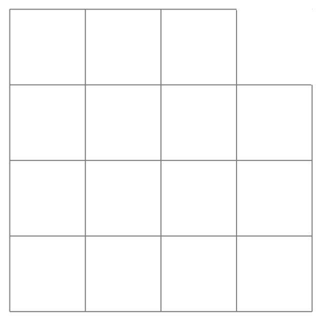
   Proof: Assume that this statement is false and there are only finitely many prime numbers. Label these numbers $p_1, p_2, p_3, \ldots, p_n$. Consider the number $n = p_1 p_2 p_3 \ldots p_n + 1$ (in words, n is the product of all of the primes, plus 1). Because n is one more than a multiple of each prime, n is not divisible by any prime number. Because n is not divisible by any prime number, then n itself must be prime, but n did not appear in the list of prime numbers. Therefore, our initial assumption is false (so "this statement is false" is false) and thus the statement is true.

5. *Statement:* Start with a $2^n \times 2^n$ chessboard, where $n \geq 1$ is a whole number. Remove the upper right corner. For every n, the resulting grid can be covered evenly by Ls, shapes occupying 3 squares on the grid that look like:

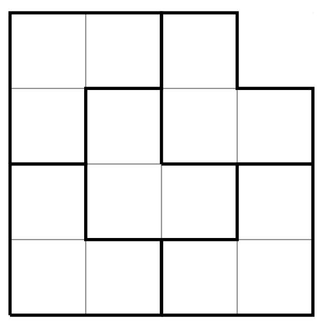


Proof: If $n = 1$, the grid is exactly the shape of an L, so can be covered by a single L.

If $n = 2$, then we have a grid of the form:



This is just four 2×2 grids glued together, with an extra L shape in the center:



Because we already know how to do the 2×2 (the $n = 1$ case) grid, we can cover the board four copies of the $n = 1$ grid, and then add in one L in the middle. So, we can do the $n = 2$ case (and it is built out of the $n = 1$ case).

Now, we repeat. If we have a tiling of the $2^{n-1} \times 2^{n-1}$ grid, then we can cover the $2^n \times 2^n$ grid with four $2^{n-1} \times 2^{n-1}$ grids leaving one L in the middle, so we get a tiling of the $2^n \times 2^n$ grid by placing the Ls as prescribed for the $2^{n-1} \times 2^{n-1}$ grids and then placing one additional L in the center.

This argument shows that, because we know how do to the $n = 2$ case, we can do the $n = 3$ case. Because the $n = 3$ case is possible, the $n = 4$ case is possible. This repeats forever, telling us that the nth case is always possible.