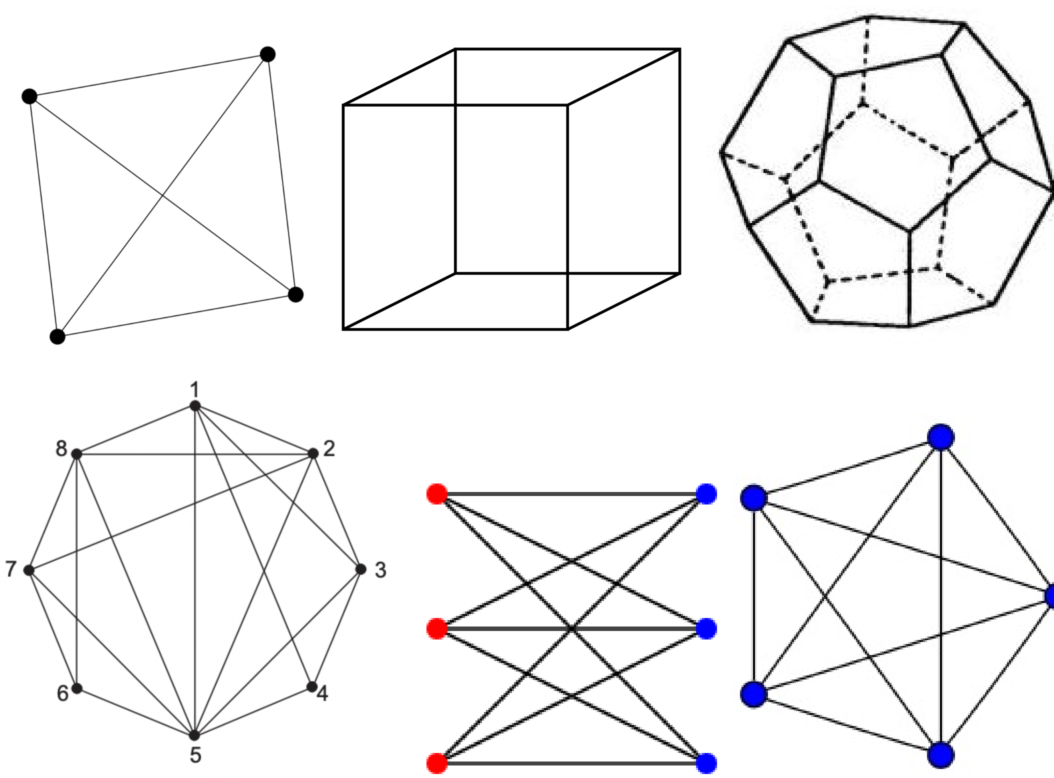
Honors Discovery Seminar: Graph Theory, Part II

Definition. A graph is **planar** if we can draw it in the plane without any of the edges crossing. A *face* of a planar graph is a region bounded by the edges. We say that the region outside a graph is also a face. (For a more senisble version of this: draw your graph on a sphere, and then count the faces.)

1. Which of the following graphs are planar? For each planar drawing that you find, find:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces}.$$

   Comment: for the cube/dodecahedron ('soccer ball'), the vertices are at the corners of each square/pentagon.



   For any graph that you think is not planar, what is the fewest number of crossings necessary in its drawing?

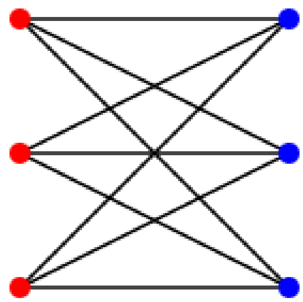
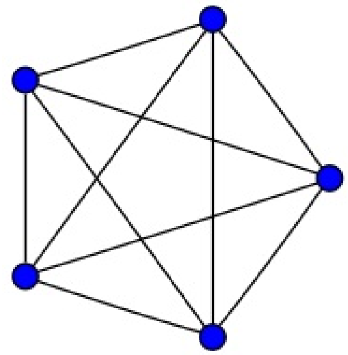
2. For a planar graph without multiple edges between vertices and with more than 3 edges, explain why $2\#$ of edges $\geq 3\#$ of faces (hint: how many edges are needed to bound a face? how many faces touch each edge?).

3. For a planar graph, explain why $\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2$.

4. For a planar graph, use the previous two problems to show $\#\text{edges} \leq 3\#\text{vertices} - 6$. Use this to show the last graph is not planar! (If you'd like: for the second-to-last graph, show that: if there are no cycles of length 3 (meaning, a path from one vertex back to itself using three edges), prove that $2\#$ of edges $\geq 4\#$ of faces, and plug that in.)

5. Non-planar graphs can be drawn without crossings on surfaces with more holes. For example, draw the following two graphs on a torus, and count the number

$$\#\text{vertices} - \#\text{edges} + \#\text{faces}.$$

6. It turns out that we can use graphs as a way to count the *number of holes* that a surface has! Can you find a relationship between the quantity

$$\#\text{vertices} - \#\text{edges} + \#\text{faces}.$$

   and the number of holes a surface has?

   For a challenge/to verify your relationship, draw the following graph on two-holed torus (the picture below) without any edges crossing and count

$$\#\text{vertices} - \#\text{edges} + \#\text{faces}.$$