Honors Discovery Seminar: Real Analysis

Real analysis is the study of real numbers, functions, convergence, limits, continuity, differentiation, integration, sequences, series, ... Calculus falls under the umbrella of real analysis!

Properties of the real numbers.

The real numbers are denoted by $\mathbb{R}$, and consist of all possible decimal numbers: $1, 2, 3, \dots,$
but also $1.5, 2.789898989..., 3.11111111...,$ and also $e, \pi, \sqrt{2}, \dots$.

Below, we will periodically encounter $\mathbb{Q}$, which denotes the rational numbers, i.e. all possible fractions a/b where a and b are nonzero whole numbers.

Check-in: Which of the following numbers are rational?

$$11/22 \qquad e \qquad \pi \qquad 0.1111111111\dots$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots \qquad \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots$$

The real numbers are defined as 'the unique complete ordered field' which means:

1. **complete**: 'limits exist' (for example, the rational numbers are not complete, because I could take a sequence of rational numbers getting closer and closer to π, but π is not rational). Alternatively, 'no gaps' (the rational numbers have gaps in them)
2. **ordered**: there is a notion of $<$ and $>$
3. **field**: we can add, multiply, and subtract real numbers, and we can divide nonzero real numbers

Because the real numbers are *complete*, we know limits of certain sequences of real numbers exist. This is incredibly important: we define both derivatives and integrals as limits!

Reminders:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(a+i\frac{(b-a)}{n})\frac{(b-a)}{n}$$

Topics of study in real analysis

1. Sequences and series
   (a) What is
   $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots?$$
   (b) What is
   $$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots?$$
   (c) What is
   $$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}?$$
   Or, what is $\lim_{n\to\infty} \frac{1}{n^2 \sin n}$?

(d) What is

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{p_n}$$

where p_n is the nth prime number?

We care about sequences and series because they show up a lot in probability, are how we define integrals, and generally describe steady-state solutions to different problems.

2. Integration

   (a) How do we define the following antiderivatives?

$$\int \frac{1}{\ln x} dx \qquad \int e^{x^2} dx \qquad \int \frac{e^x}{x} dx$$

   (b) For more: look up 'non-elementary integrals'

We care about integrals because they compute actual, real-life quantities: probability, standard deviation, areas, volumes, center of mass, electric flux, ...

3. Metric spaces

A **metric** is a notion of distance between two points. It is just a function that satisfies:

- If $x = y$, then the distance from x to y is 0.
- If $x \neq y$, then the distance from x to y is a positive real number.
- The distance from x to z is less than or equal to the sum of the distances from x to y and from y to z ('the shortest distance between two points is a straight line')

Examples of metrics:

(a) The usual thing: if x and y are real numbers, the distance between them is $|x - y|$

(b) The usual thing in higher dimensions: if (x_1, y_1) and (x_2, y_2) are two points, the distance between them is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(c) The 'taxicab' metric: if (x_1, y_1) and (x_2, y_2) are two points, the distance between them is $|x_2 - x_1| + |y_2 - y_1|$

(d) The discrete metric: if p and q are any two points, then the distance from p to q is 0 if $p = q$, and 1 if $p \neq q$

We care about metrics because they are notions of distance, and we might want algorithms to find the shortest distance between two points in some space.

4. Measure theory

A **measure** on a set is a notion of 'how big it is' relative to other sets, or the 'probability' of picking a number out of a set.

Example:

(a) The probability of picking out a rational number from the set of real numbers (randomly) is 0 because there are so many more real numbers than rational numbers. So, we say the rational numbers have measure 0 in the real numbers.

We care about measures because they are related to probability and ultimately tell how likely things are to satisfy certain numerical conditions.