COMPUTING SOME EXAMPLES OF BLOW-UPS

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1. Introduction

This problem arose from studying families of pairs of cameras, projections \( \phi_i : \mathbb{P}^3 \rightarrow \mathbb{P}^2 \) from points \( p_i \in \mathbb{P}^3 \) for \( i = 1, 2 \), called the camera centers. These rational maps are resolved by blowing up the camera centers. We study a family of such maps over \( \mathbb{A}^1 \) where the camera centers are distinct for \( t \neq 0 \) but come together when \( t = 0 \) and the resolution of the family of maps.

2. Set-Up

Consider the family of rational maps \( A_i : \mathbb{P}^3 \times \mathbb{A}^1 \rightarrow \mathbb{P}^2 \) given by

\[
A_1 ([x : y : z : w], t) = [x : y : z] \\
A_2 ([x : y : z : w], t) = [x : y : z + t]
\]

The locus of indeterminacy of the map \((A_1, A_2) : \mathbb{P}^3 \times \mathbb{A}^1 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2\) is at least set-theoretically given by the union of the individual loci: \( W = V(x, y, z) \cup V(x, y, z + t) = V(x, y, z(z + t)) \).

To resolve the rational map, we let \( f \) be the composition of \((A_1, A_2)\) with the Segre embedding \( \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^8 \), giving the rational map \( f : \mathbb{P}^3 \times \mathbb{A}^1 \rightarrow \mathbb{P}^8 \)

\[
f ([x : y : z : w], t) = [x^2 : xy : (x(z + t)) : yx : y^2 : y(y(z + t)) : zx : zy : z(z + t)]
\]

From this, we see that, scheme-theoretically, the locus of indeterminacy is given by \( Z = V(x^2, xy, x(z + t), y^2, y(z + t), xz, yz, z(z + t)) = V(x^2, y^2, xy, xz, yz, xt, yt, z(z + t)) \). This is not the same as the previously mentioned locus \( W = V(x, y, z(z + t)) \). They coincide for \( t \neq 0 \), but for \( t = 0 \), \( Z_0 = V(x^2, y^2, z^2, xy, xz, yz) \).

Ultimately, we’ll blow up \( Z \) to resolve the map, noting that on each fiber of the map, this amounts to just blowing up the locus of indeterminacy of the fiber. This is clear for \( t \neq 0 \) and for \( t = 0 \), \( Z_0 = V((x, y, z)^2) \), and the blow up of \( \mathbb{P}^3 \) at \( Z_0 \) isomorphic to the blow up of \( \mathbb{P}^3 \) at the point \((0, 0, 0)\).

For comparison, we’ll blow up \( W \) and see that the rational map is not actually resolved.

3. Blowing Up \( W \)

On \( \mathbb{P}^3 \), \( V(w) \) is completely contained in the domain of definition of \( f \), so we only compute the blow up on \( D(w) \). Abusing notation with the same coordinates, our map becomes \( f : \mathbb{A}^3_{xyz} \times \mathbb{A}^1_t \rightarrow \mathbb{P}^8 \)

\[
f ((x, y, z), t) = [x^2 : xy : (x(z + t)) : yx : y^2 : y(y(z + t)) : zx : zy : z(z + t)]
\]

We then compute \( X_W = Bl_W (\mathbb{A}^3 \times \mathbb{A}^1) \): because \( W = V(x, y, z(z + t)) \),

\[
X_W = V(xT_2 - yT_1, xT_3 - (z + t)T_1, yT_3 - (z + 2)T_2) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{P}^2_{T_1T_2T_3}.
\]

To resolve the map, we need to determine (locally) the defining equation of the exceptional divisor and divide the components of \( f \) by it. On \( D(T_1) \), the blow-up has the equation

\[
X_W = V(xT_2 - y, xT_3 - (z + t)) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{A}^2.
\]
The exceptional divisor $E$ lives above $W$ and has equation $V(x) \subset X_W$. Therefore, we take our rational map
\[ f((x, y, z), t) = [x^2 : xy : x(z + t) : yx : y(z + t) : zx : zy : z(z + t)] \]
and divide by $x$ to get
\[ \hat{f}(x, y, z, t, T_2, T_3) = [x : y : (z + t) : y : y^2/x : y(z + t)/x : z : zy/x : z(z + t)/x]. \]
We can further simplify this using the equations of the blow up, since $y = xT_2$ and $z(z + t) = xT_3$:
\[ \hat{f}(x, y, z, t, T_2, T_3) = [x : y : (z + t) : y : yT_2 : (z + t)T_2 : z : zT_2 : T_3]. \]

One can check what happens on the other patches: on $D(T_2)$, $E$ is given by $V(x)$, and on $D(T_3)$, $E$ is given by $V(z(z + t))$. Doing the same computation (and projectivizing), we get an extension of $f$:
\[ \hat{f}(x, y, z, t, [T_1 : T_2 : T_3]) = [xT_1 : yT_1 : (z + t)T_1 : yT_2 : (z + t)T_2 : z : zT_2 : T_3]. \]

This does extend the rational map across any point of the original domain, but is still undefined along the curve $x = y = z = t = T_3 = 0$ (the curve in the exceptional divisor above the special fiber of the family $t = 0$ given by $T_3 = 0$).

4. Blowing Up $Z$

To completely resolve the map, we need to blow up the ideal of $Z$, which we do in the same way (working with the map $f : \mathbb{A}^3_{xyz} \times \mathbb{A}^1 \rightarrow \mathbb{P}^8$). Because $Z = V(x^2, y^2, xy, xz, yt, z(z + t))$, we compute $X_Z = Bl_Z(\mathbb{A}^3 \times \mathbb{A}^1)$. The blow up is defined by equations that generate the kernel of the map
\[ k[x, y, z, t][T_1, \ldots, T_8] \rightarrow k[x, y, z, t] \]

\[ \begin{align*}
T_1 & \mapsto x^2 \\
T_2 & \mapsto y^2 \\
T_3 & \mapsto xy \\
T_4 & \mapsto xz \\
T_5 & \mapsto yz \\
T_6 & \mapsto xt \\
T_7 & \mapsto yt \\
T_8 & \mapsto z(z + t)
\end{align*} \]

One then sees that
\[ X_Z = V(y^2T_1 - x^2T_2, yT_1 - xT_3, zT_1 - xT_4, yzT_1 - x^2T_5, tT_1 - xT_6, ytT_1 - x^2T_7, z(z + t)T_1 - x^2T_8, \]
\[ xT_2 - yT_3, xzT_2 - y^2T_4, zT_2 - yT_5, xtT_2 - y^2T_6, tT_2 - yT_7, z(z + t)T_2 - y^2T_8, \]
\[ zT_3 - zT_4, zT_3 - xT_5, tT_3 - yT_6, tT_3 - xT_7, z(z + t)T_3 - xyT_8, \]
\[ yT_4 - xT_5, tT_4 - zT_6, ytT_4 - xzT_7, (z + t)T_4 - xT_8, \]
\[ xtT_5 - yzT_6, tT_5 - zT_7, (z + t)T_5 - yT_8, \]
\[ yT_6 - xT_7, z(z + t)T_6 - xtT_8, z(z + t)T_7 - ytT_8, \]
\[ T_1T_2 - T_3^2, T_4T_7 - T_5T_6, T_2T_4 - T_3T_5, T_3T_8 - T_4(T_5 + T_7) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{P}^7. \]

Now, we can look locally to extend the map:
On $D(T_1)$, this simplifies to

$$X_Z = V(y^2 - x^2 T_2, y - x T_3, z - x^2 T_4, yz - x^2 T_5, t - x T_6, yt - x^2 T_7, z(z + t) - x^2 T_8, T_2 - T_3^2, T_4 T_7 - T_5 T_6, \ T_2 T_4 - T_3 T_5, T_3 T_8 - T_4(T_5 + T_7))$$

Then, our exceptional divisor has equation $E = V(x^2)$, so we can extend $f$ to

$$\hat{f}(x, y, z, t, T_2, \ldots, T_8) = [1 : xy/x^2 : x(z + t)/x^2 : yx/x^2 : y(z + t)/x^2 : zx/x^2 : zy/x^2 : z(z + t)/x^2]$$

and using the equations of the blow-up, we can write this as

$$\hat{f}(x, y, z, t, T_2, \ldots, T_8) = [1 : T_3 : T_4 + T_6 : T_3 : T_2 : T_5 + T_7 : T_4 : T_5 : T_8]$$

Similarly, we can check this on the other patches and projectivize to find one global extension

$$\hat{f}(x, y, z, t, [T_1 : \ldots : T_8]) = [T_1 : T_3 : T_4 + T_6 : T_3 : T_2 : T_5 + T_7 : T_4 : T_5 : T_8].$$

This is defined everywhere and one can check that it agrees with the given map $f$ on its domain of definition.

Furthermore, for each fiber of the family, this map agrees with the fiber-wise resolutions of the map! One can see this by noting that the subvariety $Z$ restricts on each fiber to the locus of indeterminacy for $t \neq 0$ and a power of it for $t = 0$.

5. One More Blow-up

For good measure, we also blow up the ideal $(x, y, z, t)$ of the point $p = (0, 0, 0, 0)$ in the family and show that it does separate the lines $L_1 = V(x, y, z)$ and $L_2 = V(x, y, z + t)$:

$$X_p = Bl_p(\mathbb{A}^3 \times \mathbb{A}^1) = V(xT_2 - yT_1, xT_3 - zT_1, T_4 - tT_1, yT_3 - zT_2, yT_4 - tT_2, zT_4 - tT_3).$$

One can check that the lines intersect the exceptional divisor on the patch $D(T_4)$:

$$X_p = Bl_p(\mathbb{A}^3 \times \mathbb{A}^1) = V(x - tT_1, y - tT_2, z - tT_3)$$

The strict transform of the line $L_1$ is the closure of its preimage where $t \neq 0$, but for $t \neq 0$, the preimage of $V(x, y, z)$ is $V(T_1, T_2, T_3)$, so $L_1$ intersects the exceptional divisor $E$ at $(T_1, T_2, T_3) = (0, 0, 0)$. Similarly for $L_2$, for $t \neq 0$, the preimage of $V(x, y, z + t)$ is $V(T_1, T_2, T_3 + 1)$, so the line $L_2$ intersects $E$ at $(T_1, T_2, T_3) = (0, 0, -1)$. Therefore, in projective coordinates, $L_1$ intersects $E$ at $[0 : 0 : 0 : 1]$ and $L_2$ intersects $E$ at $[0 : 0 : -1 : 1]$.

Last but not least, in this blow up, we compute the extension of the map $f$:

$$\hat{f}(x, y, z, t, T_1, T_2, T_3) = [x^2/t : xy/t : x(z + t)/t : yx/t : y(z + t)/t : zx/t : zy/t : z(z + t)/t]$$

or

$$\hat{f}(x, y, z, t, T_1, T_2, T_3) = [xT_1 : yT_1 : (z + t)T_1 : yT_1 : T_2 : (z + t)T_2 : zT_2 : (z + t)T_3].$$

This is still not defined along the exceptional divisor, but if we divide one more time by $t$ we get something that is defined:

$$\hat{f}(x, y, z, t, [T_1 : T_2 : T_3 : T_4]) = [T_2^2 : T_1 T_2 : T_1(T_3 + T_4) : T_1 T_2 : T_2(T_3 + T_4) : T_1 T_3 : T_2 T_3 : T_3(T_3 + T_4)].$$

This is undefined at the points $[0 : 0 : 0 : 1]$ and $[0 : 0 : -1 : 1]$ (which happen to be the intersection points of $L_1$ and $L_2$ with $E$).