NOTES ON K-STABLITY

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I took these notes attending an introductory series of talks on K-stability as MSRI during the Birational Geometry and Moduli Spaces program in 2019. I tried to include as many references as possible.

1. Introduction to K-stability

Definition 1.1 ([Tia97, Don02]). *Let (X, L) be a polarized projective variety of dimension n. Because L is ample, for $m >> 0$, there is an embedding $|L^m| : X \to \mathbb{P}^N$. For any action $\mathbb{C}^*$ on PGL_{N+1}, there in an induced action $\mathbb{C}^*$ on the class $[X] \in \mathrm{Hilb}(\mathbb{P}^N)$.*

Let $[X_0] = \lim_{t\to 0} t \cdot [X]$.

*A **test configuration** is the induced family*

$$\begin{array}{ccc} \mathbb{C}^* \times (X, L^m) & \longrightarrow & (\mathcal{X}, \mathcal{L}) \\ \downarrow & & \downarrow \\ \mathbb{C}^* & \longrightarrow & \mathbb{A}^1 \end{array}$$

Given a test configuration, by Riemann-Roch, we can compute

$$\begin{aligned} h^0(X, L^k) &= a_0 k^n + a_1 k^{n-1} + \dots \\ &= \frac{L^n}{n!} k^n - \frac{L^{n-1} \cdot K_X}{2(n-1)!} k^{n-1} + \dots . \end{aligned}$$

We can also compute the total weight w_k of the $\mathbb{C}^*$ action on the determinant line bundle $\det H^0(X_0, L_0^k)$, so

$$w_k = b_0 k^{n+1} + b_1 k^n + \dots .$$

To compute the b_i, we can complete the family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{A}^1$ to a family $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ over $\mathbb{P}^1$ by adding the trivial fiber (X, L) over $\infty \in \mathbb{P}^1$: we achieve this by gluing the family $(\mathcal{X}, \mathcal{L})$ to the trivial family $X \times \mathbb{P}^1 \setminus \infty$ along $\mathbb{A}^1 \setminus 0$.

We then have a test configuration $(\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$ with a $\mathbb{C}^*$ action and can use equivariant Riemann-Roch to compute

$$\begin{aligned} w_k &= b_0 k^{n+1} + b_1 k^n + \dots \\ &= \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!} k^{n+1} - \frac{\overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{2n!} k^n + \dots . \end{aligned}$$

Definition 1.2 ([Tia97, Don02]). *The **generalized Futaki invariant** $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$ of the test configuration $(\mathcal{X}, \mathcal{L})$ is*

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - a_0 b_1}{a_0^2}.$$

In practice, for X Fano, we typically use $L = -rK_X$. We will exclusively use this case in what follows, and $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$ takes on a particularly nice form:

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{1}{2(-K_X)^n} \left(\left(\frac{1}{r}\overline{\mathcal{L}} \right)^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \frac{n}{n+1} \left(\frac{1}{r}\overline{\mathcal{L}} \right)^{n+1} \right).$$

Definition 1.3 ([Tia97, Don02]). *Let X be a variety such that $-K_X$ is ample. X is*

1. *K-semistable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$.*
2. *K-polystable if X is K-semistable and, if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$, then outside of a codimension 2 locus,[1] $\mathcal{X} \cong X \times \mathbb{A}^1$.*
3. *K-stable if X is K-polystable and there is no $\mathbb{G}_m$ in $\mathrm{Aut}(X)$.*

Remark 1.4. *To test if a variety is K-(semi/poly)stable, we must test infinitely many test configurations, which depend on the $\mathbb{C}^*$ action and the power m used in the embedding $|L^m| : X \to \mathbb{P}^N$. One must show this definition is actually checkable in some reasonable sense.*

Although nothing about singularities appears in this definition, asking that a variety is K-semistable has (surprising) consequences on the singularities of X.

Theorem 1.5 ([Oda12]). *If X is normal and $-K_X$ is ample, then K-semistability of X implies that X has klt singularities. In other words, if X is K-semistable, it is $\mathbb{Q}$-Fano.*

Definition 1.6. *A test configuration $(\mathcal{X}, \mathcal{L})$ is called a **special test configuration** if $\mathcal{X}$ is a $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties, i.e. $\mathcal{L} \sim -rK_{\mathcal{X}}$ and X_0 has klt singularities.*

A general test configuration can have arbitrarily bad singularities (non-reduced X_0, etc). Special test configurations have better behavior and one might hope that we can check Definition 1.3 only on special test configurations. In fact, this is true:

Theorem 1.7 ([LX14]). *To test K-(semi/poly)stability, one only needs to test special test configurations.*

Proof: (Sketch). Starting with a test configuration $(\mathcal{X}, \mathcal{L})$, we can perform some birational modifications like MMP operations or finite base change, which only changes the Futaki invariant by the degree of the map (so doesn't affect positivity). We can produce a special test configuration $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ by these modifications such that

$$d\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq \mathrm{Fut}(\mathcal{X}^s, -rK_{\mathcal{X}^s})$$

where d is the degree of the base change. So, if $\mathrm{Fut}(\mathcal{X}^s, -rK_{\mathcal{X}^s}) \geq 0$, we also have $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$. $\square$

Remark 1.8. *This was conjectured by Tian in his original definition: he required X_0 to be normal. However, this theorem is still limited because Fano varieties can have infinitely many special test configurations, so it is hard to check in practice.*

Remark 1.9. *If $(\mathcal{X}, \mathcal{L})$ is a special test configuration,*

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = -\frac{1}{2(-K_X)^n(n+1)} \left(-K_{\overline{\mathcal{X}}/\mathbb{P}^1} \right)^{n+1}.$$

So, for special test configurations, the Futaki invariant is proportional to just the first term of the weight of the $\mathbb{C}^*$ action; we don't have to keep track of the first two terms.

[1] The generalized Futaki invariant only sees codimension 1 data, so we can only ask for isomorphism in codimension 1.

Because Theorem 1.5 ties K-semistability to the singularities and birational geometry of X, one might hope for a definition of K-(semi/poly)stability in more algebro-geometric terms. In fact, we can connect the Futaki invariant to Fujita's β-invariant, defined below.

Definition 1.10. *Let X be a $\mathbb{Q}$-Fano variety and D a prime divisor over X. Let $\mu : Y \to X$ be any morphism such that $D \subset Y$. Fujita's* ***β-invariant*** *is*

$$\beta_X(D) = A_X(D)(-K_X)^n - \int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tD)dt$$

where $A_X(D)$ is the log discrepancy of the divisor D, or the number

$$A_X(D) = 1 + \mathrm{ord}_D(K_Z - f^*K_X)$$

in any resolution of singularities $f : Z \to X$.

One can check that $\beta_X(D)$ does not depend on the choice of μ and Y, so we will often write

$$\beta_X(D) = A_X(D)(-K_X)^n - \int_0^\infty \mathrm{vol}(-K_X - tD)dt.$$

With this definition, we can relate the K-(semi/poly)stability of X intrinsically to the birational geometry of X. The following theorem is usually called the **valuative criterion** for K-(semi/poly)stability.

Theorem 1.11 ([Fuj19, Li17])**.** *A variety X is K-semistable if and only if $\beta_X(D) \geq 0$ for all prime divisors D over X.*

Proof: (Sketch.)

First we show the implication $\beta(D) \geq 0 \implies X$ is K-semistable. By Theorem 1.7, we only need to show that $\mathrm{Fut}(\mathcal{X}) \geq 0$ for all special test configurations.

Let $\mathcal{X} \to \mathbb{A}^1$ be a special test configuration. If X_0 is the fiber over $0 \in \mathbb{A}^1$, ord_{X_0} defines a divisorial valuation on $K(\mathcal{X})$. Consider the valuation $v := \mathrm{ord}_{X_0}|_{K(X) \subset K(X \times \mathbb{A}^1) \cong K(\mathcal{X})}$. It is easy to show that

$$v = \begin{cases} \text{trivial val. if } \mathcal{X} \text{ is a trivial special test config.} \\ b \cdot \mathrm{ord}_D \text{ for some } b \in \mathbb{Z}_{>0}, D \text{ a divisor over } X \end{cases}$$

We consider X_0 as a divisor over $X \times \mathbb{A}^1$ as in the following diagram:

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \times \mathbb{A}^1 \arrow[rr, dashed] & & \mathcal{X} \supset X_0 \end{tikzcd}$$

and can compute the discrepancy of X_0, which is the log discrepancy of X_0 in the pair $(X \times \mathbb{A}^1, X \times \{0\})$, which is the log discrepancy of $b\mathrm{ord}_D$ with respect to X.

Consider a section $s \in H^0(X, -rK_X)$ and the section $s \times \mathbb{A}^1$ of $X \times \mathbb{A}^1$. Then,

$$q_*p^*(s \times \mathbb{A}^1) = \tilde{s} + cX_0$$

for some c, where $\tilde{s}$ is the birational transform of X_0. One can compute that

$$c = a_1 + a_2$$

where $a_1 = \mathrm{ord}_{X_0}(s \times \mathbb{A}^1) =: V(s)$ (how much s vanishes on X_0 and a_2 can be computed as follows. First, compute $a(X_0)$ the discrepancy of X_0 in the pullback

$$p^*(-K_{X \times \mathbb{A}^1}) - a(X_0)^{2} X_0 = -K_{\mathcal{Y}}.$$

[2]Note that $a(X_0) = a(X \times \mathbb{A}^1, X_0)$

Then, $a_2 = -ra(X_0) := A_X(v)$.

Now, we consider the total weight (degree $n+1$ part) of this test configuration to compute the Futaki invariant.

By computation, where the first term comes from the a_2 part and the second from the limit over sum of all a_1 parts, we have that the total weight is

$$\frac{-A_X(v)(-K_X)^n}{n!} + \frac{1}{n!}\int_0^\infty \mathrm{vol}(-K_X - tD)dt$$

which by definition is equal to

$$-\frac{1}{n!}\beta(b \cdot \mathrm{ord}_D) = -\frac{b}{n!}\beta(\mathrm{ord}_D).$$

Because we only need to consider the total weight for special test configurations, this implies that

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{\beta(b \cdot \mathrm{ord}_D)}{2(-K_X^n)}$$

so if $\beta(D) \geq 0$ for all D, we get that $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$.

The converse is harder. We must show that if X is K-semistable, then $\beta(D) \geq 0$ for all divisors D over X. However, only looking at special test configurations (where the above formula applies), we don't get *all* valuations $\beta(D)$. To conclude this, we would need an analogue of Theorem 1.7 for the computation of $\beta(D)$.

What we will actually show is that K-semistability implies a condition $(\star)$ that only relies on special test configurations, and then that $(\star)$ implies $\beta(D) \geq 0$.

First, we recall the definition of **volume** for general valuations.

Definition 1.12 ([ELS03])**.** *Let $x \in X = \mathrm{Spec} R$ be a klt singularity and $v \in \mathrm{Val}_{x,X}$ be a valuation centered at x. The **volume** of v is*

$$\mathrm{vol}(v) = \lim_{k\to\infty} \frac{\mathrm{length}(R/\mathfrak{a}_k)}{k^n/n!}$$

where $\mathfrak{a}_k = \{f \mid v(f) \geq k\}$.

There is also a definition of **log discrepancy** $A_X(v)$ for general valuations due to Jonsson and Mustaţă, see [JM12].

With these ingredients, we can define Li's **normalized volume**. For a recent survey on normalized volume, see [LLX18].

Definition 1.13 ([Li18])**.** *With the above set up, the **normalized volume** is*

$$\widehat{\mathrm{vol}}(v) := A_X(v)^n \cdot \mathrm{vol}(v)$$

*and the **local volume** at x is*

$$\widehat{\mathrm{vol}}(x, X) := \inf_{v \in \mathrm{Val}_{x,X}} \widehat{\mathrm{vol}}(v).$$

Remark 1.14. *It is a result of Blum (see [Blu18]) that this inf is in fact a min, i.e.*

$$\widehat{\mathrm{vol}}(x, X) := \min_{v \in \mathrm{Val}_{x,X}} \widehat{\mathrm{vol}}(v).$$

Now consider V a $\mathbb{Q}$-Fano variety and $x \in X$ the vertex of the cone $X = C(V, -rK_V)$. Because V is Fano, $x \in X$ is klt, and X has a standard partial resolution $\mu : Y \to X$ by blowing up the vertex with exceptional divisor $V_0 \cong V \subset Y$.

With this framework, we can finally define $(\star)$ to be

$$(\star) : V_0 \text{ is a minimizer of } \widehat{\mathrm{vol}}(x, X).$$

Indeed, we will show that K-semistability of a variety V implies that V_0 is a minimizer which implies that $\beta(D) \geq 0$ for all divisors D over V.

We begin with the second implication. Given a divisor D over V, from the canonical projection $\pi : Y \to V$, we pullback D to Y to get a valuation V_∞. Define V_t to be the quas-monimoial valuation with weight $(1, t)$ along V_0 (the divisorial valuation from the exceptional divisor V_0) and V_∞, respectively. For $t = 0$, we get V_0, and for $t = \infty$, we get V_∞. In other words, we connect V_0 and V_∞ in the space of valuations.

Li's derivative formula [Li18] says that

$$\frac{d}{dt}\widehat{\mathrm{vol}}(V_t)|_{t=0} = (n+1)\beta(D)$$

so, if we know that $\widehat{\mathrm{vol}}(V_0) \leq \widehat{\mathrm{vol}}(V_t)$ for all t, this implies that $\beta(D) \geq 0$. Indeed, by assumption, V_0 is a minimizer for the local volume, so $\beta(D) \geq 0$.

Now, we show the first implication. It is a result of Liu [Liu18] that

$$\inf \widehat{vol}(v) = \inf_{\mathfrak{a} \text{ an } m-\text{primary ideal}} \mathrm{mult}(\mathfrak{a})\mathrm{lct}(X, \mathfrak{a})$$

and in fact this inf can be taken over m-primary ideals $\mathfrak{a}'$ such that $\mathfrak{a}'$ is $\mathbb{C}^*$-equivariant.

Now let $\mathfrak{a}'$ be such an ideal and let $c = \mathrm{lct}(X, \mathfrak{a}')$. Let $Y \to (X, c\mathfrak{a}')$ be a $\mathbb{C}*$-equivariant dlt modification. From steps of the MMP, from Y we can produce Y^s with a map $f^s : Y^s \to (X, x)$ such that $\mathrm{Exc}(f^s) =: S$ is irreducible, $\mathbb{C}$-equivariant, and maps to $x \in X$. Furthermore, S satisfies (Y^s, S) is plt and $(-K_{Y^s} - S)|_S = -(K_S + \mathrm{Diff})$ is ample.

Definition 1.15. *An S as above is called a **Kollár component** in the space of valuations.*

Remark 1.16. *It is a result of Li-Xu [LX16] that, in computing the normalized volume, one can restrict to valuations coming from Kollár components, i.e.*

$$\widehat{\mathrm{vol}}(x, X) := \inf_{E \text{ Kollár component over } x \in X} \widehat{\mathrm{vol}}(v).$$

One can show that there is a one-to-one correspondence between special test configurations (up to base change) and rays of $\mathbb{C}^*$-equivariant valuations V_t containing a Kollár component. In general, there is a one-to-one correspondence between test configurations and rays, and special test configurations correspond to ones containing a Kollár component.

Finally, to finish the proof, it suffices to show that $\widehat{\mathrm{vol}}(V_0) \leq \widehat{vol}(S)$ for any $\mathbb{C}^*$-equivariant Kollár component S.

Because the ray V_t contains a Kollár component, we take $S \to V_t$ and we can compute

$$\frac{d}{dt}(\widehat{\mathrm{vol}}(V_t)|_{t=0} = c \cdot \mathrm{Fut}(\mathcal{X}) \geq 0$$

because X was K-semistable, and in fact $\widehat{\mathrm{vol}}(V_t)$ is convex, so $\widehat{\mathrm{vol}}(S) = \widehat{\mathrm{vol}}(V_t) \geq \widehat{\mathrm{vol}}(V_0)$, as desired.

□

A slight modification of the β invariant is the δ invariant.

Definition 1.17. *The δ**-invariant** or **stability threshold** of a $\mathbb{Q}$-Fano variety X is*

$$\delta(X) = \inf_{D \text{ divisor}} \frac{A_X(D)(-K_X)^n}{\int_0^\infty \mathrm{vol}(-K_X - tD)dt}.$$

Theorem 1.18. $\delta(X) \geq 1 \iff X$ *is K-semistable.*

The definitions above can also extend to the case of log Fano pairs (X, D). Two fundamental goals, areas of active current research, are to

(1) Build up *projective* moduli of K-polystable $\mathbb{Q}$-Fano varieties and of log Fano pairs (X, D).
(2) Determine which Fano varieties are K-stable.

In the next section, we will hint at a relationship between K-stability and GIT for hypersurfaces (related to Goal (1)), and in the final section, we will provide some techniques for achieving Goal (2).

2. Local to global volume comparisons

Theorem 2.1 ([Fuj19], [Liu18])**.** *Assume X is a K-semistable $\mathbb{Q}$ Fano variety of dimension n. Then,*

$$(-K_X)^n \leq (n+1)^n.$$

Furthermore, equality holds if and only if $X \cong \mathbb{P}^n$.

Remark 2.2. *The assumption that X is K-semistable is necessary: if X is a Fano manifold of dimension $n \geq 4$, then the inequality $(-K_X)^n \leq (n+1)^n$ may fail.*

Proof: We prove only the first statement. Choose a smooth point $x \in X$ and let $Y = \mathrm{Bl}_x X$ be the blow up of the point x, with birational morphism $\mu : Y \to X$ and exceptional divisor $E \subset Y$.

By assumption and the valuative criteria (Theorem 1.11), we must have $\beta(E) \geq 0$, i.e.

$$A_X(E)(-K_X)^n \geq \int_0^\infty \mathrm{vol}(-K_X - tE)dt.$$

Furthermore,

$$A_X(E) = 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X) = n.$$

We want to estimate $\mathrm{vol}(-K_X - tE) := \mathrm{vol}(\mu^*(-K_X) - tE)$. Assume for the moment that $t \in \mathbb{Q}_{\geq 0}$. Take an integer $m \in \mathbb{Z}_{\geq 0}$ such that $mt \in \mathbb{Z}_{\geq 0}$. Then,

$$\mathrm{vol}(\mu^*(-K_X) - tE) = \lim_{m\to\infty} \frac{h^0(Y, \mathcal{O}_Y(m\mu^*(-K_X) - mtE))}{m^n/n!}.$$

By construction,

$$\mu_*\mathcal{O}_Y(m\mu^*(-K_X) - mtE) = \mathcal{O}_X(-mK_X) \cdot \mathfrak{a}_{mt}$$

where $\mathfrak{a}_{mt} := m_x^{mt} = \{f \in \mathcal{O}_{x,X} \mid \mathrm{ord}_E(f) \geq mt\}$. Therefore,

$$\begin{aligned} h^0(Y, \mathcal{O}_Y(m\mu^*(-K_X) - mtE)) &= h^0(X, \mu_*\mathcal{O}_Y(m\mu^*(-K_X) - mtE)) \\ &= h^0(X, \mathcal{O}_X(-mK_X) \cdot \mathfrak{a}_{mt}) \\ &\geq h^0(X, \mathcal{O}_X(-mK_X)) - \mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{vol}(-K_X - tE) &\geq (-K_X)^n - \lim_{m\to\infty} \frac{\mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt}}{m^n/n!}) \\ &= (-K_X)^n - \lim_{m\to\infty} \frac{\mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt})}{m^n t^n/n!} \cdot t^n \\ &= (-K_X)^n - \mathrm{vol}(\mathrm{ord}_E) \cdot t^n \\ &= (-K_X)^n - t^n. \end{aligned}$$

Finally, plugging this into the inequality

$$A_X(E)(-K_X)^n \geq \int_0^\infty \mathrm{vol}(-K_X - tE)dt,$$

we find that

$$n(-K_X)^n \geq \int_0^\infty \max\{(-K_X)^n - t^n, 0\} dt$$

so

$$n(-K_X)^n \geq \frac{n}{n+1}(-K_X)^n \sqrt[n]{(-K_X)^n}$$

or

$$(-K_X)^n \leq (n+1)^n.$$

□

Recall the notion of **normalized volumes** from the previous section. With this set-up, we have a *Local to Global* Theorem on the volume of K-semistable varieties.

Theorem 2.3 ([LL19]). *Let X be a K-semistable $\mathbb{Q}$ Fano variety. Then, for any $x \in X$,*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x, X).$$

Proof: The proof is the same as the proof of Theorem 2.1, keeping $\mathrm{vol}(\mathrm{ord}_E)$ and $A_X(E)$ as in their definitions (without replacing them by 1 and n). □

We summarize some properties of the local volume function $\widehat{\mathrm{vol}}(x, X)$.

(1) [dFEM04, Li18] If X has dimension n and $x \in X$ is smooth, then

$$\widehat{\mathrm{vol}}(x, X) = n^n$$

(2) [LX19] If X has dimension n, then for any $x \in X$,

$$\widehat{\mathrm{vol}}(x, X) \leq n^n$$

and equality holds if and only if x is smooth.

(3) [LX19] If X has dimension $n = 2$ or $n = 3$ and $x \in X$ is **not** a smooth point, then

$$\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n$$

and equality holds if and only if x is an ordinary double point. Conjecturally, the ordinary double point *always* gives the second largest volume.

We can use the previous properties to study the K-stability of cubic threefolds.

Theorem 2.4 ([LX19]). *Let $X \subset \mathbb{P}^4$ be a cubic hypersurface. Then, X is K-(poly/semi) stable if and only if it is GIT-(poly/semi) stable. In particular, any smooth cubic threefold is K-stable.*

Proof: (Sketch.) K $\implies$ GIT: First, we show K $\implies$ GIT (this is a general idea due to Paul-Tian [PT06] for hypersurfaces.) By assumption, if X is K-(semi)stable, we have $\mathrm{Fut}(\mathcal{X}, \mathcal{L})(\geq) > 0$ for any test configuration. But, given a hypersurface $X \subset \mathbb{P}^4$ and a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(5)$, this induces a test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$.

Paul-Tian show that

$$\mathrm{Fut}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) = a\mu^{\mathcal{O}(1)}([X], \lambda)$$

where $a > 0$ is a positive constant and $\mu^{\mathcal{O}(1)}$ is the GIT weight. Therefore, K-semistability of X implies that the GIT weight is ≥ 0 for every one-parameter subgroup, hence the Hilbert-Mumford criterion implies that X is GIT-semistable.

GIT $\implies$ K: This is the harder direction. Suppose $X \subset \mathbb{P}^4$ is GIT polystable.[3] We want to show that X is K-polystable.

[3]This case is the most difficult. K-(semi) stability follows from some known results once we can show this.

Step 1: Openness of K-moduli. [Tia87, LWX19] A general (smooth) cubic threefold is K-stable. Tian shows that the Fermat cubic is K-stable and Li-Wang-Xu show that, in the smooth case, the K-stable locus is Zarisiki open.[4]

Step 2: Properness of K-moduli. [DS14] Take a smoothing $\mathcal{X} \to C$ over a pointed curve $0 \in C$ such that $\mathcal{X}_0 \cong X$ is the cubic hypersurface in question and $\mathcal{X}_t$ is a smooth cubic threefold. From the previous step, we can assume $\mathcal{X}_t$ is K-stable. Then, after a finite base change, by properness of K-moduli, there exits a family $\mathcal{X}' \to C$ such that $\mathcal{X}' \setminus \mathcal{X}'_0 \cong \mathcal{X} \setminus \mathcal{X}_0$ and $X' := \mathcal{X}'_0$ is K-polystable. It suffices to show that $X' \cong X$.

Step 3: Local to Global Volume Comparison. [LX19] It suffices to show that $X' \hookrightarrow \mathbb{P}^4$ as a cubic hypersurface. Then, X' being K-polystable implies that X' is GIT-polystable (from the first half of the proof). Hence, by separatedness of GIT moduli, we get that $X' \cong X$. To show that $X' \hookrightarrow \mathbb{P}^4$, we appeal to work of T. Fujita on del Pezzos. Consider $\mathcal{X}' \to C$ and take a divisor $\mathcal{H}$ on $\mathcal{X}' \setminus \mathcal{X}'_0$ such that $\mathcal{H}_t$ is a hyperplane section for $t \neq 0$. Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathcal{X}'$. Because $-K_{\mathcal{X}'_t} \sim 2\mathcal{H}_t$ and $\mathcal{X}'$ is $\mathbb{Q}$-Gorenstein, this implies that $-K_{\mathcal{X}'_0} \sim 2\overline{\mathcal{H}}_0$. We are done if $\overline{\mathcal{H}}_0$ is Cartier.

By Theorem 2.3, for all $x \in X' := \mathcal{X}'_0$, we have
$$\widehat{\mathrm{vol}}(x, X) \geq \frac{27}{64}(-K_{X'})^3.$$
Because X' is a $\mathbb{Q}$-Gorenstein limit, $(-K_{X'})^3 = (-K_{\mathcal{X}'_t})^3 = 24$, hence
$$\widehat{\mathrm{vol}}(x, X) \geq \frac{27}{64} \cdot 24 = 10.125.$$
If $\overline{\mathcal{H}}_0$ is not Cartier at $x \in X'$, take the index one cover
$$Z \to X'$$
$$z \mapsto x$$
and, work of Li-Xu implies that
$$\widehat{\mathrm{vol}}(z, Z) = \mathrm{ind}_x \overline{\mathcal{H}}_0 \cdot \widehat{\mathrm{vol}}(x, X) \geq 2(10.125) = 20.25.$$
But, by Property 3,
$$\widehat{\mathrm{vol}}(z, Z) \leq 16$$
unless $z \in Z$ is smooth. Therefore, we must have $z \in Z$ is smooth so $x \in X$ is locally analytically isomorphic to $\mathbb{C}^2/(\mathbb{Z}/2) \times \mathbb{C}$. However, local Lefschetz implies that $\overline{\mathcal{H}}_0$ is Cartier, a contradiction. $\square$

Remark 2.5. *If Property 3 is true in dimension n, the same proof shows that cubic hypersurfaces in $\mathbb{P}^n$ are K-(poly/semi) stable if and only if they are GIT-(poly/semi) stable. However, this is false in higher degree.*

3. Checking K-stability of Fano varieties

In this section, we will discuss methods for checking K-stability of Fano varieties. Our motivating examples will be del Pezzo surfaces, some Fano threefolds, and complete intersections of index 1.

Definition 3.1. *Let (X, D) be a pair. The* ***log canonical threshold*** *of (X, D) is*
$$\mathrm{lct}(X, D) = \sup\{t \geq 0 \mid (X, tD) \text{ is log canonical }\}.$$

[4]Recently, Blum-Liu-Xu [BLX19] proved openness of K-semistablity in generality, not just in the smoothable case.

Definition 3.2 ([Tia87]). *Let X be a $\mathbb{Q}$-Fano variety. Tian's α-invariant is*

$$\alpha(X) = \inf_{0 \leq D \sim_{\mathbb{Q}} -K_X} \operatorname{lct}(X, D).$$

Example 3.3. *If $X = \mathbb{P}^n$, because $-K_{\mathbb{P}^n} = (n+1)H$,*

$$\alpha(\mathbb{P}^n) = \frac{1}{n+1}.$$

Theorem 3.4 ([Tia87, DK01, OS12]). *Let X be a $\mathbb{Q}$-Fano variety of dimension n. If*

$$\alpha(X) > (\geq) \frac{n}{n+1}$$

then X is K-(semi) stable.

Proof: (Sketch.) By the valutative criterion (Theorem 1.11, we need to show that if $\alpha > 0$ (resp. ≥ 0), then $\beta > 0$ (resp. ≥ 0). Let E be a prime divisor over X. First, we will show that $\beta(E) \geq 0 \iff$ the interval $[0, \tau(E)]$ with density function $V_t = \dfrac{1}{(-K_X)^n} \operatorname{vol}_{Y|E}(-K_X - tE)$ has barycenter $< A_X(E)$, where

$$\tau(E) = \sup\{t \geq 0 \mid \operatorname{vol}(-K_X - tE) > 0\}$$

and

$$\operatorname{vol}_{Y|E}(L) = \limsup \frac{\dim(\operatorname{Im}(H^0(Y, mL) \to H^0(E, mL)))}{m^{n-1}/(n-1)!}$$

where $E \subset Y \to X$ is a birational model of X extracting the divisor E.

It is a result of Lazarsfeld-Mustaţă and Boucksom-Favre-Jonsson that, for $t_0 < \tau(E)$,

$$\operatorname{vol}_{Y|E}(-K_X - t_0 E) = -\frac{1}{n} \frac{d}{dt}|_{t=t_0} \operatorname{vol}(-K_X - tE),$$

so applying integration by parts to the integral in $\beta(E)$, we see that $\beta(E)$ can be related to $\operatorname{vol}_{Y|E}$. Then, the inequality $\beta(E) \geq 0$ can be related to the barycenter of V_t as follows.

By a result attributed to Ein-Lazarsfeld-Mustaţă, Nakayama, Popa, the function $V_t^{1/n-1}$ is a concave function. Because the barycenter of the n-simplex has coordinate $\frac{n}{n+1}$, this implies that the barycenter of V_t on the interval $[0, \tau(E)]$ is less than $\frac{n}{n+1}\tau(E)$. Combining these results, we can show the desired equivalence.

Finally, to prove the theorem, a calculation shows that if $\alpha(X) > \frac{n}{n+1}$, then $\tau(E) < \frac{n+1}{n} A_X(E)$, hence the previous paragraph shows that the barycenter is $< A_X(E)$. Therefore, $\beta(E) > 0$, as desired. (Respectively, one can use ≥ 0.) □

The upshot now is that we can use the α invariant to check K-stability of Fano varieties.

Example 3.5. *Let X be a del Pezzo surface of degree 1. We will compute $\alpha(X)$ and show that X is K-stable.*

Consider the linear system $|-K_X|$. For these surfaces, this is a pencil of cubic plane curves. Now we make the following claim: if $D \sim_{\mathbb{Q}} -K_X$ such that $\operatorname{Supp} D \notin |-K_X|$, then $\operatorname{lct}(X, D) \geq 1$. Indeed, for any $x \in X$, we can choose an effective curve $C \in |-K_X|$ such that $x \in C$. Because $C, D \sim_{\mathbb{Q}} -K_X$, this implies that $C \cdot D = 1$. However, $C \cdot D \geq \operatorname{mult}_x D$, hence (X, D) is log canonical at every $x \in X$.

As a consequence, to compute $\alpha(X)$, it suffices to check only curves $D \in |-K_X|$. But, for cubic plane curves in a smooth surface,

$$\operatorname{lct}(X, D) = \begin{cases} 1 \text{ if } D \text{ is nodal} \\ 5/6 \text{ if } D \text{ is cuspidal} \end{cases}$$

hence

$$\alpha(X) = \frac{5}{6} > \frac{2}{3}$$

and by Theorem 3.4, X is K-stable.

Remark 3.6. *More generally, Cheltsov has shown that $\alpha(X) \geq \frac{2}{3}$ for X a del Pezzo surface of degree ≤ 4, hence is K-semistable.*

The following is a refinement of Tian's criteria.

Theorem 3.7 (Fujita)**.** *If X is a surface, or smooth of dimension ≥ 3, and*

$$\alpha(X) \geq \frac{n}{n+1},$$

then X is K-stable.

Corollary 3.8. *All del Pezzo surfaces of degree ≤ 4 are K-stable.*

Example 3.9. *If X is a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1$, then Cheltsov-Park have shown $\alpha(X) \geq \frac{n}{n+1}$. Therefore, by Theorem 3.7, all such X are K-stable.*

If X has a group action by a compact group G, then one can define a variant $\alpha_G(X)$ of $\alpha(X)$.

Definition 3.10. *Let X be a smooth Fano variety with a group action by a compact group G. Define*

$$\alpha_G(X) = \inf_{0 \leq D \sim_{\mathbb{Q}} -K_X, D \textit{ is } G\textit{-invariant}} \operatorname{lct}(X, D).$$

Theorem 3.11 ([Tia87])**.** *Let X be a smooth[5] Fano variety with a group action by a compact group G. If*

$$\alpha_G(X) > \frac{n}{n+1},$$

then X is K-polystable.

Example 3.12. *Let $X = \mathrm{Bl}_{p,q,r}\mathbb{P}^2$ be a del Pezzo surface of degree 6, the blow up of $\mathbb{P}^2$ at three points. This is toric and has an action of $G = (S^1)^3 \times S_3$. The only G-invariant divisor is $-K_X \sim \sum_{E_i^2=-1} E_i$, and $(X, \sum E_i)$ is log canonical, hence $\alpha_G(X) = 1$. Therefore, X is K-polystable.*

Example 3.13. *Let X be a del Pezzo surface of degree 5, so $X \cong \overline{\mathcal{M}}_{0,5}$ and $G = S_5$ acts on X. Chelstov has shown that $\alpha_G(X) = 2$, hence is K-polystable. Furthermore, because X has a finite group of automorphisms, X is actually K-stable.*

Example 3.14. *Let $X = \mathbb{P}^n$ and $G = \mathrm{U}(n+1)$, the maximal compact subgroup of $\mathrm{PGL}(n+1)$. There are no G-invariant divisors, hence $\alpha_G(X) = \infty$, so $\mathbb{P}^n$ is K-polystable.*

We introduce another notion useful for checking K-stability and a few related results about stability of varieties of Fano index 1.

Definition 3.15. *Let X be a $\mathbb{Q}$-Fano variety. We say X is **free of log maximal singularity** or **LMS free** if, for all $m \in \mathbb{N}_{\geq 0}$, for all movable $\mathcal{M} \in |-mK_X|$, the pair $(X, \frac{1}{m}\mathcal{M})$ is log canonical.*

Remark 3.16. *If X is birationally superrigid,[6] it is LMS free.*

[5] There is not currently an algebraic proof of this equivariant criterion: Tian's proof uses analytic methods, so smoothness is necessary.

[6] We expect that 'most' mildly singular Fanos of index 1 are birationally superrigid.

Theorem 3.17 ([SZ19]). *If X is LMS free and $\alpha(X) > \frac{1}{2}$ (resp. $\geq$), then X is K-stable (resp. K-semistable).*

Proof: The proof is similar to that of Tian's α criterion, Theorem 3.4. □

Example 3.18. *Let $X = X_6 \subset \mathbb{P}(1,1,1,2,3)$, a general degree 6 hypersurface. Because $\mathcal{O}(6)$ is Cartier on $\mathbb{P}(1,1,1,2,3)$, a general X_6 is smooth. Note that $-K_X = 2H$, where H is a hyperplane section of $\mathbb{P}(1,1,1,2,3)$.*

We claim that X is LMS free and $\alpha(X) = \frac{1}{2}$. As a consequence of the previous theorem, this implies that X is K-semistable. In fact, one can do further analysis to conclude it is K-stable.

To verify the claim, let $M = \frac{1}{m}\mathcal{M}$ where $\mathcal{M} \subset |-mK_X|$ is a movable pencil. To show that (X, M) is log canonical, compute $M^2 \cdot H = 4 \geq \mathrm{mult}_x(M^2)$ where M^2 is considered as a codimension 2 cycle on X. Then, a result of Corti-de Fernex-Mustaţă implies that if X has dimension n, $x \in X$ is smooth, and $\mathfrak{a}$ an ideal cosupported at x, then $\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}_x\mathfrak{a} \geq n^n$, hence $\mathrm{lct}(X, M) = \mathrm{lct}(M^2) \geq 1$. Furthermore, $\alpha(X) = \frac{1}{2}$ because $-\frac{1}{2}K_X \sim H$ and $H^3 = 1$, using the same proof.

Theorem 3.19 ([LZ18]). *Let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of Fano index 1 of dimension n such that $n \geq 10r$. Then, X is birationally superridid, LMS free, and K-stable.*

Theorem 3.20 ([LZ18]). *Let $X \subset \mathbb{P}^{n+1}$ be a degree $n+1$ hypersurface with isolated ordinary[7] singularities of multiplicity $\leq n$. If $n \geq 250$, then X is K-stable.*

While we have given many criteria for checking when X is K-stable, one can ask how to determine if a variety is K-unstable. Here is one related result.

Theorem 3.21 ([Mat57]). *If X is K-polystable and smooth, then $\mathrm{Aut}(X)$ is reductive.*[8]

Example 3.22. *As a consequence, if X is a del Pezzo surface of degree 7, $\mathrm{Aut}(X)$ is not reductive, so X is K-unstable. One can find a torus invariant divisor E on X such that $\beta(E) < 0$.*

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[7] We say a singularity is ordinary if the projective tangent cone is smooth.

[8] Recently, Alper-Blum-Halpern-Leistner-Xu [ABHLX19] proved reductivity in general for log Fano K-polystable pairs (X, D).

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