Enumerating exceptional collections on some surfaces of general type with $p_g = 0$

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Abstract

We use constructions of surfaces as abelian covers to write down exceptional collections of maximal length for every surface X in certain families of surfaces of general type with $p_g = 0$ and $K_X^2 = 3, 4, 5, 6, 8$. We also compute the algebra of derived endomorphisms for an appropriately chosen exceptional collection, and the Hochschild cohomology of the corresponding quasiphantom category. As a consequence, we see that the subcategory generated by the exceptional collection does not vary in the family of surfaces.

1 Introduction

Exceptional collections of maximal length on surfaces of general type with $p_g = 0$ have been constructed for Godeaux surfaces [12] and [14], primary Burniat surfaces [2], and Beauville surfaces [23] and [35]. Recently, progress has also been made for some fake projective planes [24] and [22]. In this article, we present a method which can be applied uniformly to produce exceptional collections of line bundles on several different surfaces with $p_g = 0$, including Burniat surfaces with $K^2 = 6$ (cf. [2]), $5, 4, 3$, Kulikov surfaces with $K^2 = 6$ and some Beauville surfaces with $K^2 = 8$ (cf. [23], [35]). In fact we do more: we enumerate all exceptional collections of line bundles corresponding to any choice of numerical exceptional collection. We can use this enumeration process to find those exceptional collections that are particularly well-suited to studying the surface itself, and possibly its moduli space.

Both [2] and [23] hinted that it should be possible to produce exceptional collections of line bundles on a wide range of surfaces of general type with $p_g = 0$. This inspired us to build the approaches of [2] and [23] into a larger framework (see especially Sec. 2), an important part of which is a new formula for the pushforward of a line bundle on an abelian cover, generalising formulas in [39]. We believe that this is a step in the right direction, even though there are many families of surfaces which remain just out of reach (for example, see Sec. 6).

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Let X be a surface of general type with $p_g = 0$, and let Y be a del Pezzo surface with $K_Y^2 = K_X^2$. The groups $\operatorname{Pic} X/\operatorname{Tors} X$ and $\operatorname{Pic} Y$ are both isomorphic to $\mathbb{Z}^{1,N}$, where $N = 9 - K_X^2$, and moreover, the cohomology groups $H^2(X,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are completely algebraic. By exploiting this relationship between X and Y, we can study exceptional collections of line bundles on X. Indeed, exceptional collections on del Pezzo surfaces are well understood after [38], [30], and we sometimes refer to X as a fake del Pezzo surface, to emphasise this analogy.

Suppose now that X is a fake del Pezzo surface that is constructed as a branched Galois abelian cover $\varphi\colon X \to Y$, where Y is a (weak) del Pezzo surface with $K_Y^2 = K_X^2$. Many fake del Pezzo surfaces can be constructed in this way (cf. [9]), but we require certain additional assumptions on the branch locus and Galois group (see Sec. 3.1). These assumptions ensure that there is an appropriate choice of lattice isometry $\operatorname{Pic} Y \to \operatorname{Pic} X/\operatorname{Tors} X$. This isometry is combined with our pushforward formula to calculate the coherent cohomology of any line bundle on X.

Theorem 1.1 *Let X be a fake del Pezzo surface satisfying our assumptions, and let L be any line bundle on X. We have an explicit formula for the line bundles M_χ appearing in the pushforward $\varphi_* L = \bigoplus_{\chi \in G^*} M_\chi$, where G is the Galois group of the cover $\varphi\colon X \to Y$.*

Working modulo torsion, we can lift any exceptional collection of line bundles from Y to a numerical exceptional collection on X. We then incorporate Theorem 1.1 into a systematic computer search, to find those combinations of torsion twists which correspond to an exceptional collection on X.

Let $\mathbb{E}$ be an exceptional collection on X, and suppose $H_1(X,\mathbb{Z})$ is nontrivial. Then $\mathbb{E}$ can not be full, for K-theoretic reasons (see Sec. 4). Hence we have a semiorthogonal decomposition of the bounded derived category of coherent sheaves on X:

$$D^b(X) = \langle \mathbb{E}, \mathcal{A} \rangle.$$

If $\mathbb{E}$ is of maximal length, then $\mathcal{A}$ is called a quasiphantom category; that is, $K_0(\mathcal{A})$ is torsion and the Hochschild homology $HH_*(\mathcal{A})$ is trivial. Even when $H_1(X,\mathbb{Z})$ vanishes, an exceptional collection of maximal length need not be full (see [14]), and in this case $\mathcal{A}$ is called a phantom category, because $K_0(\mathcal{A})$ is trivial.

On the other hand, the Hochschild cohomology does detect the quasiphantom category $\mathcal{A}$; in fact, $HH^*(\mathcal{A})$ measures the formal deformations of $\mathcal{A}$. We calculate $HH^*(\mathcal{A})$ by considering the A_∞-algebra of endomorphisms of $\mathbb{E}$, together with the spectral sequence developed in [33]. Indeed, one of the advantages of our systematic search, is that we can find exceptional collections for which the higher multiplications in the A_∞-algebra of $\mathbb{E}$ are as simple as possible. Theorem 1.2 below serves as a prototype statement of our results for a good exceptional collection on a fake del Pezzo surface. More precise statements can be found in the text.

Theorem 1.2 *Let $\mathcal{X} \to T$ be a family of fake del Pezzo surfaces satisfying our assumptions. Then for any t in T, there is an exceptional collection $\mathbb{E}$ of line bundles on $X = \mathcal{X}_t$ which has maximal length $12 - K_X^2$. Moreover, the subcategory of $D^b(X)$ generated by $\mathbb{E}$ does not vary with t, and the Hochschild cohomology of X agrees with that of the quasi-phantom category $\mathcal{A}$ in degrees less than or equal to two.*

The significance of Theorem 1.2 is amplified by the reconstruction theorem of [16]: if X and X' are smooth, $\pm K_X$ is ample, and $D^b(X)$ and $D^b(X')$ are equivalent bounded derived categories, then $X \cong X'$. In conjunction with Theorem 1.2, we see that if K_X is ample, then X can be reconstructed from the quasi-phantom category $\mathcal{A}$. Currently, it is not clear whether there is any practical way to extract information about X from $\mathcal{A}$, although some interesting ideas are discussed in [2]. It would be interesting to know whether this "rigidity" of $\mathbb{E}$ is a general phenomenon, or just a coincidence for good choices of exceptional collection.

The study of exceptional collections of line bundles on fake del Pezzo surfaces leads naturally to the question of how to characterise effective divisors on X. For example, in [1], there is an explicit description of the semigroup of effective divisors on the Burniat surface with $K^2 = 6$, as well as possible descriptions for the other Burniat surfaces. We believe that Theorem 1.1 can be used to prove similar characterisations for the other fake del Pezzo surfaces considered in this article. Indeed, as a test case, we describe the semigroup of effective divisors on certain Beauville surfaces with $K^2 = 8$ in App. C.

In Section 2 we review abelian covers, and prove our result on pushforward of line bundles, which is used throughout. In Section 3.1, we explain our assumptions on X and its Galois covering structure $\varphi\colon X \to Y$, and describe our approach to enumerating exceptional collections on the surface of general type. Section 3.2 is an extended treatment of the Kulikov surface, which is a fake del Pezzo surface with $K^2 = 6$. We give a cursory review of dg-categories and A_∞-algebras in Section 4, as background to our discussion of quasi-phantom categories and the theory of heights from [33]. We then show to compute the A_∞-algebra and height of an exceptional collection on the Kulikov surface.

Section 5 considers the families of Burniat surfaces with $K^2 = 6, 5, 4, 3$. Exceptional collections of line bundles on Burniat surfaces with $K^2 = 6$ have already appeared in [2]. As the size of $\operatorname{Tors} X$ decreases, it becomes more difficult to find well-behaved exceptional collections. Thus we need to use the action of the Weyl group on $\operatorname{Pic} X/\operatorname{Tors} X$ (cf. Sec. 3.1) to find suitable exceptional collections on Burniat surfaces with $K^2 = 4, 3$. Finally, section 6 is a discussion of the Keum–Naie surface with $K^2 = 4$, showing what can go wrong when we tweak our assumptions.

Appendix A lists certain data relevant to the Kulikov surface example of Section 3.2. Appendix B is a reference for the calculations on Burniat surfaces in Section 5. The last Appendix C treats two Beauville surfaces with $K^2 = 8$, and should be compared with [23], [35].

In order to use results on deformations of each fake del Pezzo surface, we work over $\mathbb{C}$.

Remark 1.1 The calculation of $\varphi_* L$ according to Theorem 1.1 is elementary but repetitive; we include a few sample calculations to illustrate how to do it by hand, but when the torsion group becomes large, it is more practical to use computer algebra. Our enumerations of exceptional collections are obtained by simple exhaustive computer searches. We use Magma [11], and the annotated scripts are available from [19].

Acknowledgements I would like to thank Valery Alexeev, Ingrid Bauer, Gavin Brown, Fabrizio Catanese, Paul Hacking, Al Kasprzyk, Anna Kazanova, Alexander Kuznetsov, Miles Reid and Jenia Tevelev for helpful conversations or comments about this work.

2 Preliminaries

We collect together the relevant material on abelian covers. See especially [39], [7] or [31] for details. Unless stated otherwise, X and Y are normal projective varieties, with Y nonsingular. Let G be a finite abelian group acting faithfully on X with quotient $\varphi\colon X \to Y$. Write $\Delta = \sum \Delta_i$ for the branch locus of φ, where each Δ_i is a reduced, irreducible effective divisor on Y. The cover φ is determined by the group homomorphism

$$\Phi\colon \pi_1(Y - \Delta) \to H_1(Y - \Delta, \mathbb{Z}) \to G,$$

which assigns an element of G to the class of a loop around each irreducible component Δ_i of Δ. If Φ is surjective, then X is irreducible. The factorisation through $H_1(Y-\Delta, \mathbb{Z})$ arises because G is assumed to be abelian, so we only need to consider the map $\Phi\colon H_1(Y-\Delta,\mathbb{Z}) \to G$. For brevity, we refer to the loop around Δ_i by the same symbol, Δ_i.

Let $\widetilde{Y}$ be the blow up of Y at a point P where several branch components $\Delta_{i_1}, \ldots, \Delta_{i_k}$ intersect. Then there is an induced cover of $\widetilde{Y}$, and the image of the exceptional curve E under Φ is given by

$$\Phi(E) = \sum_{j=1}^{k} \Phi(\Delta_{i_j}). \tag{1}$$

Fix an irreducible reduced component Γ of Δ and denote $\Phi(\Gamma)$ by γ. Then the inertia group of Γ is the cyclic group $H \subset G$ generated by γ. Choosing the generator of $H^* = \mathrm{Hom}(H, \mathbb{C}^*)$ to be the dual character γ^*, we may identify H^* with $\mathbb{Z}/n$, where n is the order of γ. Composing the restriction map $\mathrm{res}\colon G^* \to H^*$ with this identification gives

$$G^* \to \mathbb{Z}/n,\ \chi \mapsto k,$$

where $\chi|_H = (\gamma^*)^k$ for some $0 \le k \le n-1$. On the other hand, given χ in G^* of order d, the evaluation map $\chi\colon G \to \mathbb{Z}/d$ satisfies

$$\chi(\gamma) = \tfrac{d}{n}\chi|_H(\gamma) = \tfrac{dk}{n}$$

as a residue class in $\mathbb{Z}/d$ (or as an integer between 0 and $d-1$).

The pushforward of $\varphi_*\mathcal{O}_X$ breaks into a direct sum of eigensheaves

$$\varphi_*\mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}. \tag{2}$$

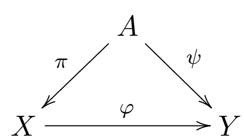
Moreover, the $\mathcal{L}_\chi$ are line bundles on Y and by Pardini [39], their associated (integral) divisors L_χ are given by the formula

$$dL_\chi = \sum_i \chi \circ \Phi(\Delta_i)\Delta_i. \tag{3}$$

The line bundles $\mathcal{L}_\chi$ play a pivotal role in the sequel, and we refer to them as the *character sheaves* of the cover $\varphi\colon X \to Y$.

2.1 Line bundles on X

We develop tools for calculating with torsion line bundles on X. Let $\pi\colon A \to X$ be the étale cover of X associated to the torsion subgroup $T = \operatorname{Tors} X = \pi_1(X)^{\mathrm{ab}} = H_1(X, \mathbb{Z})$. We call A the maximal abelian cover of X, and we have the following commutative diagram



Suppose the big cover $\psi\colon A \to Y$ is Galois with group $\widetilde{G}$, ramified over the same branch divisor Δ as $\varphi\colon X \to Y$. Then the original group G is the quotient $\widetilde{G}/T$, so we get short exact sequences

$$0 \to T \to \widetilde{G} \to G \to 0 \tag{4}$$

and

$$0 \leftarrow T^* \leftarrow \widetilde{G}^* \leftarrow G^* \leftarrow 0 \tag{5}$$

where $G^* = \operatorname{Hom}(G, \mathbb{C}^*)$, etc. In fact, for each surface that we consider, these exact sequences are split (although see sec. 6 for a slightly more tricky example), so that

$$\widetilde{G} = G \oplus T,\ \widetilde{G}^* = G^* \oplus T^*. \tag{6}$$

Let $\overline{\Gamma}$ be a reduced irreducible component of the branch locus Δ of an abelian cover $\varphi\colon X \to Y$ and suppose the inertia group of $\overline{\Gamma}$ is cyclic of order n. Then

Definition 2.1 *(see also [2]) The* reduced pullback Γ *of* $\overline{\Gamma}$ *is the (integral) divisor* $\Gamma = \frac{1}{n}\varphi^*(\overline{\Gamma})$ *on* X.

Remark 2.1 The reduced pullback extends to arbitrary linear combinations $\sum_i k_i\Delta_i$ in the obvious way. We use a bar to denote divisors on Y and remove the bar when taking the reduced pullback. In other situations, it is convenient to use D_i to denote the reduced pullback of a branch divisor Δ_i.

The remainder of this section is dedicated to calculating the pushforward $\varphi_*(L\otimes\tau)$, where $L=\mathcal{O}_X(\sum_i k_iD_i)$ is the line bundle associated to the reduced pullback of $\sum_i k_i\Delta_i$, and τ is any torsion line bundle on X. We do this by exploiting the association of the free part L with $\varphi\colon X\to Y$, and the torsion part τ with $\pi\colon A\to X$. The formulae that we obtain are a natural extension of results in [39]. It may be helpful to skip ahead to Examples 2.2.1 and 2.4.1 before reading this section in detail.

2.2 Free case

Until further notice, we write $\overline{\Gamma}\subset Y$ for an irreducible component of the branch divisor Δ of $\varphi\colon X\to Y$. By Pardini [39], the inertia group $H\subset G$ of $\overline{\Gamma}$ is cyclic, and H is generated by $\Phi(\overline{\Gamma})$ of order n. Let $\Gamma\subset X$ be the reduced pullback of $\overline{\Gamma}$, so that $n\Gamma=\varphi^*(\overline{\Gamma})$. We start with cyclic covers.

Lemma 2.1 *Let $\alpha\colon X\to Y$ be a cyclic cover with group $H\cong\mathbb{Z}/n$, and suppose that $\overline{\Gamma}$ is an irreducible reduced component of the branch divisor. Let Γ be the reduced pullback of $\overline{\Gamma}$, and suppose $0\le k\le n-1$. Then*

$$\alpha_*\mathcal{O}_X(k\Gamma)=\bigoplus_{i\in H^*-S}\mathcal{M}_i^{-1}\oplus\bigoplus_{i\in S}\mathcal{M}_i^{-1}(\overline{\Gamma}),$$

where $\mathcal{M}_i$ is the character sheaf associated to α with character $i\in H^*$, and

$$S=\{n-k,\dots,n-1\}\subset H^*\cong\mathbb{Z}/n.$$

Remark 2.2 If k is a multiple of n, say $k=pn$, the projection formula gives

$$\alpha_*\mathcal{O}_X(k\Gamma)=\alpha_*(\alpha^*\mathcal{O}_Y(p\overline{\Gamma}))=\alpha_*\mathcal{O}_X\otimes\mathcal{O}_Y(p\overline{\Gamma})=\bigoplus_{i\in H^*}\mathcal{M}_i^{-1}(p\overline{\Gamma}).$$

Thus the lemma extends to any integer multiple of Γ.

Proof After removing a finite number of points from $\overline{\Gamma}$, we may choose a neighbourhood U of $\overline{\Gamma}$ such that U does not intersect any other irreducible components of Δ. Then since X and Y are normal we may calculate $\alpha_*\mathcal{O}_X(k\Gamma)$ locally on $\alpha^{-1}(U)$ and U. In what follows, we do not distinguish U (respectively $\alpha^{-1}(U)$) from Y (resp. X).

Let $g = \Phi(\overline{\Gamma})$ so that $H = \langle g \rangle \cong \mathbb{Z}/n$, and identify H^* with $\mathbb{Z}/n$ via $g^* = 1$. Locally, write $\alpha\colon \alpha^{-1}(U) \to U$ as $z^n = b$ where $b = 0$ defines $\overline{\Gamma}$ in U. Then

$$\alpha_* \mathcal{O}_X = \bigoplus_{i=0}^{n-1} \mathcal{O}_Y z^i = \bigoplus_{i=0}^{n-1} \mathcal{O}_Y(-\tfrac{i}{n}\overline{\Gamma}) = \bigoplus_{i=0}^{n-1} \mathcal{M}_i^{-1},$$

where the last equality is given by (3). Thus $\alpha_*\mathcal{O}_X$ is generated by $1, z, \ldots, z^{n-1}$ as an $\mathcal{O}_Y$-module, and the $\mathcal{O}_Y$-algebra structure on $\alpha_*\mathcal{O}_X$ is induced by the equation $z^n = b$.

The calculation for $\mathcal{O}_X(k\Gamma)$ is similar,

$$\alpha_*\mathcal{O}_X(k\Gamma) = \alpha_*\mathcal{O}_X \frac{1}{z^k} = \bigoplus_{i=-k}^{n-k-1} \mathcal{O}_Y z^i = \bigoplus_{i=0}^{n-k-1} \mathcal{O}_Y z^i \oplus \bigoplus_{i=-k}^{-1} \mathcal{O}_Y \frac{z^{n+i}}{b}$$

where we use $z^n = b$ to remove negative powers of z. Thus

$$\begin{aligned} \alpha_*\mathcal{O}_X(k\Gamma) &= \bigoplus_{i=0}^{n-k-1} \mathcal{O}_Y(-\tfrac{i}{n}\overline{\Gamma}) \oplus \bigoplus_{i=n-k}^{n-1} \mathcal{O}_Y(-\tfrac{i}{n}\overline{\Gamma})(\overline{\Gamma}) \\ &= \bigoplus_{i \in H^*-S} \mathcal{M}_i^{-1} \oplus \bigoplus_{i \in S} \mathcal{M}_i^{-1}(\overline{\Gamma}), \end{aligned}$$

where $S = \{n-k, \ldots, n-1\}$. □

The lemma can be extended to any abelian group using arguments inspired by Pardini [39] sections 2 and 4.

Proposition 2.1 *Let $\varphi\colon X \to Y$ be an abelian cover with group G, and let $k = np + \overline{k}$, where $0 \le \overline{k} \le n-1$. Then*

$$\varphi_*\mathcal{O}_X(k\Gamma) = \bigoplus_{\chi \in G^* - S_{k\overline{\Gamma}}} \mathcal{L}_\chi^{-1}(p\overline{\Gamma}) \oplus \bigoplus_{\chi \in S_{k\overline{\Gamma}}} \mathcal{L}_\chi^{-1}((p+1)\overline{\Gamma}),$$

where

$$S_{k\overline{\Gamma}} = \{\chi \in G^* : n - \overline{k} \le \chi|_H \le n-1\}.$$

Proof By the projection formula, we only need to consider the case $k = \overline{k}$ (cf. Remark 2.2). As in the proof of Lemma 2.1, after removing a finite number of points, we may take a neighbourhood U of $\overline{\Gamma}$ which does not intersect any other components of Δ. We work on U and its preimages $\varphi^{-1}(U)$, $\beta^{-1}(U)$.

Factor $\varphi\colon X \to Y$ as

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y,$$

where α is a cyclic cover ramified over Γ with group $H = \langle g \rangle \cong \mathbb{Z}/n$, and β is unramified by our assumptions. As in Lemma 2.1 we denote the character sheaves of α by $\mathcal{M}_i$, and those of the composite map $\varphi = \beta \circ \alpha$ by $\mathcal{L}_\chi$. Now

$$\beta_* \mathcal{M}_i = \bigoplus_{\chi \in [i]} \mathcal{L}_\chi \tag{7}$$

where the notation $[i]$ means the preimage of i in H^* under the restriction map res: $G^* \to H^*$. That is,

$$[i] = \{\chi \in G^* : \chi|_H = i\},$$

where d is the order of χ. Since β is not ramified we combine Lemma 2.1 and (7) to get

$$\varphi_* \mathcal{O}_X(k\Gamma) = \bigoplus_{\chi \in G^* - S_{k\overline{\Gamma}}} \mathcal{L}_\chi^{-1} \oplus \bigoplus_{\chi \in S_{k\overline{\Gamma}}} \mathcal{L}_\chi^{-1}(\overline{\Gamma})$$

where

$$S_{k\overline{\Gamma}} = \{\chi \in G^* : n - k \leq \chi|_H \leq n - 1\}$$

is the preimage of $S = \{n-k, \ldots, n-1\} \subset H^*$ under res: $G^* \to H^*$. □

2.2.1 Example (Campedelli surface)

Let $\varphi \colon X \to \mathbb{P}^2$ be a $G = (\mathbb{Z}/2)^3$-cover branched over seven lines in general position. We label the lines $\Delta_1, \ldots, \Delta_7$, and define Φ to induce a set-theoretic bijection between $\{\Delta_i\}$ and $(\mathbb{Z}/2)^3 - \{0\}$. We make the definition of Φ more precise later (see Ex. 2.4.1). It is well known that X is a surface of general type with $p_g = 0$, $K^2 = 2$ and $\pi_1 = (\mathbb{Z}/2)^3$.

Choose generators g_1, g_2, g_3 for $(\mathbb{Z}/2)^3$ so that $\Phi(\Delta_1) = g_1$. There are eight character sheaves for the cover, which we calculate using formula (3),

$$\mathcal{L}_{(0,0,0)} = \mathcal{O}_{\mathbb{P}^2},\ \mathcal{L}_\chi = \mathcal{O}_{\mathbb{P}^2}(2) \text{ for } \chi \neq (0,0,0).$$

Write D_1 for the reduced pullback of Δ_1, so that $\varphi^*(\Delta_1) = 2D_1$. Then

$$S_{\Delta_1} = \{\chi : \chi|_{\langle g_1 \rangle} = 1\} = \{(1,0,0), (1,1,0), (1,0,1), (1,1,1)\},$$

so that by Proposition 2.1, we have

$$\varphi_* \mathcal{O}_X(D_1) = \mathcal{O}_{\mathbb{P}^2} \oplus 4\mathcal{O}_{\mathbb{P}^2}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^2}(-2).$$

2.3 Torsion case

In this section we use the maximal abelian cover A to calculate the pushforward of a torsion line bundle on X. We assume that the composite cover $A \to X \to Y$ is Galois with group $\widetilde{G}$.

Proposition 2.2 *Let τ be a torsion line bundle on X. Then*

$$\varphi_* \mathcal{O}_X(-\tau) = \bigoplus_{\chi \in G^*} \mathcal{L}_{\chi+\tau}^{-1}.$$

where addition $\chi + \tau$ takes place in $\widetilde{G}^* = G^* \oplus T^*$.

Remark 2.3 Note that $\mathcal{L}_{\chi+\tau}$ is a character sheaf for the $\widetilde{G}$-cover $\varphi\colon A \to Y$, and the proposition allows us to interpret $\mathcal{L}_{\chi+\tau}$ as a character sheaf for the G-cover $\varphi\colon X \to Y$. Unfortunately, there is still some ambiguity, because we do not determine which character in G^* is associated to each $\mathcal{L}_{\chi+\tau}$ under the splitting of exact sequence (5). On the other hand, the special case $\tau = 0$ gives

$$\varphi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_{\chi}^{-1}.$$

Proof The structure sheaf $\mathcal{O}_A$ decomposes into a direct sum of the torsion line bundles when pushed forward to X

$$\pi_* \mathcal{O}_A = \bigoplus_{\tau \in \operatorname{Tors} X} \mathcal{O}_X(-\tau).$$

Thus $\mathcal{O}_X(\tau)$ is the character sheaf with character τ under the identification $T^* \cong \operatorname{Tors} X$. The composite $\varphi_* \pi_* \mathcal{O}_A$ breaks into character sheaves according to (2), and the image of $\mathcal{O}_X(-\tau)$ is the direct sum of those character sheaves with character contained in the coset $G^* + \tau$ of τ in $\widetilde{G}^*$ under (6). □

2.4 General case

Now we combine Propositions 2.1 and 2.2 to give our formula for pushforward of line bundles $\mathcal{O}_X(\sum_i D_i) \otimes \tau$. The formula looks complicated, but most of the difficulty is in the notation.

Definition 2.2 Let n_i be the order of $\Psi(\Delta_i)$ in $\widetilde{G}$, and write $k_i = n_i p_i + \overline{k}_i$, where $0 \le \overline{k}_i \le n_i - 1$. Then given any subset $I \subset \{1, \dots, m\}$, we define

$$S_I[\tau] = \bigcap_{i \in I} S_{k_i \Delta_i}[\tau] \cap \bigcap_{j \in I^c} S_{k_j \Delta_j}[\tau]^c,$$

where

$$S_{k\overline{\Gamma}}[\tau] = \{\chi \in G^* \mid n - \overline{k} \leq \tfrac{n}{d}(\chi + \tau)(\Psi(\overline{\Gamma})) \leq n - 1\}$$

for any reduced irreducible component $\overline{\Gamma}$ of the branch locus Δ. Note that for fixed τ in T^*, the collection of all $S_I[\tau]$ partitions G^*.

Theorem 2.1 *Let $D = \sum_{i=1}^m k_i D_i$ be the reduced pullback of the linear combination of branch divisors $\sum_{i=1}^m k_i \Delta_i$ on Y. Then*

$$\varphi_* \mathcal{O}_X(D - \tau) = \bigoplus_I \bigoplus_{\chi \in S_I[\tau]} \mathcal{L}^{-1}_{\chi+\tau}(\Delta_I),$$

where I is any subset of $\{1, \ldots, m\}$ and $\Delta_I = \sum_{i \in I} \Delta_i$.

Remark 2.4 For simplicity, we have assumed that $k_i = \overline{k}_i$ for all i in the statement and proof of the theorem. When this is not the case, by the projection formula (cf. Remark 2.2) we twist by $\mathcal{O}_Y(\sum_{i=1}^m p_i \Delta_i)$.

Proof Fix i and let D_i be the reduced pullback of an irreducible component Δ_i of the branch divisor. Choose a neighbourhood of Δ_i which does not intersect any other Δ_j. This may also require us to remove a finite number of points from D_i. We work locally in this neighbourhood and its preimages under φ, π.

Now by the projection formula,

$$\pi_* \pi^* \mathcal{O}_X(k_i D_i) = \pi_* \mathcal{O}_A \otimes \mathcal{O}_X(k_i D_i),$$

and thus

$$\psi_* \pi^* \mathcal{O}_X(k_i D_i) = \bigoplus_{\tau \in \operatorname{Tors} X} \varphi_* \mathcal{O}_X(k_i D_i - \tau).$$

Then we combine Propositions 2.1 and 2.2 to obtain

$$\varphi_* \mathcal{O}_X(k_i D_i - \tau) = \bigoplus_{\chi \in G^* - S_{k_i \Delta_i}[\tau]} \mathcal{L}^{-1}_{\chi+\tau} \oplus \bigoplus_{\chi \in S_{k_i \Delta_i}[\tau]} \mathcal{L}^{-1}_{\chi+\tau}(\Delta_i),$$

where the indexing is explained in Definition 2.2.

To extend to the global setting and linear combinations $\sum k_i D_i$, we just need to keep track of which components of Δ should appear as a twist of each $\mathcal{L}^{-1}_{\chi+\tau}$ in the direct sum. This book-keeping is precisely the purpose of Definition 2.2. □

Using the formula

$$K_X = \varphi^*\big(K_Y + \sum_i \tfrac{n_i - 1}{n_i} \Delta_i\big) \tag{8}$$

and the Theorem, we give an alternative proof of the decomposition of $\varphi_* \mathcal{O}_X(K_X)$.

Corollary 2.1 *[39, Proposition 4.1] We have*

$$\varphi_* \mathcal{O}_X(K_X) = \bigoplus_{\chi \in G^*} \mathcal{L}_{\chi^{-1}}(K_Y).$$

Proof Let D_i be the reduced pullback of Δ_i. Then by (8) and the projection formula, we have

$$\begin{aligned}\varphi_*(\mathcal{O}_X(K_X)) &= \varphi_*\Big(\varphi^*\mathcal{O}_Y(K_Y)\otimes\mathcal{O}_X\Big(\sum_i(n_i-1)D_i\Big)\Big)\\ &= \mathcal{O}_Y(K_Y)\otimes\varphi_*\mathcal{O}_X\Big(\sum_i(n_i-1)D_i\Big).\end{aligned}$$

Now by definition,

$$S_{(n_i-1)\Delta_i} = \{\chi\in G^* : 1\le \tfrac{n_i}{d}\chi(\Phi(\Delta_i))\le n_i-1\} = \{\chi\in G^* : \chi(\Phi(\Delta_i))\neq 0\}.$$

Thus in the decomposition of $\varphi_*\mathcal{O}_X\big(\sum_i(n_i-1)D_i\big)$ given by Theorem 2.1, the summand $\mathcal{L}_\chi^{-1}$ is twisted by $\sum_{j\in J}\Delta_j$, where J is the set of indices j with $\chi(\Phi(\Delta_j))\neq 0$. Then by (3),

$$\mathcal{L}_\chi^{-1}\Big(\sum_{i\in J}\Delta_i\Big) = \sum_i(1-\tfrac{1}{d})\chi(\Phi(\Delta_i))\Delta_i = \mathcal{L}_{\chi^{-1}},$$

where the last equality is because $\chi^{-1}(g) = -\chi(g) = d-\chi(g)$ for any g in G. Thus we obtain

$$\varphi_*\Big(\mathcal{O}_X\Big(\sum_i(n_i-1)D_i\Big)\Big) = \bigoplus_{\chi\in G^*}\mathcal{L}_{\chi^{-1}},$$

and the Corollary follows. □

2.4.1 Example 2.2.1 continued

We resume our discussion of the Campedelli surface. The fundamental group of X is $(\mathbb{Z}/2)^3$, and so the maximal abelian cover $\pi\colon A\to X$ is a $(\mathbb{Z}/2)^6$-cover $\psi\colon A\to\mathbb{P}^2$ branched over Δ. Choose generators $g_1,\dots,g_6$ of $(\mathbb{Z}/2)^6$. As promised in Ex. 2.2.1, we now fix Φ and Ψ:

Δ_i	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7
$\Phi(\Delta_i)$	g_1	g_2	g_3	g_1+g_2	g_1+g_3	g_2+g_3	$g_1+g_2+g_3$
$\Psi(\Delta_i)-\Phi(\Delta_i)$	0	0	0	g_4	g_5	g_6	$g_4+g_5+g_6$

For clarity, the table displays the difference between $\Psi(\Delta_i)$ and $\Phi(\Delta_i)$. In order that A be the maximal abelian cover, Ψ is defined so that each $\Psi(\Delta_i)$ generates a distinct summand of $(\mathbb{Z}/2)^6$, excepting $\Psi(\Delta_7)$, which is chosen so that $\sum_i\Psi(\Delta_i)=0$. This last equality is induced by the relation $\sum_i\Delta_i = 0$ in $H_1(\mathbb{P}^2-\Delta,\mathbb{Z})$.

The torsion group $\operatorname{Tors}X$ is generated by g_4^*, g_5^*, g_6^*. As an illustration of Theorem 2.1, we calculate $\varphi_*\mathcal{O}_X(D_1)\otimes\tau$, where τ is the torsion line bundle on X associated to g_4^*.

Suppose $\varphi_* \mathcal{O}_X(D_1) \otimes \tau = \bigoplus_{\chi \in G^*} \mathcal{M}_\chi$, where $\mathcal{M}_\chi$ are the line bundles to be calculated. In the table below, we collect the data relevant to Theorem 2.1.

χ	$\mathcal{L}_{\chi+\tau}^{-1}$	$(\chi + \tau) \circ \Psi(D_1)$	Twist by Δ_1?	$\mathcal{M}_\chi$
$(0,0,0)$	$\mathcal{O}_{\mathbb{P}^2}(-1)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$(1,0,0)$	$\mathcal{O}_{\mathbb{P}^2}(-1)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}$
$(0,1,0)$	$\mathcal{O}_{\mathbb{P}^2}(-1)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$(0,0,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$(1,1,0)$	$\mathcal{O}_{\mathbb{P}^2}(-3)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$(1,0,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$(0,1,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	0	No	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$(1,1,1)$	$\mathcal{O}_{\mathbb{P}^2}(-2)$	1	Yes	$\mathcal{O}_{\mathbb{P}^2}(-1)$

Summing the last column of the table, we get

$$\varphi_* \mathcal{O}_X(D_1) \otimes \tau = \mathcal{O}_{\mathbb{P}^2} \oplus 4\mathcal{O}_{\mathbb{P}^2}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^2}(-2).$$

In particular, we see that the linear system on X associated to the line bundle $\mathcal{O}_X(D_1) \otimes \tau$ contains a single effective divisor.

3 Exceptional collections on surfaces with $p_g = 0$

3.1 Overview and definitions

We outline our method for producing exceptional collections, starting with some definitions and fundamental observations.

Definition 3.1 *An object E in $D^b(X)$ is called* exceptional *if*

$$\mathrm{Ext}^k(E, E) = \begin{cases} \mathbb{C} & \textit{if } k = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

An exceptional collection $\mathbb{E} \subset D^b(X)$ *is a sequence of exceptional objects* $\mathbb{E} = (E_0, \ldots, E_n)$ *such that if* $0 \leq i < j \leq n$ *then*

$$\mathrm{Ext}^k(E_j, E_i) = 0 \textit{ for all } k.$$

It follows from Def. 3.1 that any line bundle on a surface with $p_g = q = 0$ is exceptional. Moreover, if $\mathbb{E}$ is an exceptional collection of line bundles, and L is any line bundle, then $\mathbb{E} \otimes L = (E_0 \otimes L, \ldots, E_n \otimes L)$ is again an exceptional collection, so we always renormalise $\mathbb{E}$ so that $E_0 = \mathcal{O}_X$.

Let $\mathcal{E} = \langle \mathbb{E} \rangle$ denote the smallest full triangulated subcategory of $D^b(X)$ containing all objects in $\mathbb{E}$. Then $\mathcal{E}$ is an admissible subcategory of $D^b(X)$, and so we have a *semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{E}, \mathcal{A} \rangle,$$

where $\mathcal{A}$ is the left orthogonal to $\mathcal{E}$. That is, $\mathcal{A}$ consists of all objects F in $D^b(X)$ such that $\operatorname{Ext}^k(F, E) = 0$ for all k and for all E in $\mathcal{E}$. The K-theory is additive across semiorthogonal decompositions:

Proposition 3.1 *If $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then*

$$K_0(X) = K_0(\mathcal{A}) \oplus K_0(\mathcal{B}).$$

Moreover, if $\mathbb{E}$ is an exceptional collection of length n, then $K_0(\mathcal{E}) = \mathbb{Z}^n$. Thus if $K_0(X)$ is not free, then X can never have a full exceptional collection. The maximal length of an exceptional collection on X is less than or equal to the rank of $K(X)$.

3.1.1 Exceptional collections on del Pezzo surfaces

Let Y be the blow up of $\mathbb{P}^2$ in n points, and write H for the pullback of the hyperplane section, $\overline{E}_i$ for the ith exceptional curve. Then by work of Kuleshov and Orlov [38], [30] there is an exceptional collection of sheaves on Y

$$\mathcal{O}_{\overline{E}_1}(-1), \ldots, \mathcal{O}_{\overline{E}_n}(-1), \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H).$$

Note that the blown up points do not need to be in general position, and can even be infinitely near. We prefer an exceptional collection of line bundles on Y, so we mutate past $\mathcal{O}_Y$ to get

$$\mathcal{O}_Y,\ \mathcal{O}_Y(\overline{E}_1), \ldots, \mathcal{O}_Y(\overline{E}_n),\ \mathcal{O}_Y(H),\ \mathcal{O}_Y(2H). \tag{9}$$

In fact, we only use the numerical properties of a given exceptional collection of line bundles on Y. Choose a basis $e_0, \ldots, e_n$ for the lattice $\operatorname{Pic} Y \cong \mathbb{Z}^{1,n}$ with intersection form $\operatorname{diag}(1, -1^n)$. Then we write equation (9) numerically as

$$0,\ e_1, \ldots, e_n,\ e_0,\ 2e_0.$$

3.1.2 From del Pezzo to general type

Let X be a surface of general type with $p_g = 0$ which admits an abelian cover $\varphi\colon X \to Y$ of a del Pezzo surface Y with $K_Y^2 = K_X^2$. In addition, we suppose the maximal abelian cover $A \to X \to Y$ is also Galois. The branch divisor is $\Delta = \sum_i \Delta_i$ and we assume that Δ is sufficiently reducible so that

(A1) $\operatorname{Pic} Y$ is generated by integral linear combinations of Δ_i.

Now the Picard lattices of X and Y are isomorphic. Thus if G is not too complicated, e.g. of the form $\mathbb{Z}/p \times \mathbb{Z}/q$, we might hope to have:

(A2) The reduced pullbacks D_i of Δ_i (see Def. 2.1) generate $\operatorname{Pic} X/\operatorname{Tors} X$.

In very good cases, reduced pullback actually induces an isometry of lattices

(A3) $f\colon \operatorname{Pic} Y \to \operatorname{Pic} X/\operatorname{Tors} X$, such that $f(K_Y) = -K_X$ modulo $\operatorname{Tors} X$.

We say that a surface satisfies assumption (A) if (A1), (A2) and (A3) hold. These conditions are quite strong, and are not strictly necessary for our methods. For example, we can replace (A3) with an isometry of lattices from the abstract lattice $\mathbb{Z}^{1,n}$ to $\operatorname{Pic} X/\operatorname{Tors} X$. See Section 6 for such an example.

Definition 3.2 *A sequence $\mathbb{E} = (E_0, \dots, E_n)$ of line bundles on X is called* numerically exceptional *if $\chi(E_j, E_i) = 0$ whenever $0 \le i < j \le n$.*

Assume X satisfies (A), and let $(\Lambda_i) = (\Lambda_0, \dots, \Lambda_n)$ be an exceptional collection on Y. Now define $(L_i) = (L_0, \dots, L_n)$ by $L_i = f(\Lambda_i)^{-1}$. A calculation with the Riemann–Roch formula shows that (L_i) is a numerically exceptional collection on X. This is explained in [2].

Given a numerically exceptional collection (L_i) of line bundles on X, the remaining obstacle is to determine whether (L_i) is genuinely exceptional rather than just numerically so. Indeed, most numerically exceptional collections on X are not exceptional. The standard trick (see [12]) is to choose torsion line bundles τ_i in such a way that the twisted sequence $(L_i \otimes \tau_i)$ is an exceptional collection. We examine these choices of τ_i more carefully in what follows.

3.1.3 Acyclic line bundles

We discuss acyclic line bundles following [23].

Definition 3.3 *Let L be a line bundle on X. If $H^i(X, L) = 0$ for all i, then we call L an* acyclic *line bundle. We define the acyclic set associated to L to be*

$$\mathcal{A}(L) = \left\{ \tau \in \operatorname{Tors} X \mid h^i(L(\tau)) = 0 \text{ for all } 0 \le i \le 2 \right\}.$$

We call L numerically acyclic *if $\chi(X, L) = 0$. Clearly, an acyclic line bundle must be numerically acyclic.*

Remark 3.1 In the notation of [23], $\tau = -\chi$.

Lemma 3.1 *[23, Lemma 3.4] A numerically exceptional collection $L_0 = \mathcal{O}_X, L_1(\tau_1), \dots, L_n(\tau_n)$ on X is exceptional if and only if*

$$\begin{aligned} -\tau_i &\in \mathcal{A}(L_i^{-1}) \text{ for all } i, \text{ and} \\ \tau_i - \tau_j &\in \mathcal{A}(L_j^{-1} \otimes L_i) \text{ for all } j > i. \end{aligned} \tag{10}$$

Thus to enumerate all exceptional collections on X of a particular numerical type, it suffices to calculate the relevant acyclic sets, and systematically test the above conditions (10) on all possible combinations of τ_i.

3.1.4 Calculating cohomology of line bundles

Given a torsion twist $L(\tau)$, Theorem 2.1 gives a decomposition

$$\varphi_* L(\tau) = \bigoplus_{\chi \in G^*} \mathcal{M}_\chi,$$

for some line bundles $\mathcal{M}_\chi$ on Y, which may be computed explicitly. Since φ is finite, we have

$$h^p(L(\tau)) = \sum_{\chi \in G^*} h^p(\mathcal{M}_\chi)$$

for all p.

Thus $L(\tau)$ is acyclic if and only if each summand $\mathcal{M}_\chi$ is acyclic on Y. Now if $\chi(Y, \mathcal{M}_\chi) = 0$ and $h^0(\mathcal{M}_\chi) = h^2(\mathcal{M}_\chi) = 0$, we see that $h^1(\mathcal{M}_\chi) = 0$. Thus by Serre duality and the Riemann–Roch theorem, we are reduced to calculating Euler characteristics and determining effectivity for (lots of) divisor classes on the del Pezzo surface Y.

3.1.5 Coordinates on $\operatorname{Pic} X / \operatorname{Tors} X$

Under assumption (A), we make the following definition.

Definition 3.4 *Choose a basis $B_1, \ldots, B_n$ for $\operatorname{Pic} X / \operatorname{Tors} X$ consisting of linear combinations of reduced pullbacks. Then any line bundle L on X may be written uniquely as*

$$L = \mathcal{O}_X(d_1, \ldots, d_n)(\tau)$$

so that $L = \mathcal{O}_X\big(\sum_{i=1}^n d_i B_i\big)(\tau)$. We call d (respectively τ) the multidegree (resp. torsion twist) of L with respect to the chosen basis.

The torsion twist associated to any line bundle on X may be calculated using Theorem 2.1 and the following lemma. See Lemma 3.4 for an example.

Lemma 3.2 *If τ is a torsion line bundle, then $h^0(\tau) \neq 0$ implies $\tau = 0$.*

Remark 3.2 Definition 3.4 fixes a basis for $\operatorname{Pic} Y = \mathbb{Z}^{1,9-K^2}$ via the isometry with $\operatorname{Pic} X / \operatorname{Tors} X$. This basis corresponds to a geometric marking on the del Pezzo surface Y, and the multidegree d of L is just the image of L in $\operatorname{Pic} Y$ under the isometry. In fixing our basis, we break some of the symmetry of the coordinates. This is necessary in order to use the computer to search for exceptional collections. We can recover the symmetry later using the Weyl group action (see section 3.1.7).

3.1.6 Determining effectivity of divisor classes

Method 1 If Y is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ and D is a divisor on Y, it is not difficult to write an algorithm which determines whether D is effective or not, using only the (bi)degree of D. As $(-K_Y)^2$ decreases, the question gets more complicated. Our computer scripts [19] produce a list of a few thousand "small" positive integral linear combinations of (-1)-curves. Since the effective cone of a del Pezzo surface of degree ≤ 7 is polyhedral with extremal rays generated by the (-1)-curves (cf. [21], Chapter 8), this list serves as a criterion for checking effectivity of line bundles on X (see Sec. 3.1.4). Weak del Pezzo surfaces are treated in a similar way.

Method 2 An alternative approach is to try to characterise effective divisors on the surface X of general type (see Remark 3.3 and [1]). Let $\mathfrak{E}$ denote the semigroup generated by the reduced pullbacks D_i of the irreducible branch components Δ_i, and pullbacks of the other (-1)-curves on Y. Then $\mathfrak{E}$ approximates the semigroup of all effective divisors on X (the two are proven to be equal in some cases, cf. [1] and App. C). Moreover, $\mathfrak{E}$ is graded by multidegree, and we have a homomorphism $\mathfrak{E} \to \operatorname{Tors} X$ sending D_i to its torsion twist under Def. 3.4. The image of the graded summand $\mathfrak{E}_d$ of multidegree d approximates the set of torsion twists τ for which $\mathcal{O}_X(\sum d_i B_i)(\tau)$ is effective. We have implemented this method for various surfaces, and checked that the output is consistent with that of Method 1 above.

See the proofs of Proposition 3.2 below for a comparison of the two methods described above, and also Appendix C.

3.1.7 Group actions on the set of exceptional collections

We consider a dihedral group action and the Weyl group action on the set of exceptional collections on X. Mutations are not considered systematically in this article, since a mutation of a line bundle need not be a line bundle.

Let $\mathbb{E} = (E_1, \dots, E_n)$ be an exceptional collection of line bundles on X. Then the sequence $(E_2, \dots, E_n, E_1(-K_X))$ is also an exceptional collection, and if we renormalise the first line bundle of any exceptional collection to be $\mathcal{O}_X$, then this operation has order n.

There is also an involution on the set of exceptional collections of line bundles, which sends $\mathbb{E} = (E_1, \dots, E_n)$ to $\mathbb{E}^{-1} = (E_n^{-1}, \dots, E_1^{-1})$. Clearly, $\mathbb{E}^{-1}$ is an exceptional collection, and the two operations described generate a dihedral group action on the set of exceptional collections of length n.

The Weyl group of $\operatorname{Pic} Y$ is generated by reflections in (-2)-classes. That is, suppose α is a class in $\operatorname{Pic} Y$ with $K_Y \cdot \alpha = 0$ and $\alpha^2 = -2$. Then

$$r_\alpha \colon L \mapsto L + (L \cdot \alpha)\alpha$$

is a reflection on $\operatorname{Pic} Y$ which fixes K_Y. Any reflection sends an exceptional collection on Y to another exceptional collection. Thus by assumption (A), the Weyl group acts on *numerically* exceptional collections on X. This action accounts for the choices made in giving Y a geometric marking (see Def. 3.4).

3.2 The Kulikov surface with $K^2 = 6$

For details on the Kulikov surface (first described in [31]), its torsion group and moduli space, see [18]. The Kulikov surface X is a $(\mathbb{Z}/3)^2$-cover of the del Pezzo surface Y of degree 6. Figure 1 shows the associated cover of $\mathbb{P}^2$ branched over six lines in special position. The configuration has just one free parameter, and in fact, the Kulikov surfaces form a 1-dimensional, irreducible, connected component of the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 6$.

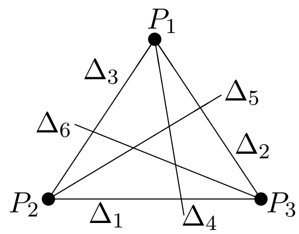


Figure 1: The Kulikov configuration

To obtain a nonsingular cover, we blow up the plane at three points P_1, P_2, P_3, giving a $(\mathbb{Z}/3)^2$-cover of a del Pezzo surface of degree 6. The exceptional curves are denoted $\overline{E}_i$. By results of [18], the torsion group $\operatorname{Tors}(X)$ is isomorphic to $(\mathbb{Z}/3)^3$, so the maximal abelian cover $\psi\colon A \to Y$ has group $\widetilde{G} \cong (\mathbb{Z}/3)^5$. Let g_i generate $\widetilde{G}$, and write g_i^* for the dual generators of $\widetilde{G}^*$. As explained in Section 2, the covers are determined by $\Phi\colon H_1(\mathbb{P}^2 - \Delta, \mathbb{Z}) \to G$ and $\Psi\colon H_1(\mathbb{P}^2 - \Delta, \mathbb{Z}) \to \widetilde{G}$, which are defined in the table below.

D	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
$\Phi(D)$	g_1	g_1	g_1	g_2	$g_1 + g_2$	$2g_1 + g_2$
$\Psi(D) - \Phi(D)$	0	g_3	$2g_3 + g_4$	$2g_4$	g_5	$2g_5$

The images of the exceptional curves $\overline{E}_i$ under Φ and Ψ are computed using formula (1):

$$\Phi(\overline{E}_1) = 2g_1 + g_2,\ \Phi(\overline{E}_2) = g_2,\ \Phi(\overline{E}_3) = g_1 + g_2,\ \text{etc.}$$

Lemma 3.3 *The Kulikov surface satisfies assumptions (A1) and (A2). That is, the free part of* $\operatorname{Pic} X$ *is generated by the reduced pullbacks of* $\Delta_1 + \overline{E}_2 + \overline{E}_3$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, *and the intersection pairing* $\operatorname{diag}(1, -1, -1, -1)$ *is inherited from* Y.

Proof Define $e_0 = D_1 + E_2 + E_3$, $e_1 = E_1$, $e_2 = E_2$, $e_3 = E_3$ in $\operatorname{Pic} X$. These are integral divisors, since they are reduced pullbacks, and the intersection pairing is $\operatorname{diag}(1, -1, -1, -1)$, which is unimodular. For example, by definition of reduced pullback, $3e_0 = \varphi^*(\Delta_1 + \overline{E}_2 + \overline{E}_3)$, and so

$$(3e_0)^2 = \varphi^*(\Delta_1 + \overline{E}_2 + \overline{E}_3)^2 = 9 \cdot 1,$$

or $e_0^2 = 1$. Hence we have an isomorphism of lattices. □

Using the basis chosen in this lemma, we compute the coordinates (Def. 3.4) of the reduced pullback D_i of each irreducible branch component Δ_i.

Lemma 3.4 *We have*

$$\mathcal{O}_X(D_1) = \mathcal{O}_X(1, 0, -1, -1), \qquad \mathcal{O}_X(D_4) = \mathcal{O}_X(1, -1, 0, 0)[2, 1, 2],$$
$$\mathcal{O}_X(D_2) = \mathcal{O}_X(1, -1, 0, -1)[1, 0, 2], \qquad \mathcal{O}_X(D_5) = \mathcal{O}_X(1, 0, -1, 0)[2, 1, 0],$$
$$\mathcal{O}_X(D_3) = \mathcal{O}_X(1, -1, -1, 0)[2, 0, 2], \qquad \mathcal{O}_X(D_6) = \mathcal{O}_X(1, 0, 0, -1)[2, 1, 1],$$

where $[a, b, c]$ *in* $(\mathbb{Z}/3)^3$ *denotes a torsion line bundle on* X*.*

Proof We prove that $\mathcal{O}_X(D_2) = \mathcal{O}_X(1, -1, 0, -1)[1, 0, 2]$. The other cases are similar. It is clear that $\Delta_2 \sim \Delta_1 - \overline{E}_1 + \overline{E}_2$ on Y, so the multidegree is correct. It remains to check the torsion twist, by showing that $\mathcal{F} = \mathcal{O}_X(D_2 - D_1 + E_1 - E_2 - \tau)$ has a global section when $\tau = [1, 0, 2]$. Then by Lemma 3.2, we have the desired equality.

The pushforward $\varphi_*\mathcal{F}$ splits into a direct sum of line bundles $\bigoplus \mathcal{M}_\chi$, one for each character $\chi = (a, b)$ in G^*. The following table collects the data required to calculate each $\mathcal{M}_\chi$ via Theorem 2.1. The second column is calculated using equation (3), and the next four columns evaluate $\chi + \tau$ on each $\Psi(\Gamma)$, where Γ is any one of Δ_1, Δ_2, $\overline{E}_1$ and $\overline{E}_2$. The final column is explained below.

χ	$\mathcal{L}_{\chi+\tau}^{-1}$	$(\chi + \tau) \circ \Psi(\Gamma)$				$\mathcal{M}_\chi$
		Δ_1	Δ_2	$\overline{E}_1$	$\overline{E}_2$	
$(0,0)$	$\mathcal{O}_Y(-2,1,1,0)$	0	1	0	1	$\mathcal{O}_Y(-3,1,2,1)$
$(1,0)$	$\mathcal{O}_Y(-1,0,0,1)$	1	2	2	1	$\mathcal{O}_Y$
$(0,1)$	$\mathcal{O}_Y(-2,1,0,1)$	0	1	1	2	$\mathcal{O}_Y(-3,1,1,2)$
$(2,0)$	$\mathcal{O}_Y(-2,0,1,1)$	2	0	1	1	$\mathcal{O}_Y(-2,0,1,1)$
$(1,1)$	$\mathcal{O}_Y(-2,1,0,1)$	1	2	0	2	$\mathcal{O}_Y(-1,0,0,0)$
$(0,2)$	$\mathcal{O}_Y(-2,1,1,0)$	0	1	2	0	$\mathcal{O}_Y(-3,2,1,1)$
$(2,1)$	$\mathcal{O}_Y(-2,0,1,0)$	2	0	2	2	$\mathcal{O}_Y(-2,1,1,0)$
$(1,2)$	$\mathcal{O}_Y(-3,1,1,1)$	1	2	1	0	$\mathcal{O}_Y(-2,0,0,0)$
$(2,2)$	$\mathcal{O}_Y(-2,1,1,1)$	2	0	0	0	$\mathcal{O}_Y(-2,1,0,1)$

Now by the projection formula (cf. Remark 2.2),

$$\varphi_*\mathcal{F} = \varphi_*\mathcal{O}_X(2D_1 + D_2 + E_1 + 2E_2 - \tau) \otimes \mathcal{O}_Y(-\Delta_1 - \overline{E}_2).$$

So according to Thm. 2.1 and the remark following it, each $\mathcal{M}_\chi$ is a twist of $\mathcal{L}^{-1}_{\chi+\tau}(-\Delta_1-\overline{E}_2)$ by a certain combination of Δ_1, Δ_2, $\overline{E}_1$ and $\overline{E}_2$. By Def. 2.2, the rules governing the twists are:

$$\begin{aligned}
&\text{twist by } \Delta_1 \text{ iff } (\chi+\tau)\circ\Psi(\Delta_1) = 1 \text{ or } 2\\
&\text{twist by } \Delta_2 \text{ iff } (\chi+\tau)\circ\Psi(\Delta_2) = 2\\
&\text{twist by } \overline{E}_1 \text{ iff } (\chi+\tau)\circ\Psi(\overline{E}_1) = 2\\
&\text{twist by } \overline{E}_2 \text{ iff } (\chi+\tau)\circ\Psi(\overline{E}_2) = 1 \text{ or } 2.
\end{aligned}$$

Thus $\varphi_*\mathcal{F}$ is given by the direct sum of the line bundles $\mathcal{M}_\chi$ listed in the final column. Note that $\mathcal{M}_{(1,0)} = \mathcal{O}_Y$, so $h^0(\varphi_*\mathcal{F}) = 1$. Hence $D_2 - D_1 + E_1 - E_2 - \tau \sim 0$. □

Corollary 3.1 *By formula* (8)*, we have*

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(3,-1,-1,-1)[0,0,2].$$

Thus the Kulikov surface satisfies (A3).

Proof The multidegree is clear by (8), but the torsion twist requires some care. Since $\mathcal{O}_X(K_X)$ is the pullback of an integral divisor on X, it should be torsion-neutral with respect to our coordinate system on $\operatorname{Pic} X$. Thus by Lemma 3.4, we see that the required twist is $[0,0,2]$. □

Remark 3.3 It is likely that every effective divisor on the Kulikov surface is a positive integral linear combination of

$$D_1, \ldots, D_6, E_1, E_2, E_3.$$

See Section 3.1.6 for some discussion of this, App. C for analogous results on two Beauville surfaces with $K^2 = 8$, and [1] for the primary Burniat surface with $K^2 = 6$.

3.2.1 Acyclic line bundles on the Kulikov surface

Let us start with the following numerical exceptional collection on Y, which has 3-block structure according to [26]:

$$\Lambda\colon 0,\ e_0 - e_1,\ e_0 - e_2,\ e_0 - e_3,\ 2e_0 - \textstyle\sum_{i=1}^3 e_i,\ e_0.$$

Given assumptions (A), we see that Λ corresponds to the following numerically exceptional sequence of line bundles on X:

$$\begin{aligned} L_0 = \mathcal{O}_X,\ L_1 = \mathcal{O}_X(-1,1,0,0), L_2 = \mathcal{O}_X(-1,0,1,0), \\ L_3 = &\mathcal{O}_X(-1,0,0,1),\ L_4 = \mathcal{O}_X(-2,1,1,1),\ L_5 = \mathcal{O}_X(-1,0,0,0). \end{aligned} \tag{11}$$

We find all collections of torsion twists $L_i(\tau_i)$ which are exceptional collections on X. The first step is to find the acyclic sets associated to the various $L_j^{-1} \otimes L_i$.

Proposition 3.2 *The acyclic sets $\mathcal{A}(L_j^{-1} \otimes L_i)$ for $j > i \geq 0$ are listed in Appendix A.*

First Proof Assuming the assertion of Remark 3.3 is correct, it is an easy exercise to check each entry in the table. As an illustration, we calculate $\mathcal{A}(L_1^{-1})$. The effective divisors on X of multidegree $(1,-1,0,0)$ are

$$D_2 + E_3, D_3 + E_2, D_4.$$

Thus by Lemma 3.4, $[1,0,2], [2,0,2], [2,1,2]$ do not appear in $\mathcal{A}(L_1^{-1})$. Next we consider degree two cohomology via Serre duality. The effective divisors of multidegree $(2,0,-1,-1)$ are

$$\begin{gathered} 2D_1 + E_2 + E_3, D_1 + D_2 + E_1 + E_3, D_1 + D_3 + E_1 + E_2, \\ D_2 + D_3 + 2E_1, D_1 + D_4 + E_1, D_1 + D_5 + E_2, D_1 + D_6 + E_3, \\ D_2 + D_5 + E_1, D_3 + D_6 + E_1, D_5 + D_6. \end{gathered}$$

Thus $[0,0,2]$, $[2,0,0]$, $[1,0,0]$, $[0,0,1]$, $[1,2,0]$, $[1,2,2]$, $[1,2,1]$, $[0,2,0]$, $[2,2,2]$, $[2,1,1]$ can not appear in $\mathcal{A}(L_1^{-1})$. The acyclic set is made up of those elements of $\operatorname{Tors} X$ which do not appear in either of the two lists above. $\square$

Remark 3.4 This first proof implicitly uses the homomorphism of semigroups $\mathfrak{E} \to \operatorname{Tors} X$ described in sec. 3.1.6. We explain this approach in more detail for the Beauville surfaces in Appendix C.

Second Proof Given that the above argument assumed the claim of Remark 3.3 without proof, we should check that each element of $\mathcal{A}(L_1^{-1})$ is really acyclic. To do this, we use Theorem 2.1 repeatedly to calculate the cohomology of all possible torsion twists of L_1. This is done in our computer script [19]. $\square$

3.2.2 Exceptional collections on the Kulikov surface

We now find all exceptional collections on X which are numerically of the form (11). Lemma 3.1 reduces us to a simple search, which can be done systematically [19].

Theorem 3.1 *There are nine exceptional collections $L_0 = \mathcal{O}_X,\ L_1(\tau_1), \ldots, L_5(\tau_5)$ on X which are numerically of the form (11). They are given in Table 1 below. Each row lists the required torsion twists τ_i for $i = 1, \ldots, 5$ as elements of $(\mathbb{Z}/3)^3$.*

	τ_1	τ_2	τ_3	τ_4	τ_5
1	$[0,0,0]$	$[0,2,2]$	$[2,2,1]$	$[2,2,1]$	$[0,0,1]$
2	$[2,2,0]$	$[2,1,2]$	$[0,0,1]$	$[1,1,1]$	$[2,2,1]$
3	$[2,2,1]$	$[2,1,2]$	$[0,0,1]$	$[1,1,1]$	$[2,0,2]$
4	$[2,2,0]$	$[2,0,1]$	$[0,2,0]$	$[2,2,1]$	$[2,1,2]$
5	$[1,1,0]$	$[1,0,2]$	$[2,2,0]$	$[1,1,1]$	$[2,2,1]$
6	$[1,1,0]$	$[1,0,2]$	$[0,0,1]$	$[1,1,1]$	$[2,2,1]$
7	$[1,1,0]$	$[1,0,2]$	$[2,2,1]$	$[1,1,1]$	$[0,0,1]$
8	$[2,0,2]$	$[2,2,0]$	$[0,1,2]$	$[1,1,1]$	$[2,2,1]$
9	$[2,0,2]$	$[2,2,1]$	$[0,1,2]$	$[1,1,1]$	$[1,0,2]$

Table 1: Exceptional collections on the Kulikov surface

Remark 3.5 By Lem. 3.4, each line bundle in Table 1 can be written as a linear combination of $D_1, \ldots, D_6, E_1, E_2, E_3$. For instance, the second row of the table is

$$\mathcal{O}_X,\ \mathcal{O}_X(-D_2 + D_3 - D_4 + D_5 - D_6 + 2E_2 - 2E_3),$$
$$\mathcal{O}_X(-D_2 + 2D_3 - D_4 - D_6 + 3E_2 - 2E_3),\ \mathcal{O}_X(D_2 - 2D_3 + D_4 + D_5 - 2D_6 - E_2),$$
$$\mathcal{O}_X(-D_1 + D_3 - 2D_4 + E_2),\ \mathcal{O}_X(-D_2 + D_3 - D_4 - E_1 + E_2 - E_3).$$

Remark 3.6

1. The precise number of exceptional collections is not important. Rather, the fact that we have definitively enumerated all exceptional collections of numerical type Λ, means that we can sift through the list to find one with the most desirable properties.

2. Let Λ' be any translation of Λ under the Weyl group action of $A_1 \times A_2$ on $\operatorname{Pic} Y$. Then Λ' is another numerical exceptional collection on X (see Sec. 3.1.7), so we may enumerate exceptional collections on X of numerical type Λ'. For the Kulikov surface, each element of the orbit corresponds to either 9, 14, 18 or 24 exceptional

collections on X. Thus, the Weyl group action does not "lift" to X in a way which is compatible with the covering $X \to Y$. On occasion, this incompatibility is used to our advantage (see Sec. 5). We return to these exceptional collections in section 4.

4 Heights of exceptional collections

Let X be a surface of general type with $p_g = q = 0$, $\operatorname{Tors} X \neq 0$ with an exceptional collection of line bundles $\mathbb{E} = (E_0, \dots, E_{n-1})$. Write $\mathcal{E}$ for the smallest full triangulated subcategory of $D^b(X)$ containing $\mathbb{E}$. In this section we calculate some invariants of $\mathbb{E}$. The invariants we consider are essentially determined by the derived category, but we must enhance the derived category in order to make computations. For completeness, we discuss some background first.

4.1 Motivation from del Pezzo surfaces

Let Y be a del Pezzo surface and let $\mathbb{E}$ be a strong exceptional collection of line bundles on Y. Recall that $\mathbb{E}$ is *strong* if $\operatorname{Ext}^k(E_i, E_j) = 0$ for all i, j and for all $k > 0$. Then the derived endomorphism ring $H^*B = \operatorname{Ext}^*(T, T) = \bigoplus_{i,j} \operatorname{Hom}(E_i, E_j)$ is an associative algebra, and we have an equivalence of categories $\mathcal{E} \cong D^b(\text{mod-}H^*B)$ (see [15]). Here we have defined $T = \bigoplus_i E_i$.

From now on, we assume that $\mathbb{E}$ is an exceptional collection on a fake del Pezzo surface X, so that we do not have the luxury of choosing a strong exceptional collection. Instead, we recover $\mathcal{E}$ by studying the higher multiplications coming from the A_∞-algebra structure on H^*B.

4.2 Digression on dg-categories

We sketch the construction of a *differential graded* (or dg) enhancement $\mathcal{D}$ of $D^b(X)$. Objects in $\mathcal{D}$ are the same as those in $D^b(X)$, but morphisms $\operatorname{Hom}^\bullet_{\mathcal{D}}(F, G)$ form a chain complex, with differential d of degree $+1$. Composition of maps $\operatorname{Hom}^\bullet_{\mathcal{D}}(F, G) \otimes \operatorname{Hom}^\bullet_{\mathcal{D}}(G, H) \to \operatorname{Hom}^\bullet_{\mathcal{D}}(F, H)$ is a morphism of complexes (the Leibniz rule), and for any object F in $\mathcal{D}$, we require $d(\mathrm{id}_F) = 0$. For a precise definition of $\operatorname{Hom}^\bullet_{\mathcal{D}}(F, G)$, one could use the Čech complex, and we refer to [33] for details. The main point is that the cohomology of $\operatorname{Hom}^\bullet_{\mathcal{D}}(F, G)$ in degree k is $\operatorname{Ext}^k_{D^b(X)}(F, G)$, so in particular, we have $H^0(\operatorname{Hom}^\bullet_{\mathcal{D}}(F, G)) = \operatorname{Hom}_{D^b(X)}(F, G)$.

4.3 Hochschild homology

We first compute some additive invariants, only making implicit use of the dg-structure. The Hochschild homology of X is given by the Hochschild–Kostant–Rosenberg isomorphism

$$HH_k(X) \cong \bigoplus_p H^{p+k}(X, \Omega^p_X),$$

so $HH_0(X) = \mathbb{C}^{12-K^2}$ and $HH_k(X) = 0$ in all other degrees. Moreover, Hochschild homology is additive over semiorthogonal decompositions.

Theorem 4.1 *[32] If $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then*

$$HH_k(X) = HH_k(\mathcal{A}) \oplus HH_k(\mathcal{B}).$$

Assuming the Bloch conjecture on algebraic zero-cycles, we have

$$K_0(X) = \mathbb{Z}^{12-K^2} \oplus \operatorname{Tors} X,$$

and we note that K-theory is also additive over semiorthogonal decompositions (see Prop. 3.1).

Now for an exceptional collection of length n, $K_0(\mathcal{E}) = \mathbb{Z}^n$ and

$$HH_k(\mathcal{E}) = \begin{cases} \mathbb{C}^n & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the maximal length of $\mathbb{E}$ is at most $12 - K_X^2$, and such an exceptional sequence of maximal length effects a semiorthogonal decomposition $D^b(X) = \langle \mathcal{E}, \mathcal{A} \rangle$ with nontrivial semiorthogonal complement $\mathcal{A}$. We say that $\mathcal{A}$ is a *quasiphantom* category; by additivity, the Hochschild homology vanishes, but $K_0(\mathcal{A}) \supseteq \operatorname{Tors} X \neq 0$, so $\mathcal{A}$ can not be trivial.

4.4 Height

The Hochschild cohomology groups of X may be computed via the other Hochschild–Kostant–Rosenberg isomorphism (cf. [32]):

$$HH^k(X) = \bigoplus_{p+q=k} H^q(X, \Lambda^p T_X).$$

Thus for a surface of general type with $p_g = 0$, we have

$$HH^0(X) \cong H^0(\mathcal{O}_X) = \mathbb{C},\ HH^1(X) = 0,\ HH^2(X) \cong H^1(T_X),$$
$$HH^3(X) \cong H^2(T_X),\ HH^4(X) \cong H^0(2K_X) = \mathbb{C}^{1+K^2}.$$

Recall that the degree two (respectively three) Hochschild cohomology is the tangent space (resp. obstruction space) to the formal deformations of a category.

In principle, [33] gives an algorithm for computing $HH^*(\mathcal{A})$ using a spectral sequence and the notion of height of an exceptional collection. Moreover, by [33, Prop. 6.1], for an exceptional collection to be full, its height must vanish. Thus the height may be used to prove existence of phantom categories without reference to the K-theory. We outline the algorithm of [33] below.

Given an exceptional collection $\mathbb{E}$ on X, there is a long exact sequence (induced by a distinguished triangle)

$$\ldots \to NHH^k(\mathbb{E}, X) \to HH^k(X) \to HH^k(\mathcal{A}) \to NHH^{k+1}(\mathbb{E}, X) \to \ldots$$

where $NHH(\mathbb{E}, X)$ is the *normal Hochschild cohomology* of the exceptional collection $\mathbb{E}$. The normal Hochschild cohomology can be computed using a spectral sequence with first page

$$\mathbf{E}^1_{-p,q} = \bigoplus_{\substack{0 \leq a_0 < \cdots < a_p \leq n-1 \\ k_0 + \cdots + k_p = q}} \mathrm{Ext}^{k_0}(E_{a_0}, E_{a_1}) \otimes \cdots \otimes \mathrm{Ext}^{k_{p-1}}(E_{a_{p-1}}, E_{a_p}) \otimes \mathrm{Ext}^{k_p}(E_{a_p}, S^{-1}(E_{a_0})).$$

The spectral sequence relies on the dg-structure on $\mathcal{D}$; the initial differentials d' and d'' are induced by the differential on $\mathcal{D}$ and the composition map respectively, while the higher differentials are related to the A_∞-algebra structure on Ext-groups, (see Sec. 4.6).

The existing examples of exceptional collections on surfaces of general type with $p_g = 0$ suggest that $NHH^k(\mathbb{E}, X)$ vanishes for small k. Thus the *height* $h(\mathbb{E})$ of an exceptional collection $\mathbb{E} = (E_0, \ldots, E_{n-1})$ is defined to be the smallest integer m for which $NHH^m(\mathbb{E}, X)$ is nonzero. Alternatively, m is the largest integer such that the canonical restriction morphism $HH^k(X) \to HH^k(\mathcal{A})$ is an isomorphism for all $k \leq m-2$ and injective for $k = m-1$.

4.5 Pseudoheight

The height may be rather difficult to compute in practice, requiring a careful analysis of the Ext-groups of $\mathbb{E}$ and the maps in the spectral sequence. The pseudoheight is easier to compute and sometimes gives a good lower bound for the height.

Definition 4.1 *The* pseudoheight $ph(\mathbb{E})$ *of an exceptional collection* $\mathbb{E} = (E_0, \ldots, E_{n-1})$ *is*

$$ph(\mathbb{E}) = \min_{0 \leq a_0 \leq a_1 < \cdots < a_p \leq n-1} \big(e(E_{a_0}, E_{a_1}) + \cdots + e(E_{a_{p-1}}, E_{a_p}) + e(E_{a_p}, E_{a_0}(-K_X)) - p + 2 \big),$$

where $e(F, F') = \min\{i : \mathrm{Ext}^i(F, F') \neq 0\}$.

The pseudoheight is just the total degree of the first nonzero term in the first page of the spectral sequence, where the shift by 2 takes care of the Serre functor.

Consider the length $2n$ anticanonical extension of the sequence $\mathbb{E}$ (see also Sec. 3.1.7):

$$E_0, \ldots, E_{n-1}, E_n = E_0(-K_X), \ldots, E_{2n-1} = E_{n-1}(-K_X). \tag{12}$$

If the E_i are line bundles, then we have a numerical lower bound for the pseudoheight.

Lemma 4.1 *[33, Lem. 4.10, Lem. 5.1] If K_X is ample and $E_i \cdot K_X \geq E_j \cdot K_X$ for all $i < j$ and for all E_i, E_j in the anticanonically extended sequence* (12)*, then* $ph(\mathbb{E}) \geq 3$.

The numerical conditions required by the Lemma are not particularly stringent. For example, all the exceptional collections we have exhibited on the Kulikov surface in Sec. 3.2 have pseudoheight at least 3, even before we consider the Ext-groups more carefully.

Remark 4.1 If L is a line bundle, then $\dim \operatorname{Ext}^k(L, L(-K_X)) = h^{2-k}(2K_X)$ by Serre duality, which is the case $p = 0$ in Def. 4.1. Thus any exceptional collection of line bundles on a surface of general type with $p_g = 0$ has pseudoheight at most 4. Moreover, if $ph(\mathbb{E}) = 4$, then $h(\mathbb{E}) = 4$ by [33].

4.6 The A_∞-algebra of an exceptional collection

Let $\mathbb{E} = (E_0, \ldots, E_{n-1})$ be an exceptional collection on X, and define $T = \oplus_{i=0}^{n-1} E_i$. Then $B = \operatorname{Hom}^\bullet_{\mathcal{D}}(T, T)$ is a differential graded algebra via the dg-structure on $\mathcal{D}$ (see Sec. 4.2). It can be difficult to compute the dg-algebra structure on B directly, so we pass to the A_∞-algebra H^*B.

We discuss A_∞-algebras, referring to [27] for details and further references. An A_∞-*algebra* is a graded vector space $A = \bigoplus_{p \in \mathbb{Z}} A^p$, together with graded multiplication maps $m_n \colon A^{\otimes n} \to A$ of degree $2-n$, for each $n \geq 1$. These multiplication maps satisfy an infinite sequence of relations, starting with

$$\begin{aligned} m_1 m_1 &= 0, \\ m_1 m_2 &= m_2(m_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes m_1). \end{aligned}$$

These first two relations ensure that m_1 is a differential on A, satisfying the Leibniz rule with respect to m_2. The third relation is

$$m_2(\mathrm{id}_A \otimes m_2 - m_2 \otimes \mathrm{id}_A) = m_1 m_3 + m_3(m_1 \otimes \mathrm{id}_A \otimes \mathrm{id}_A + \mathrm{id}_A \otimes m_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \mathrm{id}_A \otimes m_1),$$

which shows that m_2 is not associative in general, but if $m_n = 0$ for all $n \geq 3$, then A is an ordinary associative differential graded algebra.

In fact, by the above discussion, we can view B is an A_∞-algebra, with m_1 being the differential, m_2 the multiplication, and $m_n = 0$ for $n \geq 3$. By a theorem of Kadeishvili (cf. [27]), the homology $H^*B = H^*(B, m_1)$ has a canonical A_∞-algebra structure, for which $m_1 = 0$, m_2 is induced by the multiplication on B, and H^*B and B are quasi-isomorphic as A_∞-algebras. This canonical A_∞-structure is unique, and H^*B is called a *minimal model* for B. We say that B is *formal* if it has a minimal model H^*B for which $m_n = 0$ for all $n \geq 3$, so that H^*B is just an associative graded algebra.

The A_∞-algebra of $\mathbb{E}$ is

$$H^*B = \operatorname{Ext}^*(T, T) = \bigoplus_k \bigoplus_{0 \leq i,j \leq n-1} \operatorname{Ext}^k(E_i, E_j),$$

and m_2 coincides with the Yoneda product on Ext-groups. Clearly, if the exceptional collection $\mathbb{E}$ consists of sheaves, then H^*B has only three nontrivial graded summands, in degrees 0, 1 and 2. Since m_n has degree $2-n$, the summands of degree 0 and 1 are crucial in determining the A_∞-algebra structure.

4.6.1 Recovering $\mathcal{E}$ from H^*B

According to [15], [28], the subcategory $\mathcal{E}$ of $\mathcal{D}$ generated by the exceptional collection $\mathbb{E}$ is equivalent to the triangulated subcategory $\mathrm{Perf}(B) \subset D^b(\text{mod-}B)$ of perfect objects over the dg-algebra B. A perfect object is a differential graded B-module that is quasi-isomorphic to a bounded chain complex of projective and finitely generated modules. As mentioned above, it is preferable to consider the A_∞-algebra H^*B instead, noting that $\mathcal{E}$ is in turn equivalent to the triangulated category of perfect A_∞-modules over H^*B. If B is formal, the equivalence reduces to $\mathcal{E} \cong D^b(\text{mod-}H^*B)$, which should be compared with Section 4.1.

We search for exceptional collections whose Hom- and Ext^1-groups are mostly zero. In good cases, this implies that B is formal, and H^*B has no deformations. It then follows that $\mathcal{E}$ is rigid, i.e. constant in families.

4.7 Quasiphantoms on the Kulikov surface

We study some properties of the exceptional collections on the Kulikov surface from section 3.2. For the purposes of the discussion, we fix the following exceptional collection

$$\mathbb{E}\colon \mathcal{O},\ L_1[2,2,0],\ L_2[2,1,2],\ L_3[0,0,1],\ L_4[1,1,1],\ L_5[2,2,1],$$

which can be found in the second row of Table 1 in Sec. 3.2.

Using Thm 2.1, we may compute the Ext-groups of the extended sequence (12). We present the results in Table 2 below. The ijth entry of the table is the following formal polynomial in q

$$\sum_{k\in\mathbb{Z}} \dim \mathrm{Ext}^k(E_i, E_{i+j})q^k,$$

where $0 \leq i, j \leq 5$, and the zigzag delineates those entries whose target E_{i+j} is in the anticanonically extended part of (12).

Lemma 4.2 *The only nonzero* Ext^1*-groups are* $\mathrm{Ext}^1(E_1, E_4)$ *which is* 2*-dimensional, and* $\mathrm{Ext}^1(E_1, E_5)$ *which is* 1*-dimensional.* □

Remark 4.2 The lemma shows that $\mathbb{E}$ does not have 3-block structure. A 3-block structure means the exceptional collection can be split into three mutually orthogonal blocks (cf. [26]). In fact, every exceptional collection in Table 1, and every exceptional collection in the Weyl group orbit (cf. Sec. 3.1.7), has some non-zero Ext^1-groups. This is in contrast with the exceptional collections on the Burniat surface exhibited in [2], which are of the same numerical type, and have 3-block structure.

	0	1	2	3	4	5
0	1	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
1	1	0	0	$2q+3q^2$	$q+2q^2$	$4q^2$
2	1	0	q^2	q^2	$4q^2$	$6q^2$
3	1	q^2	q^2	$4q^2$	$6q^2$	$6q^2$
4	1	0	$3q^2$	$5q^2$	$5q^2$	$5q^2$
5	1	$3q^2$	$5q^2$	$5q^2$	$5q^2$	$6q^2$

Table 2: Ext-table of an exceptional collection on the Kulikov surface

Proposition 4.1 *The A_∞-algebra of $\mathbb{E}$ is formal, and the product m_2 of any two elements with strictly positive degree is trivial.*

Proof The A_∞-algebra H^*B of $\mathbb{E}$, is the direct sum of all Ext-groups appearing above the zigzag in the table. By [41, Lemma 2.1] or [34, Thm 3.2.1.1], we may assume that $m_n(\dots, id_{E_i}, \dots) = 0$ for all E_i and all $n > 2$.

We show that every product m_3 must be zero for degree reasons. By Lemma 4.2, there are only two nonzero arrows in degree 1, and they can not be composed with one another, since they have the same source. Thus the product m_3 of any 3 composable elements of H^*B has degree at least $\deg m_3 + 1 + 2 + 2 = 4$, and is therefore identically zero, because the graded piece H^4B is trivial. The same argument applies for all products m_n with $n \geq 3$. Thus H^*B is a formal A_∞-algebra. In fact, we see from the table that any product m_2 of two elements of nonzero degree also vanishes for degree reasons. □

Moreover, we calculate the Hochschild cohomology of $\mathcal{A}$ using heights.

Proposition 4.2 *We have $HH^0(\mathcal{A}) = \mathbb{C}$, $HH^1(\mathcal{A}) = 0$, $HH^2(\mathcal{A}) = \mathbb{C}$, and $HH^3(\mathcal{A})$ contains a copy of $\mathbb{C}^3$.*

Proof The pseudoheight of $\mathbb{E}$ may also be computed from the table, where now we also need the portion below the zigzag. The minimal contribution to the pseudoheight is achieved by incorporating one of the nonzero Ext^1-groups. For example,

$$e(E_1, E_4) + e(E_4, E_1 \otimes \omega_X) - 1 + 2 = 1 + 2 - 1 + 2 = 4,$$

so $ph(\mathbb{E}) = 4$. In this case, by [33], the height and pseudoheight are equal. Hence $HH^k(\mathcal{A}) = HH^k(X)$ for $k \leq 2$, and $HH^3(\mathcal{A}) \supset HH^3(X)$. By the Hochshild–Kostant–Rosenberg isomorphism, the dimensions of $HH^k(X)$ follow from the infinitesimal deformation theory of the Kulikov surface, which was studied in [18]: $H^1(T_X) = 1$ and $H^2(T_X) = 3$. □

In summary, we have

Theorem 4.2 *Every Kulikov surface X has a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{E}, \mathcal{A} \rangle$$

where $\mathcal{E}$ is generated by the exceptional collection $\mathbb{E}$, and $\mathcal{E}$ is rigid, i.e. $\mathcal{E}$ does not vary with X. The semiorthogonal complement $\mathcal{A}$ is a quasiphantom category whose formal deformation space is isomorphic to that of $D^b(X)$, and therefore X may be reconstructed from $\mathcal{A}$.

5 Burniat surfaces revisited

Exceptional collections on Burniat surfaces X_6 with $K^2 = 6$ were first constructed and studied in [2], where two 3-block exceptional collections are exhibited. In this section, we consider the other families of Burniat surfaces X_k with $K_X^2 = k$, for $2 \leq k \leq 6$, which are obtained when the branch locus degenerates. Burniat surfaces were discovered in [17], and an alternate construction is given in [25]. The torsion groups of X_k are either $(\mathbb{Z}/2)^k$, or $(\mathbb{Z}/2)^3$ when $K^2 = 2$ (cf. [40], [4]). We use the description of X_k as a $(\mathbb{Z}/2)^2$-cover of a (weak) del Pezzo surface Y with $K_Y^2 = k$. For $3 \leq k \leq 6$, X_k satisfies assumptions (A), and so we are able to enumerate exceptional collections on all these Burniat surfaces. We illustrate this using the numerical exceptional collection

$$\Lambda \colon 0,\ e_1, \ldots, e_{9-k},\ e_0,\ 2e_0$$

for $4 \leq k \leq 6$. Heuristically, as the size of the torsion group decreases it becomes more difficult to find good exceptional collections. Thus for $k = 3$, we have to be more careful, using a different choice for Λ, together with the Weyl group action to find exceptional collections. Exceptional collections of line bundles of maximal length on the Burniat–Campedelli surface X_2 with $K^2 = 2$ remain elusive, because this surface does not satisfy assumption (A1).

5.1 Primary Burniat surfaces with $K^2 = 6$

Exceptional collections on primary Burniat surfaces with $K^2 = 6$ were first constructed and studied in [2] (see also [1]). We apply our own methods here, to give new examples of exceptional collections and to put exceptional collections on the other families of Burniat surfaces into context.

We briefly explain the Burniat line configuration, see [4] for details. Take the three coordinate points P_1, P_2, P_3 in $\mathbb{P}^2$, and label the edges $\overline{A}_0 = P_1P_2$, $\overline{B}_0 = P_2P_3$, $\overline{C}_0 = P_3P_1$. Then let $\overline{A}_1$, $\overline{A}_2$ (respectively $\overline{B}_i$, $\overline{C}_i$) be two general lines passing through P_1 (resp. P_2, P_3). This gives nine lines in total, four passing through each P_i. Blow up the three points P_i to obtain a del Pezzo surface Y of degree 6. The strict transforms of these nine lines

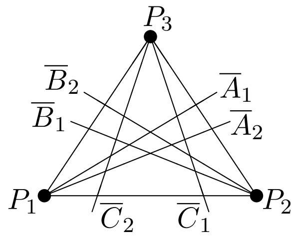


Figure 2: The Primary Burniat configuration with $K^2 = 6$

(for which we use the same labels) together with the three exceptional curves $\overline{E}_i$, are called the *Burniat line configuration*.

The Burniat surface X is a $(\mathbb{Z}/2)^2$-cover of Y branched in the Burniat line configuration, and X is a surface of general type with $p_g = 0$, $K^2 = 6$ and $\mathrm{Tors}(X) = (\mathbb{Z}/2)^6$. The maximal abelian cover A of X is a $(\mathbb{Z}/2)^8$-cover of Y.

The Burniat configuration has four free parameters, and primary Burniat surfaces form a 4-dimensional irreducible connected component of the moduli space of sufaces of general type (see [36]). In particular, $h^1(T_X) = 4$ and $h^2(T_X) = 6$.

In Appendix B.1, we show that the primary Burniat surfaces satisfy assumptions (A), exhibiting a basis for $\mathrm{Pic}\, X/\mathrm{Tors}\, X$ in terms of reduced pullbacks of irreducible branch divisors. The appendix also lists coordinates for the reduced pullback of each irreducible component of the branch divisor according to Definition 3.4.

We consider the following exceptional collection on Y

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_0,\ 2e_0, \tag{13}$$

and use assumption (A) to produce a numerical exceptional collection (L_i) on X. The computer lists acyclic sets $\mathcal{A}(L_i^{-1})$ and $\mathcal{A}(L_j^{-1} \otimes L_i)$, and a systematic search through these enumerates all exceptional collections of numerical type (13).

Theorem 5.1 *There are 81332 exceptional collections $L_0 = \mathcal{O}_X, L_1(\tau_1), \ldots, L_5(\tau_5)$ on X_6 of numerical type* (13)*. We give a sample of two below.*

	τ_1	τ_2	τ_3	τ_4	τ_5
1	$[1,0,1,0,0,0]$	$[0,0,0,1,0,0]$	$[0,1,0,0,0,1]$	$[0,0,0,0,0,1]$	$[1,1,1,1,0,1]$
2	$[1,0,1,0,0,0]$	$[0,0,0,1,0,0]$	$[0,1,0,0,0,1]$	$[1,1,1,1,0,0]$	$[1,1,1,1,0,1]$

Table 3: Exceptional collections on the primary Burniat surface

Remark 5.1 The precise number of exceptional collections is not important, especially since we have not even taken into account the action of the Weyl group. The basic observation is that there is an abundance of exceptional collections of line bundles on the primary Burniat surface, from which we may choose those with the best properties.

There are 16 exceptional collections on X of numerical type Λ which have no Ext^1-groups, and the two sample exceptional collections are taken from these 16. In all 16 cases, the Ext-groups for the anticanonically extended sequence (12) have the same dimensions, displayed in Table 4.

	0	1	2	3	4	5
0	1	q^2	q^2	q^2	q^2	$6q^2$
1	1	0	0	$2q^2$	$5q^2$	$5q^2$
2	1	0	$2q^2$	$5q^2$	$5q^2$	$6q^2$
3	1	$2q^2$	$5q^2$	$5q^2$	$6q^2$	$6q^2$
4	1	$3q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$
5	1	0	q^2	q^2	q^2	$3q^2$

Table 4: Ext-table of an exceptional collection on the primary Burniat surface

This is the best possible situation, because the product m_2 of any two elements of degree 2 must be identically zero for degree reasons, and all higher products m_n are also zero. Moreover the quasiheight of $\mathbb{E}$ is 4. To summarise, we have:

Proposition 5.1 *Let $\mathbb{E}$ be any one of the 16 exceptional collections on the primary Burniat surface for which there are no* Ext^1*-groups. Then the A_∞-algebra $H^*\mathbb{E}$ is formal, and the product of any two elements of positive degree vanishes. The Hochschild cohomology of each of the corresponding quasi-phantom categories $\mathcal{A}$ is*

$$HH^0(\mathcal{A}) = \mathbb{C},\ HH^1(\mathcal{A}) = 0,\ HH^2(\mathcal{A}) = \mathbb{C}^4,\ HH^3(\mathcal{A}) \supset \mathbb{C}^6.$$

5.2 Secondary Burniat surfaces with $K^2 = 5$

The secondary Burniat surfaces arise when the branch configuration has one or two triple points. We first impose a single triple point P_4 on the three branch lines $\overline{A}_1$, $\overline{B}_1$, and $\overline{C}_2$ (see Figure 3). The $(\mathbb{Z}/2)^2$-cover would then have a $\frac{1}{4}(1,1)$ singularity over P_4, so we blow up at P_4, to obtain a del Pezzo surface Y of degree 5. The induced nonsingular $(\mathbb{Z}/2)^2$-cover X of Y is called a secondary Burniat surface with $K^2 = 5$. Since the cover is unramified over P_4, the torsion group of X is only $(\mathbb{Z}/2)^5$ as opposed to $(\mathbb{Z}/2)^6$ for the primary Burniat surface.

The configuration in Figure 3 has three free parameters, and secondary Burniat surfaces with $K^2 = 5$ form a 3-dimensional irreducible connected component of the moduli space of surfaces of general type (see [5]), so $h^1(T_X) = h^2(T_X) = 3$.

In Appendix B.2, we give a basis for $\operatorname{Pic} X/\operatorname{Tors} X$ and coordinates for $\operatorname{Pic} X$. In particular, X satisfies assumptions (A). We consider the following exceptional collection of

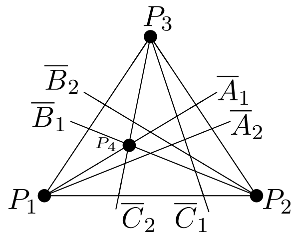


Figure 3: The Secondary Burniat configuration with $K^2 = 5$

line bundles on Y

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_0,\ 2e_0.$$

As usual, we get a numerical exceptional collection (L_i) on X, and we enumerate all exceptional collections on X_5 corresponding to our chosen numerical exceptional collection.

Theorem 5.2 *There are 2597 exceptional collections on X_5 corresponding to $L_0, \dots, L_6$. We give a sample*

$$L_0,\ L_1[0,0,1,1,1],\ L_2[0,1,0,0,1],\ L_3[1,1,1,0,1],\ L_4[0,1,1,0,0],\ L_5[0,1,1,0,1],\ L_6[1,0,0,1,1],$$

whose Ext-table is found in Table 5.

The sample exceptional collection was chosen because it is the only one of numerical type Λ for which the four line bundles $E_1, \dots, E_4$ corresponding to the (-1)-curves on Y are mutually orthogonal. There are many other exceptional collections of numerical type Λ with very few non-zero Ext^1-groups, but unlike X_6, we do not find any exceptional collections that have no non-zero Ext^1-groups.

Nevertheless, from the table we see that there is no nontrivial composition of two elements of degree 1 in H^*B. Moreover, the elements of degree 1 do not compose with any element below the zigzag.

Proposition 5.2 *The A_∞-algebra of the displayed exceptional collection on the secondary Burniat surface with $K^2 = 5$ is formal, and the product of any two elements of nonzero degree is zero. Moreover, the corresponding quasi-phantom category has Hochschild cohomology*

$$HH^0(\mathcal{A}) = \mathbb{C},\ HH^1(\mathcal{A}) = 0,\ HH^2(\mathcal{A}) = \mathbb{C}^3,\ HH^3(\mathcal{A}) \supset \mathbb{C}^3.$$

5.3 Secondary Burniat surfaces with $K^2 = 4$

There are two ways to impose a second triple point P_5 on the Burniat configuration, leading to two different secondary Burniat surfaces with $K_X^2 = 4$ (see Figure 4). If P_4 and P_5 do not lie on the same branch line, then the blow up Y of $\mathbb{P}^2$ at $P_1, \dots, P_5$ is a del Pezzo

	0	1	2	3	4	5	6
0	1	$q+2q^2$	$q+2q^2$	$q+2q^2$	q^2	$3q^2$	$6q^2$
1	1	0	0	0	$2q^2$	$5q^2$	$4q^2$
2	1	0	0	$2q^2$	$5q^2$	$4q^2$	$5q^2$
3	1	0	$2q^2$	$5q^2$	$4q^2$	$5q^2$	$5q^2$
4	1	$2q^2$	$5q^2$	$4q^2$	$5q^2$	$5q^2$	$5q^2$
5	1	$3q^2$	$q+3q^2$	$3q^2$	$3q^2$	$3q^2$	$3q^2$
6	1	q	$q+q^2$	$q+q^2$	$q+q^2$	0	$2q^2$

Table 5: Ext-table of an exceptional collection on the secondary Burniat surface with $K^2 = 5$

surface of degree 4, and the $(\mathbb{Z}/2)^2$-cover is called *non-nodal*. If P_4 and P_5 do lie on a single branch line (in Fig. 4, this line is $\overline{A}_1$), then Y is a weak del Pezzo surface, and the $(\mathbb{Z}/2)^2$-cover is called *nodal* because the canonical model of X has a $\frac{1}{2}(1,1)$ singularity. In both cases, the second triple point causes the torsion group of X to drop to $(\mathbb{Z}/2)^4$.

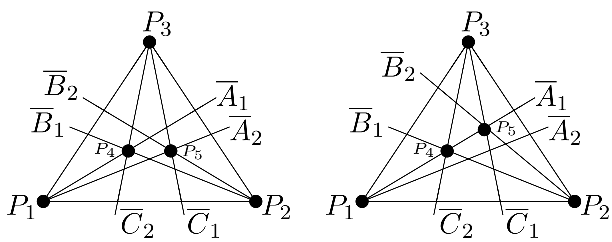


Figure 4: The secondary Burniat configurations with $K^2 = 4$ (nodal configuration is on the right)

Both configurations in Figure 4 have two free parameters, so that we obtain two 2-dimensional families of secondary Burniat surfaces with $K^2 = 4$. We recall some facts from [5] and [6]. The non-nodal case again forms an irreducible connected component of the moduli space, with $h^1(T_X) = 2$ and $h^2(T_X) = 0$. The nodal case is more interesting: the 2-dimensional family is a proper subset of a 3-dimensional irreducible connected component of the moduli space. In fact, $h^1(T_{X^n}) = 3$ and $h^2(T_{X^n}) = 1$, and there is a 3-dimensional family of *extended Burniat surfaces* (see [5]), each of which is a $(\mathbb{Z}/2)^2$-cover of a generalisation of the nodal Burniat configuration. We do not directly consider extended Burniat surfaces in this article.

The data listed in Appendix B.3 shows that both X_4 and X_4^n satisfy assumption (A). Choosing the numerical exceptional collection

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_0,\ 2e_0,$$

we enumerate all exceptional collections on X_4^n corresponding to Λ.

Theorem 5.3 *There are 13 exceptional collections on X_4^n of numerical type Λ. Here is a sample exceptional collection*

$$L_0,\ L_1[1,0,1,0],\ L_2[0,1,0,1],\ L_3[0,0,1,1],\ L_4[0,1,1,0],\ L_5,\ L_6[0,1,0,1],\ L_7[1,0,1,1], \quad (14)$$

whose Ext*-table is found in Table 6.*

	0	1	2	3	4	5	6	7
0	1	q^2	q^2	q^2	q^2	q^2	$3q^2$	$6q^2$
1	1	$q+q^2$	0	$q+q^2$	$q+q^2$	$2q^2$	$5q^2$	$3q^2$
2	1	0	0	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$
3	1	0	$q+q^2$	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$
4	1	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$
5	1	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$	$4q^2$
6	1	$3q^2$	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$2q^2$	$2q^2$
7	1	$2q$	q	q	q	q	$2q+q^2$	q^2

Table 6: Ext-table of an exceptional collection on the nodal Burniat surface with $K^2 = 4$

The non-nodal surface X_4 has six exceptional collections of numerical type Λ, but it is difficult to find one for which the A_∞-algebra is obviously formal, because there are too many nonzero Ext^1-groups. We use the Weyl group action on the del Pezzo surface Y to obtain the following numerical exceptional collection

$$\Lambda'\colon 0,\ e_4,\ e_2,\ e_5,\ e_1,\ e_3,\ e_0,\ 2e_0.$$

This is just a permutation of the order in which we blow up the points in $\mathbb{P}^2$ to construct Y.

Theorem 5.4 *There are 40 exceptional collections on X_4 of numerical type Λ', and we exhibit one with the minimum number of* Ext^1*-groups*

$$L_0,\ L_1[0,0,0,1],\ L_2[0,1,1,0],\ L_3[1,0,1,0],\ L_4[0,1,0,1],\ L_5[1,0,0,0],\ L_6[1,1,1,1],\ L_7[1,1,1,0]. \quad (15)$$

The Ext*-table of* (15) *is displayed in Table 7.*

	0	1	2	3	4	5	6	7
0	1	q^2	q^2	q^2	q^2	q^2	$3q^2$	$6q^2$
1	1	0	$q+q^2$	0	$q+q^2$	$2q^2$	$5q^2$	$3q^2$
2	1	$q+q^2$	$q+q^2$	$q+q^2$	$2q^2$	$5q^2$	$3q^2$	$4q^2$
3	1	0	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$
4	1	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$
5	1	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$	$4q^2$
6	1	$3q^2$	q^2	$2q^2$	$2q^2$	$2q^2$	$2q^2$	$2q^2$
7	1	$2q$	q	q	q	q	q	q^2

Table 7: Ext-table of an exceptional collection on the non-nodal Burniat surface with $K^2 = 4$

We see that both the non-nodal and nodal secondary Burniat surfaces with $K^2 = 4$ have quite a few nonzero Ext^1-groups, since we do not have as much freedom to search for "good" exceptional collections. Nevertheless, a careful examination of the tables shows that no two elements of degree 1 are composable. Thus in both cases, the A_∞-algebra is formal, and the height is 4.

We summarise our results on Burniat surfaces with $K^2 = 6, 5, 4$.

Theorem 5.5 *Every primary or secondary Burniat surface has at least one exceptional collection of maximal length whose A_∞-algebra is formal. Moreover, the product of any two elements of positive degree vanishes, and the height is 4. Thus the Hochschild cohomology of each corresponding quasiphantom category is*

$$HH^0(\mathcal{A}) = H^0(\mathcal{O}_X),\ HH^1(\mathcal{A}) = 0,\ HH^2(\mathcal{A}) = H^1(T_X),\ HH^3(\mathcal{A}) \supset H^2(T_X).$$

5.4 Tertiary Burniat surface with $K^2 = 3$

Imposing a third triple point on the branch configuration (see Fig. 5) gives a tertiary Burniat surface X_3 with $K_X^2 = 3$. The weak del Pezzo surface Y has three (-2)-curves, $\overline{A}_1$, $\overline{B}_1$ and $\overline{C}_1$, and the canonical model of X_3 is a $(\mathbb{Z}/2)^2$-cover of a 3-nodal cubic. The torsion group of X_3 is $(\mathbb{Z}/2)^3$.

Here the moduli space gets quite involved, and we follow the description of [6]. The tertiary Burniat surfaces form a 1-dimensional irreducible family, inside a 4-dimensional irreducible component of the moduli space. The *extended tertiary Burniat surfaces* form an open subset of this irreducible component, and the remainder consists of $(\mathbb{Z}/2)^2$-covers of certain singular cubic surfaces. Our main point of interest is that $h^1(T_X) = 4$, and $h^2(T_X) = 0$.

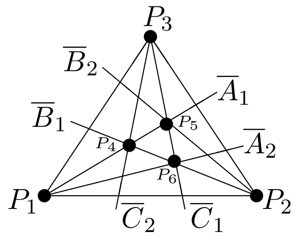


Figure 5: The tertiary Burniat configuration with $K^2 = 3$

In Appendix B.4, we show that X satisfies assumption (A). We use the computer to enumerate all exceptional collections on X of numerical type

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_0,\ 2e_0.$$

Lemma 5.1 *There are no exceptional collections of line bundles of numerical type Λ on the tertiary Burniat surface.* □

Remark 5.2 Our systematic search does yield exceptional collections $\mathbb{E}'$ of length seven with numerical type $0, e_1, \ldots, e_6$, but in each case, there are no line bundles corresponding to e_0 or $2e_0$ which extend $\mathbb{E}'$. We have also checked part of the orbit of Λ under the action of the Weyl group of Y, and although we find some exceptional collections on X_3 of length eight, we do not find any of length nine.

5.4.1 The E_6-symmetry

In order to find an exceptional collection of line bundles on X_3, we choose a different numerical exceptional collection Λ_1, using the E_6-symmetry of $\operatorname{Pic} Y$ and the Borel–de Siebenthal procedure. As an example, we consider the sublattice $3A_2$ inside the extended Dynkin diagram $\widetilde{E}_6$, which corresponds to a singular del Pezzo surface Y' with $3 \times \frac{1}{3}(1,2)$ singularities. The minimal resolution $\widetilde{Y}$ is a toric surface with a cycle of nine rational curves with self-intersections

$$-(-2)-(-1)-(-2)-(-2)-(-1)-(-2)-(-2)-(-1)-(-2)-$$

To construct $\widetilde{Y}$, choose points P_1, P_2, P_3 in general position in $\mathbb{P}^2$. Blow up each P_i once, and blow up the infinitely near points Q_1, Q_2, Q_3, where Q_i is supported at P_i with tangent direction P_iP_{i+1}. Alternatively, $\widetilde{Y}$ is the minimal resolution of the quotient of $\mathbb{P}^2$ by the $\mathbb{Z}/3$-action $\frac{1}{3}(0,1,2)$, which has three fixed points. We fix a geometric marking on $\widetilde{Y}$ so that the strict transform of the exceptional curve over P_i has class $e_i - e_{3+i}$ in $\operatorname{Pic}\widetilde{Y}$, and the exceptional curve over Q_i has class e_{3+i}. Then the cycle of curves described above have numerical classes

$$e_1 - e_4,\ e_4,\ e_0 - e_1 - e_2 - e_4,\ e_2 - e_5,\ e_5,\ e_0 - e_2 - e_3 - e_5,\ e_3 - e_6,\ e_6,\ e_0 - e_1 - e_3 - e_6$$

Taking cumulative sums of these classes gives a numerical exceptional collection Λ_1 on any del Pezzo surface of degree three:

$$\Lambda_1\colon 0,\ e_1-e_4,\ e_1,\ e_0-e_2-e_4,\ e_0-e_4-e_5,\ e_0-e_4, \\ 2e_0-e_2-e_3-e_4-e_5,\ 2e_0-e_2-e_4-e_5-e_6,\ 2e_0-e_2-e_4-e_5. \quad (16)$$

We again search for exceptional collections of type Λ_1 on X, and again we do not find any. Fortunately, this time we do find exceptional collections on X using the Weyl group action on Y.

Theorem 5.6 *There are exceptional collections on X_3 corresponding to certain numerical exceptional collections in the Weyl group orbit of Λ_1. For example, let Λ_1' be the numerical exceptional collection obtained from Λ_1 by swapping e_1 and e_2. Then*

$$L_0,\ L_1[0,0,1],\ L_2[0,1,1],\ L_3[1,0,0],\ L_4[0,1,1],\ L_5[0,0,1], \\ L_6[1,1,0],\ L_7[0,0,0],\ L_8[0,1,0] \quad (17)$$

is an exceptional collection on X_3 of numerical type Λ_1', whose Ext*-table is found in Table 8.*

Remark 5.3 We have not studied the whole orbit of numerical exceptional collections, because the Weyl group is quite large, but we can give an overview based on probabilistic methods. It seems that approximately two thirds of the orbit of Λ_1 do not give any exceptional collections on X, and the remaining numerical types typically correspond to anywhere between one and 21 exceptional collections on X. We see that exceptional collections are much more scarce on X_3 than for the other Burniat surfaces.

5.4.2 The A_∞-algebra

The exceptional collection (17) was chosen to have the fewest nonzero Ext^1-groups, but there are six of them. Of these, there are no three above the zigzag that may be composed with one another under m_3. Thus m_3 is identically zero on H^*B for degree reasons, and the A_∞-algebra of $\mathbb{E}$ is formal. There is a single possible product of two elements of degree 1, coming from the chain $E_1 \to E_4 \to E_7$. It is not clear whether this product is zero.

To compute the Hochschild cohomology, we first consider the pseudoheight of $\mathbb{E}$. Examining the table, we see that the pseudoheight is 3, because

$$e(E_1,E_4)+e(E_4,E_7)+e(E_7,E_1\otimes\omega_X^{-1})+2-2=3.$$

In fact, this cycle of line bundles is the only one contributing 3 to the pseudoheight. In other words, the first page of the spectral sequence converging to $NHH^\bullet(\mathbb{E},X)$ has a single term of total degree 3:

$$\mathrm{Ext}^1(E_1,E_4)\otimes\mathrm{Ext}^1(E_4,E_7)\otimes\mathrm{Ext}^3(E_7,S^{-1}(E_1))\subset\mathbf{E}^1_{-2,5}.$$

	0	1	2	3	4	5	6	7	8
0	1	0	q^2	q^2	q^2	$2q^2$	$2q^2$	$2q^2$	$3q^2$
1	1	q^2	q^2	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
2	1	0	0	q^2	$q+2q^2$	q^2	$2q^2$	$2q^2$	$2q^2$
3	1	0	q^2	q^2	q^2	$2q^2$	$2q^2$	$2q^2$	$3q^2$
4	1	q^2	q^2	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
5	1	0	0	q^2	$q+2q^2$	q^2	$2q^2$	$2q^2$	$2q^2$
6	1	0	q^2	q^2	q^2	$2q^2$	$2q^2$	$2q^2$	$3q^2$
7	1	q^2	q^2	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
8	1	0	0	q^2	$q+2q^2$	q^2	$2q^2$	$2q^2$	$2q^2$

Table 8: Ext-table of an exceptional collection on the tertiary Burniat surface with $K^2 = 3$

The differential d_1 on the first page maps this term to the direct sum of the following three spaces

$$\begin{gathered}\operatorname{Ext}^2(E_1, E_7) \otimes \operatorname{Ext}^3(E_7, S^{-1}(E_1))\\ \operatorname{Ext}^1(E_1, E_4) \otimes \operatorname{Ext}^4(E_4, S^{-1}(E_1))\\ \operatorname{Ext}^1(E_4, E_7) \otimes \operatorname{Ext}^4(E_7, S^{-1}(E_4))\end{gathered}$$

in $\mathbf{E}^1_{-1,5}$.

If we can show that any of the maps are nonzero, then it follows that $\mathbf{E}^2_{-2,5} = 0$ and thus $h(\mathbb{E}) = 4$. We do not currently know of a practical method for computing nontrivial products in H^*B, but the following rough idea should work.

Observe that $\operatorname{Ext}^k(E_i, E_j) = H^k(E_i^{-1} \otimes E_j)$. We write $L_{ij} = E_i^{-1} \otimes E_j$, and so we are actually checking injectivity of the cup product $H^1(L_{14}) \otimes H^1(L_{47}) \xrightarrow{\cup_X} H^2(L_{17})$. It is difficult to compute $\cup_X$ explicitly on X, so we pushforward each L_{ij} to Y, and compare with the cup product $\cup_Y$ on Y. We have

$$H^1(\varphi_* L_{14}) \otimes H^1(\varphi_* L_{47}) \xrightarrow{\cup_Y} H^2(\varphi_* L_{14} \otimes \varphi_* L_{47}) \xrightarrow{\mu} H^2(\varphi_* L_{17}), \tag{18}$$

where μ is induced by the natural map $\varphi_* L_{14} \otimes \varphi_* L_{47} \to \varphi_*(L_{14} \otimes L_{47})$. By comparing the definition of $\cup_Y$ using Čech complexes with that of $\cup_X$, we see that the composite map displayed in (18) is equal to $\cup_X$.

It remains to compute the cup product on Y, which can be done by chasing exact sequences, and to check that μ is injective. We hope to finish this in the near future.

6 Keum–Naie surface with $K^2 = 4$

In this section we investigate a construction of Keum–Naie surfaces, which just fails to satisfy our assumptions from Sec. 3.1. The problem is that the maximal abelian cover $A \to X$ does not factor through a Galois cover of the del Pezzo surface Y. Thus while we can describe the free part of $\operatorname{Pic} X$ in terms of reduced pullbacks of branch divisors, we can only describe an index 2 subgroup of $\operatorname{Tors} X$ using our approach. We have used various numerical exceptional collections to search for exceptional collections of maximal length on X, but without success.

6.1 Construction and basic properties of the surface

Keum–Naie surfaces were discovered independently in [37] and [29] as branched double covers of Enriques surfaces with eight nodes. The connected component of the moduli space containing Keum–Naie surfaces has dimension 6, and the torsion group is $(\mathbb{Z}/2)^3 \times \mathbb{Z}/4$.

Following [9] and [3], we consider a special 2-dimensional subfamily of Keum–Naie surfaces. Each surface X in the subfamily admits a singular $\mathbb{Z}/2 \times \mathbb{Z}/4$-cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over eight lines, four in each ruling. The branch configuration is shown in Figure 6.

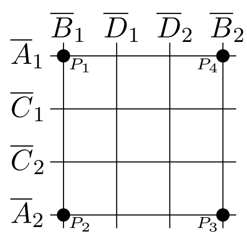


Figure 6: The Keum–Naie configuration with $K^2 = 4$

The map $\Phi \colon H_1(\mathbb{P}^1 \times \mathbb{P}^1 - \Delta, \mathbb{Z}) \to \mathbb{Z}/2 \times \mathbb{Z}/4$ governing the cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ is described in the table below.

Γ	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_1$	$\overline{C}_2$	$\overline{D}_1$	$\overline{D}_2$
$\Phi(\Gamma)$	h	$3h$	$g_1 + h$	$g_1 + 3h$	g_1	g_1	$g_1 + 2h$	$g_1 + 2h$
$\Psi'(\Gamma) - \Phi(\Gamma)$	0	g_2	g_3	g_4	0	g_2	g_5	$g_3 + g_4 + g_5$

In the table, g_i have order 2 while h has order 4. We see that the four divisors $\overline{A}_i$, $\overline{B}_i$ have inertia group $\mathbb{Z}/4$ under φ, while the other branch divisors have inertia group $\mathbb{Z}/2$. Moreover, the four points $P_1, \dots, P_4$ correspond to $\frac{1}{2}(1,1)$ singularities on X. We blow up the P_i to get a nonsingular cover of a weak del Pezzo surface Y of degree 4. Write $\overline{E}_i$ for the (-1)-curve corresponding to P_i. We use the same labels for the strict transforms under the blow up, so $\overline{A}_i$, $\overline{B}_i$ are (-2)-curves on Y. Note that by formula (1) the curves $\overline{E}_i$ have inertia group $\mathbb{Z}/2$.

The following proposition explains why we can not find exceptional collections on the Keum–Naie surface.

Proposition 6.1 *Let A be the maximal abelian cover of X. The composite map $A \to X \to Y$ is not Galois.*

Proof The torsion group of X is $(\mathbb{Z}/2)^3 \times \mathbb{Z}/4$, so if $\psi\colon A \to Y$ is Galois, we have a surjective homomorphism $\Psi\colon H_1(Y-\Delta,\mathbb{Z}) \to (\mathbb{Z}/2)^4 \times (\mathbb{Z}/4)^2$. Now consider $\Psi(\overline{E}_1) = \Psi(\overline{A}_1 + \overline{B}_1)$. The order of $\Psi(\overline{E}_i)$ must be 2, while the orders of $\Psi(\overline{A}_1)$ and $\Psi(\overline{B}_1)$ must be 4. We have similar requirements coming from the other $\overline{E}_i$. There is no surjective homomorphism Ψ satisfying these conditions. □

We define A' to be the intermediate Galois cover $A' \to X$ corresponding to the subgroup $(\mathbb{Z}/2)^4$ of index 2 in $\operatorname{Tors} X$, generated by $g_2,\dots,g_5$. The composite map $A' \to X \to Y$ is Galois, with defining map $\Psi'\colon H_1(Y-\Delta,\mathbb{Z}) \to \mathbb{Z}/4 \times (\mathbb{Z}/2)^5$ in the table above.

6.2 The Picard group

Lemma 6.1 *The reduced pullbacks $e_0 = A_1 + B_1 + E_1$, $e_1 = A_2 + B_1 + E_2$, $e_2 = A_1$, $e_3 = A_2$, $e_4 = B_1$, $e_5 = B_2$ generate the Picard lattice of X with intersection matrix $U \oplus \operatorname{diag}(-1,-1,-1,-1)$, where $U = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.*

Proof We first show that linear combinations of the quoted divisors generate $\operatorname{Pic} Y$. We use the following linear equivalences on Y

$$\begin{aligned}\overline{C}_1 \sim \overline{C}_2 \sim \overline{A}_1 + \overline{E}_1 + \overline{E}_4 \sim \overline{A}_2 + \overline{E}_2 + \overline{E}_3\\ \overline{D}_1 \sim \overline{D}_2 \sim \overline{B}_1 + \overline{E}_1 + \overline{E}_2 \sim \overline{B}_2 + \overline{E}_3 + \overline{E}_4,\end{aligned} \tag{19}$$

to express E_3 and E_4 in terms of the basis. The rest of the proof is similar to that of Lemma 3.3. Checking the intersection matrix requires some care with the definition of reduced pullback, because the inertia groups of φ are not uniform. For example, we actually have $4e_0 = \varphi^*(\overline{A}_1 + 2\overline{E}_1 + \overline{B}_1)$ and $4e_1 = \varphi^*(\overline{A}_2 + 2\overline{E}_2 + \overline{B}_1)$, so that $16 e_0 \cdot e_1 = \deg(\varphi)\cdot 2$, and hence $e_0 \cdot e_1 = 1$. □

Remark 6.1 By Lemma 6.1, we see that A_1 is an elliptic curve of self-intersection -1 on X, even though $\overline{A}_1$ is a (-2)-curve on Y. In other words, assumptions (A1) and (A2) hold for the Keum–Naie surface, but (A3) does not. Instead we get an isometry from the abstract lattice $\mathbb{Z}^{1,5} \to \operatorname{Pic} X/\operatorname{Tors} X$, under which the image of $2e_0 + 2e_1 - \sum_{i=2}^{5} e_i$ is the class of $\mathcal{O}_X(K_X)$ modulo torsion.

We compute the coordinates of the reduced pullback of each irreducible branch component using the basis provided by Lemma 6.1.

Lemma 6.2 *We have*

$$\begin{aligned} \mathcal{O}_X(A_1) &= \mathcal{O}_X(0,0,1,0,0,0) & \mathcal{O}_X(D_1) &= \mathcal{O}_X(1,1,-1,-1,0,0)[0,1,0,0] \\ \mathcal{O}_X(A_2) &= \mathcal{O}_X(0,0,0,1,0,0) & \mathcal{O}_X(D_2) &= \mathcal{O}_X(1,1,-1,-1,0,0)[0,1,0,1] \\ \mathcal{O}_X(B_1) &= \mathcal{O}_X(0,0,0,0,1,0) & \mathcal{O}_X(E_1) &= \mathcal{O}_X(1,0,-1,0,-1,0) \\ \mathcal{O}_X(B_2) &= \mathcal{O}_X(0,0,0,0,0,1) & \mathcal{O}_X(E_2) &= \mathcal{O}_X(0,1,0,-1,-1,0) \\ \mathcal{O}_X(C_1) &= \mathcal{O}_X(1,1,0,0,-1,-1)[1,1,1,0] & \mathcal{O}_X(E_3) &= \mathcal{O}_X(1,0,0,-1,0,-1)[0,0,0,1] \\ \mathcal{O}_X(C_2) &= \mathcal{O}_X(1,1,0,0,-1,-1)[1,0,0,1] & \mathcal{O}_X(E_4) &= \mathcal{O}_X(0,1,-1,0,0,-1)[0,1,1,1] \end{aligned}$$

Proof This is similar to Lemma 3.3. One minor point, in computing the multidegrees. The linear equivalences (19) on Y pull back to X giving numerical equivalences

$$\begin{aligned} C_1 \equiv C_2 \equiv 2A_1 + E_1 + E_4 \equiv 2A_2 + E_2 + E_3 \\ D_1 \equiv D_2 \equiv 2B_1 + E_1 + E_2 \equiv 2B_2 + E_3 + E_4. \end{aligned}$$

These can be rearranged to give

$$A_1 + B_1 + E_1 \equiv A_2 + B_2 + E_3, \;\; A_2 + B_1 + E_2 \equiv A_1 + B_2 + E_4,$$

which is used to express each reduced pullback in terms of the basis from Lemma 6.1. □

Lemma 6.3 *By formula* (8) *and Lem. 6.1,* $\mathcal{O}_X(K_X) = \mathcal{O}_X(2,2,-1,-1,-1,-1)$. □

We conclude by noting that we have searched for, but not found any exceptional collections of maximal length on the Keum–Naie surface. It seems that our subgroup of $\operatorname{Tors} X$ is too small to allow us the freedom to find any. On a related note, exceptional collections of maximal length have not been discovered on the Burniat–Campedelli surface with $K^2 = 2$ (see [1]), and some Beauville surfaces considered in [35]. Here the situation is more straightforward, because these surfaces fail to satisfy assumption (A1) and (A2) respectively.

A Appendix: Acyclic bundles on the Kulikov surface

For reference, here are the acyclic line bundles on the Kulikov surface used in section 3.2.

L	$\mathcal{A}(L)$
L_1^{-1}	[0, 0, 0], [0, 1, 0], [1, 1, 0], [2, 1, 0], [2, 2, 0], [1, 0, 1], [2, 0, 1], [0, 1, 1], [1, 1, 1], [0, 2, 1], [2, 2, 1], [0, 1, 2], [1, 1, 2], [0, 2, 2]
L_2^{-1}	[0, 1, 0], [1, 1, 0], [2, 2, 0], [2, 0, 1], [0, 1, 1], [1, 1, 1], [2, 1, 1], [1, 2, 1], [2, 2, 1], [0, 0, 2], [1, 0, 2], [0, 1, 2], [1, 1, 2], [1, 2, 2]
L_3^{-1}	[0, 1, 0], [1, 1, 0], [1, 0, 1], [0, 1, 1], [1, 1, 1], [0, 2, 1], [1, 2, 1], [0, 0, 2], [2, 0, 2], [0, 1, 2], [1, 1, 2], [2, 1, 2], [0, 2, 2], [1, 2, 2]
L_4^{-1}	[0, 0, 0], [0, 1, 0], [2, 1, 0], [0, 2, 0], [2, 2, 0], [1, 0, 1], [2, 0, 1], [0, 1, 1], [1, 1, 1], [2, 1, 1], [0, 2, 1], [2, 2, 1], [1, 1, 2], [0, 2, 2], [2, 2, 2]
L_5^{-1}	[0, 1, 0], [1, 1, 0], [2, 2, 0], [1, 0, 1], [2, 0, 1], [0, 1, 1], [1, 1, 1], [0, 2, 1], [1, 2, 1], [2, 2, 1], [0, 0, 2], [0, 1, 2], [1, 1, 2], [0, 2, 2], [1, 2, 2]
$L_2^{-1} \otimes L_1$	[1, 0, 0], [2, 0, 0], [2, 1, 0], [0, 1, 1], [0, 1, 2], [2, 1, 2], [0, 2, 2]
$L_3^{-1} \otimes L_1$	[0, 0, 0], [1, 0, 0], [2, 0, 0], [1, 1, 0], [2, 1, 0], [2, 2, 0], [1, 1, 2], [2, 1, 2], [2, 2, 2]
$L_4^{-1} \otimes L_1$	[0, 1, 0], [1, 1, 0], [0, 1, 1], [1, 1, 1], [1, 2, 1], [0, 0, 2], [1, 0, 2], [2, 0, 2], [0, 1, 2], [1, 1, 2], [1, 2, 2]
$L_5^{-1} \otimes L_1$	[1, 0, 0], [2, 0, 0], [1, 1, 0], [2, 1, 0], [2, 2, 0], [0, 1, 1], [0, 0, 2], [0, 1, 2], [1, 1, 2], [2, 1, 2], [0, 2, 2], [2, 2, 2]
$L_3^{-1} \otimes L_2$	[1, 0, 1], [1, 1, 1], [2, 1, 1], [2, 0, 2], [1, 1, 2], [2, 1, 2], [1, 2, 2]
$L_4^{-1} \otimes L_2$	[0, 0, 0], [0, 1, 0], [1, 1, 0], [1, 0, 1], [0, 1, 1], [1, 1, 1], [0, 2, 1], [2, 0, 2], [0, 1, 2], [1, 1, 2], [0, 2, 2]
$L_5^{-1} \otimes L_2$	[0, 1, 0], [1, 0, 1], [0, 1, 1], [1, 1, 1], [2, 1, 1], [0, 2, 1], [0, 0, 2], [2, 0, 2], [1, 1, 2], [2, 1, 2], [0, 2, 2], [1, 2, 2]
$L_4^{-1} \otimes L_3$	[0, 0, 0], [0, 1, 0], [1, 1, 0], [2, 2, 0], [2, 0, 1], [0, 1, 1], [1, 1, 1], [2, 2, 1], [1, 0, 2], [0, 1, 2], [1, 1, 2]
$L_5^{-1} \otimes L_3$	[0, 1, 0], [1, 1, 0], [2, 1, 0], [2, 2, 0], [2, 0, 1], [1, 1, 1], [2, 1, 1], [1, 2, 1], [0, 0, 2], [1, 0, 2], [0, 1, 2], [1, 2, 2]
$L_5^{-1} \otimes L_4$	[1, 0, 0], [2, 0, 0], [1, 1, 0], [2, 2, 0], [0, 0, 2], [0, 1, 2], [2, 1, 2], [2, 2, 2]

B Appendix: Burniat surface data

B.1 Primary Burniat surface

The maps determining the covers $A \to X \to Y$ are $\Phi\colon H_1(Y-\Delta,\mathbb{Z}) \to G$ and $\Psi\colon H_1(Y-\Delta,\mathbb{Z}) \to G \oplus T$. We tabulate them below.

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Phi(\Gamma)$	g_1	g_1	g_1	g_2	g_2	g_2	g_1+g_2	g_1+g_2	g_1+g_2
$\Psi(\Gamma)-\Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	g_7	g_8	$\sum_{i=3}^{8} g_i$

The images of the exceptional curves are obtained in the usual way from equation (1) of Sec. 2,

$$\Phi(\overline{E}_1) = g_2,\ \Phi(\overline{E}_2) = g_1+g_2,\ \Phi(\overline{E}_3) = g_1,\ \text{etc.}$$

The following reduced pullbacks are a basis for $\operatorname{Pic} X/\operatorname{Tors} X$,

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3.$$

According to these generators, the coordinates on $\operatorname{Pic} X$ are

	Multidegree				Torsion
$\mathcal{O}_X(A_0)$	1	-1	-1	0	$[1,1,0,0,0,1]$
$\mathcal{O}_X(A_1)$	1	-1	0	0	$[1,0,0,0,1,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	$[0,1,0,0,1,0]$
$\mathcal{O}_X(B_0)$	1	0	-1	-1	$[0,0,1,1,0,1]$
$\mathcal{O}_X(B_1)$	1	0	-1	0	$[0,0,1,0,1,0]$
$\mathcal{O}_X(B_2)$	1	0	-1	0	$[0,0,0,1,1,0]$
$\mathcal{O}_X(C_0)$	1	-1	0	-1	0
$\mathcal{O}_X(C_1)$	1	0	0	-1	$[0,0,0,0,1,1]$
$\mathcal{O}_X(C_2)$	1	0	0	-1	$[0,0,0,0,1,0]$

and $\mathcal{O}_X(K_X) = \mathcal{O}_X(3,-1,-1,-1)[0,0,0,0,0,1]$ by equation (8).

B.2 Secondary Burniat surface with $K^2 = 5$

The map $\Phi\colon H_1(\mathbb{P}^2-\Delta,\mathbb{Z}) \to (\mathbb{Z}/2)^2$ is the same as for the primary Burniat surface, but the triple point at P_4 changes Ψ. Indeed, we have $\Psi_5(\overline{A}_1+\overline{B}_1+\overline{C}_2)=0$, which kills one factor of the torsion group. Thus the maximal abelian cover $\psi_5\colon A \to Y$ is determined by

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Psi_5(\Gamma)-\Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	g_7	$g_4+g_6+g_7$	g_3+g_5

and the torsion group is generated by $g_3^*, \ldots, g_7^*$.

The following reduced pullbacks are a basis for the free part of $\operatorname{Pic} X$

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3,\ e_4 = C_0 - C_2 + E_1,$$

with intersection form $\operatorname{diag}(1, -1, -1, -1, -1)$. Note that $\overline{E}_4$ is not a branch divisor, which explains the funny choice for e_4.

The coordinates of $\operatorname{Pic} X$ according to this basis are:

	Multidegree					Torsion
$\mathcal{O}_X(A_0)$	1	−1	−1	0	0	[1, 1, 0, 0, 0]
$\mathcal{O}_X(A_1)$	1	−1	0	0	−1	[1, 0, 0, 0, 0]
$\mathcal{O}_X(A_2)$	1	−1	0	0	0	[0, 1, 0, 0, 1]
$\mathcal{O}_X(B_0)$	1	0	−1	−1	0	[0, 0, 1, 1, 0]
$\mathcal{O}_X(B_1)$	1	0	−1	0	−1	[0, 0, 1, 0, 0]
$\mathcal{O}_X(B_2)$	1	0	−1	0	0	[0, 0, 0, 1, 1]
$\mathcal{O}_X(C_0)$	1	−1	0	−1	0	0
$\mathcal{O}_X(C_1)$	1	0	0	−1	0	[0, 0, 0, 0, 1]
$\mathcal{O}_X(C_2)$	1	0	0	−1	−1	0

Thus $\mathcal{O}_X(K_X) = \mathcal{O}_X(3, -1, -1, -1, -1)[0, 0, 0, 0, 1]$.

B.3 Secondary Burniat surfaces with $K^2 = 4$

The maps $\Psi_4, \Psi_4^n \colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/2)^6$ determining respectively the non-nodal and nodal Burniat surfaces, differ from one another slightly. We tabulate them below.

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Psi_4(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	0	$g_4 + g_6$	$g_3 + g_5$
$\Psi_4^n(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	$g_3 + g_4$	$g_3 + g_6$	$g_3 + g_5$

The restriction imposed by P_5 is $\Psi_4(\overline{A}_2 + \overline{B}_2 + \overline{C}_1) = 0$ in the non-nodal case, and $\Psi_4^n(\overline{A}_1 + \overline{B}_2 + \overline{C}_1) = 0$ in the nodal case. Either way, g_7 is eliminated, so the torsion group is $(\mathbb{Z}/2)^4$, generated by $g_3^*, \ldots, g_6^*$.

We extend the basis chosen for the free part of $\operatorname{Pic}(X_5)$. The basis is the same for non-nodal and nodal surfaces

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3,$$
$$e_4 = C_0 - C_2 + E_1,\ e_5 = B_0 - B_2 + E_3.$$

Coordinates for non-nodal surface:

	Multidegree						Torsion
$\mathcal{O}_X(A_0)$	1	-1	-1	0	0	0	$[1,1,0,0]$
$\mathcal{O}_X(A_1)$	1	-1	0	0	-1	0	$[1,0,0,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	0	-1	$[0,1,1,0]$
$\mathcal{O}_X(B_0)$	1	0	-1	-1	0	0	$[0,0,1,1]$
$\mathcal{O}_X(B_1)$	1	0	-1	0	-1	0	$[0,0,1,0]$
$\mathcal{O}_X(B_2)$	1	0	-1	0	0	-1	$[0,0,1,1]$
$\mathcal{O}_X(C_0)$	1	-1	0	-1	0	0	0
$\mathcal{O}_X(C_1)$	1	0	0	-1	0	-1	$[0,0,1,0]$
$\mathcal{O}_X(C_2)$	1	0	0	-1	-1	0	0

Coordinates for nodal surface are the same (with same multidegrees) except for the following:

	Multidegree						Torsion
$\mathcal{O}_X(A_1)$	1	-1	0	0	-1	-1	$[1,0,1,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	0	0	$[0,1,0,0]$

In both cases, $\mathcal{O}_X(K_X) = \mathcal{O}(3,-1,-1,-1,-1,-1)[0,0,1,0]$.

B.4 Tertiary Burniat surfaces with $K^2 = 3$

The map $\Psi_3\colon H_1(Y-\Delta,\mathbb{Z}) \to (\mathbb{Z}/2)^5$ is similar to Ψ_4^n, with an extra restriction due to the triple point at P_6: $\Psi(\overline{A}_2 + \overline{B}_1 + \overline{C}_1) = 0$. This gives

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Psi_3(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	$g_3+g_4+g_5$	g_3+g_4	g_4+g_5	g_3+g_5

The basis of the free part of $\mathrm{Pic}(X_3)$ extends that of the secondary Burniat surfaces:

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3,$$
$$e_4 = C_0 - C_2 + E_1,\ e_5 = B_0 - B_2 + E_3,\ e_6 = A_0 - A_2 + E_2.$$

The coordinates for each reduced pullback are

	Multidegree							Torsion
$\mathcal{O}_X(A_0)$	1	−1	−1	0	0	0	0	$[1,1,0]$
$\mathcal{O}_X(A_1)$	1	−1	0	0	−1	−1	0	$[1,0,1]$
$\mathcal{O}_X(A_2)$	1	−1	0	0	0	0	−1	$[1,1,0]$
$\mathcal{O}_X(B_0)$	1	0	−1	−1	0	0	0	$[0,0,1]$
$\mathcal{O}_X(B_1)$	1	0	−1	0	−1	0	−1	$[1,0,1]$
$\mathcal{O}_X(B_2)$	1	0	−1	0	0	−1	0	$[0,0,1]$
$\mathcal{O}_X(C_0)$	1	−1	0	−1	0	0	0	0
$\mathcal{O}_X(C_1)$	1	0	0	−1	0	−1	−1	$[1,0,1]$
$\mathcal{O}_X(C_2)$	1	0	0	−1	−1	0	0	0

Thus $\mathcal{O}_X(K_X) = \mathcal{O}_X(3,-1,-1,-1,-1,-1,-1)[1,0,1]$.

C Appendix: Beauville surfaces

In this appendix we apply our methods to two Beauville surfaces. Each is an abelian cover of $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying assumptions (A). Thus we may write any line bundle on X as $\mathcal{O}_X(a,b)(\tau)$. We recall some facts about numerical exceptional collections on such abelian covers of $\mathbb{P}^1 \times \mathbb{P}^1$ from [23].

Lemma C.1 *1. A sequence $\mathcal{O}, L_1, L_2, L_3$ of line bundles on X is numerically exceptional if and only if it belongs to one of the four numerical types:*

$$\begin{aligned}
(I_c)\ & \mathcal{O},\ \mathcal{O}(-1,0),\ \mathcal{O}(c-1,-1),\ \mathcal{O}(c-2,-1),\\
(II_c)\ & \mathcal{O},\ \mathcal{O}(0,-1),\ \mathcal{O}(-1,c-1),\ \mathcal{O}(-1,c-2),\\
(III_c)\ & \mathcal{O},\ \mathcal{O}(-1,c),\ \mathcal{O}(-1,c-1),\ \mathcal{O}(-2,-1),\\
(IV_c)\ & \mathcal{O},\ \mathcal{O}(c,-1),\ \mathcal{O}(c-1,-1),\ \mathcal{O}(-1,-2),
\end{aligned}$$

where c is any integer.

2. For fixed c, the dihedral group action on numerically exceptional collections (see Sec. 3.1.7) has two orbits:

$$\begin{aligned}
I_c \to IV_c \to I_{-c} \to IV_{-c} \to I_c,\\
II_c \to III_c \to II_{-c} \to III_{-c} \to II_c.
\end{aligned}$$

As explained in Sec. 3.1.7, the Weyl group of $\mathbb{P}^1 \times \mathbb{P}^1$ acts on numerical exceptional collections on X, interchanging I_c with II_c and III_c with IV_c. The difference is that the Weyl group action does not lift to exceptional collections, so there are two orbits. Thus we need only consider numerically exceptional collections of line bundles of type I_c or II_c.

C.1 $(\mathbb{Z}/3)^2$-Beauville surface

This surface was discovered by Beauville and first described in [20], but for similar examples see also [8]. Let X be a $(\mathbb{Z}/3)^2$-cover of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ branched over eight lines, four in each ruling. We label these branch divisors $\Delta_1, \dots, \Delta_8$ (see Figure 7). Clearly, the branch locus has two free parameters, and in fact, the $(\mathbb{Z}/3)^2$-Beauville surfaces form a two dimensional irreducible connected component of the moduli space [8], so $h^1(T_X) = 2$ and $h^2(T_X) = 8$.

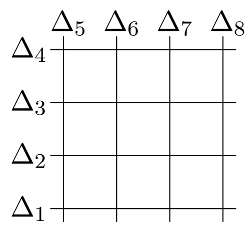


Figure 7: The branch locus for a Beauville surface with $G = (\mathbb{Z}/3)^2$

The cover $\varphi \colon X \to Y$ and maximal abelian cover $\psi \colon A \to Y$ are determined by the maps $\Phi \colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/3)^2$ and $\Psi \colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/3)^6$ as shown in the table

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
$\Phi(D)$	g_1	g_2	$2g_1$	$2g_2$	$g_1 + g_2$	$g_1 + 2g_2$	$2g_1 + 2g_2$	$2g_1 + g_2$
$\Psi(D) - \Phi(D)$	0	0	g_3	$2g_3$	g_4	g_5	g_6	$2(g_4 + g_5 + g_6)$

The small quotient group $G \cong (\mathbb{Z}/3)^2$ is generated by g_1, g_2 and T is generated by $g_3, \dots, g_6$.

Remark C.1 The original construction [20] of X is to take the free $(\mathbb{Z}/3)^2$-quotient of a product $C_1 \times C_2$ of two special curves of genus 6. This realises a subgroup $(\mathbb{Z}/3)^2$ of the full torsion group $\operatorname{Tors} X = (\mathbb{Z}/3)^4$. Using this quotient construction, many exceptional collections of line bundles on X with numerical type I_1 were constructed and studied in [35]. We use abelian covers to completely enumerate all exceptional collections of line bundles on X, of any numerical type.

Let D_i denote the reduced pullback of Δ_i. Then the torsion free part of $\operatorname{Pic} X$ is based by D_1 and D_5, with intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma C.2 *The coordinates of each $\mathcal{O}_X(D_i)$ are*

$$\begin{aligned}
\mathcal{O}_X(D_1) &= \mathcal{O}_X(1,0) & \mathcal{O}_X(D_5) &= \mathcal{O}_X(0,1) \\
\mathcal{O}_X(D_2) &= \mathcal{O}_X(1,0)[1,0,1,0] & \mathcal{O}_X(D_6) &= \mathcal{O}_X(0,1)[0,1,2,0] \\
\mathcal{O}_X(D_3) &= \mathcal{O}_X(1,0)[0,2,2,1] & \mathcal{O}_X(D_7) &= \mathcal{O}_X(0,1)[0,1,0,2] \\
\mathcal{O}_X(D_4) &= \mathcal{O}_X(1,0)[1,2,2,1] & \mathcal{O}_X(D_8) &= \mathcal{O}_X(0,1)[0,1,0,0]
\end{aligned}$$

With this basis, using (8) we have

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(2,2)[1,2,2,2]. \tag{20}$$

C.1.1 The semigroup of effective divisors

In this section we prove:

Proposition C.1 *The semigroup of effective divisors on X is generated by $D_1, \ldots, D_8$.*

This should be compared with the results of [1] on primary Burniat surfaces and the discussion of Sec. 3.1.6.

We introduce some notation. Define $\mathfrak{E}$ to be the semigroup generated by $D_1, \ldots, D_8$. It is convenient to consider $\mathfrak{E}$ as the image of the multiplicative semigroup M of monomials in the bigraded polynomial ring $\mathbb{Z}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$ under the homomorphism $x_i \mapsto D_i$, $y_i \mapsto D_{4+i}$. The x_i have bidegree $(1,0)$, and y_i have bidegree $(0,1)$. We abuse notation to consider monomials in M as elements of $\mathfrak{E}$ when appropriate. Let $t\colon \mathfrak{E} \to \operatorname{Tors} X$ be the semigroup homomorphism defined in Sec. 3.1.6, sending each D_i to its associated torsion twist according to Lem. C.2.

Since K_X is ample, we have

Lemma C.3 *If $\mathcal{O}(a,b)(\tau)$ is an effective line bundle on X, then $a \geq 0$ and $b \geq 0$.*

We analyse the possible values for a and b.

Lemma C.4 *If $a, b \geq 2$, then $\mathcal{O}_X(a,b)(\tau)$ is effective for all τ in $\operatorname{Tors} X$, unless $a = b = 2$ and $\tau = [1,2,2,2]$.*

Proof Consider the set $M_{(2,2)}$ of monomials of bidegree $(2,2)$. We use the computer [19] to check that the image of $M_{(2,2)}$ under t is precisely $\operatorname{Tors} X - \{[1,2,2,2]\}$. Moreover, the missing torsion twist is that of K_X, which is not effective, because $p_g(X) = 0$.

On the other hand, we also check that $t(M_{(3,2)}) = t(M_{(2,3)}) = \operatorname{Tors} X$, so every line bundle of bidegree $(3,2)$ or $(2,3)$ is effective. Now for any $a \geq 3$, we see that $x_1^{a-3} y_1^{b-2} M_{(3,2)}$ gives a global section for each $\mathcal{O}_X(a,b)(\tau)$. A similar argument works for $b \geq 3$. □

It remains to check what happens if $a \leq 1$ or $b \leq 1$. We suppose the latter (the case $a \leq 1$ is similar).

Lemma C.5 *Suppose $b \leq 1$. The line bundle $\mathcal{O}_X(a,b)(\tau)$ is effective if and only if there is a monomial m in $M_{(a,b)}$ such that $t(m) = \tau$.*

Proof **Case $b = 0$.** For $a < 6$, we check effectivity of each line bundle directly. This is a finite number of line bundles, and so we use the computer [19]. Note that if $a < a'$, then $t(M_{(a,0)}) \subseteq t(M_{(a',0)})$. Moreover, for $a \geq 6$, $t(M_{(a,0)})$ stabilises to $H = \{[\alpha, \beta, \gamma, 2\beta] : \alpha, \beta, \gamma \in \mathbb{Z}/3\}$. Indeed, H is a subgroup of $\operatorname{Tors} X$, so it is closed under composition of torsion elements. Thus if $a \geq 6$ then $\mathcal{O}_X(a, 0)(\tau)$ is effective for any τ in H.

Now fix $a \geq 6$ and τ in $\operatorname{Tors} X - H$. We show that $\mathcal{O}_X(a, 0)(\tau)$ is not effective. Write $a = 6 + 3j + k$ where $j \geq 0$ and $0 \leq k \leq 2$. Then by Lemma C.2,

$$\varphi_* \mathcal{O}_X(a, 0)(\tau) = \varphi_* L(kD_1) \otimes \mathcal{O}_Y(j\Delta_1) = \varphi_* L(kD_1) \otimes \mathcal{O}_Y(j, 0),$$

where $L = \mathcal{O}_X(6, 0)(\tau)$. Thus if each summand of $\varphi_* L(kD_1)$ for $0 \leq k \leq 2$ has negative degree in the second factor, we see that $\mathcal{O}_X(a, 0)(\tau)$ can not be effective for any $a \geq 6$. We have again reduced the problem to checking a finite number of line bundles, and this is done by computer in [19].

Case $b = 1$. The argument is similar to the previous case, so we give only a sketch. First check $a < 4$ directly. Then for $a \geq 4$, the image $t(M_{(a,1)})$ stabilises to $H \cup [0, 1, 0, 0]H$, the union of two cosets of H in $\operatorname{Tors} X$. This can be seen directly from Lemma C.2. The other torsion twists are ineffective for any $a \geq 4$, by a similar computation to that of case $b = 0$ above. □

C.1.2 Acyclic line bundles

Now, by the Riemann–Roch theorem, the numerically acyclic line bundles on X are $\mathcal{O}(1, k)(\tau)$ and $\mathcal{O}(k, 1)(\tau)$. Thus we may use Proposition C.1 to find all acyclic line bundles.

Proposition C.2 *For $k \geq 4$ or $k \leq -2$, the acyclic sets on X are*

$$\mathcal{A}(\mathcal{O}_X(1, k)) = \mathcal{S}, \quad \mathcal{A}(\mathcal{O}_X(k, 1)) = \mathcal{T},$$

where

$$\mathcal{S} = \{[2, \alpha, \beta, \gamma] : \alpha, \beta, \gamma \in \mathbb{Z}/3\}, \ \mathcal{T} = \{[\alpha, \beta, \gamma, 2 - \beta] : \alpha, \beta, \gamma \in \mathbb{Z}/3\}.$$

Proof We prove that $\mathcal{A}(\mathcal{O}(k, 1)) = \mathcal{T}$ for $k \geq 4$. The acyclic set $\mathcal{A}(\mathcal{O}(1, k))$ for $k \geq 4$ can be calculated in the same way, and the negative cases follow by Serre duality.

Fix an integer $k \geq 4$. Then by Lem. C.5, $\mathcal{O}_X(k, 1)(\tau)$ is not effective if and only if τ is an element of $\operatorname{Tors} X - (H \cup [0, 1, 0, 0]H) = \mathcal{T}$. Now by Serre duality and (20),

$$H^2(\mathcal{O}_X(k, 1)[\alpha, \beta, \gamma, 2 - \beta]) = H^0(\mathcal{O}_X(2 - k, 1)([1 - \alpha, 2 - \beta, 2 - \gamma, \beta]),$$

which also vanishes by Lem. C.5, or if $k \geq 3$, we can use Lem. C.3. Thus $\mathcal{A}(\mathcal{O}_X(1, k)) = \mathcal{T}$ for all $k \geq 4$. □

In fact, the same proof shows that $\mathcal{S} \subset \mathcal{A}(\mathcal{O}_X(1,k))$ and $\mathcal{T} \subset \mathcal{A}(\mathcal{O}_X(k,1))$ for any integer k. For values of k between -1 and 3, there are a few extra acyclic twists, because the image of t has not yet stabilised to its maximum. These can be checked directly, using Prop. C.1 as before.

L	$\mathcal{A}(L)$
$\mathcal{O}_X(1,-1)$	$\mathcal{S}, [1,1,1,0], [0,1,2,0]$
$\mathcal{O}_X(1,0)$	$\mathcal{S}, [0,2,0,0], [1,2,0,0], [1,1,1,0], [0,2,1,0], [1,2,1,0], [0,1,2,0], [0,2,2,0], [1,2,2,0], [1,0,1,1], [0,0,2,1], [1,2,1,2], [0,2,2,2]$
$\mathcal{O}_X(1,1)$	$\mathcal{S} \cup \mathcal{T}, [1,0,0,0], [0,0,1,0], [1,2,1,2], [0,2,2,2]$
$\mathcal{O}_X(1,2)$	$\mathcal{S}, [1,0,0,0], [0,0,1,0], [1,2,0,1], [0,2,1,1], [0,0,0,2], [1,0,0,2], [1,1,0,2], [0,0,1,2], [1,0,1,2], [0,1,1,2], [0,0,2,2], [1,0,2,2]$
$\mathcal{O}_X(1,3)$	$\mathcal{S}, [1,1,0,2], [0,1,1,2]$

In the other direction,

L	$\mathcal{A}(L)$
$\mathcal{O}_X(-1,1)$	$\mathcal{T}, [1,0,2,1], [1,1,1,2]$
$\mathcal{O}_X(0,1)$	$\mathcal{T}, [1,0,0,0], [2,0,0,0], [1,1,0,0], [2,1,2,0], [2,0,0,1], [2,2,1,1], [1,0,2,1], [1,2,2,1], [1,1,1,2], [1,2,1,2], [2,2,1,2], [2,1,2,2]$
$\mathcal{O}_X(2,1)$	$\mathcal{T}, [2,1,0,0], [0,0,1,0], [2,0,1,0], [0,1,1,0], [0,0,0,1], [0,2,0,1], [2,0,1,1], [2,2,2,1], [2,1,0,2], [0,1,2,2], [0,2,2,2], [2,2,2,2]$
$\mathcal{O}_X(3,1)$	$\mathcal{T}, [0,1,1,0], [0,2,0,1]$

Many exceptional collections on X of numerical type I_1 and with formal A_∞-algebra were constructed in [35]. We can classify all exceptional collections of line bundles on X, of any numerical type. The enumeration is summarised below, but see [19] for details.

Proposition C.3 *Exceptional collections of line bundles on the $\mathbb{Z}/3$-Beauville surface are enumerated in the table below. The integer $c \geq 0$ determines the numerical type of the exceptional collection, either I_c or II_c. The number of type I_c is equal to the number of type II_c.*

c	0	1	2	≥ 3
#(*Exceptional collections*)	6661	3613	2213	$2187 = 3^7$

We display a sample exceptional collection of type I_1

$$\mathcal{O}_X,\ \mathcal{O}_X(-1,0)[0,1,0,0],\ \mathcal{O}_X(0,-1)[2,2,0,0],\ \mathcal{O}_X(-1,-1)[1,0,1,0]$$

	0	1	2	3
0	1	q^2	q^2	$4q^2$
1	1	0	0	$6q^2$
2	1	$2q^2$	$6q^2$	$8q^2$
3	1	$4q^2$	$6q^2$	$6q^2$

Table 9: Ext-table of an exceptional collection on the $(\mathbb{Z}/3)^2$-Beauville surface

Table 9 is the Ext-table of this exceptional collection. We see that there are no nonzero Ext^1-groups. Hence the A_∞-algebra is formal, and the height is 4. Thus the Hochschild cohomology of the corresponding quasiphantom category is $HH^0(\mathcal{A}) = \mathbb{C}$, $HH^1(\mathcal{A}) = 0$, $HH^2(\mathcal{A}) = \mathbb{C}^2$, $HH^3(\mathcal{A}) \supset \mathbb{C}^8$.

C.2 $(\mathbb{Z}/5)^2$-Beauville surface

We consider the $(\mathbb{Z}/5)^2$-Beauville surface, which was first described in [10] and [20]. Exceptional collections of line bundles on this surface were classified by Galkin and Shinder [23], which was a major influence on our overall approach. We recover the results of [23] as a test case for our methods.

This time X is a $(\mathbb{Z}/5)^2$-cover of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ branched over six lines, three in each ruling. This branch configuration is rigid, and in fact the moduli space of such Beauville surfaces is zero dimensional and smooth. The torsion group of X is $\operatorname{Tors} X \cong (\mathbb{Z}/5)^2$, which is fully realised by the standard construction of X as a free $(\mathbb{Z}/5)^2$-quotient of $C_1 \times C_2$, where C_i are Fermat quintic curves. Thus $C_1 \times C_2$ is the maximal abelian cover A (this description of A is not necessary for our approach).

The maps $\Phi\colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/5)^2$ and $\Psi\colon H_1(Y - B, \mathbb{Z}) \to \widetilde{G} \cong (\mathbb{Z}/5)^4$ determining the covers are defined in the following table

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
$\Phi(D)$	g_1	g_2	$4g_1 + 4g_2$	$g_1 + 2g_2$	$3g_1 + 4g_2$	$g_1 + 4g_2$
$\Psi(D) - \Phi(D)$	0	0	0	g_3	g_4	$4g_3 + 4g_4$

The reduced pullbacks D_1 (respectively D_4) of Δ_1 (resp. Δ_4) are a basis for the free part of $\operatorname{Pic} X$. As usual, the other reduced pullbacks may be written in terms of this basis, and we have

Lemma C.6

$$\mathcal{O}_X(D_1) = \mathcal{O}_X(1,0), \qquad \mathcal{O}_X(D_5) = \mathcal{O}_X(0,1),$$
$$\mathcal{O}_X(D_2) = \mathcal{O}_X(1,0)[1,1], \qquad \mathcal{O}_X(D_5) = \mathcal{O}_X(0,1)[1,4],$$
$$\mathcal{O}_X(D_3) = \mathcal{O}_X(1,0)[4,2], \qquad \mathcal{O}_X(D_6) = \mathcal{O}_X(0,1)[1,0].$$

By (8), we have

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(2,2)[3,3].$$

Lemma C.7 *The semigroup $\mathfrak{E}$ of effective divisors on X is the set of positive integer linear combinations of $D_1, \ldots, D_6$.*

The proof of this Lemma is similar to that of Prop. C.1.

As in Sec. 3.1.6, we define a semigroup homomorphism $t\colon \mathfrak{E} \to \operatorname{Tors} X$ using the torsion twists from Lem. C.6. Using Lem. C.7, we list all acyclic line bundles on X in the following table. We note that the restrictions $t|_{\mathfrak{E}_{(1,j)}}$ and $t|_{\mathfrak{E}_{(k,1)}}$ are surjective for $j \geq 5$ and $k \geq 4$.

L	$\mathcal{A}(L)$
$\mathcal{O}_X(1,-2)$	$[2,0]$
$\mathcal{O}_X(1,-1)$	$[2,0], [3,0], [3,4]$
$\mathcal{O}_X(1,0)$	$[2,0], [3,0], [4,0], [0,1], [4,3], [3,4], [4,4]$
$\mathcal{O}_X(1,1)$	$[3,0], [4,0], [0,3], [4,3], [0,4], [3,4], [4,4]$
$\mathcal{O}_X(1,2)$	$[4,0], [3,2], [0,3], [1,3], [4,3], [0,4], [4,4]$
$\mathcal{O}_X(1,3)$	$[0,3], [1,3], [0,4]$
$\mathcal{O}_X(1,4)$	$[1,3]$
$\mathcal{O}_X(-1,1)$	$[4,2]$
$\mathcal{O}_X(0,1)$	$[4,2], [0,3], [4,3], [3,4]$
$\mathcal{O}_X(2,1)$	$[3,0], [4,0], [4,1], [0,4]$
$\mathcal{O}_X(3,1)$	$[4,1]$

Up to choices of coordinates, these are precisely the acyclic line bundles listed in [23], and there are no others. It seems that the rigidity of X is reflected in the small number of acyclic line bundles.

Using this list of acyclic line bundles, and Lemma C.1, we can classify all exceptional collections of line bundles of length four on X. Here is the complete list, which form two orbits, replicating results of [23].

$$\begin{cases} I_{-1} & \mathcal{O}, & \mathcal{O}(-1,0)[0,4], & \mathcal{O}(-2,-1)[1,0], & \mathcal{O}(-3,-1)[1,4] \\ IV_{-1} & \mathcal{O}, & \mathcal{O}(-1,-1)[1,1], & \mathcal{O}(-2,-1)[1,0], & \mathcal{O}(-1,-2)[2,3] \\ I_1 & \mathcal{O}, & \mathcal{O}(-1,0)[0,4], & \mathcal{O}(0,-1)[1,2], & \mathcal{O}(-1,-1)[1,1] \\ IV_1 & \mathcal{O}, & \mathcal{O}(1,-1)[1,3], & \mathcal{O}(0,-1)[1,2], & \mathcal{O}(-1-2)[2,3] \end{cases}$$

$$\begin{cases} II_0 & \mathcal{O}, & \mathcal{O}(0,-1)[1,2], & \mathcal{O}(-1,-1)[1,1], & \mathcal{O}(-1,-2)[2,3] \\ III_0 & \mathcal{O}, & \mathcal{O}(-1,0)[0,4], & \mathcal{O}(-1,-1)[1,1], & \mathcal{O}(-2,-1)[1,0] \end{cases}$$

We do not continue the analysis of quasi-phantoms, since it appears in [23]. We only verify that our results are consistent.

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