Topological Twists of Supersymmetric Factorization Algebras

Chris Elliott

May 3rd, 2018

Abstract

The idea of topologically twisting a supersymmetric field theory was introduced in the physics literature in order to generate interesting new examples of topological field theories. The idea is very general, but systematically realising the examples it produces using mathematical models for topological quantum field theory (such as the functorial axioms of Atiyah-Segal or the theory of $\mathbb{E}_n$ -algebras) is not always possible. In this talk Ill explain what it means to twist a supersymmetric field theory in the factorization algebra framework developed by Costello and Gwilliam, and address the question of just how topological these topologically twisted theories really are. This is based on joint work with Pavel Safronov.

Introduction

1 Lecture 1 – Supersymmetric Field Theory

1.1 Classical Field Theory

We'll begin with an introduction to the mathematical formalism underlying classical (perturbative – meaning we study perturbations of a fixed classical configuration) field theory following the approach of Costello and Gwilliam. Everything I'll discuss today can be generalised, with some work, to perturbative *quantum* field theory, and while I'll mention that from time to time I won't discuss it in any detail. We'll see that there's a rich and interesting story to tell even at the classical level. There are two points of view on classical field theory that I'll discuss:

- Fields: modelled by a sheaf of dg Lie algebras.
- Observables: modelled by a factorization algebra with a (-1)-shifted Poisson algebra structure.

These two models are Koszul dual to one another with respect to the Lie structure on the fields and the multiplication on the local observables. The Poisson bracket on the observables arises from an additional piece of data that we fix on the fields, a shadow of the action functional.

1.1.1 The BV-BRST Complex

The BV-BRST formalism is a model for the space of fields in a classical field theory, modulo the action of symmetries, and after imposing the condition that the fields are critical points for an action functional. In the BV formalism we impose these equations in a derived sense, so the BV complex will in particular be a cochain complex where H^1 describes solutions to the linearised equations of motion modulo gauge, but where the cohomology outside of degree 1 is non-zero, modelling derived directions in the moduli space of solutions.

Definition 1.1. A *classical field theory* on $\mathbb{R}^n$ is a sheaf of dg Lie algebras L on $\mathbb{R}^n$ (or more generally an L_∞-algebra) equipped with an invariant antisymmetric pairing of degree -3 valued in the sheaf of densities on $\mathbb{R}^n$ (sometimes called the *antibracket*). In order to match up with the physics terminology we'll refer to this sheaf of Lie algebras as the *BV-BRST complex* of the theory.

Remark 1.2. This definition is a shadow of a "non-perturbative" model for a classical field theory, which we might model by a sheaf of (-1)-shifted Poisson stacks, which we obtain as the derived critical locus of an action functional. The complex above models the (-1)-shifted tangent complex of such a stack.

Example 1.3. To give a flavour of the sort of dg Lie algebra we have in mind, let's give the example of *Chern-Simons theory*. This is a 3d classical field theory whose BV-BRST complex is the de Rham complex with coefficients in a simple dg Lie algebra: $L(U) = (\Omega^\bullet(U; \mathfrak{g}), \mathrm{d}_{\mathrm{dR}})$. This has a Lie bracket by wedging the forms and bracketing the Lie algebra factors, and a pairing of degree -3 by wedging and applying the Killing form. One can obtain this sheaf on any oriented 3-manifold M as the -1-shifted tangent complex of the stack $\mathrm{Flat}_G(M)$ of flat G-bundles.

1.1.2 The Classical Factorization Algebra

To introduce the other model of a field theory, let me first explain what exactly a factorization algebra is. The idea is that we can model observables in a field theory on each open subset of spacetime (i.e. of $\mathbb{R}^n$), along with the data of how to extend observables from a smaller open set to a larger one. That is, we'll describe a kind of precosheaf.

Definition 1.4. A *prefactorization algebra* on $\mathbb{R}^n$ is a "multiplicative" precosheaf Obs of vector spaces on $\mathbb{R}^n$. That is, an assignment of a vector space $\mathrm{Obs}(U)$ to each open set with the property that $\mathrm{Obs}(U \sqcup V) \cong \mathrm{Obs}(U) \otimes \mathrm{Obs}(V)$, along with structure maps $\bigotimes_{i=1}^n \mathrm{Obs}(U_i) \to \mathrm{Obs}(V)$ for each pairwise disjoint collection of open subsets $U_1, \ldots, U_n \subseteq V$ satisfying the natural compatibility conditions. A *factorization algebra* on $\mathbb{R}^n$ is a prefactorization algebra which satisfies descent for covers $\{U_i\}$ of open sets U satisfying the condition that for any sequence of points $x_1, \ldots, x_n$ in U there is an element U_i of the cover so that $\{x_1, \ldots, x_n\} \subseteq U_i$. Such covers are known as *Weiss covers.*

The idea here is that the local sections $\mathrm{Obs}(U)$ on an open set U model the (classical or quantum) observables of a theory that depend only on measurements inside of U, so in particular there's a canonical extension (our structure maps) $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$ for every pair of open sets $U \subseteq V$. The sheaf condition is intended to convey the idea that all observables are determined by their values on arbitrarily small neighbourhoods of finite sets of points.

Now, of course there's a connection between these two constructions. Given a classical field theory – modelled by a sheaf of Lie algebras – one can construct a factorization algebra which models its classical observables. The antibracket pairing on the fields turns into a Poisson bracket on the observables.

Definition 1.5. Recall that the *Chevalley-Eilenberg cochain complex* of a dg Lie algebra $\mathfrak{g}$ is the cochain complex $C^\bullet(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}^*[-1])$ with differential coming from the dual of the differential on $\mathfrak{g}$ and the dual to the Lie bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, extended as a derivation. The algebra of *classical observables* of a sheaf L of Lie algebras is the prefactorization algebra that assigns to an open set V the complex

$$\mathrm{Obs}_L(V) := C^\bullet(L(V)).$$

In fact, in reasonable examples this prefactorization algebra is a factorization algebra (see [CG16, Chapter 6 Theorem 5.2.1] for the precise statement).

If L is a classical field theory, i.e. carries a degree -3 invariant pairing, then the classical observables Obs_L has the structure of a factorization algebra in the category of $\mathbb{P}_0$-algebras. A $\mathbb{P}_0$ structure on a commutative dga A is a Poisson bracket of degree 1: that is a Lie bracket $\{,\}\colon A \otimes A[-1] \to A$ of degree 1 which is a derivation for the product.

Remark 1.6. In this talk I won't say very much about quantization. The twisting story I'll describe will make sense for *any* classical field theory with an action of a supersymmetry algebra. It's not too difficult to construct quite a few examples of classical supersymmetric field theories – we'll discuss some examples of this form later in the talk – but one needs to take care when quantizing: it's possible for the supersymmetry action to be anomalous, i.e. to not lift to an action at the quantum level. This is one source of "non-topological" behaviour for a topologically twisted quantum field theory: even though one might have a perfectly topological theory at the classical level which makes sense on all oriented manifolds, there could be an anomaly that prevents the theory from being globally defined at the quantum level.

1.2 Supersymmetry Actions

Having defined a classical field theory, let me talk about supersymmetry. In order to explain what it means for a field theory to be supersymmetric I'll need to discuss more broadly what it means for a Lie algebra to act on a field theory (either viewed in terms of its BV-BRST complex or as a factorization algebra of observables), and then I'll need to tell you what a "supersymmetry algebra" is.

1.2.1 Lie Algebra Actions

Since we gave two dual descriptions of a classical field theory, there will be correspondingly two dual descriptions of a Lie algebra action on a field theory. We'll describe these notions and how they relate to one another, and we'll explain the slightly more subtle notion of *Poincaré invariance* (the correct notion for our purposes will involve more data than just an infinitesimal action of the Poincaré algebra).

First let's talk about the action of a dg Lie algebra on a classical field theory. This is fairly straightforward – we just need to say what it means for one Lie algebra to act on another.

Definition 1.7. We say a dg Lie algebra $\mathfrak{g}$ acts on a classical field theory L on $\mathbb{R}^n$ if there is a homomorphism of sheaves of dg Lie algebras $\underline{\mathfrak{g}} \to \mathrm{Der}(L(U))$, where $\underline{\mathfrak{g}}$ denotes the constant sheaf with fiber $\mathfrak{g}$.

Remark 1.8. This definition needs a bit of modification for L_∞-structures. Costello and Gwilliam define an action by defining the semidirect product. That is, they say an L_∞ algebra action is an L_∞ structure on $\mathfrak{g} \oplus L(\mathbb{R}^n)$ such that the short exact sequence

$$0 \to L(\mathbb{R}^n) \to \mathfrak{g} \oplus L(\mathbb{R}^n) \to \mathfrak{g} \to 0$$

is a sequence of Lie algebras.

Remark 1.9. There's no reason why we had to consider a single Lie algebra $\mathfrak{g}$ and not a *sheaf* of Lie algebras – we could've considered local symmetries instead of global symmetries. It's worth mentioning one local Lie algebra that we see acting on field theories that depend on a metric (i.e. families of field theories over the moduli space of metrics modulo diffeomorphisms). There's a sheaf of dg Lie algebras on $\mathbb{R}^n$ associated to a choice g_0 of metric given by

$$L_{\mathrm{Kill}} = T_{\mathbb{R}_n} \to \mathrm{Sym}^2 T_{\mathbb{R}_n}$$

placed in degrees 0 and 1, where the differential sends a vector field X to the Lie derivative $\mathcal{L}_X(g_0)$ and the bracket is also defined as the Lie derivative. We call this the Lie algebra of *Killing vector fields* after the kernel of the differential. One observes that this is the -1-shifted tangent complex to the moduli space of metrics modulo diffeomorphisms.

Now let's discuss what it means for a Lie algebra to act on a factorization algebra.

Definition 1.10 ([CG16, 4.8.1.2])**.** A *degree k derivation* of a factorization algebra Obs on $\mathbb{R}^n$ is a cohomological degree k endomorphism $F_U \colon \mathrm{Obs}(U) \to \mathrm{Obs}(U)$ for each open set U in $\mathbb{R}^n$ that collectively satisfy a Leibniz rule. That is, if $U_1, \ldots, U_m \subseteq V$ are disjoint open subsets of an open set, and $m^V_{U_1,\ldots,U_m}$ is the associated factorization map, we require that

$$F_V \circ m^V_{U_1,\ldots,U_m}(\mathcal{O}_1, \ldots, \mathcal{O}_m) = \sum_{i=1}^m (-1)^{k(|\mathcal{O}_1|+\cdots+|\mathcal{O}_m|)} m^V_{U_1,\ldots,U_m}(\mathcal{O}_1, \ldots, F_{U_i}(\mathcal{O}_i), \ldots, \mathcal{O}_m).$$

The algebra $\mathrm{Der}^\bullet(\mathrm{Obs})$ of derivations of arbitrary degree is naturally a dg Lie algebra, with the bracket defined on each open set, and the differential defined by $(dF)_U = [d_U, F_U]$.

Definition 1.11. If $\mathfrak{g}$ is a dg Lie algebra and Obs is a factorization algebra on $\mathbb{R}^n$, an *action* of $\mathfrak{g}$ on Obs is a morphism $\mathfrak{g} \to \mathrm{Der}^\bullet(\mathrm{Obs})$ of dg Lie algebras.

In the context of classical field theory we'll often restrict attention to *inner* actions of dg Lie algebras on the classical observables. If Obs is a $\mathbb{P}_0$ factorization algebra then there is a map of dg Lie algebras $\mathrm{Obs}(\mathbb{R}^n)[-1] \to \mathrm{Der}^\bullet(\mathrm{Obs})$ defined by $\mathcal{O} \mapsto \{\mathcal{O}, -\}$.

Definition 1.12. A $\mathfrak{g}$ action $\mathfrak{g} \to \mathrm{Der}^\bullet(\mathrm{Obs})$ is *inner* if it factors through a dg Lie map $\mathfrak{g} \to \mathrm{Obs}(\mathbb{R}^n)[-1]$. The shift here makes the right hand side into an unshifted dg Lie algebra.

Of course, a $\mathfrak{g}$ action on a classical field theory L and a $\mathfrak{g}$ action on its factorization algebra of observables are closely related.

Proposition 1.13. Given a dg Lie algebra $\mathfrak{g}$ and a classical field theory L with factorization algebra of classical observables Obs, any $\mathfrak{g}$-action on L induces a canonical $\mathfrak{g}$-action on Obs.

From now on for simplicity we'll consider examples where the induced $\mathfrak{g}$-action on Obs is inner. In general there's an obstruction, so any $\mathfrak{g}$-action on L induces an inner action of a central extension of $\mathfrak{g}$ on Obs. In just a moment I'll remark on why this issue won't be important for the ideas I'll be discussing today.

Definition 1.14. A factorization algebra on $\mathbb{R}^n$ is *Poincaré invariant* if

1. It receives an inner action of the Poincaré algebra $\mathfrak{iso}(n)$ as in definition 1.11: we'll denote the derivations corresponding to translation and rotation generators by ∂_i and α_{ij} respectively.
2. There are isomorphisms $\tau_A\colon \mathrm{Obs}(U) \to \mathrm{Obs}(A(U))$ for each isometry $A \in \mathrm{ISO}(n)$ and each open set U compatible with the factorization structure, and where $\tau_A \circ \tau_B = \tau_{AB}$. Given disjoint open sets $U_1, \ldots, U_k \subseteq V$ and isometries $A_1, \ldots, A_k$ such that the $\tau_{A_i}(U_i)$ are still disjoint and contained in V, if we denote by $m_{A_1,\ldots,A_k}\colon \mathrm{Obs}(U_1) \times \cdots \times \mathrm{Obs}(U_k) \to \mathrm{Obs}(V)$ the map that first acts on each local observable by an isometry then composes using the factorization algebra structure, we require that the maps $m_{A_1,\ldots,A_k}$ vary smoothly in $(A_1, \ldots, A_k)$.
3. These two structures are compatible in the following sense:
$$\begin{aligned} (1,\ldots,1,\partial_i,1,\ldots,1)m_{A_1,\ldots,A_k}(\mathcal{O}_1,\ldots,\mathcal{O}_k) &= m_{A_1,\ldots,A_k}(\mathcal{O}_1,\ldots,\partial_i\mathcal{O}_\ell,\ldots,\mathcal{O}_k) \\ \text{and } (1,\ldots,1,\alpha_{ij},1,\ldots,1)m_{A_1,\ldots,A_k}(\mathcal{O}_1,\ldots,\mathcal{O}_k) &= m_{A_1,\ldots,A_k}(\mathcal{O}_1,\ldots,\alpha_{ij}\mathcal{O}_\ell,\ldots,\mathcal{O}_k) \end{aligned}$$
where ∂_i and α_{ij} are placed in the ℓ^{th} slot on the left-hand side.

Remark 1.15. So why don't we need to be careful about the distinction between inner actions and more general actions? In general a non-inner action of the Poincaré group defines an inner action of a central extension of the Poincaré group on the classical observables. While Poincaré groups do admit non-trivial central extensions, the groups $\mathrm{SO}(n)$ do not, which means that there's always at least a smooth inner $\mathrm{SO}(n)$-action on the observables. As we'll see, today we'll only need to consider the actions of $\mathbb{R}^n$ and of $\mathrm{SO}(n)$ separately, which are necessarily always inner.

Definition 1.16. The Lie algebra L_{Kill} of Killing vector fields includes the Poincaré algebra as a subalgebra. We say a Poincaré invariant factorization algebra on $\mathbb{R}^n$ is *locally covariant* if the infinitesimal Poincaré action extends to an inner action of L_{Kill}. The image of $\mathrm{Sym}^2 T_{\mathbb{R}^n}$ under the inner action is called the (gravitational) *stress-energy tensor* of the theory. It corresponds to the variation of the action functional with respect to a deformation of the background metric on $\mathbb{R}^n$.

1.2.2 Supersymmetry Algebras

An n-dimensional *superpoincaré algebra* of signature (p, q) where $p + q = n$ is a Lie superalgebra extending the algebra $\mathfrak{iso}(p, q)$ of isometries of $\mathbb{R}^{p,q}$. For simplicity, from now on we'll focus on the case of Euclidean signature, $q = 0$. There is a superpoincaré algebra associated to each choice of spinorial representation of $\mathfrak{so}(n)$. Let's be more precise.

Choose a real spinorial representation S of the group $\mathrm{SO}(n)$ along with an symmetric $\mathfrak{so}(n)$-equivariant bilinear pairing $\Gamma\colon S \otimes S \to \mathbb{R}^n$. There is a lot of terminology associated to this choice; one typically finds that there is either one minimal representation that admits such a pairing, in which case $S = S_{\min}^N$, or there are two such minimal representations, in which case $S = S_1^{N_1} \oplus S_2^{N_2}$. We say that there are N *or* (N_1, N_2) *supersymmetries* and often talk about e.g. the "$N = 4$ supersymmetry algebra". Once such an S is chosen the pairing Γ is determined uniquely up to rescaling.

Definition 1.17. The *superpoincaré algebra* associated to the representation S is the super Lie algebra $\mathfrak{iso}(n) \ltimes \Pi S$ [1] with one additional bracket from $S \otimes S$ to $\mathbb{R}^n$ defined by the pairing Γ.

Definition 1.18. The group of *R-symmetries* is the group of outer automorphisms of the superpoincaré algebra that act trivially on the bosonic part. Typical supersymmetric field theories will carry an action of some subalgebra $\mathfrak{g}_R$ of the Lie algebra of R-symmetries compatible with the superpoincaré algebra, but *not* necessarily an action of all R-symmetries. One often considers not the superpoincaré algebra but the *supersymmetry algebra*

$$\mathfrak{g}_R \ltimes (\mathfrak{iso}(p, q) \ltimes \Pi S)$$

associated to this choice of R-symmetry action, where $\mathfrak{g}_R$ acts on S.

With both of these ingredients in hand, we can finally say what a supersymmetric field theory is.

Definition 1.19. A classical field theory on $\mathbb{R}^n$ is *supersymmetric* (for the superpoincaré algebra $\mathcal{A}$) if it is Poincaré invariant and the infinitesimal inner action of $\mathfrak{iso}(n)$ extends to an inner action of the supersymmetry algebra $\mathcal{A}$.

2 Lecture 2 – Topological Twists

2.1 The Idea of Twisting

Now, we can finally say what it means to *twist* a classical field theory. On the level of the BV-BRST complex this construction is actually quite simple.

Definition 2.1. Let Q be a *square-zero supercharge*, i.e. an element of S where $Q^2 = 0$, or equivalently $[Q, Q] = 0$. We say Q is *topological* if the map $[Q, -]\colon S \to \mathbb{R}^n$ is surjective. We say Q is *holomorphic* if n is even and the image of the map $[Q, -]$ has dimension $n/2$.

A set of *twisting data* is a pair (Q, α) where Q is a square-zero supercharge and α is a $U(1)$ subgroup of the R-symmetry group so that Q has α-weight 1. Given a supersymmetric classical field theory with an action of the group of R-symmetries, a choice of twisting data defines an action of the group $\mathbb{C}^\times \ltimes \Pi\mathbb{C}$ of automorphisms of the odd affine line.

The following can be argued straighforwardly.

Proposition 2.2. Let L be a supersymmetric field theory on $\mathbb{R}^n$, and let (Q, α) be a set of twisting data acting on L. The action of the group $\mathbb{C}^\times \ltimes \Pi\mathbb{C}$ defines a new grading and differential on L (the new grading is the α-weight and the differential is the Q-action). Form the total complex with respect to the original dg-structure and this new dg-structure. The resulting complex $L^{(\alpha,Q)}$ is still a classical field theory – we call this the *twisted* field theory associated to our choice of twisting data.

[1] The notation Π means that S is placed in fermionic degree.

Remark 2.3. If we didn't choose the $U(1)$-action α the above proposition would still have been true, but we wouldn't have obtained a sheaf of dg Lie algebras with a degree -3 pairing, but rather a sheaf of super Lie algebras with an odd pairing. In other words our $\mathbb{Z}$-grading would have collapsed to a $\mathbb{Z}/2$-grading.

Remark 2.4. There's a way of producing topological supercharges which is very important both physically and (as we'll see) mathematically. Suppose we have a supersymmetric theory with an action of an R-symmetry group G_R. We can "twist" the supersymmetry algebra by choosing a homomorphism $\phi\colon \mathfrak{so}(n) \to \mathfrak{g}_R$ and changing the action of $\mathfrak{so}(n)$ on S according to this action. If a square-zero supercharge Q is fixed by this new $\mathfrak{so}(n)$ action then the image of $[Q,-]$ in $\mathbb{R}^n$ is an irreducible $\mathfrak{so}(n)$-representation; it's non-trivial because the pairing on spinors was non-degenerate, which means it must be all of $\mathbb{R}^n$, meaning that Q was topological. We say such a Q is a topological supercharge associated to the *twisting homomorphism* ϕ. We'll talk a bit more about the role played by these twisting homomorphisms later.

At this point an obvious but important question presents itself: what can we say about the factorization algebra associated to this twisted theory? In particular, what does it mean for Q to be a topological supercharge from this point of view? In the rest of the talk I'll talk about the answer to this question. In what remains of this section I'll give some straightforward initial observations. We'll denote the map $\mathcal{A} \to \mathrm{Obs}[-1]$ defining our supersymmetry action by ν. The following claim follows immediately from the definition of the factorization algebra of classical observables.

Proposition 2.5. Up to regrading, the classical observables of the twisted theory are equivalent to the algebra of classical observables of the original theory where we add the operator $\{\nu(Q), -\}$ to the differential.

If we bear this in mind, along with the fact that ν is a Lie algebra homomorphism, if Q is a topological supercharge then for any translation x_i there exists a supersymmetry Q_i where

$$\nu(x_i) = \nu([Q, Q_i]) = \{\nu(Q), \nu(Q_i)\}.$$

Therefore, since $\nu(Q_i)$ is closed for the differential in Obs, in the Q-twisted field theory the element $\nu(x_i)$ is homotopically trivial. This is our simple key observation.

Observation 2.6. In a topologically twisted theory, all translations act homotopically trivially. We call such a theory *de Rham translation invariant.*

2.2 Twisted Theories and $\mathbb{E}_n$-Algebras

Let's run with observation 2.6 and see where it leads us. The story I'll tell here will follow very closely the construction of vertex algebras from holomorphic field theories on $\mathbb{C}$ demonstrated by Huang [Hua94] and Costello-Gwilliam [CG16, Chapter 5, Theorem 2.2.1]. I'll sketch an argument which says that there's a canonical way – under certain fairly mild assumptions – of associating an $\mathbb{E}_n$-algebra to any topologically twisted theory.

Remark 2.7. You might be aware of a theorem of Lurie's [Lur] which says that an $\mathbb{E}_n$-algebra is equivalent to a factorization algebra on $\mathbb{R}^n$ which is *locally constant*, meaning that given any inclusion $U \hookrightarrow V$ of contractible open subsets of $\mathbb{R}^n$ the factorization map $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$ is a quasi-isomorphism. Lurie's result however is not constructive, whereas the result I'll discuss today produces an algebra over a specific model for the $\mathbb{E}_n$ operad from a suitably topological factorization algebra.

To any Poincaré invariant (or just translation invariant) factorization algebra Obs we can assign an algebra over a certain coloured operad, or dually a coalgebra over a coloured cooperad, as explained in [CG16, Chapter 4, Section 8.2].

Definition 2.8. Define $\mathrm{Disc}_n^{\mathrm{col}}$ to be the $\mathbb{R}_{>0}$-coloured operad whose space of degree k morphisms from $(r_1, \ldots, r_k)$ to s is the space $\mathrm{Disc}_n^{\mathrm{col}}(r_1, \ldots, r_k|s)$ of isometric embeddings $\overline{B}_{r_1} \sqcup \cdots \sqcup \overline{B}_{r_k} \to B_s$, where B_r is the open ball of radius r with closure $\overline{B}_r$, and with the obvious composition maps.

Construction 2.9. Given a translation invariant factorization algebra Obs, define A_r to be $\mathrm{Obs}(B_r(0))$: the algebra assigned to the unit ball of radius r about 0. The set $\{A_r\}_{r\in\mathbb{R}_{>0}}$ can be made into a $\mathrm{Disc}_n^{\mathrm{col}}$-algebra by sending an embedding $F \in \mathrm{Disc}_n^{\mathrm{col}}(r_1,\dots,r_k|s)$ where the k balls have centers $x_1,\dots,x_k$ to the operation

$$A_{r_1}\times\cdots\times A_{r_k} \to \mathrm{Obs}(B_{r_1}(x_1))\times\cdots\times\mathrm{Obs}(B_{r_k}(x_k)) \to A_s,$$

where the first arrow is the equivalence coming from translation invariance, and the second arrow is given by the factorization structure.

If we prefer, we can view $\{A_r\}$ as defining a *coalgebra* over the $\mathbb{R}_{>0}$-coloured cooperad $C^\infty(\mathrm{Disc}_n^{\mathrm{col}})$. We'll take this point of view, and make a simplifying assumption.

Definition 2.10. We say a factorization algebra is *rescaling invariant* if the factorization map $\mathrm{fact}_{r,R}$ associated to an embedding of concentric balls around the origin is a quasi-isomorphism.

Proposition 2.11. If a translation invariant factorization algebra is actually *de Rham* translation invariant, then the corresponding $C^\infty(\mathrm{Disc}_n^{\mathrm{col}})$-coalgebra lifts to an $\Omega^\bullet(\mathrm{Disc}_n^{\mathrm{col}})$-coalgebra, or dually to a $C_\bullet(\mathrm{Disc}_n^{\mathrm{col}})$-algebra.

Theorem 2.12. There is a fully faithful embedding

$$\mathbb{E}_n\text{-alg} \to C_\bullet(\mathrm{Disc}_n^{\mathrm{col}})\text{-alg}$$

whose essential image consists of those $C_\bullet(\mathrm{Disc}_n^{\mathrm{col}})$-algebras which are rescaling invariant.

So the upshot is that, given a topologically twisted factorization algebra, its observables form an $\mathbb{E}_n$ algebra, in a concrete way, as long as we verify a single condition: rescaling invariance. It is possible to verify this condition in examples, for instance when we can define an action of the *dilation* group $\mathbb{R}_{>0}$ on the twisted theory.

Example 2.13 (Superconformal Field Theories)**.** Superconformal groups are supersymmetric extensions of the groups of conformal transformations of $\mathbb{R}^n$. If a supersymmetric field theory is actually superconformal then, in particular, it admits an action of the group of dilations. If additionally the supercharge Q is dilation invariant then the action of dilations survives twisting. In this context one always ends up with a homotopically trivial dilation action.

However, usually the supercharge Q will *not* be dilation invariant. One needs to modify the dilation action using the R-symmetry group in order to make it dilation invariant. In superconformal theories (always in dimensions ≥ 3 and in the interesting examples in dimension < 3) this is possible, fairly explicitly. So topological twists of superconformal field theories always define $\mathbb{E}_n$-algebras.

2.3 Twisting Homomorphisms and G-Structured Theories

Now, factorization homology as defined by Ayala and Francis produced a *framed* functorial field theory from an $\mathbb{E}_n$-algebra. When is this theory actually oriented? When the $\mathbb{E}_n$-algebra admits a compatible action of the group $\mathrm{SO}(n)$. In our setup we can construct such an $\mathrm{SO}(n)$ action, again, in a very general context, motivated by the constructions appearing in the physics literature.

Definition 2.14. A square-zero supercharge Q is compatible with a *twisting homomorphism* $\phi\colon \mathrm{SO}(n)\to G_R$ if Q is fixed by the subgroup $\mathrm{SO}(n)\subseteq \mathrm{SO}(n)\times G_R$ defined by the map $(1,\phi)$. More generally one can consider supercharges compatible with a twisting homomorphism from a subgroup $G\subseteq \mathrm{SO}(n)$.

If Q is compatible with a twisting homomorphism ϕ then the associated G-action survives to a G-action on the Q-twisted factorization algebra. This is, however, not quite enough to obtain a G-action on the associated $\mathbb{E}_n$-algebra. For that we need something more.

Definition 2.15. A *smooth* G_{dR}*-action* is a smooth G-action along with a homotopy trivialization of the infinitesimal $\mathfrak{g}$ action.

Theorem 2.16. If a de Rham translation invariant and rescaling invariant factorization algebra admits a compatible smooth G_{dR} action for some subgroup $G \subseteq \mathrm{SO}(n)$ then the corresponding $\mathbb{E}_n$-algebra lifts to a $G \ltimes \mathbb{E}_n$-algebra.

So when can we lift the twisting G-action to a G_{dR}-action? Well, to first order we can answer this question using the gravitational stress-energy tensor.

Definition 2.17. Let Obs be a locally covariant factorization algebra as in definition 1.9. The *gravitational stress-energy tensor* is the map $\mathrm{Sym}^2 T_{\mathbb{R}^n} \to L_{\mathrm{Kill}} \to \mathrm{Obs}[-1]$.

Theorem 2.18. If Obs is a locally covariant factorization algebra on $\mathbb{R}^n$, Q is a topological supercharge compatible with a G-twisting homomorphism ϕ, and the ϕ-twisted gravitational stress-energy tensor is Q-exact, then the G-action on the Q-twisted theory is Q-exact.

This isn't enough alone to get a G_{dR}-action – for that we need some conditions on the potential guaranteeing that we have a Lie trivialization. An alternative origin of G_{dR}-actions is from superconformal theories.

Theorem 2.19. If Obs is an $\mathcal{N} = 2$ or more superconformal theory in dimension 2 or an $\mathcal{N} = 4$ or more superconformal theory in dimension 3 or 4, and Q is a topological supercharge of maximal rank compatible with a twisting homomorphism from $\mathrm{Spin}(n)$, then the twisted $\mathrm{SO}(n)$-action extends to an $\mathrm{SO}(n)_{\mathrm{dR}}$-action.

The proof of this theorem proceeds by direct calculation in the superconformal algebra.

2.4 Examples

I'll conclude with a discussion of a few interesting examples.

Examples 2.20.

1. **2d Superconformal Theories – The A- and B-models.**
   One can do this analysis explicitly in the example of *2d $N = 2$ superconformal field theories*. These are 2d $N = 2$ supersymmetric field theories where the supersymmetry action extends to an action of the $N = 2$ super Virasoro algebra. One checks that one can twist the $N = 2$ super Virasoro algebra so that not only the translations but also the rotation is Q-exact. One can use this to define a twisted $N = 2$ superconformal field theory where the gravitational stress-energy tensor vanishes, and therefore an oriented 2d field theory. Witten's 2d A-model and B-model arise from this $N = 2$ superconformal story as twists of a supersymmetric sigma model (see Ben-Zvi–Heluani–Szczesny [BZHS08] where a vertex algebra construction is presented as the chiral de Rham complex, following work of Malikov–Schechtman–Vaintrob [MSV99] who proved that the chiral de Rham complex carries an action of the super Virasoro algebra).

2. **4d $N = 4$ theories – Vafa-Witten and Kapustin-Witten.**
   [VW94, KW07, Set13] In the 4d $N = 4$ supersymmetry algebra there are several natural twisting homomorphisms that we might choose. The R-symmetry group acting on 4d $N = 4$ is SL_4, so working with complexified Lie algebras twisting homomorphisms are homomorphisms $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_4(\mathbb{C})$. The two most important examples are the *Vafa-Witten twist* ϕ_{VW} which sends (A, B) to the block diagonal matrix $\mathrm{diag}(A, A)$, and the *Kapustin-Witten twist* ϕ_{KW} which sends (A, B) to the block diagonal matrix $\mathrm{diag}(A, B)$. With respect to the Vafa-Witten twist there is a single topological supercharge, and with respect to the Kapustin-Witten twist there is a $\mathbb{CP}^1$-family of topological supercharges. The Vafa-Witten twisted theory [VW94] is obtained from the Vafa-Witten twisting homomorphism and supercharge, so defines a 4d oriented theory.

   In their paper [KW07], Kapustin and Witten define a $\mathbb{CP}^1$ of topological twists of 4d $N = 4$ theory. These twists can be viewed as the $\mathbb{CP}^1$ of topological twists that factor through a given *holomorphic* twist, which is automatically invariant under $\mathrm{U}(2) \subseteq \mathrm{SO}(4)$ so defines a theory on arbitrary complex surfaces. However only a single point in this $\mathbb{CP}^1$-family (the so-called "A" point) is invariant for the twisted $\mathrm{SO}(4)$-action. In general the family is only invariant under $\mathrm{SO}(2) \times \mathrm{SO}(2)$, so defines a theory on products of oriented surfaces. We should emphasise that this is a purely perturbative story. If one works with the whole stack of local solutions to the equations of motion then there are additional subtleties relevant for the geometric Langlands program. I'll talk about some of these issues in my lecture series this week (or you can see [EY15, EY17]).

Remark 2.21. In his thesis [Set13], Setter calculates the Vafa-Witten and Kapustin-Witten twisted theories explicitly, and in particular finds an agreement between the Vafa-Witten twisted theory and the Kapustin-Witten B-twist after reducing on a circle. One can also see this from our perspective without needing to do an explicit calculation, since the Vafa-Witten and Kapustin-Witten twisting supercharges coincide, and while the twisted SO(4)-actions are different they agree when one restricts to an SO(3) subgroup.

3. **8d $N = 1$ theories and their dimensional reductions.**
[AOS97] The 8d $N = 1$ supersymmetry algebra is interesting in that it admits topological supercharges – it's the only non-trivial minimal supersymmetry algebra with this property (with 16 or fewer supercharges). There are various ways of seeing this, one way is to check that one can identify the two semispin representations with copies of the complexified octonions, and the complex Γ-pairing $S_+ \otimes S_- \to \mathbb{C}^8$ with complexified octonion multiplication. From this point of view, clearly all Weyl spinors square to zero and all complexified octonions which are not zero divisors are topological.

The topological twist of 8d $N = 1$ super Yang-Mills theory has been studied by Acharya, O'Loughlin and Spence [AOS97]. They computed the twist and verified that the gravitational stress-energy tensor vanishes. They argue that this 8d theory is defined on arbitrary Spin(7)-manifolds. From our point of view this is easy to see: under the defining embedding Spin(7) → SO(8) the semispin representations split off a one-dimensional summand. If we twist by a topological supercharge in this summand we obtain a Spin(7)-invariant theory, and therefore a Spin(7)-topological field theory in 8-dimensions. This theory has interesting dimensional reductions: to a theory in 7d defined on G_2-manifolds and a theory in 6d defined on Calabi-Yau 3-folds. It would be interesting to connect these topologically twisted theories to topics studied in mathematics, perhaps Donaldson-Thomas theory and its categorifications?

References

[AOS97] Bobby S Acharya, M O'Loughlin, and B Spence. Higher-dimensional analogues of Donaldson-Witten theory. *Nuclear Physics B*, 503(3):657–674, 1997.

[BZHS08] David Ben-Zvi, Reimundo Heluani, and Matthew Szczesny. Supersymmetry of the chiral de Rham complex. *Compositio Mathematica*, 144(2):503–521, 2008.

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory, Volume I*. Number 31 in New Mathematical Monographs. Cambridge University Press, 2016.

[EY15] Chris Elliott and Philsang Yoo. Geometric Langlands twists of $N = 4$ gauge theory from derived algebraic geometry. *arXiv preprint arXiv:1507.03048*, 2015.

[EY17] Chris Elliott and Philsang Yoo. A physical origin for singular support conditions in geometric langlands theory. *arXiv prepring arXiv:1707.01292*, 2017.

[Hua94] Yi-Zhi Huang. Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras. *Communications in mathematical physics*, 164(1):105–144, 1994.

[KW07] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *Communications in Number Theory and Physics*, 1:1–236, 2007.

[Lur] Jacob Lurie. Higher algebra. 2014. *Draft available from author's website at http://www.math.harvard.edu/˜lurie/papers/HigherAlgebra.pdf.*

[MSV99] Fyodor Malikov, Vadim Schechtman, and Arkady Vaintrob. Chiral de Rham complex. *Communications in mathematical physics*, 204(2):439–473, 1999.

[Set13] Kevin Setter. *Topological quantum field theory and the geometric Langlands correspondence*. PhD thesis, California Institute of Technology, 2013.

[VW94] Cumrun Vafa and Edward Witten. A strong coupling test of S-duality. *Nuclear Physics B*, 431(1):3–77, 1994.

Institut des Hautes Études Scientifiques
35 Route de Chartres, Bures-sur-Yvette, 91440, France
celliott@ihes.fr