JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 117 & 118 (1997) 119–144

A global view of residues in the torus

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Abstract

We study the total sum of Grothendieck residues of a Laurent polynomial relative to a family $f_1, \ldots, f_n$ of sparse Laurent polynomials in n variables with a finite set of common zeroes in the torus $T = (C^*)^n$. Under appropriate assumptions we may embed T in a toric variety X in such a way that the total residue may be computed by a global object in X, the toric residue. This yields a description of some of its properties and new symbolic algorithms for its computation.
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1991 Math. Subj. Class.: Primary 14M25; Secondary 32A27, 68Q40

1. Introduction

Let $f_1, \ldots, f_n$ be Laurent polynomials in n variables with a finite set V of common zeroes in the torus $T = (C^*)^n$. Given another Laurent polynomial q, the global residue of the differential form

$$\phi_q = \frac{q}{f_1 \cdots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n},$$

i.e., the sum of the local Grothendieck residues of ϕ_q at each of the points in V, with respect to $f_1, \ldots, f_n$, is a rational function of the coefficients of the f's which has interesting applications in a number of different contexts such as: integral representations formulae in complex analysis; explicit division formulae and effective bounds for problems in commutative algebra (see [5] which contains an extensive bibliography); duality methods in the study of algebraic questions such as the ideal

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[1] Supported by NSF Grant DMS-9404642; part of the work on this paper was done while visiting the University of Buenos Aires.
[2] Supported by UBACYT and CONICET, Argentina.

PII S0022-4049(97)00008-X

membership problem, complexity bounds, etc. [4, 14, 15, 17, 30]. We are interested in the properties of the global residue and in finding symbolic algorithms for its computation.

In a previous joint paper with Sturmfels [8], we considered the case of Taylor polynomials $f_1, \ldots, f_n$ and showed that it was possible to use Gröbner bases methods to compute global residues when the leading terms of the f_j's, relative to a positive weight, intersect only at the origin. This assumption means that the closures of the zeroes of the f_j's in a weighted projective compactification of C^n have no common intersection with the divisor at infinity.

In this paper, we generalize the above situation to consider Laurent polynomials $f_1, \ldots, f_n$ for which there exists a toroidal compactification X of T such that the closures of the zeroes of the f_j's have no common intersection outside of T (see Assumption 3 in Section 4). It is known, by the fundamental work of Khovanskii [24], that this assumption holds generically for Laurent polynomials with fixed Newton polyhedra. We then show, in Theorem 4, that the global residue of ϕ_q equals the toric residue of the rational n-form Φ_q which extends ϕ_q to X. The toric residue [11] is a global object defined as the trace over X of a suitable Čech cohomology class $[\Phi_q] \in H^n(X, \Omega_X^n)$ which coincides, when X is simplicial, with the integral over X of the Dolbeault representative of $[\Phi_q]$. Its main properties are studied in [11, 7] and summarized in Theorem 1.5. This global view allows us to show and bound the polynomial dependence of the global residue on coefficients corresponding to points in the interior of the Newton polyhedron. This follows from a differentiation formula for toric residues, together with the toric version of the Euler–Jacobi vanishing conditions [25], which we recover with our methods.

In Section 4 we restrict ourselves to the case when the polynomials f_j are supported in an integral multiple k_jP of a fixed polytope P (see Assumption 3). This is a variation on the "unmixed" case where one considers polynomials with the same Newton polyhedron. Now the divisors defined by the closure in the toric variety associated to P of the zeroes of the f_j's are all multiples of a single ample divisor. This has very important algebraic consequences which allow us to give a simple algorithm for the evaluation of global residues by means of Gröbner bases computations. We propose also a new method to compute traces of Laurent polynomials over V. These results extend to the case of certain rational functions q on T, regular at the points of V. We should point out that Assumption 4 is essentially disjoint from the notion of "reduced system" considered by Zhang [35], and a similar assumption in the work of Gel'fond and Khovanskii [20].

Finally, in Section 5, we consider the case of Taylor polynomials satisfying an appropriate version of Assumption 3. We obtain a generalization of the (Gröbner basis) normal form algorithm of [8], as well as results concerning the quotient ring $C[z_1, \ldots, z_n]/\langle f_1, \ldots, f_n \rangle$.

This global approach is pursued further in a forthcoming joint paper with Bernd Sturmfels, where we express the denominator of the global and toric residues in terms of sparse resultants.

2. Preliminaries on toric varieties and residues

We begin with some background on toric geometry mostly to establish notation and terminology. We refer the reader to [10, 12, 18, 28] for details and proofs.

A complete toric variety X of dimension n is determined by a complete fan Σ in an n-dimensional real vector space N_R. There is a distinguished lattice of maximal rank $N \subset N_R$ and we let M denote the dual lattice. The N-generators of the one-dimensional cones in Σ will be denoted $\eta_1, \ldots, \eta_s$. We will denote by S the polynomial ring $S = C[x_1, \ldots, x_s]$. It is convenient to think of the variable x_i as being associated to the generator η_i and hence to a torus-invariant irreducible divisor D_i of X. The Chow group of invariant Weil divisors in X will be denoted by $A_{n-1}(X)$. It is an abelian group of rank $r := s - n$.

Given $a \in Z^s$, we will let x^a stand for the (Laurent) monomial $\prod_{i=1}^{s} x_i^{a_i}$. As in [10], we define an $A_{n-1}(X)$-valued grading of S by

$$\deg(x^a) := \left[\sum_{i=1}^{s} a_i D_i \right] \in A_{n-1}(X).$$

More explicitly, *deg* is deduced from the right morphism in the exact sequence

$$0 \to M \to Z^s \to A_{n-1}(X) \to 0\,, \tag{1}$$

where the left morphism sends an element $m \in M$ to the s-tuple $\langle m, \eta \rangle := (\langle m, \eta_1 \rangle, \ldots, \langle m, \eta_s \rangle)$. In particular, for any $m \in M$, the monomial $x^{\langle m, \eta \rangle} := \prod_{i=1}^{s} x_i^{\langle m, \eta_i \rangle}$ has degree 0. We will denote by S_α the graded piece of S of polynomials of degree α.

Let $Z(\Sigma)$ denote the algebraic subvariety of C^s associated to the radical monomial ideal

$$B(\Sigma) := \left\langle \prod_{\eta_i \notin \sigma} x_i, \sigma \text{ a cone of } \Sigma \right\rangle.$$

The algebraic group $G := \mathrm{Hom}_Z(A_{n-1}(X), C^*) \hookrightarrow (C^*)^s$ acts naturally on C^s leaving $Z(\Sigma)$ invariant. The toric variety X may be realized as the categorical quotient $C^s - Z(\Sigma)/G$. When X is simplicial, i.e., all the cones in Σ are generated by linearly independent vectors, the G-orbits are closed and X is the geometric quotient $C^s - Z(\Sigma)/G$.

We recall (see [3]) the definition of the Euler form on X generalizing the classical construction in P^n. Let $m_1, \ldots, m_n$ be a Z-basis of M. For any subset $I = \{\rho_1, \ldots, \rho_n\}$ of $\{1, \ldots, s\}$ define

$$\det(\eta_I) := \det(\langle m_i, \eta_{\rho_j} \rangle_{1 \le i,j \le n})\,, \quad \mathrm{d}x_I = \mathrm{d}x_{\rho_1} \wedge \cdots \wedge \mathrm{d}x_{\rho_n}\,, \quad \hat{x}_I = \prod_{\rho \notin I} x_\rho\,.$$

Note that the product $\det(\eta_I)\,\mathrm{d}x_I$ is independent of the ordering of $\rho_1, \ldots, \rho_n$. Let $\Omega = \Omega_X$ be defined by the formula

$$\Omega = \sum_{|I|=n} \det(\eta_I)\, \hat{x}_I\, \mathrm{d}x_I,$$

where the sum is over all n-element subsets $I \subset \{1,\dots,s\}$. This form is well-defined up to ± 1 (depending on the choice of basis of M).

Given homogeneous polynomials $F_j \in S_{\alpha_j}$, $j = 0,\dots,n$, we define their *critical degree* to be

$$\rho = \rho(F_0,\dots,F_n) := \left(\sum_{j=0}^{n} \alpha_j\right) - \beta_0 \in A_{n-1}(X),$$

where $\beta_0 = \sum_{i=1}^{s} \deg(x_i) \in A_{n-1}(X)$ is the sum of the degrees of the variables, i.e., β_0 is the anticanonical class of X. Each $H \in S_\rho$ determines a meromorphic n-form on X

$$\omega_F(H) = \frac{H\,\Omega}{F_0 \cdots F_n}.$$

Next, we recall the definition and basic properties of the *toric residue* [7, 11]). Let $F_0,\dots,F_n$, and H be homogeneous polynomials as above, and suppose, in addition, that $F_0,\dots,F_n$ do not vanish simultaneously on X. The form $\omega_F(H)$ defines a Čech cohomology class $[\omega_F(H)] \in H^n(\mathscr{U}, \widehat{\Omega}^n_X)$, where $\mathscr{U}$ is the covering $\{U_j := \{x \in X : F_j(X) \neq 0\};\ j = 0,\dots,n\}$ and $\widehat{\Omega}^n_X$ is the sheaf of Zariski n-forms on X. It is not hard to see that $[\omega_F(H)]$ is alternating on the order of $F_0,\dots,F_n$ and that if H is in the ideal $\langle F_0,\dots,F_n\rangle$, then $\omega_F(H)$ is a Čech coboundary. Thus, $[\omega_F(H)]$ depends only on the equivalence class of H modulo the ideal generated by $F_0,\dots,F_n$. The toric residue

$$\mathrm{Res}_F : S_\rho / \langle F_0,\dots,F_n\rangle_\rho \longrightarrow C$$

is given by the formula $\mathrm{Res}_F(H) = \mathrm{Tr}_X([\omega_F(H)])$, where Tr_X denotes the trace map $\mathrm{Tr}_X : H^n(X, \widehat{\Omega}^n_X) \to C$. When there is no danger of confusion, we will write $\mathrm{Res}(H)$ instead of $\mathrm{Res}_F(H)$.

The main properties of the toric residue are summarized in the following theorem.

Theorem 1. *Let X be a simplicial, complete toric variety. Let $F_j \in S_{\alpha_j}$, $j = 0,\dots,n$, be homogeneous polynomials which do not vanish simultaneously on X and let ρ be the critical degree.*

(i) *Suppose the set $Z_0 := \{x \in X : F_j(x) = 0,\ j \neq 0\}$ is finite. Then, for any $H \in S_\rho$,*

$$\mathrm{Res}_F(H) = \sum_{x \in Z_0} \mathrm{Res}_{\{F_1,\dots,F_n\},x}\left(\frac{(H/F_0)\,\Omega}{F_1 \cdots F_n}\right),$$

where $\mathrm{Res}_{\{F_1,\dots,F_n\},x}$ denotes the local Grothendieck residue associated to $F_1,\dots,F_n$ at the point x.

(ii) *Suppose that $\alpha_0,\dots,\alpha_n \in \mathrm{Pic}(X)$ are ample divisors, then the toric residue map defines an isomorphism*

$$\mathrm{Res}_F : S_\rho / \langle F_0,\dots,F_n\rangle_\rho \to C$$

(iii) *Assuming again that $\alpha_0,\dots,\alpha_n \in \mathrm{Pic}(X)$ are ample divisors, let σ be a cone of dimension n in the fan defining X. Suppose that σ is generated by $\eta_1,\dots,\eta_n$, ordered* *in such a way that* $\det(\langle m_i, \eta_j \rangle) > 0$. *There exists an* $(n+1) \times (n+1)$ *matrix of homogeneous polynomials* $A = (A_{ij})$ *such that*

$$F_j = A_{0j} \prod_{i>n} x_i + \sum_{i=1}^{n} A_{ij} x_i.$$

Let Δ_σ *denote both the polynomial* $\det(A) \in S_\rho$ *and its class in the above quotient. Then*,

$$\mathrm{Res}_F(\Delta_\sigma) = 1.$$

(iv) *Let X now be a non-necessarily simplicial complete toric variety*, *but assume that the divisors* α_j *are all integral multiples of a fixed ample divisor* $\alpha = [\sum_{i=1}^{s} a_i D_i] \in \mathrm{Pic}(X)$, *i.e.*, $\alpha_j = k_j \alpha$, $k_j \in N$, $j = 0, \ldots, n$. *Let* $J_F(\alpha)$ *denote the* toric Jacobian [11]

$$J_F(\alpha) = \frac{1}{\det(\eta_I)\ \hat{x}_I} \det \begin{pmatrix} k_0 F_0 & \cdots & k_n F_n \\ \partial F_0/\partial x_{\rho_1} & \cdots & \partial F_n/\partial x_{\rho_1} \\ \vdots & & \vdots \\ \partial F_0/\partial x_{\rho_n} & \cdots & \partial F_n/\partial x_{\rho_n} \end{pmatrix},$$

for any choice of linearly independent $\eta_{\rho_1}, \ldots, \eta_{\rho_n}$. *Then*, $J_F(\alpha) \in S_\rho$ *and*

$$\mathrm{Res}_F(J_F(\alpha)) = \left(\prod_{j=0}^{n} k_j \right) n!\, \mathrm{vol}(P_\alpha),$$

where $\mathrm{vol}(P_\alpha)$ *denotes the volume of the polytope*

$$P_\alpha := \{ m \in M : \langle m, \eta_i \rangle \geq -a_i \}.$$

Proof. The first three items are proved in [7]. When all $k_j = 1$, part (iv) is contained in [11, Theorem 5.1]. The extension to this slightly more general situation is straightforward. Let $K := \prod_{j=0}^{n} k_j$ and set $K_j = K/k_j$, $j = 0, \ldots, n$, $\tilde{F} := (F_0^{K_0}, \ldots, F_n^{K_n})$. As all $F_j^{K_j}$ have the same ample degree $K\alpha$, we know by Theorem 5.1 of [11] that $\mathrm{Res}_{\tilde{F}}(J_{\tilde{F}}(\alpha)) = \deg(\tilde{F}) = n!\mathrm{vol}(KP_\alpha) = n!K^n \mathrm{vol}(P_\alpha)$. On the other hand,

$$\frac{J_F(\alpha)\,\Omega}{F_0 \ldots F_n} = \frac{K}{\prod_{j=0}^{n} K_j} \frac{J_{\tilde{F}}(K\alpha)\,\Omega}{F_0^{K_0} \ldots F_n^{K_n}}.$$

Therefore,

$$\mathrm{Res}_F(J_F(\alpha)) = \frac{K}{\prod_{j=0}^{n} K_j}\, n! K^n\, \mathrm{vol}(P_\alpha) = n! K\, \mathrm{vol}(P_\alpha),$$

as asserted. □

Remark 2. (i) In [7, 3.12], it is conjectured that the isomorphism in (ii) of Theorem 1 holds under the weaker assumption that $F_0, \ldots, F_n \in B(\Sigma)$. This conjecture is true, for example, when X is a weighted projective space. Here $F \in B(\Sigma)$ simply means that $\deg(F)$ is positive, but it need not be Cartier (though a multiple will be Cartier and ample).

(ii) Note that P_α is defined up to translations, but its volume is well defined.

We recall that a polytope $P \subset R^n$ is said to be *integral* whenever all of its vertices have integer coordinates. One can associate to an integral polytope P a fan $\Sigma(P)$ and, consequently, a toric variety X_P (see [18, 28]). We take as lattice $N = Z^n$ and the one-dimensional cones in $\Sigma(P)$ are spanned by the first integer vectors $\eta_1, \dots, \eta_s$ in the inner normals to the facets of P. The variety X_P comes endowed with an ample divisor defined as follows: Let $b_i = -\min_{m \in P} \langle m, \eta_i \rangle$ and set

$$\beta = \left[\sum_{i=1}^{s} b_i \cdot D_i \right] \in A_{n-1}(X_P). \tag{2}$$

Then, β is ample and the variety X_P may be reconstructed from β as $X_P = \mathrm{Proj}(S_{*\beta})$, $S_{*\beta} = \sum_{k=0}^{\infty} S_{k\beta}$.

We denote by $e_1, \dots, e_n$ the standard basis of Z^n and by $\xi_1, \dots, \xi_n$ the dual basis. The monomials

$$t_j := x^{\langle \xi_j, \eta \rangle} = \prod_{i=1}^{s} x_i^{\eta_{ij}}; 1 \le j \le n, \tag{3}$$

where $\eta_i = (\eta_{i1}, \dots, \eta_{in})$, $i = 1, \dots, s$, define coordinates in the torus $T \subset X_P$. Note that for any $a \in Z^n$, $t^a = x^{\langle a, \eta \rangle}$.

For any $k \in N$ there is a natural isomorphism $\varphi_k : C[kP \cap Z^n] \to S_{k\beta}$ defined by

$$t^m \mapsto \prod_{i=1}^{s} x_i^{\langle m, \eta_i \rangle + k b_i}, \quad m \in kP \cap Z^n.$$

We say that an integral polytope P is *prime at one of its vertices* v if v belongs to precisely n facets of P. Thus, P is prime at some vertex if and only if $\Sigma(P)$ has (at least) one simplicial cone of dimension n. We say P is *prime* if it is prime at all of its vertices, i.e., if X_P is a simplicial toric variety.

Given a Laurent polynomial $f = \sum a_\alpha t^\alpha \in C[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, we denote by $\Delta(f)$ its *Newton polyhedron*, i.e., the convex hull of $\{\alpha \in Z^n : a_\alpha \neq 0\}$. Given a polytope Q, we say that f is *Q-supported* if $\Delta(f) \subseteq Q$ and $\Delta(f)$ intersects all the facets of Q.

Given an integral polytope P and a Laurent polynomial $f = \sum_{m \in \Delta(f)} a_m t^m$, the *P-homogenization* of f is the polynomial

$$F(x_1, \dots, x_s) := \sum_{m \in \Delta(f)} a_m \prod_{i=1}^{s} x_i^{\langle m, \eta_i \rangle + c_i}; \quad c_i = -\min_{\mu \in \Delta(f)} \langle \mu, \eta_i \rangle. \tag{4}$$

We note that, relative to the degree associated to the toric variety X_P, F is homogeneous of degree $\gamma = [\sum_{i=1}^{s} c_i D_i]$. Moreover, the closure in X_P of the divisor of zeroes of f in the torus is equal to the zero set of F.

Remark 3. (i) In the particular case when f is kP-supported, for some $k \in N$, then $c_i = kb_i$ and

$$F(x_1,\dots,x_s) = \sum_{m\in\Delta(f)} a_m \prod_{i=1}^{s} x_i^{\langle m,\eta_i\rangle + kb_i} = \varphi_k(f) \in S_{k\beta}\,.$$

Thus, F has ample degree equal to $k\beta$.

(ii) Note that the P-homogenization of a Laurent polynomial depends only on the primitive vectors $\eta_1,\dots,\eta_s$ in the one-dimensional cones of the fan $\Sigma(P)$. Thus, we may similarly define the X-homogenization of a Laurent polynomial relative to any toric variety X.

3. Global residues in the torus

Let $f_1,\dots,f_n \in C[t_1,t_1^{-1},\dots,t_n,t_n^{-1}]$ be Laurent polynomials whose common set of zeroes Z in $T = (C^*)^n$ is finite. Given $a \in Z^n$, denote by ϕ_a the meromorphic form on the torus

$$\phi_a = \frac{t^a}{f_1\cdots f_n}\frac{\mathrm{d}t_1}{t_1}\wedge\cdots\wedge\frac{\mathrm{d}t_n}{t_n}.$$

We are interested in computing the *global residue*

$$R_f^T(a) = \sum_{t\in Z} \mathrm{Res}_t(\phi_a),$$

where $\mathrm{Res}_t(\phi_a)$ stands for the Grothendieck residue of ϕ_a with respect to $f_1,\dots,f_n$. We refer to [1, 22, 34] for the classical theory of residues and to [26, 27, 31, 32] for an algebraic approach.

Let Δ_j denote the Newton polyhedron of f_j. Then Khovanskii [24] has shown that for generic polynomials with the given Newton polyhedra Δ_j there exists an integral polytope P (in fact, P may be taken to be the Minkowski sum $\Delta_1 + \cdots + \Delta_n$) for which the following property holds.

Assumption 1. Let F_j denote the P-homogenization of f_j, then the set of common zeroes

$$V = \{x \in X_P : F_j(x) = 0; \quad j = 1,\dots,n\} \tag{5}$$

is contained in T (consequently, finite).

The next theorem shows that, under this assumption, the global residue $R_f^T(a)$ equals a toric residue in X_P.

Theorem 4. *Let $f_1,\dots,f_n \in C[t_1,t_1^{-1},\dots,t_n,t_n^{-1}]$ be Laurent polynomials and P an n-dimensional integral polytope for which Assumption* 3 *holds. Let β be the ample* *divisor in* X_P *defined by* (2). *Then, given* $a \in Z^n$, *there exist monomials* $H(a), F_0(a) \in S$, *and a non-negative integer* $k_0(a)$ *such that*:

(i) $\deg F_0(a) = k_0(a)\beta$ *and* $\deg H(a) = k_0(a)\beta + \sum_{j=1}^n \deg(F_j) - \beta_0 = \rho(F(a))$, *where* $F(a)$ *stands for the vector* $(F_0(a), F_1, \ldots, F_n)$.

(ii) $R_f^T(a) = \mathrm{Res}_{F(a)}(H(a))$.

Proof. Set, as above, $b_i = -\min_{m \in P}\langle m, \eta_i\rangle$, and $c_{ji} = -\min_{m \in \Delta_j}\langle m, \eta_i\rangle$. Let m_0 be a rational interior point of P. Let $k_0(a)$ be the smallest non-negative integer such that $k_0(a)m_0 \in Z^n$ and

$$e_i(a) := \langle a, \eta_i\rangle - 1 + \sum_{j=1}^n c_{ji} + k_0(a)(\langle m_0, \eta_i\rangle + b_i) \geq 0 \tag{6}$$

for all $i = 1, \ldots, s$. The existence of such $k_0(a)$ is guaranteed by the fact that $\langle m_0, \eta_i\rangle + b_i > 0$, for all $i = 1, \ldots, s$, since m_0 is in the interior of $P = \{x \in R^n : \langle x, \eta_i\rangle + b_i \geq 0, \ i = 1, \ldots, s\}$. Set

$$H(a) = \prod_{i=1}^s x_i^{e_i(a)} \ ; \quad F_0(a) := \prod_{i=1}^s x_i^{k_0(a)(b_i + \langle m_0, \eta_i\rangle)}. \tag{7}$$

Clearly, $H(a)$ and $F_0(a)$ satisfy (i). Consider then the meromorphic form in X_P

$$\Phi_a := \frac{H(a)\,\Omega}{F_0(a)F_1 \cdots F_n},$$

where Ω is defined using the standard basis $\xi_1, \ldots, \xi_n$ of M. Then, in terms of the coordinates t_i defined in (3), the restriction of Φ_a to the torus is the form ϕ_a. Indeed, it follows from (3) and (4) that for $i = 1, \ldots, n$

$$F_j(x_1, \ldots, x_s) = \left(\prod_{i=1}^s x_i^{c_{ji}}\right) f_j(x^{\langle \xi_1, \eta\rangle}, \ldots, x^{\langle \xi_n, \eta\rangle}).$$

Therefore, we may rewrite

$$\Phi_a = \frac{x^{\langle a, \eta\rangle}}{f_1(x^{\langle \xi_1, \eta\rangle}, \ldots, x^{\langle \xi_n, \eta\rangle}) \cdots f_n(x^{\langle \xi_1, \eta\rangle}, \ldots, x^{\langle \xi_n, \eta\rangle})} \ \frac{\Omega}{x_1 \ldots x_s}.$$

This implies that ϕ_a is the restriction of Φ_a to T since, as shown in the proof of Proposition 9.5 of [3], the form $\Omega/x_1 \cdots x_s$ restricts to $(\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n)/(t_1 \cdots t_n)$.

Now, since $\langle m_0, \eta_i\rangle + b_i > 0$ for all $i = 1, \ldots, s$, the support of the divisor $\{F_0(a) = 0\}$ coincides with that of the divisor at infinity $D \subset X_P$. The fact that $f_1, \ldots, f_n$ satisfy Assumption 1 then means that $F_0(a), F_1, \ldots, F_n$ have empty intersection in X_P. Note that (5) also implies that

$$\{t \in T : f_1(t) = \cdots = f_n(t) = 0\} = \{x \in X_P : F_1(x) = \cdots = F_n(x) = 0\}.$$

Thus, for P prime, i.e., X_P simplicial, (ii) in Theorem 4 is a consequence of (i) in Theorem 1.

In order to prove the general case, we let Σ' be a simplicial refinement of $\Sigma(P)$, and denote by X' the corresponding simplicial toric variety. There exists a birational morphism $\pi : X' \to X$ which is the identity on the torus. The fact that the closures in X_P of the divisors in the torus defined by $f_1, \ldots, f_n$ do not intersect at infinity, implies that their closures in X' do not intersect at infinity either. The form $\Phi'_a = \pi^*(\Phi_a)$ extends ϕ_a rationally to X' and may be written as

$$\Phi'_a := \frac{H'(a)\,\Omega'}{F'_0 \cdot F'_1 \cdots F'_n},$$

where F'_j is the X'-homogenization of f_j, $j = 1, \ldots, n$, and the divisor of zeroes of F'_0 is supported in the divisor at infinity of X'. Therefore, it follows from (i) in Theorem 1 that

$$R^T_f(a) = \mathrm{tr}_{X'}[\Phi'_a] = \mathrm{tr}_X[\Phi_a] = \mathrm{Res}_{F(a)}(H(a)),$$

which completes the proof of the theorem. □

The following corollary is a toric version of the Euler–Jacobi vanishing theorem [25].

Corollary 5. *Let $f_1, \ldots, f_n$ be as in Theorem* 4 *and let $a \in (\Delta_1 + \cdots + \Delta_n)^\circ$, then $R^T_f(a) = 0$.*

Proof. The assumption on a is equivalent to $\langle a, \eta_i \rangle + \sum_{j=1}^n c_{ji} > 0$ for all $i = 1, \ldots, s$ and consequently (6) is satisfied with $k_0(a) = 0$. This means that $F_0(a) = 1$ and Φ_a is a global meromorphic form on X_P with poles on the union of $n = \dim X_P$ divisors. When X is simplicial, the vanishing of the toric residue of Φ_a is a consequence of an extension of a theorem of Griffiths (see [7, 9, 21]). In the general case, we may argue as in the proof of Theorem 4 passing to a simplicial refinement $\pi : X' \to X$. □

Theorem 4 and its Corollary 5 may be used to study the dependence of the global residue $R^T_f(a)$ on the coefficients of $f_1, \ldots, f_n$.

Lemma 6. *Let X be a smooth, complete toric variety, $S = C[x_1, \ldots, x_s]$ its homogeneous coordinate ring, $F_j \in S_{\alpha_j}$ homogeneous polynomials. Let x^θ be a monomial of degree α_n and set $F_n^{(\lambda)} = F_n + \lambda x^\theta$; $\lambda \in C$. Assume that for $|\lambda| < 2\varepsilon$, the polynomials $F_0, \ldots, F_n^{(\lambda)}$ do not vanish simultaneously on X. Let H be a polynomial of degree $\rho = \rho(F_0, \ldots, F_n) = \rho(F_0, \ldots, F_n^{(\lambda)})$ and let*

$$R(\lambda) = \mathrm{Res}\left(\frac{H\Omega}{F_0 \cdots F_n^{(\lambda)}}\right)$$

Then, for $|\lambda| < \varepsilon$, the derivative $\mathrm{d}^\ell R/\mathrm{d}\lambda^\ell$ exists and

$$\frac{\mathrm{d}^\ell R}{\mathrm{d}\lambda^\ell} = (-1)^\ell\, \ell!\, \mathrm{Res}\left(\frac{H\, x^{\ell\theta}\, \Omega}{F_0 \cdots (F_n^{(\lambda)})^{\ell+1}}\right).$$

Proof. Let $U_j = \{x \in X : F_j(x) \neq 0\}$, $U_n^{(\lambda)} = \{x \in X : F_n^{(\lambda)}(x) \neq 0\}$, and

$$U_n^{(\varepsilon)} = \{x \in X : F_n^{(\lambda)}(x) \neq 0 \text{ for all } |\lambda| \leq \varepsilon\}\,.$$

It is easy to see that $U_n^{(\varepsilon)}$ is open and therefore we have coverings $\mathscr{U}^{(\lambda)} = \{U_0, \ldots, U_n^{(\lambda)}\}$ and $\mathscr{U}^{(\varepsilon)} = \{U_0, \ldots, U_n^{(\varepsilon)}\}$. Let $\{\sigma_j : j = 0, \ldots, n\}$, be a partition of unity subordinated to the covering $\mathscr{U}^{(\varepsilon)}$. Since $U_n^{(\varepsilon)} \subset U_n^{(\lambda)}$ for $|\lambda| < \varepsilon$, $\{\sigma_j\}$ is also subordinated to $\mathscr{U}^{(\lambda)}$ for $|\lambda| < \varepsilon$.

We recall now that the toric residue of

$$\omega^{(\lambda)} = \frac{H\Omega}{F_0 \cdots F_n^{(\lambda)}} \in \Omega^n(U_0 \cap \cdots \cap U_n^{(\lambda)})$$

is defined as

$$R(\lambda) = \mathrm{Res}(\omega^{(\lambda)}) = \int_X \eta^{(\lambda)},$$

where $\eta^{(\lambda)} \in \mathscr{A}^{n,n}(X)$ is obtained from $\omega^{(\lambda)}$ via the Dolbeault isomorphism. We recall also that $\eta^{(\lambda)}$ may be constructed from $\omega^{(\lambda)}$ by the following processes: multiplication by a σ_j; extension by zero to a larger open set; $\bar{\partial}$-differentiation. Consequently:

- $\eta^{(\lambda)}$ depends smoothly on λ in the sense that its expression in the local coordinates associated to the open cones in the fan defining X has coefficients which depend smoothly on λ.
- Since X is compact,

$$\frac{\mathrm{d}^\ell R}{\mathrm{d}\lambda^\ell} = \int_X \frac{\partial^\ell}{\partial \lambda^\ell}(\eta^{(\lambda)}).$$

- Since the operation of taking derivative with respect to λ commutes with all the operations involved in the Dolbeault isomorphism, we have

$$\frac{\mathrm{d}^\ell R}{\mathrm{d}\lambda^\ell} = \mathrm{Res}\left(\frac{\partial^\ell}{\partial \lambda^\ell}(\omega^{(\lambda)})\right) = \mathrm{Res}\left(\frac{(-1)^\ell \ell!\, H\, x^{\ell\theta}\, \Omega}{F_0 \cdots (F_n^{(\lambda)})^{\ell+1}}\right). \qquad \square$$

Let $f_j \in C[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$, $j = 1, \ldots, n$, be Laurent polynomials. Let $m \in \Delta(f_j)$ and set $f_j^{(\lambda)}(t) = f_j(t) + \lambda t^m$. Suppose now that:

Assumption 2. There exists an n-dimensional integral polytope P and $\varepsilon > 0$ such that the Laurent polynomials $f_1, \ldots, f_j^{(\lambda)}, \ldots, f_n$ satisfy Assumption 1 for the toric variety X_P and $|\lambda| < 2\varepsilon$.

By the results of Khovanskii [24], this assumption is satisfied, for example, for generic polynomials with fixed Newton polyhedra $\Delta_1, \ldots, \Delta_n$ whose Minkowski sum, $P = \Delta_1 + \cdots + \Delta_n$, is n-dimensional.

Given $a \in Z^n$, let $R_f^T(\lambda, a)$ denote the rational function of λ defined as global residue in the torus of the form

$$\phi_a^{(\lambda)} = \frac{t^a}{f_1 \cdots f_j^{(\lambda)} \cdots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n}.$$

The following theorem is proved by Zhang [35] in the case when $f_1, \ldots, f_n$ form a *reduced system*. This is a geometric condition on the Newton polyhedra $\Delta(f_1), \ldots, \Delta(f_n)$ which is stronger than Khovanskii's non-degeneracy assumption [24]. In fact, Zhang shows that, in the reduced case, the global toric residue is a Laurent polynomial on the coefficients of the f_j.

Theorem 7. *Let* $f_1, \ldots, f_n \in C[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ *be Laurent polynomials satisfying Assumption* 3. *Then, for* $|\lambda| < \varepsilon$,

$$\frac{\mathrm{d}^\ell}{\mathrm{d}\lambda^\ell} R_f^T(\lambda, a) = (-1)^\ell\, \ell!\, R^T_{f_{(j)}^{\ell+1}}(\lambda, a + \ell m),$$

where $f_{(j)}^{\ell+1}$ *stands for the n-tuple* $(f_1(t), \ldots, (f_j^{(\lambda)}(t))^{\ell+1}, \ldots, f_n(t))$.

Proof. We assume, without loss of generality, that $j = n$. Passing, if necessary, to a smooth refinement of the fan $\Sigma(P)$ and arguing as in the proof of Theorem 4, we may assume that there exists a smooth complete toric variety X such that:

- If $F_1, \ldots, F_n^{(\lambda)}$ denote the X-homogenizations of $f_1, \ldots, f_n^{(\lambda)}$, then, for $|\lambda| < 2\varepsilon$, the set of common zeroes, $V^{(\lambda)} = \{x \in X : F_1(x) = \cdots = F_n^{(\lambda)}(x) = 0\}$, is finite and contained in T.
- The meromorphic form $\phi_a^{(\lambda)}$ extends to a rational form in X, $\Phi_a^{(\lambda)}$, which may be written as

  $$\Phi_a^{(\lambda)} = \frac{H(a)\Omega}{F_0(a) F_1 \ldots F_{n^{(\lambda)}}},$$

  where $H(a)$ and $F_0(a)$ are monomials in $S = C[x_1, \ldots, x_s]$.

Consequently, it follows from (i) in Theorem 1 that $R_f^T(\lambda) = \mathrm{Res}(\Phi_a^{(\lambda)})$. But then we can apply Lemma 6 to conclude that

$$\frac{\mathrm{d}^\ell}{\mathrm{d}\lambda^\ell} R_f^T(\lambda, a) = (-1)^\ell\, \ell!\, \mathrm{Res}\left(\frac{H(a)\, x^{\ell\theta}\, \Omega}{F_0 \cdots (F_n^{(\lambda)})^{\ell+1}}\right),$$

where $\theta_i = \langle m, \eta_i \rangle + c_{ni}$, $c_{ni} = -\min_{\mu \in \Delta(f_n)} \langle \mu, \eta_i \rangle$.

But, using again (i) in Theorem 1 and the fact that the X-homogenization of $(f_n^{(\lambda)})^{\ell+1}$ is equal to $(F_n^{(\lambda)})^{\ell+1}$, we see that this last toric residue may be computed as the global residue in T of the form

$$\frac{t^a\, t^{\ell m}}{f_1 \cdots (f_n^{(\lambda)})^{\ell+1}} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n}. \qquad \square$$

Theorem 8. *Let $\Delta_1, \ldots, \Delta_n$ be integral polytopes in R^n whose Minkowski sum, P, is n-dimensional. Let $f_j(u;t) = \sum_{m \in \Delta_j} u_m^{(j)} t^m$ be a generic Laurent polynomial with Newton polyhedron Δ_j and suppose $m \in \Delta_j^{\circ}$. Then, given $a \in Z^n$, the global residue $R_f^T(u;a)$ depends polynomially on $u_m^{(j)}$ and its degree is bounded by the smallest non-negative integer δ such that*

$$\langle a, \eta_i \rangle + b_i + (\delta + 1)(\langle m, \eta_i \rangle + c_{ji}) > 0 \quad \textit{for all } i = 1 \ldots, s,$$

where $\eta_1, \ldots, \eta_s$ are the primitive vectors in the one-dimensional cones in $\Sigma(P)$, $b_i = -\min_{\mu \in P} \langle \mu, \eta_i \rangle$, and $c_{ji} = -\min_{\mu \in \Delta_j} \langle \mu, \eta_i \rangle$.

Proof. Since $R_f^T(u;a)$ is a rational function of the coefficients u, it suffices to show that the derivative

$$\frac{\partial^{\delta+1}}{\partial (u_m^{(j)})^{\delta+1}} R_f^T(u,a) = 0.$$

But, according to Theorem 7,

$$\frac{\partial^{\delta+1}}{\partial (u_m^{(j)})^{\delta+1}} R_f^T(u,a) = (-1)^{\delta+1} (\delta+1)! \, R^T_{f_{(j)}^{\delta+2}}(u, a + (\delta+1)m).$$

On the other hand, the Minkowski sum of the Newton polyhedra of $f_{(j)}^{\delta+2}$ is $P + (\delta+1)\Delta_j$ and, since our choice of δ guarantees that $a + (\delta+1)m \in (P + (\delta+1)\Delta_j)^{\circ}$, the assertion of the theorem follows from Corollary 5. □

4. Computation of global residues and applications

In this section we will apply the results of Section 3 to describe methods for the computation of global residues in the torus. Although, in principle, these methods apply generically they are particularly useful when the given Laurent polynomials satisfy Assumption 3 below.

The realization of the global residue as a toric residue has computational applications in the case when the degrees α_j correspond to ample divisors since, by (ii) in Theorem 1, the quotient $S_\rho / \langle F_0(a), \ldots, F_n \rangle_\rho$ is one dimensional. Unfortunately, given Laurent polynomials $f_1, \ldots, f_n$ it is not always possible to find a complete toric variety X so that the closures in X of the divisor of zeroes $\{f_j(t) = 0\}$ define ample divisors. This is, however, the case if $\Delta(f_j) = P$, $j = 1, \ldots, n$, since then the X_P-homogenizations $F_1, \ldots, F_n$ will all have ample degree β, where β is defined by (2).

Given Laurent polynomials $q, f_1, \ldots, f_n$, such that the set V of common zeroes in T of $f_1, \ldots, f_n$ is finite, we can reduce the computation of the global residue $R_f^T(q)$ to the "*unmixed*" case by the following procedure. Let $g_j(t)$ be generic Laurent polynomials with Newton polyhedron

$$Q_j = \Delta(f_1) + \cdots + \widehat{\Delta}(f_j) + \cdots + \Delta(f_n), \quad j = 1, \ldots, n,$$

and set $g(t) = g_1(t)\cdots g_n(t)$, $h_j(t) = f_j(t)g_j(t)$. Then

$$\Delta(h_j) = P = \Delta(f_1) + \cdots + \Delta(f_n), \quad j = 1\ldots, n.$$

Generically, the polynomials $h_1, \ldots, h_n$ satisfy Assumption 3 relative to P and $R_f^T(q) = R_h^T(q \cdot g)$.

In what follows, we will make the following assumption which is weaker than equal Newton polyhedron but still allows for efficient computations.

Assumption 3. There exists an integral polytope P of dimension n and positive integers k_j such that f_j is k_jP-supported, $j = 1, \ldots, n$.

If $\Delta_j = k_jP$ for all $j = 1, \ldots, n$, then Assumption 3 is satisfied (for X_P) for generic choices of the coefficients. In any case, if the above hypothesis holds, then Assumption 3 is equivalent to the non-vanishing of the facet resultants $R^{\eta_i}(f_1, \ldots, f_n)$, relative to the polytopes $k_1P, \ldots, k_nP$, for all $i = 1, \ldots, s$ (see [23]). Sparse resultants are the generalization of classical homogeneous resultants to generic over determined sparse polynomial systems. We refer the reader to [19, 29, 33] for the definition and main properties and to [16] for an algorithm to compute them. We note, also, that Assumption 4 is essentially disjoint from Zhang's notion of a reduced system.

Theorem 9. *Let $f_1, \ldots, f_n$ be Laurent polynomials and P an n-dimensional integral polytope for which Assumptions* 3 *and* 4 *are satisfied. Given $a \in Z^n$, let $H(a)$, $F_0(a)$, and $k_0(a)$ be as in* Theorem 4. *Set $K = k_1 + \cdots + k_n$. Then*:

(i) *If $k_0(a) = 0$, i.e., $a \in (KP)^\circ$, then $R_f^T(a) = 0$.*

(ii) *There exists a constant $c_a \in C$ such that $H(a) \equiv c_a J_F$ modulo the ideal $\langle F_0(a), F_1, \ldots, F_n\rangle$, where J_F is the toric Jacobian of $F_0(a), F_1, \ldots, F_n$. Consequently,*

$$R_f^T(a) = c_a \left(\prod_{j=0}^{n} k_j\right) n!\,\mathrm{vol}\,(P), \;\; k_0 = k_0(a).$$

(iii) *Suppose P is prime at a vertex v and let $\sigma(v)$ be the corresponding simplicial cone in $\Sigma(P)$. Let $\Delta_{\sigma(v)} \in S_\rho$ be as in* (iii) *of* Theorem 1 (*for an appropriate ordering of the variables corresponding to the facets meeting at v*). *Then $H(a) \equiv r_a \Delta_{\sigma(v)}$, modulo $\langle F_0(a), \ldots, F_n\rangle$, for some $r_a \in C$, and $R_f^T(a) = r_a$.*

(iv) *Let $m \in \Delta(f_j) \cap (k_jP)^\circ$, then the global residue $R_f^T(a)$ depends polynomially on the coefficient $u_m^{(j)}$ and its degree is bounded by the smallest non-negative integer δ such that*

$$\langle a, \eta_i\rangle + Kb_i + (\delta + 1)(\langle m, \eta_i\rangle + k_jb_i) > 0 \text{ for all } i = 1, \ldots, s,$$

where $\eta_1, \ldots, \eta_s$ are the primitive vectors in the one-dimensional cones in $\Sigma(P)$, and $b_i = -\min_{\mu \in P}\langle \mu, \eta_i\rangle$.

Proof. The proof of the first statement is analogous to that of Corollary 5. Assertions (ii) and (iii) follow from Theorem 1 since the P-homogenizations F_j have ample degree

$k_j\beta$ and, given that we may assume $k_0(a) > 0$ (otherwise, $R_f^T(a) = 0$ by (i)), $F_0(a)$ has ample degree as well.

In order to prove (iv) we observe that the verification of Assumption 3 involves only the terms in the facets of k_jP and, consequently, given $m \in \Delta(f_j) \cap (k_jP)^\circ$, $f_j^{(\lambda)}(t) = f_j(t) + \lambda t^m$, the Laurent polynomials $f_1, \ldots, f_n$ satisfy Assumption 3 if and only if $f_1, \ldots, f_j^{(\lambda)}, \ldots, f_n$ satisfy it for all $\lambda \in C$. Hence Assumption 3 holds and we may argue as in the proof of Theorem 8. □

Remark 10. Under the assumptions of (iii) in Theorem 9 we also have that $J_F(\beta) \equiv d_a \Delta_{\sigma(v)}$, modulo $\langle F_0(a), \ldots, F_n \rangle$, for some $d_a \in C$ and $\mathrm{vol}(P) = d_a/(n! \prod_{j=0}^n k_j)$.

As observed in [7, (3.11)], it is possible to use (Gröbner basis) normal forms to compute toric residues. This allows us to describe the following algorithm for the computation of the sum of Grothendieck residues on the torus associated to Laurent polynomials $f_1, \ldots, f_n$ satisfying Assumptions 3 and 4. For simplicity we will assume that the polytope P has an interior point $m_0 \in Z^n$; the passage to the general case is straightforward.

Algorithm 1.

Input: An integral polytope $P = \{x \in R^n : \langle x, \eta_i \rangle \geq -b_i, \ i = 1, \ldots, s\}$ of dimension n, where $\eta_1, \ldots, \eta_s$ are the primitive integer vectors in the inner normals to the facets of P; a point $m_0 \in P^\circ \cap Z^n$; Laurent polynomials $f_1, \ldots, f_n$ satisfying Assumptions 3 and 4; $a \in Z^n$.

Output: The global residue in the torus $R_f^T(a)$.

Step 1: Find a non-negative integer $k_0(a)$ for which the inequalities (6) are satisfied for every $i = 1, \ldots, s$. Define $F_0(a) \in S$ as in (7).

Step 2: Compute $F_j(x_1, \ldots, x_s) = \varphi_{k_j}(f_j)$ for $j = 1, \ldots, n$. Choose any term order in $C[x_1, \ldots, x_s]$ and compute a Gröbner basis of the ideal $I(k_0(a)) := \langle F_0(a), F_1, \ldots, F_n \rangle$.

Step 3: Compute J_F, or in case P has a simplicial vertex v, compute $\Delta(\sigma(v))$.

Step 4: Compute the normal form (normalf) of J_F and/or $\Delta(\sigma(v))$ modulo $I(k_0(a))$. Let x^d be the least (relative to the chosen term order) monomial of critical degree not in $I(k_0(a))$. It is easy to show that, since $\dim(S_\rho/I(k_0(a)_\rho) = 1$, the normal forms of elements of S_ρ are multiples of x^d. Thus,

$$\mathrm{normalf}(J_F) = c(J_F)\, x^d, \qquad \mathrm{normalf}(\Delta(\sigma(v)) = c(v)\, x^d$$

for appropriate non-zero constants $c(J_F), c(v) \in C$.

Step 5: Define the monomial $H_0(a) \in S$ as in (7). Compute its normal form modulo $I(k_0(a))$ and write it as $\mathrm{normalf}(H_0(a)) = c(a)\, x^d$, $c(a) \in C$.

Step 6: $R_f^T(a) = c(a)/c(v)$ or $R_f^T(a) = c(a)/c(J_F) \cdot (\prod_{j=0}^n k_j) n! \,\mathrm{vol}(P)$.

Remark 11. Note that steps 2–4 depend only on the integer $k_0(a)$ and not on a. Hence, if we fix a non-negative integer k_0, we can use these steps to compute simultaneously $R_f^T(a)$ for all $a \in Z^n$ which satisfy (6) with the given k_0.

Even though we start off with a zero-dimensional complete intersection $f_1,\dots,f_n$ in the torus or, as will be the case in Section 5, in C^n, we add $r=s-n$ variables and the resulting homogenized polynomials (together with $F_0(a)$) are neither a complete intersection – they are so only away from $Z(\Sigma)$ – nor have dimension 0 (except in the case $s=n+1$). This need not, however, have dire consequences for the complexity of the proposed algorithms.

Because the multidegree associated to X_P is defined by the exact sequence (1), if we increase s by 1, we also increase by 1 the linear relations satisfied by the exponents of monomials of a fixed degree. For example, with the same assumptions and notation as in (iii) of Theorem 9, let $\kappa=\sum_{j=0}^{n} k_j$. Then $\rho=\rho(F(a))=\kappa\beta-\beta_0$. Multiplication by $\prod_{i=1}^{s} x_i$ injects $S_\rho\to S_{\kappa\beta}\cong C[\kappa P\cap Z^n]$. Therefore, there are at most $\mathrm{Card}(\kappa P\cap Z^n)$ monomials of critical degree. On the other hand, it follows from [18, Section 5.3], that $\mathrm{Card}(jP\cap Z^n)$ is a polynomial of degree n in j with leading coefficient $\mathrm{vol}(P)$. Thus, $\dim S_\rho=\mathrm{O}(\kappa^n)$, a bound independent of r. Indeed, working with multihomogeneous polynomials in s variables is equivalent to working with polynomials in n variables and "controlled" Newton polytopes. This becomes quite apparent in the proof of Theorem 15.

Thus, given that in all the algorithms proposed in this paper we are just interested in the computation of normal forms of polynomials of a given multidegree ρ, one should modify the standard algorithms for the construction of Gröbner bases in order that the performance of the algorithms be essentially governed by the hypotheses of dimension 0 in n variables on the input polynomials. As observed in [13, Section 3; 2] this may be achieved, in the case of P^n, by removing, in the search for a Gröbner basis of the ideal $I(k_0(a))$, any polynomial of standard degree strictly greater than ρ.

Theorem 9 suggests another possible way – a variant of Algorithm 1 based on (fast) linear algebra algorithms instead of Gröbner bases computation – to compute the global residue. $R_f^T(a)$ is characterized by the equation

$$H(a)-R_f^T(a)\,\Delta(\sigma(v))\in I(k_0(a)).$$

One could then try to compute $R_f^T(a)$ by detecting the Y coordinate of a solution of the following linear system:

$$H(a)=Y\Delta(\sigma(v))+B_0F_0(a)+\sum_{j=1}^{n}B_jF_j,$$

where $B_j, j=1,\dots,n$ are polynomials with indeterminate coefficients of degree $\rho-k_j\beta$. Because of the above bound on the number of monomials of degree ρ, this is a system with $\mathrm{O}((n+1)\kappa^n)$ variables. A similar system could be defined in the more general case described by (ii) in Theorem 9.

We note that, given $f_1,\dots,f_n$, satisfying Assumption 1, the reasons why Algorithm 1 works are that: $I(k_0(a))_\rho$ has codimension 1 in S_ρ and we have a "normalizing" element – either J_F or $\Delta(\sigma(v))$ – whose residue is known. Both of these conditions are likely to be satisfied in much greater generality than the one discussed here. Indeed, as noted in Remark 2, it is conjectured that the codimension-one statement holds if P is prime and $I(k_0(a)) \subset B(\Sigma)$ (a condition easy to verify given that $B(\Sigma)$ is a radical monomial ideal), and we know from [7] that these same conditions guarantee the existence of a polynomial Δ of critical degree and toric residue 1. Thus, in practice, if $F_1, \ldots, F_n \in B(\Sigma)$, one may construct monomials $H(a)$ and $F_0(a)$, with the latter in the ideal $B(\Sigma)$, such that the corresponding form Φ_a restricts to ϕ_a, and verify – through normal form computations, for example – that the codimension-one condition holds. It is then possible to compute $R_f^T(a)$ using Algorithm 1.

Another variant of Algorithm 1 permits the computation of the total sum of Grothendieck residues in the torus of rational functions of the form q/f_0, where q is any Laurent polynomial and f_0 is a k_0P-supported Laurent polynomial for some $k_0 \in N$, which does not vanish in the set V of common zeroes of $f_1, \ldots, f_n$. By linearity, we may assume that q is a Laurent monomial. Given $a \in Z^n$, denote

$$\begin{aligned} R^T_{f_0,f}(a) &= \sum_{t\in V} \operatorname{Res}_t \left(\frac{t^a}{f_0 \cdots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n} \right) \\ &= \sum_{t\in V} \operatorname{Res}_t \left(\frac{t^a/f_0}{f_1 \cdots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n} \right). \end{aligned}$$

Then, $R^T_{f_0,f}(a)$ may be computed by a procedure similar to Algorithm 1, setting:

$$F_0(a)(x) := \left(\prod_{i=1}^{s} x_i^{\ell(b_i + \langle m_0, \eta_i \rangle)} \right) F_0(x),$$

where ℓ is the smallest non-negative integer such that

$$e_i(a) := \langle a, \eta_i \rangle - 1 + \kappa b_i + \ell(\langle m_0, \eta_i \rangle + b_i) \geq 0, \ \ \kappa = \sum_{j=0}^{n} k_j,$$

$F_0(x)$ is the P-homogenization of F_0, and $H_0(a) = \prod_{i=1}^{s} x_i^{e_i(a)}$.

Computation of traces. Given $a \in Z^n$, we consider the rational form on the torus

$$\begin{aligned} \lambda_a &= t^a \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_n}{f_n} \\ &= \frac{t^a J_f}{f_1 \cdots f_n} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n = \frac{t^a J_f^T}{f_1 \cdots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n}, \end{aligned} \tag{8}$$

where J_f denotes the affine Jacobian $J_f := \det(\partial f_i/\partial t_j)$, and $J_f^T := \det(t_j \partial f_i/\partial t_j)$ is the *affine toric Jacobian.*

If $f_1, \ldots, f_n$ satisfy Assumption 1, then $V = \{t \in T : f_1(t) = \cdots = f_n(t) = 0\}$ is a finite set. Consequently, the C-algebra $A_f := C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]/\langle f_1, \ldots, f_n \rangle$ is finite dimensional and for any Laurent polynomial q, the trace $\mathrm{tr}(q)$ of the map multiplication by q in A_f is equal to the sum of the values of q at all points in V (counted with multiplicity). It may also be computed as

$$\operatorname{tr}(q) = \sum_{t\in V} \operatorname{Res}\left(\frac{qJ_f^T}{f_1\cdots f_n}\,\frac{\mathrm{d}t_1}{t_1}\wedge\cdots\wedge\frac{\mathrm{d}t_n}{t_n}\right) = \sum_{t\in V} \operatorname{Res}\left(q\,\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_n}{f_n}\right).$$

We will use the results of Section 3 to propose a new algorithm for the computation of traces under our assumptions. It suffices to compute the traces of monomials, i.e., the power sums of the elements in V, from which the multisymmetric functions of the elements in V may be deduced (see [8] and the references therein).

As in the description of Algorithm 1, we will assume, to simplify the description, that there exists $m_0 \in P^\circ \cap Z^n$. The passage to the case m_0 rational, non-integer, is straightforward. It's clear, moreover, that after changing $P \mapsto P - m_0,\ f_j \mapsto t^{-k_j m_0} f_j$, we may assume that $m_0 = 0 \in P^\circ$. This implies, in particular, that $b_i > 0$ for all $i = 1\ldots,s$. Let $F_0 := \prod_{i=1}^s x_i^{b_i}$. Then, $\deg(F_0) = \beta$ and $F_0,\ldots,F_n$ satisfy the assumptions of (iv) in Theorem 1. Let $J_F = J_F(\beta)$ be the toric Jacobian defined by (1) and $K := \prod_{j=1}^n k_j = \prod_{j=0}^n k_j$.

Theorem 12. *Let $a \in Z^n \cap k_0P$, k_0 a non-negative integer. Then*:

(i) *The form λ_a defined in* (8) *is the restriction to the torus of the rational form Λ_a in X_P defined by*:

$$\Lambda_a := \frac{(\prod_{i=1}^s x_i^{\langle a,\eta_i\rangle + k_0 b_i})J_F\Omega}{F_0^{k_0+1}F_1\ldots F_n},$$

where, as in Section 3, *Ω is the Euler form defined relative to the standard basis $\xi_1,\ldots,\xi_n$ of M.*

(ii) *Let $J_F(k_0)$ denote the toric Jacobian, in the variety X_P of $F_0^{k_0+1},F_1,\ldots,F_n$. Then, $J_F(k_0) = (k_0+1)F_0^{k_0}J_F$ and, therefore, its toric residue relative to $\langle F_0^{k_0+1},F_1,\ldots,F_n\rangle$ is equal to $(k_0+1)Kn!\operatorname{vol}(P)$.*

Proof. The second assertion follows immediately from the definition of the toric Jacobian and (iv) in Theorem 1. Since the restriction of $\prod_{i=1}^s x_i^{\langle a,\eta_i\rangle}$ to T is t^a, statement (i) reduces to showing that the restriction to T of the form

$$\frac{J_F\Omega}{F_0\ldots F_n} \tag{9}$$

is the form $(\mathrm{d}f_1/f_1)\wedge\cdots\wedge(\mathrm{d}f_n/f_n)$. Let $K_0 = K,\ K_j = K/k_j,\ j = 1,\ldots,n$. Then the polynomials $F_j^{K_j}$ are all of degree $K\beta$ and we can define a map

$$F^K : X \to P^n.$$

On T we may define coordinates $(t_1,\ldots,t_n)$ as in (3). Thus, the restriction of F^K to T takes values in the affine open set $U_0 = \{[y_0,\ldots,y_n] : y_0 \neq 0\}$ and, for $t\in T$,

$$F^K(t) = [1, f_1^{K_1}(t),\ldots,f_n^{K_n}(t)]. \tag{10}$$

On the other hand, since all F_j, $j=0,\dots,n$, have the same degree, it follows from the proof of Theorem 5.1 in [11] that

$$\frac{J_F^K\,\Omega}{F_0^{K_0}\dots F_{n^{K_n}}}=(F^K)^*\left(\frac{\Omega_{P^n}}{y_0\dots y_n}\right),$$

where J_F^K is the toric Jacobian of $(F_0^{K_0},\dots,F_n^{K_n})$ and Ω_{P^n} is the standard Euler form in P^n. From the definition of the toric Jacobian we see that

$$\frac{J_F^K\,\Omega}{F_0^{K_0}\dots F_{n^{K_n}}}=K^{n-1}\,\frac{J_F\,\Omega}{F_0\dots F_n}.$$

Thus, the restriction of (9) to T will be, up to a multiple, equal to the pullback by (10) of the restriction of $\Omega_{P^n}/y_0\cdots y_n$ to U_0. This restriction, in turn, is equal to

$$\left(\prod_{j=1}^{n}K_j\right)\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_n}{f_n}=K^{n-1}\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_n}{f_n}$$

which proves the assertion. □

From Theorem 12 we deduce the following algorithm for the computation of the trace $\mathrm{tr}(t^a)$:

Algorithm 2.

Input: An integral polytope $P=\{x\in R^n : \langle x,\eta_i\rangle \geq -b_i,\ i=1,\dots,s\}$ of dimension n, where $\eta_1,\dots,\eta_s$ are the primitive integer vectors in the inner normals to the facets of P and we assume, for simplicity, that all $b_i>0$, i.e., $0\in P^\circ$; Laurent polynomials $f_1,\dots,f_n$ satisfying Assumptions 3 and 4; $a\in Z^n$.

Output: The trace $\mathrm{tr}(t^a)$.

Step 1: Find a non-negative integer k_0 such that $k_0\geq -\langle a,\eta_i\rangle/b_i$ for every $i=1,\dots,s$.

Step 2: Compute $F_j(x_1,\dots,x_s)=\varphi_{k_j}(f_j)$ for $j=1,\dots,n$. Set $F_0:=\prod_{i=1}^{s}x_i^{b_i}$. Choose any term order in $C[x_1,\dots,x_s]$ and compute a Gröbner basis of the ideal $I(k_0):=\langle F_0^{k_0+1},F_1,\dots,F_n\rangle$.

Step 3: Compute the toric Jacobian $J_F:=J_F(\beta)$ of $F_0,\dots,F_n$.

Step 4: Compute the normal form of $(\prod_{i=1}^{s}x_i^{\langle a,\eta_i\rangle+k_0b_i})J_F$ modulo $I(k_0)$. Let x^d be the least (relative to the chosen term order) monomial of critical degree not in $I(k_0)$. Then,

$$\mathrm{normalf}(J_F(k_0))=c(k_0)\,x^d;\quad \mathrm{normalf}\left(\left(\prod_{i=1}^{s}x_i^{\langle a,\eta_i\rangle+k_0b_i}\right)J_F\right)=c(a)\,x^d$$

for appropriate constants $c(k_0),c(a)\in C$.

Step 5: The complex number $c(k_0)$ is different from 0 and

$$\mathrm{tr}(t^a)=\frac{c(a)}{c(k_0)\,(k_0+1)\,Kn!\,\mathrm{vol}(P)}.$$

5. Global residues in C^n

The purpose of this section is to show how toric residues may be used to study global residues in C^n. As it will be apparent, the results presented below are not the most general possible; however, they should suffice to illustrate the basic ideas without undue notational confusion. Assumption 4 is as in [20, Section 9].

Let $f_1, \ldots, f_n \in C[z_1, \ldots, z_n]$ be polynomials for which the set,

$$Z_f^{\mathscr{A}} = \{z \in C^n : f_1(z) = \cdots = f_n(z) = 0\}$$

of common zeroes in C^n is finite. Given $a \in Z^n_{\geq 0}$, denote by ψ_a the meromorphic form

$$\psi_a = \frac{z^a \, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{f_1 \cdots f_n}.$$

We are interested in computing the *affine global residue*

$$R_f^{\mathscr{A}}(a) = \sum_{z \in Z_f^{\mathscr{A}}} \mathrm{Res}_z(\psi_a),$$

where, as before, $\mathrm{Res}_z(\psi_a)$ stands for the Grothendieck residue of ψ_a with respect to $f_1, \ldots, f_n$.

Throughout this section we will fix an n-dimensional integral polytope $P \subset R^n \cong M_R$ with the following property.

Assumption 4. P is contained in the positive orthant; it contains the origin and has a vertex in each coordinate axis.

Given such a polytope, we may order the inner normals, $\eta_1, \ldots, \eta_s$, to its facets so that $\eta_j = e_j$, for $j = 1, \ldots, n$. Then, the polytope P may be written as $P = \{x \in R^n_{\geq 0} : \langle x, \eta_{n+\ell}\rangle + b_\ell \geq 0,\ \ell = 1, \ldots, r = s - n\}$, with all $b_\ell > 0$. The positive orthant in $N_R \cong R^n$ is an n-dimensional cone σ_0 in the fan $\Sigma(P)$ and, thus, it defines an affine open set $X_{\sigma_0} \subset X_P$.

Let $U_{\sigma_0} = \{x \in C^s : x_{n+1} = \cdots = x_s = 1\}$, then it is clear from the definition of $Z(\Sigma(P))$ in Section 2 that $U_{\sigma_0} \cap Z(\Sigma(P)) = \emptyset$ and, moreover, G acts trivially on U_{σ_0}. We will identify $C^n \cong U_{\sigma_0} = X_{\sigma_0} \subset X_P$ and rename the inner normals $e_1, \ldots, e_n,\ \eta_1, \ldots, \eta_r$. Accordingly, we rename the coordinates $x_1, \ldots, x_s$ as $z_1, \ldots, z_n,\ \zeta_1, \ldots, \zeta_r$. Note that $T \subset X_P$ is contained in U_{σ_0} and may be identified to $(C^*)^n \subset C^n$. Moreover, $z_1, \ldots, z_n$ may be seen as the extension to C^n of the toric coordinates $t_1, \ldots, t_n$ defined by (3).

Given $f(z) = \sum_{m \in \Delta(f)} a_m z^m \in C[z_1, \ldots, z_n]$, we will denote by

$$F^P(z;\zeta) = \sum_{m \in \Delta(f)} a_m z^m \left(\prod_{\ell=1}^{r} \zeta_\ell^{\langle m, \eta_\ell\rangle + c_\ell} \right); \quad c_\ell = -\min_{\mu \in \Delta(f)} \langle \mu, \eta_\ell \rangle. \tag{11}$$

Note that $F^P(z;\zeta)$ differs from the X_P-homogenization of f by a factor $\prod_{i=1}^n z_i^{v_i}$, where $v_i = -\min_{\mu \in \Delta(f)} \langle \mu, e_i \rangle$. The divisor $\{F^P = 0\} \subset X_P$ is the closure of the zero set

$\{f=0\} \subset C^n \cong U_{\sigma_0}$. Clearly, the restriction of F^P to $C^n \cong U_{\sigma_0}$ is f. Unless there is a possibility of confusion we will drop the superscript P.

Suppose now that P is as in Assumption 4 and that $f_1,\ldots,f_n \in C[z_1,\ldots,z_n]$ satisfy the following condition.

Assumption 5. There exist positive integers $k_1,\ldots,k_n$ such that $\Delta(f_j) \subseteq k_j P$ and

$$-\min_{\mu\in\Delta(f_j)} \langle \mu, \eta_\ell\rangle = k_j b_\ell, \quad \text{for all } \ell = 1,\ldots,r.$$

Since this condition means that $\Delta(f_j)$ intersects the facets associated to $\eta_1,\ldots,\eta_r$ we will say that f_j is *k_jP-supported at infinity*.

Set $F_j(z;\zeta) = F_j^P(z;\zeta)$ as in (11), $K = k_1 + \cdots + k_n$ and, given $a \in Z^n_{\geq 0}$, let $k_0(a)$ be the smallest non-negative integer such that

$$e_\ell(a) := \langle a+1, \eta_\ell\rangle + (K + k_0(a))b_\ell - 1 \geq 0, \tag{12}$$

for all $\ell = 1,\ldots,r$, where 1 denotes the vector $(1,\ldots,1)$. Note that (12) is equivalent to requiring that $(a+1) \in ((k_0(a)+K)P)^\circ$. Define:

$$H_0(a) = z^a \prod_{\ell=1}^r \zeta_\ell^{e_\ell(a)}; \quad F_0(a) = \prod_{\ell=1}^r \zeta_\ell^{k_0(a)b_\ell}. \tag{13}$$

Let $\beta \in A_{n-1}(X_P)$ denote the ample divisor defined as in (2). In the notation of this section $\beta = [\sum_{\ell=1}^r b_\ell D_\ell]$ where D_ℓ is the T-invariant divisor associated to the one-dimensional cone spanned by η_ℓ. It is easy to check that $\deg F_j = k_j\beta$, $\deg F_0(a) = k_0(a)\beta$, $\deg H_0(a) = (k_0(a)+K)\beta - \beta_0 = \rho(F_0(a), F_1,\ldots,F_n)$, and the rational form on X_P

$$\Psi_a = \frac{H_0(a)\,\Omega}{F_0(a)\cdot F_1\cdots F_n},$$

where, as before, Ω is the Euler form associated to the standard basis $\xi_1,\ldots,\xi_n$, restricts to ψ_a in U_{σ_0}.

Theorem 13. *Let $f_1,\ldots,f_n \in C[z_1,\ldots,z_n]$ be k_jP-supported at infinity and suppose that the zero set $\{F_1 = \cdots = F_r = 0\} \subset X_P$ is contained in U_{σ_0}. Let $a \in Z^n_{\geq 0}$ and let $k_0(a)$, $F_0(a)$, and $H(a)$ be as above. Then*:

(i) *If $k_0(a) = 0$, i.e., if $a+1 \in (KP)^\circ$, $R_f^{\mathscr{A}}(a) = 0$.*

Moreover, for $k_0(a) > 0$,

(ii) *$R_f^{\mathscr{A}}(a)$ is equal to the toric residue $\mathrm{Res}_F(H(a))$, where $F = (F_0(a),\ldots,F_n)$.*

(iii) *There exist homogeneous polynomials A_{ij} such that for any $j = 1,\ldots,n$*

$$F_j = A_{0j}\prod_{\ell=1}^r \zeta_\ell + \sum_{j=1}^n A_{ij} z_i.$$

(iv) *For any choice of polynomials A_{ij} as in* (iii), *let*

$$\Delta := \prod_{\ell=1}^r \zeta_\ell^{k_0(a)b_\ell - 1}\cdot \det((A_{ij})_{i,j=1,\ldots,n}).$$

Then, $R_f^{\mathscr{A}}(a)$ is the unique complex number such that $H(a) \equiv R_f^{\mathscr{A}}(a) \cdot \Delta$, modulo the ideal $\langle F_0(a), F_1, \ldots, F_n \rangle$.

Proof. We note, first of all, that by assumption $F_0(a), F_1, \ldots F_n$ do not vanish simultaneously on X_P. Therefore, $V_f^{\mathscr{A}}$ is finite since it coincides with the set of common zeroes in X_P of $F_1, \ldots, F_n$, which is compact. Thus, it makes sense to consider the global residue $R_f^{\mathscr{A}}(a)$.

The first assertion follows as in the proof of Corollary 5, while (ii) is a consequence of (i) in Theorem 1. If we assume now that $k_0(a) > 0$, $F_0(a), F_1, \ldots, F_n$ all have ample degree and (iii) follows from the corresponding statement in Theorem 1. Alternatively, we may just observe that according to Assumption 5, the only term in F_j which contains none of the z-variables comes from the constant term in f_j and is thus a multiple of $\prod_{\ell=1}^r \zeta_\ell^{k_j b_\ell}$.

The final statement follows from the fact that, according to Theorem 1(ii), the quotient $S_\rho / \langle F_0(a), \ldots, F_n \rangle_\rho$ is one-dimensional and, by (iv) in Theorem 1, Δ is a polynomial of critical degree ρ and toric residue 1. □

Remark 14. Note that given polynomials $f_1, \ldots, f_n \in C[z_1, \ldots, z_n]$ which are $k_j P$-supported at infinity, the condition that the set of common zeroes in X_P, $\{F_1 = \cdots = F_n = 0\}$, be contained in U_{σ_0}. is equivalent to the non-vanishing of all facet resultants $R^{\eta_\ell}(f_1, \ldots, f_n)$, relative to $k_1 P, \ldots, k_n P$, $\ell = 1, \ldots, n$.

Theorem 13 implies the following algorithm for the computation of $R_f^{\mathscr{A}}(a)$.

Algorithm 3.

Input: An integral polytope $P = \{x \in R^n_{\geq 0} : \langle x, \eta_\ell \rangle \geq -b_\ell,\ \ell = 1, \ldots, r\}$ of dimension n, where $\eta_1, \ldots, \eta_r$ are the primitive integer vectors in the inner normals to those facets of P which do not lie in the coordinate hyperplanes; polynomials $f_1, \ldots, f_n$ satisfying (11) and such that $R^{\eta_\ell}(f_1, \ldots, f_n) \neq 0$ for all $\ell = 1, \ldots, r$; $a \in Z^n_{\geq 0}$.

Output: The global residue in C^n, $R_f^{\mathscr{A}}(a)$.

Step 1: Find a non-negative integer $k_0(a)$ such that $a + 1 \in ((k_0(a) + K)P)^\circ$, $K = k_1 + \cdots + k_n$. Define $F_0(a) \in C[\zeta_1, \ldots, \zeta_r]$ as in (13).

Step 2: Compute $F_j(z, \zeta)$ for $j = 1, \ldots, n$. Choose any term order in the polynomial ring $C[z_1 \ldots, \zeta_r]$ and compute a Gröbner basis of $I(k_0(a)) := \langle F_0(a), F_1, \ldots, F_n \rangle$.

Step 3: Compute Δ as in (iv) of Theorem 13.

Step 4: Compute the normal form of Δ and of $H(a)$ modulo $I(k_0(a))$. Let x^d be the least (relative to the chosen term order) monomial of critical degree not in $I(k_0(a))$. Then,

$$\mathrm{normalf}(\Delta) = c(\Delta) x^d; \quad \mathrm{normalf}(H(a)) = c(a) x^d; \quad c(\Delta), c(a) \in C.$$

Step 5: The complex number $c(\Delta) \neq 0$ and $R_f^{\mathscr{A}}(a) = c(a)/c(\Delta)$.

Examples: (i) Let $P = \{x \in R^n_{\geq 0} : -x_1 - \cdots - x_n \geq -1\}$ be the standard simplex. Then $X_P = P^n$ and given $f \in C[z_1, \ldots, z_n]$, $F^P(z; \zeta)$ is its usual homogenization. Any such f

is kP-supported at infinity, for $k = deg(f)$. A family $f_1,\dots,f_n$ satisfies the assumption in Theorem 13 if and only if they do not intersect at infinity in the usual sense.

(ii) Let $w = (w_1,\dots,w_n)$ be an n-tuple of positive integers with no common factors, and $P_w = \{x \in R^n_{\geq 0} : -w_1x_1 - \cdots - w_nx_n \geq -W;\ W = \prod_{j=1}^n w_j\}$; $X_{P_w} = P^n_w$ is the weighted projective space with weights $(1, w_1,\dots,w_n)$. For $f \in C[z_1,\dots,z_n]$, the polynomial $F^{P_w}(z;\zeta)$ is its weighted homogenization. It is no longer true that an arbitrary $f \in C[z_1,\dots,z_n]$ will be kP_w-supported at infinity for some integer k; indeed, this will only happen if the weighted degree of f is a multiple of W. However, as it is clear from the proof of Theorem 13, this is only used to appeal to (ii) in Theorem 1 which, as noted in Remark 2, holds in greater generality for weighted projective space. Thus, we may apply Theorem 13 and Algorithm 3 to a family $f_1,\dots,f_n$ which do not intersect at infinity in P^n_w in the sense of [8, (1.3)].

It is convenient to introduce the following notation: Given an integral polytope P and $m \in Z^n$ we will denote by $P[m]$ the translated polytope $\{\mu + m : \mu \in P\}$. The following theorem expresses Theorem 13 purely in terms of $C[z_1,\dots,z_n]$.

Theorem 15. *Let $f_1,\dots,f_n \in C[z_1,\dots,z_n]$ be as in Theorem 13 and write*

$$f_j(z) - f_j(0) = \sum_{i=1}^n a_{ji}(z)\, z_i,$$

with $a_{ji}(z) \in C[z_1,\dots,z_n]$ and $\Delta(a_{ji}) \subset (k_jP)[-\xi_i]$. Let $\delta := \det(a_{ji})$.

(i) *Given $a \in Z^n_{\geq 0}$, there exists $g_0(z) \in C[z_1,\dots,z_n]$ with $\Delta(g_0)[1] \subset (KP)^\circ$ and*

$$z^a - R^{\mathscr{A}}_f(a) \cdot \delta \equiv g_0(z) \,(\mathrm{mod}\langle f_1,\dots,f_n\rangle).$$

(ii) *Let $k_0 \in Z_{\geq 0}$ be such that $a + 1 \in ((K + k_0)P)^\circ$. Then $R^{\mathscr{A}}_f(a) \in C$ is uniquely determined by the property that there exist $g_1,\dots,g_n \in C[z_1,\dots,z_n]$ with $\Delta(g_j)[1]$ contained in $((K + k_0 - k_j)P)^\circ$ and*

$$z^a - R^{\mathscr{A}}_f(a) \cdot \delta = g_0 + \sum_{j=1}^n g_j \cdot f_j. \tag{14}$$

Proof. Suppose $a_{ji}(z) = \sum_{m \in \Delta(a_{ji})} \lambda_m z^m$ and let $\xi_1,\dots,\xi_n$ denote the standard basis of R^n. We set

$$A_{ji}(z;\zeta) = \sum_{m \in \Delta(a_{ji})} \lambda_m\, z^m \left(\prod_{\ell=1}^r \zeta_\ell^{\langle m+\xi_i, \eta_\ell\rangle + k_j b_\ell} \right).$$

The assumption on the Newton polyhedron of a_{ji} guarantees that all exponents are positive and we have

$$F^P_j(z;\zeta) = f_j(0) \left(\prod_{\ell=1}^r \zeta_\ell^{k_j b_\ell} \right) + \sum_{i=1}^n A_{ji}(z;\zeta)\, z_i$$

and $A_{ji}(z;1) = a_{ji}(z)$. Clearly, if Δ is as in Theorem 13, we have $\Delta(z;1) = \delta(z)$. According to (iv) in Theorem 13 we may write

$$H_0(a) - R_f^{\mathscr{A}}(a) \cdot \Delta = G_0\, F_0(a) + \sum_{j=1}^{n} G_j\, F_j. \tag{15}$$

We may assume, without loss of generality, that G_0 is homogeneous of degree $K\beta - \beta_0$ and G_j is homogeneous of degree $(K + k_0(a) - k_j)\beta - \beta_0$. Let $g_j(z) = G_j(z;1)$, then setting $\zeta = (1,\dots,1)$ in (15) we have the identity (14).

Suppose now that $z^m \zeta^v$ is a monomial of degree $K\beta - \beta_0$. Then the monomial $z^{m+1}\zeta^{v+1}$ has degree $K\beta$ and, therefore, for every $\ell = 1,\dots,r$

$$-\langle m+1, \eta_\ell\rangle + v_\ell + 1 = Kb_\ell.$$

Consequently, $\langle m+1, \eta_\ell\rangle + Kb_\ell > 0$, for all $\ell = 1,\dots,r$, and $m + 1 \in (KP)^\circ$. This shows that $\Delta(g_0)[1] \subset (KP)^\circ$ as asserted by (i). A similar argument yields the statement $\Delta(g_j)[1] \subset ((K + k_0 - k_j)P)^\circ$.

To complete the proof of (ii), suppose there exist $c \in C$ and polynomials $g_0,\dots,g_n$ with $\Delta(g_j)[1] \subset ((K + k_0 - k_j)P)^\circ$ and such that

$$z^a = c \cdot \delta + g_0 + \sum_{j=1}^{n} g_j\, f_j. \tag{16}$$

Set

$$\zeta^{\langle a,\eta\rangle} = \prod_{\ell=1}^{r} \zeta_\ell^{\langle a,\eta_\ell\rangle}; \quad H_1(a) := \prod_{\ell=1}^{r} \zeta_\ell^{e_\ell(a) - \langle a,\eta_\ell\rangle}.$$

If in (16) we change $z_j \mapsto z_j \zeta^{\langle \xi_j,\eta\rangle}$ and multiply both sides of the equation by $H_1(a)$ we obtain $H_0(a) = c\Delta + G_0F_0(a) + \sum_{j=1}^{n} G_j F_j$, with

$$G_j(z;\zeta) = g_j(z_1\zeta^{\langle \xi_1,\eta\rangle},\dots,z_n\zeta^{\langle \xi_n,\eta\rangle}) \cdot \zeta^{\langle 1,\eta\rangle} \cdot \prod_{\ell=1}^{r} \zeta_\ell^{K_j b_\ell - 1},$$

where $K_j = K + k_0 - k_j$, $j = 0,\dots,n$. Because of the assumptions on the Newton polyhedra of the g_j's, the exponents of $\zeta_1,\dots,\zeta_r$ in the above expression are all non-negative and we may apply (iv) in Theorem 13 to conclude that $c = R_f^{\mathscr{A}}$. □

Theorem 15 has the following immediate consequence which should be seen as a converse of the Euler–Jacobi vanishing conditions.

Corollary 16. *With the same assumptions as in Theorem* 13, *let* $a \in Z^n_{\geq 0}$ *be such that* $R_f^{\mathscr{A}}(a) = 0$. *Then,* z^a *coincides, modulo the ideal* $\langle f_1,\dots,f_n\rangle$ *with a polynomial whose Newton polyhedron is contained in the interior of* $(KP)[-1]$.

It is easy to check that the polynomial $\delta = det(a_{ji})$ has Newton polyhedron contained in $(KP)[-1]$. We deduce that $C[z_1,\dots,z_n]/\langle f_1,\dots,f_n\rangle$ has a C-basis which consists of polynomials with Newton polyhedra contained in $(KP)[-1]$. More precisely:

Corollary 17. *Suppose the polynomials $f_1,\dots,f_n$ satisfy the assumptions of Theorem* 13. *Then the ring $C[z_1,\dots,z_n]/\langle f_1,\dots,f_n\rangle$ has a C-basis of the form $\{\delta, h_1,\dots,h_\gamma\}$, where $\gamma < (\prod_{i=1}^n k_i)n!\mathrm{vol}(P)$ and $\Delta(h_i) \subseteq ((KP)[-1])^\circ$ for all $i = 1,\dots,\gamma$.*

Proof. It follows from (14) that the quotient ring $C[z_1,\dots,z_n]/\langle f_1,\dots,f_n\rangle$ has a basis consisting of (the class of) δ together with a suitable collection of monomials corresponding to points in $(KP)[-1]$.

The complex dimension of the quotient is less or equal than the mixed volume of the polytopes $\Delta(f_1),\dots,\Delta(f_n)$, which is less or equal that the mixed volume of $k_1P,\dots,k_nP$. Therefore, $\gamma < (\prod_{i=1}^n k_i)n!\mathrm{vol}(P)$. □

Note that, since as observed in the proof of Theorem 15, $\Delta(z;1) = \delta(z)$, the global residue in C^n of the form

$$\frac{\delta(z)}{f_1\cdots f_n}\,\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$$

is equal to 1. We can improve the choice of δ to get a polynomial with similar properties but fewer monomials. Given $f_1,\dots,f_n$ as in Theorem 15, $f_j = \sum_{m\in\Delta(f_j)} a_m^j z^m$, $j = 1,\dots,n$, let

$$f_j^\infty = \sum_{m\in C_j^\infty} a_m^j z^m,$$

where C_j^∞ denotes the union of the facets of k_jP not lying in the coordinate hyperplanes. As $0 \notin C_j^\infty$, we can find polynomials $\alpha_{ji},\ i,j = 1,\dots,n$

$$f_j^\infty = \sum_{i=1}^n \alpha_{ji} z_i. \tag{17}$$

Corollary 18. *Let $f_1,\dots,f_n$ satisfy the assumptions of Theorem* 13 *and let α_{ji}, for $i,j = 1,\dots,n$ be as in* (17). *Assume, moreover, that each monomial in $\alpha_{ji}z_i$ is a monomial in f_j^∞. Define $\delta^\infty := \det(\alpha_{ji})$, $\mathrm{d}z = \mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$. Then*:

(i) *The total sum residues in C^n of the form $[\delta^\infty/(f_1\cdots f_n)]\,\mathrm{d}z$ is equal to* 1.

(ii) *All monomials appearing in δ^∞ lie in $(KP)[-1] - ((KP)[-1])^\circ$.*

(iii) *There exists a C-basis of the quotient ring $C[z_1,\dots,z_n]/\langle f_1,\dots,f_n\rangle$ of the form $\{\delta^\infty, h_1,\dots,h_\gamma\}$, where $\Delta(h_i) \subseteq ((KP)[-1])^\circ$ for all $i = 1,\dots,\gamma$.*

(iv) *Given any polynomial q, let $c, c_1,\dots,c_\gamma \in C$ be such that $q = c\cdot\delta^\infty + \sum_{i=1}^s c_ih_i$ in the quotient ring. Then the sum of residues in C^n of $[q/(f_1\cdots f_n)]\,\mathrm{d}z$ is equal to c.*

Proof. We have already observed that the Newton polyhedron of δ is contained in $(KP)[-1]$. The polynomial δ^∞ consists of exactly those terms that lie in the facets of $(KP)[-1]$ not contained in the translates of the coordinate hyperplanes. Thus, the monomials in $\delta - \delta^{\infty}$ lie in $((KP)[-1])^{\circ}$ and the first assertion follows from (i) in Theorem 13. The other statements follow immediately. □

Remark 19. Let P be, as in the first example following Algorithm 3, the standard simplex and suppose $f_j(z_1,\ldots,z_n)=z_j^{r_j+1}+g_j(z_1,\ldots,z_n)$ with $\deg(g_j)\le r_j$, for any $j=1,\ldots,n$, or, with the previous notations, $f_j^{\infty}=z_j^{r_j+1}$. In this case, δ^{∞} is just the monomial z^r. We can then view Corollary 18 as an extension of Lemma 4.2 in [8] (see also [4]).

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