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Frobenius Modules and Hodge Asymptotics

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Abstract: We exhibit a direct correspondence between the potential defining the $H^{1,1}$ small quantum module structure on the cohomology of a Calabi-Yau manifold and the asymptotic data of the A-model variation of Hodge structure. This is done in the abstract context of polarized variations of Hodge structure and Frobenius modules.

1. Introduction

The even cohomology of a compact smooth manifold is a Frobenius algebra with respect to the cup product and the intersection form. For a compact, Kähler manifold X, multiplication by a Kähler class defines a representation of the Lie algebra $\mathfrak{sl}(2)$ on the full cohomology $H^*(X, \mathbb{C})$, whose semisimple element induces the standard $\mathbb{Z}$-grading. This is the content of the Hard Lefschetz Theorem. Beginning with the formulation of the Mirror Symmetry phenomenon [5], there has been considerable interest in studying the simultaneous action on cohomology of the Kähler cone $\mathcal{K}$ of X. Looijenga and Lunts [22] have shown that the copies of $\mathfrak{sl}(2)$ associated with the elements of $\mathcal{K}$ generate a semisimple Lie algebra and have studied some of their properties. Another point of view, introduced in [10], consists in studying $H^*(X, \mathbb{C})$ as a mixed Hodge structure which splits over $\mathbb{R}$ and is polarized by the action of every Kähler class. Hence, the crucial information is contained in the structure of $H^*(X, \mathbb{C})$ as a Sym $H^{1,1}$-module. In particular, it follows from [9, Prop. 4.66] that we may define a polarized variation of Hodge structure on $H^*(X, \mathbb{C})$ parametrized by the complexified Kähler cone of X. If a polyhedral cone of Kähler classes is chosen, this variation becomes a nilpotent orbit in the sense of Schmid [27]. This approach has proved fruitful in the study of mixed Lefschetz theorems [10].

Quantum cohomology is a deformation of the cup product on $H^*(X, \mathbb{C})$ defined in terms of the Gromov-Witten potential – a generating function for certain enumerative

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invariants. If X is a Calabi-Yau manifold, the action of $H^{1,1}$ on $\oplus H^{p,p}(X)$, with respect to the small quantum product, leads to a variation of Hodge structure, called the A-model variation by Morrison [25]. A local variation of Hodge structure is described by an algebraic component – the nilpotent orbit – and an analytic part described by a holomorphic map with values in a graded component of a nilpotent Lie algebra. For the A-model variation the nilpotent orbit is the one described in the previous paragraph.

Both Frobenius algebras and polarized variations of Hodge structure have been extensively studied in the recent physics literature. Variations of Hodge structure appear, for instance, in connection with the tree level amplitudes of twisted $N = 2$ theories – the B-model– and, for Calabi-Yau threefolds, as special geometry ([4, 11, 12]). On the other hand, $2D$ topological field theories are equivalent to Frobenius algebras. Families of these algebras were also considered: the tangent bundle of the moduli space of topological conformal field theories has, on each fiber, a Frobenius algebra structure ([17, 18]). A relation between the two objects arises in mirror symmetry via the equivalence of the A and B model correlation functions ([5, 20, 14, 25]). What is perhaps not so well known is a direct construction due to Morrison of a variation of Hodge structure based on the A-model [25]. In this paper we show a correspondence between any polarized variation of Hodge structure with appropriate degenerating behavior and a certain sub-structure of a family of Frobenius algebras. Our main result is to exhibit a simple, direct correspondence between the holomorphic data of the variation and the (small) quantum potential in such a way that the horizontality equation of a variation of Hodge structure corresponds to a graded component of the WDVV equations.

We will work throughout in the setting of abstract variations of Hodge structure. The analogous abstract notion on the "quantum" side is that of a Frobenius module introduced in Sect. 3 and their deformations defined by potentials encoding the essential properties of a graded portion of the Gromov-Witten potential.

The paper is organized as follows. In §2 we review the asymptotic description of variations. Theorem 2.2 contains the algebraic and analytic characterization of local variations. We also recall the notion of maximally unipotent boundary points and of canonical coordinates [5, 23, 16]. In Sect. 3 we define Frobenius modules and their deformations. Sect. 4 is devoted to the proof of our main result, Theorem 4.1, which establishes an equivalence between local variations with appropriate behavior at the boundary and quantum potentials. Finally, in §5 we review the construction of the A-model variation and show that it coincides with the one constructed in Theorem 4.1. As a byproduct, we obtain a direct proof that the A-model variation is indeed a polarized variation of Hodge structure.

We note that the A-model variation involves only the small quantum module structure. In the case of Hodge structures of weights 3, 4 and 5, corresponding to threefolds, fourfolds and fivefolds, the module structure suffices to recover the full quantum algebra, so that our results extend the previously known correspondences ([26, 6]) in weights 3 and 4. Also, the full quantum algebra can be recovered if it is assumed to be generated, in the geometric context, by $H^{1,1}$. In this last case, the family of Frobenius algebras obtained from a variation of Hodge structure can be seen as a Frobenius manifold. These matters will be analyzed elsewhere [19]. S. Barannikov [1–3] has introduced the notion of semi-infinite variations of Hodge structure to deal with the full quantum algebra. He has also shown that, for projective complete intersections, the A-model variation is of geometric origin and coincides with the polarized variation of Hodge structure of the mirror family.

Finally, we wish to thank Gregory Pearlstein for his very helpful comments.

2. Hodge Theory Preliminaries

In this section we briefly review the asymptotic description of variations of Hodge structure. We refer to [21, 27, 8, 6] for details and proofs.

A (real) *variation of Hodge structure* $\mathcal{V}$ over a connected complex manifold M consists of a holomorphic vector bundle $\mathcal{V} \to M$, a flat connection ∇ on $\mathcal{V}$ with quasi-unipotent monodromy, a flat real form $\mathcal{V}_{\mathbb{R}} \subset \mathcal{V}$, and a finite decreasing filtration $\mathcal{F}$ of $\mathcal{V}$ by holomorphic subbundles – the *Hodge filtration* – satisfying

$$\nabla \mathcal{F}^p \subset \Omega^1_M \otimes \mathcal{F}^{p-1} \quad \text{(Griffiths' Transversality) and} \tag{2.1}$$

$$\mathcal{V} = \mathcal{F}^p \oplus \overline{\mathcal{F}}^{k-p+1} \tag{2.2}$$

for some integer k – the *weight* of the variation – and where barring denotes conjugation relative to $\mathcal{V}_{\mathbb{R}}$. As a C^∞-bundle, $\mathcal{V}$ may then be written as a direct sum

$$\mathcal{V} = \bigoplus_{p+q=k} \mathcal{V}^{p,q}, \qquad \mathcal{V}^{p,q} := \mathcal{F}^p \cap \overline{\mathcal{F}}^q; \tag{2.3}$$

the integers $h^{p,q} := \dim \mathcal{V}^{p,q}$ are the *Hodge numbers*. A *polarization* of the variation is a flat non-degenerate bilinear form $\mathcal{Q}$ on $\mathcal{V}$, defined over $\mathbb{R}$, of parity $(-1)^k$, whose associated flat Hermitian form $\mathcal{Q}^h(\,\cdot\,,\,\cdot\,) := i^{-k}\,\mathcal{Q}(\,\cdot\,,\,\bar{\cdot}\,)$ makes the decomposition (2.3) orthogonal and such that $(-1)^p\mathcal{Q}^h$ is positive definite on $\mathcal{V}^{p,k-p}$.

Via parallel translation to a fixed fiber V we may describe a polarized variation of Hodge structure by a holomorphic *period map* $\Phi\colon M \to D/\Gamma$, where D is the classifying space of polarized Hodge structures on V and Γ is the monodromy group. We recall that D is Zariski open in the smooth projective variety $\check{D}$ consisting of all filtrations F in V, with $\dim F^p = \sum_{r\geq p} h^{r,k-r}$, satisfying $Q(F^p, F^{k-p+1}) = 0$, where Q denotes the restriction of $\mathcal{Q}$ to V. The complex Lie group $G_{\mathbb{C}} := \mathrm{Aut}(V, Q)$ acts transitively on $\check{D}$, and D is an open orbit of $G_{\mathbb{R}} := \mathrm{Aut}(V_{\mathbb{R}}, Q)$.

Let $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebras of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$, respectively. The choice of a base point $F \in \check{D}$ defines a filtration

$$F^a\mathfrak{g} := \{T \in \mathfrak{g} : T F^p \subset F^{p+a}\}$$

compatible with the Lie bracket. In particular, $F^0\mathfrak{g}$ is the isotropy subalgebra at F and since $[F^0\mathfrak{g}, F^{-1}\mathfrak{g}] \subset F^{-1}\mathfrak{g}$, the quotient $F^{-1}\mathfrak{g}/F^0\mathfrak{g}$ defines a $G_{\mathbb{C}}$-invariant subbundle of the holomorphic tangent bundle of $\check{D}$ – the *horizontal tangent bundle*. Because of (2.1), the differential of Φ or, more precisely, of any local lifting of Φ takes values on the horizontal bundle. Such maps are called *horizontal*.

Suppose now that M has a smooth compactification $\overline{M}$ such that $X := \overline{M} \setminus M$ is a normal crossings divisor. Around a point of X, the *local variation* may be described by a horizontal map

$$\Phi : (\Delta^*)^r \times \Delta^m \to D/\Gamma, \tag{2.4}$$

where Δ is the unit disk in $\mathbb{C}$ and Δ^* the punctured disk. We shall also denote by Φ its lifting to the universal covering $U^r \times \Delta^m$, where U is the upper-half plane. We let $z = (z_j)$, $t = (t_l)$ and $s = (s_j)$ be the coordinates on U^r, Δ^m and $(\Delta^*)^r$ respectively. By definition, we have $s_j = e^{2\pi i z_j}$.

Asymptotically, a period map has an algebraic component – the *nilpotent orbit* – encoding the singularities of the connection ∇, and an analytic part described by a holomorphic map with values in a nilpotent Lie algebra. Assuming, for simplicity, that the local monodromy of the variation is unipotent, let $N_1, \ldots, N_r$ denote the monodromy logarithms. Our convention is such that $\Phi(z+e_i, t) = \exp(N_i)\Phi(z, t)$, where e_i denotes the i^{th} standard vector. It follows from Schmid's Nilpotent Orbit Theorem [27] that the $\check{D}$-valued map

$$\Psi(s, t) \;:=\; \exp\left(-\sum_{j=1}^{r} \frac{\log s_j}{2\pi i}\, N_j\right) \cdot \Phi(s, t)$$

extends holomorphically to the origin. The *limiting Hodge filtration* is $F_0 := \Psi(0, 0) \in \check{D}$. The map

$$\theta(z) \;:=\; \exp\left(\sum_{j=1}^{r} z_j\, N_j\right) \cdot F_0 \in \check{D} \tag{2.5}$$

is holomorphic, horizontal, and D-valued for $\mathrm{Im}(z_j) \gg 0$; *i.e.*, is the period map of a local variation.

A nilpotent linear transformation $N \in \mathfrak{gl}(V_{\mathbb{R}})$ defines an increasing filtration, the *weight filtration*, $W(N)$ of V, defined over $\mathbb{R}$ and uniquely characterized by requiring that $N(W_l(N)) \subset W_{l-2}(N)$ and that $N^l : \mathrm{Gr}_l^{W(N)} \to \mathrm{Gr}_{-l}^{W(N)}$ be an isomorphism. It follows from [7, Theorem 3.3] that if $N_1, \ldots, N_r$ are the monodromy logarithms of a local variation, then the weight filtration $W(\sum \lambda_j N_j)$, $\lambda_j \in \mathbb{R}_{>0}$, is independent of the choice of $\lambda_1, \ldots, \lambda_r$ and, therefore, is associated with the positive real cone $\mathcal{C} \subset \mathfrak{g}_{\mathbb{R}}$ spanned by $N_1, \ldots, N_r$.

The shifted weight filtration $W = W(\mathcal{C})[-k]$ and the limiting Hodge filtration $F_0 \in \check{D}$ define a *mixed Hodge Structure* on V; *i.e.* F_0 induces a Hodge structure of weight ℓ on Gr_ℓ^W for each ℓ. Recall ([9, Theorem 2.13]) that mixed Hodge structures are equivalent to (canonical) bigradings of V, $I^{*,*}$, satisfying $I^{p,q} \equiv \overline{I^{q,p}} \mod (\oplus_{a<p,b<q} I^{a,b})$. Thus, $W_l = \oplus_{p+q\le l} I^{p,q}$ and $F_0^a = \oplus_{p\ge a} I^{p,q}$.

A mixed Hodge structure (W, F) is said to *split* over $\mathbb{R}$ if $I^{p,q} = \overline{I^{q,p}}$; in that case the subspaces $V_l = \oplus_{p+q=l} I^{p,q}$ define a real grading of W. A structure for which $I^{p,q} = \{0\}$ if $p \neq q$ is said to be of *Hodge-Tate* type. A map $T \in \mathfrak{gl}(V_{\mathbb{R}})$ such that $T(I^{p,q}) \subset I^{p+a,q+b}$ is called a morphism of bidegree (a, b).

A *polarized mixed Hodge structure* [7, (2.4)] of weight k on $V_{\mathbb{R}}$ consists of a mixed Hodge structure (W, F) on V, a $(-1, -1)$ morphism $N \in \mathfrak{g}_{\mathbb{R}}$, and a nondegenerate, $(-1)^k$-symmetric, bilinear form Q such that

1. $N^{k+1} = 0$,
2. $W = W(N)[-k]$, where $W[-k]_j = W_{j-k}$,
3. $Q(F^a, F^{k-a+1}) = 0$ and,
4. the Hodge structure of weight $k+l$ induced by F on $\ker(N^{l+1} : \mathrm{Gr}^W_{k+l} \to \mathrm{Gr}^W_{k-l-2})$ is polarized by $Q(\cdot, N^l \cdot)$.

It follows from Schmid's SL_2-orbit theorem [27] that the mixed Hodge structure $(W(\mathcal{C})[-k], F_0)$ associated with a local variation is polarized by every $N \in \mathcal{C}$. Conversely, given commuting nilpotent elements $N_1, \ldots, N_r \in \mathfrak{g}_{\mathbb{R}}$ so that the weight filtration $W(\sum \lambda_j N_j)$, $\lambda_j \in \mathbb{R}_{>0}$, is independent of the choice of $\lambda_1, \ldots, \lambda_r$, and $F_0 \in \check{D}$ such that $(W(\mathcal{C}), F_0)$ is polarized by every element $N \in \mathcal{C}$, the map (2.5) is a period mapping for $\operatorname{Im}(z_j)$ sufficiently large [9, Prop. 4.66]. Moreover, if $(W(\mathcal{C}), F_0)$ splits over $\mathbb{R}$, then $\theta(z) \in D$ for $\operatorname{Im}(z_j) > 0$. We refer to the map θ, or equivalently, to $\{N_1, \ldots, N_r; F_0\}$ as a *nilpotent orbit*.

The following example shows the relationship between nilpotent orbits (equivalently, polarized mixed Hodge structures) and the Lefschetz structure on the cohomology of a compact Kähler manifold. This point of view was introduced in [10] where it was used to obtain relations between the Lefschetz decompositions corresponding to different Kähler classes.

Example 2.1. If X is a compact Kähler manifold of dimension k, the bigrading $I^{p,q} := H^{k-q,k-p}(X)$ defines a mixed Hodge structure (W, F) on $H^*(X, \mathbb{C})$ that splits over $\mathbb{R}$. The interest of this construction lies in the fact that this mixed Hodge structure is polarized by the Kähler cone. Indeed, the Hard Lefschetz Theorem is equivalent to the statement that if ω is a Kähler class and L_ω denotes multiplication by ω, then $W = W(L_\omega)[-k]$; while the Hodge-Riemann bilinear relations imply that L_ω polarizes (W, F) relative to the intersection form. The restriction of (W, F) to $V := \oplus_{p=0}^k H^{p,p}$ defines a mixed Hodge structure of Hodge-Tate type.

We now describe the analytic component of a local variation. The bigrading associated with the limiting mixed Hodge structure (W, F_0) defines a bigrading $I^{*,*}\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ by $I^{a,b}\mathfrak{g} := \{X \in \mathfrak{g} : X(I^{p,q}) \subset I^{p+a,q+b}\}$. Set

$$\mathfrak{p}_a := \bigoplus_q I^{a,q}\mathfrak{g} \quad \text{and} \quad \mathfrak{g}_- := \bigoplus_{a \le -1} \mathfrak{p}_a. \tag{2.6}$$

The nilpotent subalgebra $\mathfrak{g}_-$ is a complement of the stabilizer subalgebra at F_0. Hence $(\mathfrak{g}_-, X \mapsto \exp(X) \cdot F_0)$ provides a local model for the $G_{\mathbb{C}}$-homogeneous space $\check{D}$ near F_0. Thus, locally around the origin, we may write $\Psi(s, t) = \exp(\Gamma(s, t)) \cdot F_0$, where $\Gamma(s, t)$ is a holomorphic $\mathfrak{g}_-$-valued map with $\Gamma(0, 0) = 0$. We also write

$$\Phi(s, t) = \exp\left(\frac{1}{2\pi i}\sum_{j=1}^r \log(s_j) N_j\right) \cdot \exp(\Gamma(s, t)) \cdot F_0 = \exp\big(X(s, t)\big) \cdot F_0,$$

where $X(s, t) \in \mathfrak{g}_-$. The horizontality of Φ now translates, in terms of the gradings (2.6), into:

$$\exp\big(-X(s, t)\big)\, d \exp\big(X(s, t)\big) = dX_{-1} \in \mathfrak{p}_{-1} \otimes T^*((\Delta^*)^r \times \Delta^m), \tag{2.7}$$

where X_{-1} denotes the $\mathfrak{p}_{-1}$-graded part of X. In particular,

$$dX_{-1} \wedge dX_{-1} = 0, \tag{2.8}$$

where $X_{-1} = \frac{1}{2\pi i}\sum_{j=1}^r \log(s_j) N_j + \Gamma_{-1}$.

The following result, which follows from [8, Thm. 2.8] and [6, Thm. 2.7], shows that the nilpotent orbit together with the $\mathfrak{p}_{-1}$-valued holomorphic function Γ_{-1} completely determine the local variation:

Theorem 2.2. *Let* $\{N_1, \ldots, N_r; F_0\}$ *be a nilpotent orbit and* $R : \Delta^r \times \Delta^m \to \mathfrak{p}_{-1}$ *be a holomorphic map with* $R(0,0) = 0$. *Define* $X_{-1}(z,t) := \sum_{j=1}^r z_j N_j + R(s,t)$, $s_j = e^{2\pi i z_j}$, *and suppose that the differential equation* (2.8) *holds. Then, there exists a unique period mapping*

$$\Phi(s,t) = \exp\left(\frac{1}{2\pi i}\sum_{j=1}^{r}\log(s_j)N_j\right)\cdot\exp(\Gamma(s,t))\cdot F_0,$$

defined in a neighborhood of the origin in Δ^{r+m} *such that* $\Gamma_{-1} = R$.

In the ensuing sections we will be concerned with a special type of maximally degenerating variation. These are relevant to the study of mirror symmetry and, from a Hodge theoretic perspective they have the advantage of allowing us to use a canonical system of coordinates on the parameter space of the variation. Following Morrison [24, Def. 3], we consider

Definition 2.3. *Given a polarized variation of Hodge structure of weight k over* $(\Delta^*)^r$ *whose monodromy is unipotent, we say that* $0 \in \Delta^r$ *is a* maximally unipotent boundary point *if*

1. $\dim I^{k,k} = 1$, $\dim I^{k-1,k-1} = r$ *and* $\dim I^{k,k-1} = \dim I^{k-2,k} = 0$, *where* $I^{*,*}$ *is the bigrading associated to the limiting mixed Hodge structure and,*
2. $\mathrm{Span}_{\mathbb{C}}\{N_1(I^{k,k}), \ldots, N_r(I^{k,k})\} = I^{k-1,k-1}$, *where* N_j *are the monodromy logarithms of the variation.*

The limiting Hodge filtration F_0 and the holomorphic function Γ of a local variation depend on the choice of coordinates on $(\Delta^*)^r$. However, in the maximally unipotent case we may normalize our choices as follows.

Proposition 2.4. *Let* $\Phi = \exp(\sum_{j=1}^r \frac{1}{2\pi i}\log(s_j)N_j)\cdot\exp(\Gamma(s))\cdot F_0$ *be a polarized variation of Hodge structure that has a maximally unipotent boundary point at* $0 \in \Delta^r$. *Then, there is a coordinate system on* Δ^r, *unique up to scaling, where* Γ *satisfies* $\Gamma(I^{1,1}) = 0$.

For a proof of Proposition 2.4, see [6, §3]. We will refer to these as *canonical coordinates*. They are standard in the physics literature and their Hodge-theoretic interpretation is due to D. Morrison [23] and P. Deligne [16].

3. Frobenius Modules

The cohomology of even degree of a compact manifold is a graded Frobenius algebra relative to cup product and the intersection form. When X is Kähler, the Hard Lefschetz Theorem and the Hodge-Riemann bilinear relations describe the action of $H^{1,1}(X)$ on the full cohomology. We abstract these properties in the notion of a (framed) Frobenius module.

Let $V = \oplus_{p=0}^k V_{2p}$ be a graded $\mathbb{C}$-vector space and $\mathcal{B}$ a symmetric nondegenerate bilinear form on V pairing V_{2p} with $V_{2(k-p)}$. Let $\{T_a\}_{0\le a\le m}$ be a $\mathcal{B}$-self dual, graded basis of V. We will refer to $\{T_a\}$ as an *adapted basis*. For $0 \le a \le m$ define $\delta(a)$ by $\mathcal{B}(T_{\delta(a)}, T_b) = \delta_{ab}$ for all $b = 0, \ldots, m$. We also set $\tilde{a} := p$ if and only if $T_a \in V_p$ and assume that the map $\sim : \{0, \ldots, m\} \to \{0, \ldots, 2k\}$ is increasing.

Definition 3.1. $(V, \mathcal{B}, e, *)$ *is a* graded V_2-Frobenius module *of weight k if*

1. $e \neq 0$ *and* $V_0 = \langle e \rangle$.
2. V *is a graded* Sym V_2*-module under* $*$.
3. *For all* $v_1, v_2 \in V$ *and* $w \in V_2$,

$$\mathcal{B}(w * v_1, v_2) = \mathcal{B}(v_1, w * v_2). \tag{3.1}$$

4. $w * e = w$ *for all* $w \in V_2$.

Since $T_0 \in V_0$, it must be a non-zero multiple of e and we assume that an adapted basis satisfies $T_0 = e$. Clearly, the fact that V is a Sym V_2-module is equivalent to

$$T_j * (T_l * T) = T_l * (T_j * T) \text{ for all } T_j, T_l \in V_2 \text{ and } T \in V. \tag{3.2}$$

We say that V is *real* if V has a real structure, $V_{\mathbb{R}}$, compatible with its grading, $*$ is real, $e \in V_{\mathbb{R}}$, and $\mathcal{B}$ is defined over $\mathbb{R}$.

Example 3.2. If X is a compact Kähler manifold of dimension k, let $V_{2p} := H^{p,p}(X)$, $\mathcal{B}_{int}$ the intersection pairing on $V := \oplus_{p=0}^{k} V_{2p}$, and $\smile$ the restriction of the cup product to V. Then, $(V, \mathcal{B}_{int}, 1, \smile)$ defines a real Frobenius module. The real structure is induced by $H^*(X, \mathbb{R})$.

As in the case of the cohomology of a compact Kähler manifold, to any real Frobenius module we can associate a Hodge-Tate mixed Hodge structure:

$$I^{p,p} := V_{2(k-p)}. \tag{3.3}$$

The multiplication operator $L_w \in \mathrm{End}(V)$, $w \in V_2$, is an infinitesimal automorphism of the bilinear form

$$Q(v_a, v_b) := (-1)^{k+\tilde{a}/2} \mathcal{B}(v_a, v_b), \tag{3.4}$$

as well as a $(-1, -1)$-morphism of the associated mixed Hodge structure. We will say that $w \in V_2 \cap V_{\mathbb{R}}$ *polarizes* V if the mixed Hodge structure $(I^{*,*}, Q, L_w)$ is polarized. A real Frobenius module V is said to be *polarizable* if it contains a polarizing element. Given a polarizing element w, the set of polarizing elements is an open cone in $V_2 \cap V_{\mathbb{R}}$. We can then choose a basis $T_1, \ldots, T_r$ of $V_2 \cap V_{\mathbb{R}}$ spanning a simplicial cone $\mathcal{C}$ contained in the closure of the polarizing cone and with $w \in \mathcal{C}$. Such a choice of a basis of V_2 will be called a *framing* of the polarized Frobenius module.

Given an adapted basis $\{T_0, \ldots, T_m\}$ of V, let $z_0, \ldots, z_m$ be the corresponding linear coordinates on V and set $q_j := \exp(2\pi i z_j)$ for $j = 1, \ldots, r := \dim V_2$. We may identify $U^r \cong (V_2 \cap V_{\mathbb{R}}) \oplus i\,\mathcal{C}$ and view the correspondence

$$\sum_{j=1}^{r} z_j T_j \in (V_2 \cap V_{\mathbb{R}}) \oplus i\,\mathcal{C} \mapsto (q_1, \ldots, q_r) \in (\Delta^*)^r$$

as the natural covering map.

Proposition 3.3. *Framed, real Frobenius modules of weight k are equivalent to nilpotent orbits of weight k whose limiting mixed Hodge structure is of Hodge-Tate type, split over* $\mathbb{R}$*, have a marked real element in* F^k*, and have the origin as a maximally unipotent boundary point.*

Proof. Let $(V, \mathcal{B}, e, *)$ be a real Frobenius module with framing $T_1, \ldots, T_r$. Set $N_j := L_{T_j}$ and $F^p := \oplus_{a \geq p} I^{a,a}$. Then $\{N_1, \ldots, N_r; F\}$ is a nilpotent orbit. The element $e \in I^{k,k} = F^k$ is a distinguished real element and the conditions of Definition 2.3 are clearly satisfied.

Conversely, suppose $\{N_1, \ldots, N_r; F\}$ is a nilpotent orbit whose limiting mixed Hodge structure is of Hodge-Tate type, split over $\mathbb{R}$ and satisfies both conditions of Definition 2.3. Set $V_{2p} := I^{k-p,k-p}$; in particular, the marked element $e \in F^k = I^{k,k} = V_0$ and it follows from (2) in Definition 2.3 that the map

$$N \in \mathrm{Span}_{\mathbb{C}}\{N_1, \ldots, N_r\} \mapsto N(e)$$

identifies the polynomial algebra $\mathbb{C}[N_1, \ldots, N_r]$ with Sym V_2 and defines a Sym V_2-action on V. Let $\mathcal{B}$ be defined from the polarization Q as in (3.4), then since the monodromy transformations N_j are infinitesimal automorphisms of Q, (3.1) is satisfied. Thus, $(V, \mathcal{B}, e, *)$ is a Frobenius module. The equivalence between nilpotent orbits and polarized mixed Hodge structures implies that $T_j = N_j(e)$, $j = 1, \ldots, r$, are a framing of V and the fact that $N_1, \ldots, N_r$ are real implies that the Frobenius structure is real. □

A Frobenius module structure may also be encoded in a polynomial of degree 3 in the variables $z_0, \ldots, z_m$. Indeed, if we let

$$\phi_0(z_0, \ldots, z_m) := \sum_{\tilde{j}=2,\ 0 \leq \tilde{a}, \tilde{b} \leq 2k} z_j z_a z_b \, C(\tilde{a}) \, \mathcal{B}(T_j * T_a, T_b),$$

with

$$C(\tilde{a}) := \begin{cases} \frac{1}{6} \text{ if } k = 3 \text{ and } \tilde{a} = 2, \\ \frac{1}{4} \text{ if } k \neq 3 \text{ and } \tilde{a} = 2 \text{ or } \tilde{a} = 2k-4, \\ \frac{1}{2} \text{ otherwise,} \end{cases}$$

then we recover the Sym V_2-action by:

$$T_j * T_a := \sum_{\tilde{c} = \tilde{a}+2} \frac{\partial^3 \phi_0}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c \, ; \quad j = 1, \ldots, r \, .$$

The polynomial ϕ_0 is called a *(classical) potential* for the Frobenius module.

We may generalize this construction by considering deformations of the classical potential. This is motivated by the construction of the quantum product as a deformation of the cup product on the cohomology. We assume, for simplicity, that $k > 3$. Let $R := \mathbb{C}\{q_1, \ldots, q_r\}_0$ denote the ring of convergent power series vanishing for $q_1 = \cdots = q_r = 0$ and R' be its image under the map induced by $q_j \mapsto e^{2\pi i z_j}$ for $1 \leq j \leq r$.

Definition 3.4. *Let* $(V, \mathcal{B}, e, *)$ *be a Frobenius module of weight* $k > 3$ *with classical potential* ϕ_0. *A* quantum potential *on* V *is a function* $\phi : V \to \mathbb{C}$ *of the form* $\phi = \phi_0 + \phi_\hbar$, *where*

$$\phi_\hbar(z) := \sum_{\tilde{a}=2k-4} z_a \phi_h^a(z_1, \ldots, z_r) + \sum_{\substack{2 < \tilde{a} < 2k-4 \\ \tilde{a}+\tilde{b}=2k-2}} z_a z_b \phi_h^{ab}(z_1, \ldots, z_r), \tag{3.5}$$

with $\phi_h^a, \phi_h^{ab} \in R'$ and such that

$$\sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3\phi}{\partial z_l \partial z_a \partial z_{\delta(c)}} \frac{\partial^3\phi}{\partial z_j \partial z_c \partial z_{\delta(d)}} = \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3\phi}{\partial z_j \partial z_a \partial z_{\delta(c)}} \frac{\partial^3\phi}{\partial z_l \partial z_c \partial z_{\delta(d)}} \tag{3.6}$$

holds for all a, $\tilde{j} = \tilde{l} = 2$ and $\tilde{d} = \tilde{a} + 4$.

Given a quantum potential ϕ on $(V, \mathcal{B}, e, *)$, we can define a deformation of the module structure by

$$T_j \cdot_q T_a := \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3\phi}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c\,, \qquad \text{with } q = (q_1, \dots, q_r) \in \Delta^r\,. \tag{3.7}$$

We should stress that, even though the right side of (3.7) depends explicitly on the variables $z_0, \dots, z_m$, (3.5) implies that it is actually a function of $q_1, \dots, q_r$. Condition (3.6) guarantees that (3.7) defines an action of Sym V_2 for all q. Moreover, $(V, \mathcal{B}, T_0, \cdot_q)$ is a Frobenius module of weight k for all q, and $\cdot_0 = *$. We will say that a deformation of the Frobenius module V is *framed* if V is framed.

Remark 3.5. Definition 3.4 abstracts the properties of the graded portion of the Gromov-Witten potential needed to describe the action of $H^{1,1}(X, \mathbb{C})$ in the small quantum cohomology ring of a Calabi-Yau manifold X. In particular, (3.6) is a graded component of the WDVV equations. We refer to [14, §8.2, §8.3] and [6, §5] for details.

We can extend the definition of quantum potential to the weight 3 case by taking $\phi = \phi_0 + \phi_\hbar$ for $\phi_\hbar \in R'$. With this notion, all the results from Sects. 4 and 5 extend to this weight. For V of weight 1 or 2, the Frobenius module is determined by $\mathcal{B}$ and e; hence no deformations are possible.

4. Correspondence

In this section we will prove the main result of this paper, namely the correspondence between deformations of framed Frobenius modules and degenerating polarized variations of Hodge structures. In §5 we will show that when the deformation arises from the quantum product of a Calabi-Yau manifold, the associated variation of Hodge structure is the so-called A-model variation.

Theorem 4.1. *There is a one to one correspondence between*

- *Quantum potentials ϕ on a framed Frobenius module $(V, \mathcal{B}, e, *)$ of weight k, and*
- *Germs of polarized variations of Hodge structure of weight k on V degenerating at a maximally unipotent boundary point to a limiting mixed Hodge structure of Hodge-Tate type, split over $\mathbb{R}$, and together with a marked real point $e \in F^k$.*

Under this correspondence, classical potentials – equivalently, framed Frobenius modules – correspond to nilpotent orbits as in Proposition 3.3.

Proof. Let $(V, \mathcal{B}, e, *)$ be a framed Frobenius module of weight k, $\{T_0, \dots, T_m\}$ an adapted basis, and let $\{N_1, \dots, N_r; F\}$ be the nilpotent orbit associated by Proposition 3.3. Given a quantum potential $\phi = \phi_0 + \phi_\hbar$ on V define

$$\Gamma_{-1}(q)(T_a) := \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^2\phi_\hbar(q)}{\partial z_a \partial z_{\delta(c)}} T_c. \tag{4.1}$$

Notice that because of (3.5), Γ_{-1} is holomorphic on some open neighborhood of $q = 0 \in \Delta^r$, $\Gamma_{-1}(0) = 0$, and it takes values on $\mathfrak{p}_{-1}$ relative to the grading (2.6) defined by the limiting mixed Hodge structure of $\{N_1, \dots, N_r; F\}$.

As before, we set $X_{-1}(q) := \frac{1}{2\pi i}\sum_{j=1}^r \log(q_j)N_j + \Gamma_{-1}(q) \in \mathfrak{p}_{-1}$ and note that the deformed Frobenius structure may be recovered from $X_{-1}(q)$ by

$$T_j \cdot_q T_a \;=\; \frac{\partial X_{-1}}{\partial z_j}(T_a)\,; \quad \tilde{j} = 2\,,\; 0 \le a \le m\,. \tag{4.2}$$

Equations (3.6) imply that X_{-1} satisfies the integrability condition (2.8). Indeed,

$$\begin{aligned} dX_{-1} \wedge dX_{-1} \;=\; 0 \iff & \frac{\partial X_{-1}}{\partial z_j}\frac{\partial X_{-1}}{\partial z_l} \;=\; \frac{\partial X_{-1}}{\partial z_l}\frac{\partial X_{-1}}{\partial z_j} \\ \iff & T_j \cdot_q (T_l \cdot_q T_a) \;=\; T_l \cdot_q (T_j \cdot_q T_a)\,, \end{aligned} \tag{4.3}$$

which, by (3.6), holds whenever $\tilde{j} = \tilde{l} = 2$ and all a. Theorem 2.2 now implies that X_{-1} defines a unique polarized variation of Hodge structure on a neighborhood of $0 \in \Delta^r$ whose nilpotent orbit is $\{N_1, \dots, N_r; F\}$. Hence the origin is a maximally unipotent boundary point and the limiting mixed Hodge structure is of Hodge-Tate type.

Conversely, let Φ be the period map of a local variation having a maximally unipotent boundary point at the origin. Let $\{N_1, \dots, N_r; F\}$ be the corresponding nilpotent orbit and $I^{*,*}$ the limiting mixed Hodge structure, which we assume to be of Hodge-Tate type. Let $(V, \mathcal{B}, e, *)$ be the real, framed Frobenius module given by Proposition 3.3 and ϕ_0 the corresponding classical potential. Let $\{T_0, \dots, T_m\}$ be an adapted basis such that $T_j = N_j(e)$, $j = 1, \dots, r$. Using canonical coordinates q on Δ^r, we define a quantum potential from the holomorphic function $\Gamma\colon \Delta^r \to \mathfrak{g}_-$ associated with Φ by:

$$\begin{aligned} \phi_h^{ab}(q) &:= \frac{1}{2}\,\mathcal{B}(\Gamma_{-1}(T_a), T_b) \text{ for } 2 < \tilde{a} < 2k-4 \text{ and } \tilde{a} + \tilde{b} = 2k-2, \\ \phi_h^{a}(q) &:= \mathcal{B}(-\Gamma_{-2}(T_a), T_0) \text{ for } \tilde{a} = 2k-4, \\ \phi_\hbar &:= \sum_{\tilde{a} = 2k-4} z_a \phi_h^a + \sum_{\substack{2 < \tilde{a} < 2k-4 \\ \tilde{a}+\tilde{b} = 2k-2}} z_a z_b \phi_h^{ab}, \\ \phi &:= \phi_0 + \phi_\hbar. \end{aligned}$$

Clearly, $\phi_\hbar$ is as in (3.5). In order to verify that (3.6) is satisfied we consider the associated deformation (3.7) of the Frobenius module structure

$$T_j \cdot_q T_a \;:=\; \sum_{\tilde{c} = \tilde{a}+2} \frac{\partial^3 \phi}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c$$

and show that it may also be given as

$$T_j \cdot_q T_a = \frac{\partial X_{-1}}{\partial z_j}(T_a). \tag{4.4}$$

Indeed, for $2 < \tilde{a} < 2k-4$ we have $\Gamma_{-1}(T_a) = \sum_{\tilde{c}=\tilde{a}+2} \phi_h^{a\delta(c)} T_c$, so that

$$
\begin{aligned}
\frac{\partial \Gamma_{-1}}{\partial z_j}(T_a) &= \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial}{\partial z_j}\phi_h^{a\delta(c)} T_c = \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3}{\partial z_j \partial z_a \partial z_{\delta(c)}} \sum_{\tilde{u}+\tilde{v}=2k-2} \frac{1}{2}\, z_u z_v\, \phi_h^{uv} \\
&= \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi_\hbar}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c,
\end{aligned}
$$

where we have used that $\phi_h^{ab} = \phi_h^{ba}$. Then

$$
\begin{aligned}
\frac{\partial X_{-1}}{\partial z_j}(T_a) &= N_j(T_a) + \frac{\partial \Gamma_{-1}}{\partial z_j}(T_a) \\
&= \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi_0}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c + \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi_\hbar}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c \\
&= \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c = T_j \cdot_q T_a.
\end{aligned}
$$

In order to verify (4.4) when $\tilde{a} = 2k-4$ we first prove the identity

$$
\Gamma_{-1}(T_a) = \sum_{\tilde{c}=2k-2} \frac{\partial}{\partial z_{\delta(c)}} \mathcal{B}(-\Gamma_{-2}(T_a), T_0)\, T_c\,, \quad \tilde{a} = 2k-4 \tag{4.5}
$$

as a consequence of the horizontality condition (2.7). If this condition is rewritten in terms of $G(q) := \exp \Gamma(q)$ and $\Theta = \sum N_j\, dz_j$ we get

$$
dG = [G, \Theta] + G\, d\Gamma_{-1}.
$$

This equation is graded by (2.6) and its homogeneous pieces are

$$
dG_{-\ell} = [G_{-\ell+1}, \Theta] + G_{-\ell+1}\, d\Gamma_{-1}, \quad \ell \geq 2. \tag{4.6}
$$

In particular, for $\ell = 2$ we obtain

$$
d\Gamma_{-2} = [\Gamma_{-1}, \Theta + \frac{1}{2}\, d\Gamma_{-1}].
$$

Evaluating at T_a and given that the canonical coordinates $(q_1, \ldots, q_r)$ are characterized by $\Gamma_{-1}(T_b) = 0$ for all $\tilde{b} = 2k-2$, we obtain

$$
d\Gamma_{-2}(T_a) = -\Theta\big(\Gamma_{-1}(T_a)\big). \tag{4.7}
$$

By the $\mathcal{B}$-self-duality of the basis $\{T_0, \ldots, T_m\}$, we can write

$$
\Gamma_{-1}(T_a) = \sum_{\tilde{c}=2k-2} \mathcal{B}(\Gamma_{-1}(T_a), T_{\delta(c)}) T_c. \tag{4.8}
$$

Now, if $\tilde{c} = 2k-2$ and $j = 1, \ldots, r$, then $N_j(T_c) = \delta_{jc} T_m$ and, therefore, $\Theta(T_c) = T_m\, dz_{\delta(c)}$ and (4.7), (4.8) imply

$$
d\Gamma_{-2}(T_a) = -\sum_{\tilde{c}=2k-2} dz_{\delta(c)} \mathcal{B}(\Gamma_{-1}(T_a), T_{\delta(c)}) T_m,
$$

so that,

$$\frac{\partial}{\partial z_{\delta(c)}}\Gamma_{-2}(T_a) \;=\; -\mathcal{B}(\Gamma_{-1}(T_a), T_{\delta(c)})T_m$$

implying that

$$\mathcal{B}\left(\frac{\partial}{\partial z_{\delta(c)}}\Gamma_{-2}(T_a), T_0\right) \;=\; -\mathcal{B}(\Gamma_{-1}(T_a), T_{\delta(c)}). \tag{4.9}$$

Finally, (4.5) follows from applying (4.9) to (4.8).

Thus, if $\tilde{a} = 2k - 4$,

$$\begin{aligned}
\frac{\partial \Gamma_{-1}}{\partial z_j}(T_a) &= \sum_{\tilde{c}=2k-2} \frac{\partial}{\partial z_j}\,\frac{\partial}{\partial z_{\delta(c)}}\,\mathcal{B}(-\Gamma_{-2}(T_a), T_0)\; T_c \\
&= \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial}{\partial z_j}\,\frac{\partial}{\partial z_{\delta(c)}}\,\frac{\partial}{\partial z_a}\left(\sum_{\tilde{b}=2k-4} z_b \phi_\hbar^b(q)\right)\; T_c \\
&= \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi_\hbar}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c\,,
\end{aligned}$$

and (4.4) follows as before.

Given (4.4), the equivalences in (4.3) show that the integrability condition (2.8) implies that the quantum potential ϕ satisfies (3.6).

Finally, we note that (4.4) and (4.2) imply that these correspondences are inverses of each other. $\square$

5. *A*-Model Variation

Here we will show that the polarized variation of Hodge structure associated to a quantum potential by Theorem 4.1 agrees with the A-model variation defined, in the case of the cohomology on a Calabi-Yau manifold, by the Gromov-Witten potential, as in, for example, [14, Chapter 8]. As a byproduct we give a different proof of the fact that the A-model variation associated with a general potential, in the sense of Definition 3.4, is a polarized variation of Hodge structure.

We begin by recalling the definition of the A-model variation. Let $\phi = \phi_0 + \phi_\hbar$ be a quantum potential on the framed Frobenius module $(V, \mathcal{B}, e, *)$. Let $\{T_0, \ldots, T_m\}$ be an adapted basis of V and $(z_0, \ldots, z_m)$ the corresponding linear coordinates on V; set $q_j = \exp(2\pi i z_j)$ for $j = 1, \ldots, r$. We view $(q_1, \ldots, q_r)$ as coordinates on $(\Delta^*)^r$. Let ∇ be the connection on the vector bundle $\mathcal{V} := (\Delta^*)^r \times V$ defined on a constant section T by

$$\nabla_{\frac{\partial}{\partial q_j}} T \;:=\; \frac{1}{2\pi i q_j} T_j \cdot_q T. \tag{5.1}$$

Proposition 5.1. *The connection ∇ is flat. It has a simple pole at $q_j = 0$ and its residue is the nilpotent operator*

$$\mathrm{Res}_{q_j=0}(\nabla)(T_a) \;=\; \frac{1}{2\pi i}\left(\sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi_0}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c\right). \tag{5.2}$$

Proof. Given the definition of the quantum product (3.7) and (5.1), if T_a denotes a constant section,

$$\nabla_{\frac{\partial}{\partial q_j}} T_a \;=\; \frac{1}{2\pi i q_j}\left(\sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^3 \phi_0}{\partial z_j \partial z_a \partial z_{\delta(c)}} T_c\right) + H_{ja}(q)$$

for some function H, which extends holomorphically to $0 \in \Delta^r$. This implies the residue assertion.

The curvature of ∇ reduces to

$$R_\nabla\left(\frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_l}\right)(T_a) = \frac{1}{2\pi i}\left(\frac{1}{q_l}\nabla_{\frac{\partial}{\partial q_j}}(T_l \cdot T_a) - \frac{1}{q_j}\nabla_{\frac{\partial}{\partial q_l}}(T_j \cdot T_a)\right).$$

A straightforward computation shows that this last expression vanishes since ϕ satisfies (3.6). □

Remark 5.2. It follows from (5.2) that the operators $\mathrm{Res}_{q_j=0}(\nabla)$ agree, up to a constant, with the morphisms L_{T_j} of left multiplication by T_j in the Frobenius module $(V, \mathcal{B}, e, *)$.

Consider the flags of subbundles of $\mathcal{V}$:

$$\mathcal{F}^p \;:=\; (\Delta^*)^r \times (\oplus_{a \ge p} V_{2(k-a)}) \quad \text{and} \quad \mathcal{U}_\ell \;:=\; (\Delta^*)^r \times (\oplus_{b \ge \ell} V_{2b}).$$

Proposition 5.3. *The subbundles $\mathcal{F}^p$ satisfy Griffiths' horizontality* (2.1). *Moreover, for any given $\hat{q} \in (\Delta^*)^r$, there is a (multivalued) flat frame of $\mathcal{V}$, $\{T_a^\flat\}$, such that $T_a^\flat(q) \equiv T_a$* mod $\mathcal{U}_{\tilde{a}+1}$ *and $T_a^\flat(\hat{q}) = T_a$.*

Proof. Since the maps $T \mapsto T_j \cdot_q T$ are homogeneous of degree 2, the horizontality follows directly from (5.1).

Since ∇ defines a connection on the bundle $\mathcal{U}_\ell$ inducing a trivial connection on $\mathcal{U}_\ell/\mathcal{U}_{\ell+1}$, the second statement follows. □

Next, we want to compute the monodromy of ∇. We fix all the coordinates q_i for $i \neq j$ and consider the one-dimensional problem around $q_j = 0$. The flat sections $T_a^\flat$ can be written in terms of the constant sections as $T_a^\flat = \sum_b f_{ba} T_b$, and the flatness condition leads to the ODE with a regular singularity at the origin

$$\frac{\partial f_{ba}}{\partial q_j} \;=\; -\sum_c \left(\frac{1}{q_j}(\mathrm{Res}_{q_j=0}(\nabla))_{bc} + H_{jc\delta(b)}\right) f_{ca}, \tag{5.3}$$

where H_{jcd} are holomorphic at $q_j = 0$. Therefore, classical results for such an equation (see [13, Ch. 4, Thm. 4.1]) imply that the coefficients f_{ba} are of the form

$$f_{ba}(q) \;=\; \big(G(q_j)\,\exp(-\log(q_j)\,\mathrm{Res}_{q_j=0}(\nabla))\big)_{ba} \tag{5.4}$$

for some function G, holomorphic at $q_j = 0$, with $G(0) = Id_n$.

Parallel transport around $q_j = 0$, in the anti-clockwise direction, gives that the monodromy of ∇, written relative to the frame $\{T_a^\flat\}$, is

$$M_j \;:=\; \exp\big(-2\pi i\,\mathrm{Res}_{q_j=0}(\nabla)\big).$$

We let $N_j := -\log(M_j) = 2\pi i\,\mathrm{Res}_{q_j=0}(\nabla)$. Notice that, in view of (5.2), the monodromy in a flat frame can be computed purely in terms of the classical potential. All together we conclude:

Proposition 5.4. *The matrix of the local monodromy logarithm operator N_j with respect to the frame $\{T_a^\flat\}$ coincides with the matrix of left $*$-multiplication by T_j, L_{T_j}, with respect to the basis $\{T_a\}$.*

The fact that ∇ has a simple pole at $q_j = 0$ with nilpotent residue L_{T_j} allows us to construct Deligne's canonical extension $(\mathcal{V}^c, \nabla^c)$ [15] which is characterized by the fact that

$$\tilde{T}_a := \exp\left(\sum_{j=1}^{r} \frac{\log(q_j)}{2\pi i} N_j\right) T_a^\flat, \quad a = 0, \ldots, m, \tag{5.5}$$

are a flat frame of $(\mathcal{V}^c, \nabla^c)$.

Proposition 5.5. *For $a = 0, \ldots, m$, $\tilde{T}_a$ is the unique ∇^c-flat section of $\mathcal{V}^c$ such that $\tilde{T}_a \equiv T_a \mod \mathcal{U}_{\tilde{a}+1}$, and $\tilde{T}_a(\hat{q}) = T_a$. The matrix of N_j acting on the frame $\{\tilde{T}_a\}$ equals the matrix of the classical product $*$ acting on $\{T_a\}$.*

Proof. The first statement follows from Proposition 5.3 and (5.5). Since $[N_j, N_l] = 0$ for all $1 \le j, l \le r$, we have

$$\begin{aligned} N_l(\tilde{T}_a) &= N_l\left(\exp\left(\sum_{j=1}^{r} \frac{\log(q_j)}{2\pi i} N_j\right) T_a^\flat\right) = \exp\left(\sum_{j=1}^{r} \frac{\log q_j}{2\pi i} N_j\right) N_l(T_a^\flat) \\ &= \sum_b (L_{T_l})_{ba} \tilde{T}_b, \end{aligned}$$

and the second statement follows. □

Remark 5.6. In the context of the Gromov-Witten potential, the previous result reduces to [14, Prop. 8.5.4] whose proof involves the formalism of *gravitational correlators*. The elementary proof given above shows that it is a direct consequence of the definition of the connection and, in particular, of the homogeneity of the the operators L_{T_j}.

Because of Propositions 5.3 and 5.5, we know the first (graded) component of the sections $T_a^\flat$ and $\tilde{T}_a$. A lengthy but straightforward computation yields the second component of both $T_a^\flat$ and $\tilde{T}_a$.

Lemma 5.7. *The ∇-flat sections $T_a^\flat$ satisfy*

$$T_a^\flat(q) \equiv T_a - \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^2 \phi}{\partial z_a \partial z_{\delta(c)}} T_c \mod \mathcal{U}_{\tilde{a}+2}.$$

Lemma 5.8. *The ∇^c-flat sections $\tilde{T}_a$ satisfy the following formulas, for $k > 3$.*

For $\tilde{a} \ge 2k-2$, $\tilde{T}_a = T_a$.
For $\tilde{a} = 2k-4$, $\tilde{T}_a = T_a - \sum_{\tilde{c}=\tilde{a}+2} 2\pi i q_{\delta(c)} \frac{\partial}{\partial q_{\delta(c)}} \phi_h^a\, T_c + \phi_h^a\, T_m$.
For $2 < \tilde{a} < 2k-4$, $\tilde{T}_a \equiv T_a - \sum_{\tilde{c}=\tilde{a}+2} \phi_h^{a\delta(c)}\, T_c \mod \mathcal{U}_{\tilde{a}+2}$.
For $\tilde{a} = 2$, $\tilde{T}_a \equiv T_a - \sum_{\tilde{c}=\tilde{a}+2} 2\pi i q_a \frac{\partial}{\partial q_a} \phi_h^{\delta(c)}\, T_c \mod \mathcal{U}_{\tilde{a}+2}$.
For $\tilde{a} = 0$, $\tilde{T}_0 \equiv T_0 \mod \mathcal{U}_{\tilde{a}+2}$.

We can now extend trivially the form Q, defined by (3.4), to a form $\mathcal{Q}$ on $\mathcal{V}$. $\mathcal{Q}$ is flat because of (3.2).

To define a flat real structure $\mathcal{V}_{\mathbb{R}}$ on $\mathcal{V}$ we proceed as follows. Let

$$\tilde{\mathcal{V}} := \Delta^r \times V \quad \text{and} \quad \tilde{\nabla} := \nabla - \frac{1}{2\pi i}\sum_{j=1}^{r} N_j \frac{dq_j}{q_j}.$$

Then $\tilde{\nabla}$ is a flat connection on the bundle $\tilde{\mathcal{V}}$; for $v \in V$ we define $\tilde{\sigma}_v$ to be the $\tilde{\nabla}$-flat section of $\tilde{\mathcal{V}}$ such that $\tilde{\sigma}_v(0) = v$. Then $\mathcal{V}_{\mathbb{R}}$ is the local system generated by the sections $\exp(-\frac{1}{2\pi i}\sum_{j=1}^{r}\log(q_j)N_j)\,\tilde{\sigma}_v(q)$, for all $v \in V_{\mathbb{R}}$.

Definition 5.9. *Let $\phi = \phi_0 + \phi_\hbar$ be a quantum potential on the framed, real Frobenius module $(V, \mathcal{B}, e, *)$. Then $(\mathcal{V}, \nabla, \mathcal{F}, \mathcal{V}_{\mathbb{R}}, \mathcal{Q})$ is the* A-model variation *of the potential.*

Theorem 5.10. *The A-model variation is a polarized variation of Hodge structure. Moreover, it is the variation associated to the potential ϕ by Theorem 4.1.*

Proof. Let Φ be the "period map" of $(\mathcal{V}, \nabla, \mathcal{F}, \mathcal{V}_{\mathbb{R}}, \mathcal{Q})$ defined by parallel transport to the fiber $\mathcal{V}_{\hat{q}}$, $\hat{q} \in (\Delta^*)^r$. By Proposition 5.4 the local monodromy logarithms N_j are the left multiplication operators L_{T_j} and, by Proposition 5.5, the limiting Hodge filtration becomes $F^p := \oplus_{a \geq p} V_{2(k-a)}$. Thus, Proposition 3.3 implies that $\{N_1, \ldots, N_r; F\}$ is a nilpotent orbit.

Let now $\exp(-\sum_j z_j N_j) \cdot \Phi(q) = \exp\Gamma(q) \cdot F$, where Γ is a holomorphic, $\mathfrak{g}_-$-valued map defined locally around $0 \in \Delta^r$. Since the map Φ is horizontal, the $\mathfrak{p}_{-1}$-valued map $X_{-1} = \sum_j z_j N_j + \Gamma_{-1}$ satisfies the integrability condition (2.8) and it follows from Theorem 2.2 that $(\mathcal{V}, \nabla, \mathcal{F}, \mathcal{V}_{\mathbb{R}}, \mathcal{Q})$ is a polarized variation of Hodge structure.

In order to prove that this variation agrees with the one defined in Theorem 4.1 we appeal to the uniqueness statement in Theorem 2.2. Hence, it suffices to show that Γ_{-1} is related to the potential ϕ by (4.1). But, the matrix presentation of $\exp(-\Gamma(q))$ in the basis $\{T_a\}$ is the matrix expressing the ∇^c-flat frame $\{\tilde{T}_a\}$ in terms of the constant frame $\{T_a\}$. Thus, it follows from Lemma 5.8, that

$$\Gamma_{-1}(T_a) = \sum_{\tilde{c}=\tilde{a}+2} \frac{\partial^2 \phi_\hbar}{\partial z_a \partial z_{\delta(c)}} T_c,$$

as desired. □

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