

Residues in toric varieties

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Abstract. We study residues on a complete toric variety X , which are defined in terms of the homogeneous coordinate ring of X . We first prove a global transformation law for toric residues. When the fan of the toric variety has a simplicial cone of maximal dimension, we can produce an element with toric residue equal to 1. We also show that in certain situations, the toric residue is an isomorphism on an appropriate graded piece of the quotient ring. When X is simplicial, we prove that the toric residue is a sum of local residues. In the case of equal degrees, we also show how to represent X as a quotient $(Y \setminus \{0\})/\mathbf{C}^*$ such that the toric residue becomes the local residue at 0 in Y .

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Introduction

Toric residues provide a tool for the study of certain homogeneous ideals of the homogeneous coordinate ring of a toric variety – such as those appearing in the description of the Hodge structure of their hypersurfaces [BC]. They were introduced in [C2], where some of their properties were described in the special case when all of the divisors involved were linearly equivalent. The main results of this paper are: an extension of the Isomorphism Theorem of [C2] to the case of non-equivalent ample divisors, a global transformation law for toric residues, and a theorem expressing the toric residue as a sum of local (Grothendieck) residues.

Let us first establish the notation we will use. We will assume that X is a complete toric variety of dimension n . As such, X is determined by a fan Σ in an n -dimensional real vector space $N_{\mathbf{R}}$. There is a distinguished lattice of maximal rank $N \subset N_{\mathbf{R}}$ and we let M denote the dual lattice. The N -generators of the 1-dimensional cones in Σ will be denoted $\eta_1, \dots, \eta_{m+r}$. This means that r is the rank of the Chow group $A_{n-1}(X)$. We will make frequent use of the homogeneous coordinate ring S of X , which is the polynomial ring $S = \mathbf{C}[x_1, \dots, x_{n+r}]$. Here, each variable x_i corresponds to the generator η_i and hence to a torus-invariant

irreducible divisor D_i of X . As in [C1], we grade S by declaring that the monomial $\prod_{i=1}^{n+r} x_i^{a_i}$ has degree $[\sum_{i=1}^{n+r} a_i D_i] \in A_{n-1}(X)$.

We will let $\beta = \sum_{i=1}^{n+r} \deg(x_i) \in A_{n-1}(X)$ denote the sum of the degrees of the variables. As is well known, β is the anticanonical class on X . Then, given homogeneous polynomials $F_i \in S_{\alpha_i}$ for $i = 0, \dots, n$, we define their *critical degree* to be

$$\rho = \left(\sum_{i=0}^n \alpha_i \right) - \beta \in A_{n-1}(X).$$

As in [C2], each $H \in S_\rho$ determines a meromorphic n -form on X

$$\omega_F(H) = \frac{H \Omega}{F_0 \dots F_n},$$

where F stands for the vector (F_0, \dots, F_n) and Ω is a choice of an Euler form in X [BC]. If the F_i don't vanish simultaneously on X , then relative to the open cover $U_i = \{x \in X : F_i(x) \neq 0\}$ of X , this gives a Čech cohomology class $[\omega_F(H)] \in H^n(X, \widehat{\Omega}_X^n)$. Here, $\widehat{\Omega}_X^n$ is the sheaf of Zariski n -forms on X , i.e., $\widehat{\Omega}_X^n = j_* \Omega_{X_0}^n$, where X_0 is the smooth part of X and $j: X_0 \hookrightarrow X$ is the natural inclusion.

It is not hard to see that $[\omega_F(H)]$ is alternating on the order of F_0, \dots, F_n and that if H is in the ideal $\langle F_0, \dots, F_n \rangle$, then $\omega_F(H)$ is a Čech coboundary. Thus, $[\omega_F(H)]$ depends only on the equivalence class of H modulo the ideal generated by F_0, \dots, F_n . Then the *toric residue*

$$\text{Res}_F: S_\rho / \langle F_0, \dots, F_n \rangle_\rho \rightarrow \mathbf{C}$$

is given by the formula

$$\text{Res}_F(H) = \text{Tr}_X([\omega_F(H)]),$$

where $\text{Tr}_X: H^n(X, \widehat{\Omega}_X^n) \rightarrow \mathbf{C}$ is the trace map. When there is no danger of confusion, we will write $\text{Res}(H)$ instead of $\text{Res}_F(H)$.

Our first main result is the following *Global Transformation Law*.

THEOREM 0.1. *Let $F_i \in S_{\alpha_i}$ and $G_i \in S_{\beta_i}$ for $i = 0, \dots, n$. Suppose*

$$G_j = \sum_{i=0}^n A_{ij} F_i,$$

where A_{ij} is homogeneous of degree $\beta_j - \alpha_i$, and assume the G_i don't vanish simultaneously on X . Let ρ be the critical degree for F_0, \dots, F_n . Then, for each $H \in S_\rho$, $H \det(A_{ij})$ is of the critical degree for G_0, \dots, G_n , and

$$\text{Res}_F(H) = \text{Res}_G(H \det(A_{ij})).$$

The proof uses a Čech cochain argument. One application of this transformation law is that in certain cases, we can describe explicit elements of S_ρ with nonzero residue. For this purpose, assume X is complete and its fan Σ contains a n -dimensional *simplicial* cone σ . Then denote the variables of the coordinate ring as $x_1, \dots, x_n, z_1, \dots, z_r$, where x_1, \dots, x_n correspond to the 1-dimensional cones of σ . Also suppose that $\alpha_0, \dots, \alpha_n$ are \mathbf{Q} -ample classes, which means that some multiple is Cartier and ample. In this situation, we will show that each $F_j \in S_{\alpha_j}$ can be written in the form

$$F_j = A_{0j} z_1 \dots z_r + \sum_{i=1}^n A_{ij} x_i.$$

Then the $(n+1) \times (n+1)$ -determinant $\Delta_\sigma = \det(A_{ij})$ is in S_ρ and has the following important property.

THEOREM 0.2. *Assume X is complete and $\sigma \in \Sigma$ is simplicial and n -dimensional. Suppose that $F_i \in S_{\alpha_i}$ for $i = 0, \dots, n$, where α_i is \mathbf{Q} -ample and the F_i don't vanish simultaneously on X . Then*

$$\text{Res}_F(\Delta_\sigma) = \pm 1.$$

The Global Transformation Law allows us to reduce the proof of this theorem to the special case when $F_0 = z_1 \dots z_r$ and $F_i = x_i, i = 1, \dots, n$. This is done in Section 2. An alternate proof for simplicial toric varieties is given in Section 4 as an application of Theorem 0.4.

In Section 3 we prove the following *Residue Isomorphism Theorem*.

THEOREM 0.3. *Let X be complete and simplicial, and assume that $F_i \in S_{\alpha_i}$ for $i = 0, \dots, n$, where α_i is ample and the F_i don't vanish simultaneously on X . Then*

- (i) *The toric residue map $\text{Res}_F: S_\rho / \langle F_0, \dots, F_n \rangle_\rho \rightarrow \mathbf{C}$ is an isomorphism.*
- (ii) *For each variable $x_i, 0 \leq i \leq n+r$, we have $x_i \cdot S_\rho \subset \langle F_0, \dots, F_n \rangle$.*

In the case when all the α_i are equal to a fixed ample divisor α , this theorem follows from the fact that F_0, \dots, F_n are a regular sequence in the Cohen-Macaulay ring $S_{*\alpha} = \bigoplus_{k \geq 0} S_{k\alpha}$ [C2, Sect. 3]. In the general case, the proof relies on the use of the Cayley trick and results of Batyrev and Cox [BC] concerning the cohomology of projective hypersurfaces in toric varieties, to show that

$$\dim_{\mathbf{C}}(S_\rho / \langle F_0, \dots, F_n \rangle_\rho) = 1$$

when X is simplicial and the divisors $F_i = 0$ are ample with empty intersection. Then, the first (and main) part of the Residue Isomorphism Theorem follows

immediately from Theorem 0.2, and the second part is a consequence of the first using Theorem 0.2 and Cramer's Rule.

As a corollary of Theorems 0.2 and 0.3, we get a simple algorithm (see Remark 3.11) for computing toric residues in terms of normal forms. In Section 3, we will also conjecture a more general form of Theorem 0.3 and give some examples in support.

The main result of Section 4 is a theorem stating that for simplicial toric varieties, the toric residue may be computed as a sum of local Grothendieck residues. The toric setting is not essential here and, in fact, it is convenient to work with the more general notion of a V -manifold or orbifold. The proof of the following local/global theorem is based on the theory of residual currents ([CH]).

THEOREM 0.4. *Let X be a complete simplicial toric variety of dimension n , and let F_0, \dots, F_n be homogeneous polynomials which don't vanish simultaneously on X . If H is a polynomial in S_ρ , where ρ is the critical degree, and $D_{\hat{k}} = \{x \in X: F_i(x) = 0, i \neq k\}$ is finite, then the toric residue is given by*

$$\text{Res}_F(H) = (-1)^k \sum_{x \in D_{\hat{k}}} \text{Res}_{k,x} \left(\frac{H \Omega}{F_0 \dots F_n} \right).$$

Here, as we will explain in Section 4, $\text{Res}_{k,x}(H \Omega / F_0 \dots F_n)$ denotes the local Grothendieck residue

$$\text{Res}_x \left(\frac{(H/F_k) \Omega}{F_0 \dots \widehat{F_k} \dots F_n} \right).$$

Note that the finiteness condition holds automatically whenever the divisor $\{F_k = 0\}$ is \mathbf{Q} -ample. Under appropriate conditions, Theorem 0.4 gives a framework for the study of sums of local residues—both in the affine and toric cases—as a global residue defined in a suitable toric compactification. It is possible, for example, to interpret in this light the results of [CDS] which correspond to the case when the toric variety under consideration is a weighted projective space.

Finally, in Section 5, we show that, in the equal degree case, the toric residue equals a single local residue at the origin of the affine cone of X . This generalizes the observation in [PS] that toric residues on \mathbf{P}^n can be written as a residue at the origin in \mathbf{C}^{n+1} .

1. The global transformation law

This section will prove the Global Transformation Law (Theorem 0.1) for toric residues on an arbitrary n -dimensional complete toric variety X . Given F_i and $G_j = \sum_{i=0}^n A_{ij} F_i$ as in the statement of the theorem, first observe that if G_0, \dots, G_n have no common zeroes in X , then the same holds for F_0, \dots, F_n . Thus we get

open covers $U_i = \{x \in X: F_i(x) \neq 0\}$ and $V_j = \{x \in X: G_j(x) \neq 0\}$ of X , which we denote \mathcal{U} and \mathcal{V} respectively.

If the critical degrees of the F_i and G_j are $\rho(F)$ and $\rho(G)$ respectively, then

$$\rho(G) = \rho(F) + \deg(\det(A_{ij}))$$

follows easily since A_{ij} is homogeneous and $\deg A_{ij} = \deg G_j - \deg F_i$. Thus, if H is homogeneous of degree $\rho(F)$, then we get Čech cohomology classes $[\omega_F(H)] \in H^n(\mathcal{U}, \widehat{\Omega}_X^n)$ and $[\omega_G(H \det(A_{ij}))] \in H^n(\mathcal{V}, \widehat{\Omega}_X^n)$. To prove Theorem 0.1, it suffices to show that these cohomology classes have the same image in $H^n(X, \widehat{\Omega}_X^n)$.

Consider the open covering $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$. Since \mathcal{U} and \mathcal{V} can be regarded as refinements of \mathcal{W} with obvious refinement maps, we get a commutative diagram

$$\begin{array}{ccc} & H^n(\mathcal{U}, \widehat{\Omega}_X^n) & \\ & \nearrow & \searrow \\ H^n(\mathcal{W}, \widehat{\Omega}_X^n) & & H^n(X, \widehat{\Omega}_X^n) \\ & \searrow & \nearrow \\ & H^n(\mathcal{V}, \widehat{\Omega}_X^n) & \end{array}$$

Then Theorem 0.1 is an immediate consequence of the following proposition.

PROPOSITION 1.1. *There is a cohomology class $[\theta] \in H^n(\mathcal{W}, \widehat{\Omega}_X^n)$ which maps to both $[\omega_F(H)] \in H^n(\mathcal{U}, \widehat{\Omega}_X^n)$ and $[\omega_G(H \det(A_{ij}))] \in H^n(\mathcal{V}, \widehat{\Omega}_X^n)$ in the above diagram.*

Proof. We first introduce some notation for the Čech complex of $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$. Given index sets $I = \{0 \leq i_1 < \dots < i_p \leq n\}$ and $J = \{0 \leq j_1 < \dots < j_q \leq n\}$ with $p = |I|$ and $q = |J|$, let $W_{IJ} = \bigcap_{i \in I} U_i \cap \bigcap_{j \in J} V_j$. Also, let I' denote the complement of I in $\{0, \dots, n\}$, ordered increasingly.

We define a Čech cochain $\theta \in \mathcal{C}^n(\mathcal{W}, \widehat{\Omega}_X^n)$ by the assignment

$$W_{IJ} \mapsto \theta_{IJ} = \varepsilon(I) \frac{H \det(M_{I'J}) \Omega}{F_I G_J} \in \widehat{\Omega}_X^n(W_{IJ}).$$

Here, I and J are index sets with $p + q = n + 1$, $M_{I'J}$ is the $I'J$ -minor of the matrix (A_{ij}) , $\varepsilon(I)$ is the sign of the permutation (I, I') , $F_I = F_{i_1} \cdots F_{i_p}$, and $G_J = G_{j_1} \cdots G_{j_q}$.

When $I = \{0, \dots, n\}$, both I' and J are empty and θ assigns to the open set $U_0 \cap \dots \cap U_n$ the form $H \Omega / (F_0 \dots F_n)$. Thus, the refinement map $\mathcal{C}^n(\mathcal{W}, \widehat{\Omega}_X^n) \rightarrow \mathcal{C}^n(\mathcal{U}, \widehat{\Omega}_X^n)$ maps θ to the cocycle $\omega_F(H)$. Similarly, when $J = \{0, \dots, n\}$, the cochain θ assigns to the open set $V_0 \cap \dots \cap V_n$ the form $H \det(A_{ij}) \Omega / (G_0 \dots G_n)$ and, hence, the refinement map $\mathcal{C}^n(\mathcal{W}, \widehat{\Omega}_X^n) \rightarrow \mathcal{C}^n(\mathcal{V}, \widehat{\Omega}_X^n)$ maps θ to the cocycle $\omega_G(H \det(A_{ij}))$. Consequently, the proposition will follow once we show that θ is also a cocycle, i.e., $\delta(\theta) = 0$, where $\delta: \mathcal{C}^n(\mathcal{W}, \widehat{\Omega}_X^n) \rightarrow \mathcal{C}^{n+1}(\mathcal{W}, \widehat{\Omega}_X^n)$ is the Čech coboundary.

To prove that $\delta(\theta) = 0$, let I and J be index sets with $p + q = n + 2$. Then set $I_k = I - \{i_k\}$, $1 \leq k \leq p$, $I'_k = I' \cup \{i_k\}$, and $J_\ell = J - \{j_\ell\}$, $1 \leq \ell \leq q$, so that

$$\begin{aligned} (\delta\theta)_{IJ} &= \sum_{k=1}^p (-1)^{k-1} \theta_{I_k J} + (-1)^p \sum_{\ell=1}^q (-1)^{\ell-1} \theta_{I J_\ell} \\ &= \sum_{k=1}^p \frac{(-1)^{k-1} \varepsilon(I_k) \det(M_{I'_k J}) H \Omega}{F_{I_k} G_J} \\ &\quad + \sum_{\ell=1}^q \frac{(-1)^{p+\ell-1} \varepsilon(I) \det(M_{I' J_\ell}) H \Omega}{F_I G_{J_\ell}}. \end{aligned}$$

Writing the last expression with common denominator $F_I G_J$, it suffices to show that

$$\begin{aligned} &\sum_{k=1}^p (-1)^{k-1} \varepsilon(I_k) F_{i_k} \det(M_{I'_k J}) \\ &\quad + \sum_{\ell=1}^q (-1)^{p+\ell-1} \varepsilon(I) G_{j_\ell} \det(M_{I' J_\ell}) = 0. \end{aligned}$$

If we substitute G_{j_ℓ} by $\sum_{k=1}^p A_{i_k j_\ell} F_{i_k} + \sum_{u \in I'} A_{u j_\ell} F_u$, then the above equation becomes

$$\begin{aligned} &\sum_{k=1}^p \left[(-1)^{k-1} \varepsilon(I_k) \det(M_{I'_k J}) + \sum_{\ell=1}^q (-1)^{p+\ell-1} \varepsilon(I) A_{i_k j_\ell} \det(M_{I' J_\ell}) \right] F_{i_k} \\ &\quad + \varepsilon(I) (-1)^p \sum_{u \in I'} \left[\sum_{\ell=1}^q (-1)^{\ell-1} A_{u j_\ell} \det(M_{I' J_\ell}) \right] F_u = 0. \end{aligned}$$

We will show that the expressions inside the two sets of brackets are identically zero.

First, for $u \in I'$, note that $\sum_{\ell=1}^q (-1)^{\ell-1} A_{u j_\ell} \det(M_{I' J_\ell})$ is the determinant of the matrix whose first row consists of $A_{u j_\ell}$, $\ell = 1, \dots, q$ and whose remaining

rows are the same as those of the matrix $M_{I'J}$. But, since $u \in I'$, such a matrix has two identical rows and its determinant vanishes. Hence the sum inside the second set of brackets is zero.

Turning to the first set of brackets, note that expanding the determinant of the $q \times q$ -matrix $M_{I'_k J}$ along the row corresponding to i_k gives

$$\det(M_{I'_k J}) = (-1)^m \sum_{\ell=1}^q (-1)^{\ell-1} A_{i_k j_\ell} \det(M_{I' J_\ell}),$$

where m denotes the number of elements of I' which precede i_k . However, since going from (I, I') to (I_k, I'_k) requires $(m - k + p)$ transpositions, we see that

$$\varepsilon(I_k) = (-1)^{m-k+p} \varepsilon(I),$$

and it follows that the desired expression is zero. \square

2. Elements with nonzero residue

The goal of this section is to prove Theorem 0.2. We begin with $F_j \in S_{\alpha_j}$, $0 \leq j \leq n$, which don't vanish simultaneously on the complete toric variety X . As in the introduction, we have the coordinate ring $S = \mathbf{C}[x_1, \dots, x_{n+r}]$ of X , where the variables x_i correspond to the \mathbf{Z} -generators η_i of the 1-dimensional cones of Σ . For each n -dimensional cone $\sigma \in \Sigma$, set $\hat{x}_\sigma = \prod_{\eta_i \notin \sigma} x_i$ and let $B(\Sigma) \subset S$ be the ideal generated by the monomials \hat{x}_σ , $\sigma \in \Sigma$.

We also assume that α_j is \mathbf{Q} -ample. This means that $d\alpha_j$ is ample for some positive integer d , so that $S_{d\alpha_j} \subset B(\Sigma)$ by Lemma 9.15 of [BC]. Then $(S_{\alpha_j})^d \subset S_{d\alpha_j} \subset B(\Sigma)$, and since $B(\Sigma)$ is radical, we conclude that

$$S_{\alpha_j} \subset B(\Sigma), \tag{2.1}$$

when α_j is \mathbf{Q} -ample.

To see the relevance of (2.1), fix a n -dimensional cone $\sigma \in \Sigma$ and, as in Theorem 0.2, assume that σ is *simplicial*. Then we can arrange for η_1, \dots, η_n to be the generators of σ , and we make a slight notational change replacing the variable x_{n+a} by z_a , $a = 1, \dots, r$. Then each F_j can be written

$$F_j = B_j + \sum_{i=1}^n A_{ij} x_i,$$

where B_j depends only on z_1, \dots, z_r . But $F_j \in B(\Sigma)$ by (2.1) and, since $B(\Sigma)$ is a monomial ideal, it follows easily that B_j must be divisible by $\hat{x}_\sigma = z_1 \dots z_r$. Thus $B_j = A_{0j} z_1 \dots z_r$, so that

$$F_j = A_{0j} z_1 \dots z_r + \sum_{i=1}^n A_{ij} x_i, \tag{2.2}$$

as claimed in the introduction. Then we define

$$\Delta_\sigma = \det(A_{ij}), \quad 0 \leq i, j \leq n. \quad (2.3)$$

Note that

$$\deg(\Delta_\sigma) = \sum_{j=0}^n \alpha_j - \sum_{i=1}^n \deg(x_i) - \sum_{a=1}^r \deg(z_a) = \rho(F).$$

A direct application of the Global Transformation Law to (2.2) and (2.3) yields

$$\text{Res} \left(\frac{\Delta_\sigma \Omega}{F_0 \dots F_n} \right) = \text{Res} \left(\frac{\Omega}{(z_1 \dots z_r) \cdot x_1 \dots x_n} \right).$$

Thus, to prove Theorem 0.2, we need only prove the following proposition.

PROPOSITION 2.4. *With assumptions and notations as above,*

$$\text{Res} \left(\frac{\Omega}{(z_1 \dots z_r) \cdot x_1 \dots x_n} \right) = \pm 1.$$

Proof. We first study the restriction of

$$\omega = \frac{\Omega}{(z_1 \dots z_r) \cdot x_1 \dots x_n} \quad (2.5)$$

to the affine open set $X_\sigma \subset X$ determined by σ . To apply the construction of [C1] to the simplicial toric variety X_σ , we start with the exact sequence

$$0 \rightarrow M \xrightarrow{\gamma} \mathbf{Z}^n \rightarrow D(\sigma) \rightarrow 0,$$

where $\gamma(m) = (\langle m, \eta_1 \rangle, \dots, \langle m, \eta_n \rangle)$. Then $D(\sigma)$ is finite since σ is simplicial, and $G(\sigma) = \text{Hom}_{\mathbf{Z}}(D(\sigma), \mathbf{C}^*)$ is isomorphic to N/N' , where N' is the sublattice of N generated by η_1, \dots, η_n . The map $\mathbf{Z}^n \rightarrow D(\sigma)$ induces an action of $G(\sigma)$ on \mathbf{C}^n , and it follows from [C1] that the quotient is X_σ . In particular, we have a quotient map $p_\sigma: \mathbf{C}^n \rightarrow X_\sigma$.

To relate this construction to $S = \mathbf{C}[x_1, \dots, x_{n+r}] = \mathbf{C}[x_1, \dots, x_n, z_1, \dots, z_r]$, note that S is the coordinate ring of the affine space \mathbf{C}^{n+r} . Then let $Z(\Sigma) \subset \mathbf{C}^{n+r}$ be the subvariety defined by $B(\Sigma)$. In [C1], a natural map $p: \mathbf{C}^{n+r} - Z(\Sigma) \rightarrow X$ is constructed. Now consider the inclusion $\mathbf{C}^n \rightarrow \mathbf{C}^{n+r}$ defined by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1, \dots, 1). \quad (2.6)$$

One easily sees that the image of this map lies in $\mathbf{C}^{n+r} - Z(\Sigma)$, and the argument of Theorem 1.9 of [BC] shows that we have a commutative diagram

$$\begin{array}{ccc}
 \mathbf{C}^n & \xrightarrow{p_\sigma} & X_\sigma \\
 \downarrow & & \downarrow \\
 \mathbf{C}^{n+r} - Z(\Sigma) & \xrightarrow{p} & X.
 \end{array} \tag{2.7}$$

We now recall the Euler form Ω from [BC]. Fix an integer basis m_1, \dots, m_n for the lattice M . Then, given a subset $I = \{\ell_1, \dots, \ell_n\} \subset \{1, \dots, n+r\}$ with n elements, define

$$\det(\eta_I) = \det(\langle m_i, \eta_{\ell_j} \rangle_{1 \leq i, j \leq n}).$$

Also set $dx_I = dx_{\ell_1} \wedge \dots \wedge dx_{\ell_n}$ and $\hat{x}_I = \prod_{\ell \notin I} x_\ell$. Then Ω is given by the formula

$$\Omega = \sum_{|I|=n} \det(\eta_I) \hat{x}_I dx_I, \tag{2.8}$$

where the sum is over all n -element subsets $I \subset \{1, \dots, n+r\}$. Note that Ω is well-defined up to ± 1 .

From (2.6) and (2.7), we see that $p_\sigma^*(\omega)$ is computed by setting $z_1 = \dots = z_r = 1$ in the above formula for Ω . Thus

$$p_\sigma^*(\omega) = \pm \frac{|N/N'| dx_1 \wedge \dots \wedge dx_n}{x_1 \dots x_n}. \tag{2.9}$$

The next step in the proof is to study what happens when we change X slightly. Suppose that Σ' is a refinement of the fan Σ such that σ is still a cone of Σ' . Then we get a birational morphism $\pi: X' \rightarrow X$ of toric varieties which is the identity on the affine piece $X'_\sigma = X_\sigma$. For X' , we have an Euler form Ω' , and the analog of ω in (2.5) is denoted ω' . Note that x_1, \dots, x_n have the same meaning for both ω and ω' . Then, since σ is a cone for both fans and we haven't changed N , it follows from (2.9) that $\pi^*(\omega) = \omega'$. We also have open covers \mathcal{U} of X and \mathcal{U}' of X' , and we leave it to the reader to verify that these covers are compatible under π . It follows that

$$\pi^*([\omega]) = [\omega'],$$

as elements of $H^n(X', \hat{\Omega}_{X'}^n)$. Since

$$\mathrm{Tr}_{X'} \circ \pi^* = \mathrm{Tr}_X,$$

when π is birational, ω and ω' have the same toric residue. In particular,

$$\text{Res}(\omega) = \pm 1 \iff \text{Res}(\omega') = \pm 1. \tag{2.10}$$

Next, instead of changing the fan, suppose we change the lattice. If N_1 is a sublattice of N , then as explained in [O, Corollary 1.16], we get a toric variety X_1 such that N/N_1 acts on X_1 with X as quotient. Let $\pi_1: X_1 \rightarrow X$ be the quotient map. The toric varieties X and X_1 have the same coordinate ring S (though the gradings differ). Now assume that $\eta_1, \dots, \eta_{n+r}$ lie in N_1 . Then one easily sees that the Euler forms Ω and Ω_1 are related by the formula

$$\Omega = \pm |N/N_1| \Omega_1,$$

so that if ω_1 is the analog of ω for X_1 , we have

$$\pi_1^*(\omega) = \pm |N/N_1| \omega_1.$$

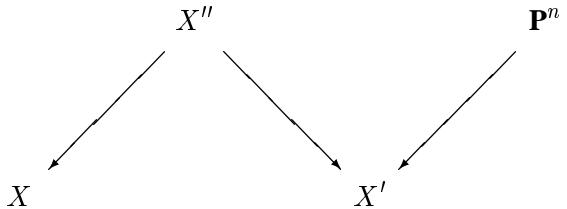
However, since $\pi_1: X_1 \rightarrow X$ is a finite map of degree $|N/N_1|$, we also have

$$\text{Tr}_{X_1} \circ \pi_1^* = |N/N_1| \text{Tr}_X.$$

From here, it follows that ω and ω_1 have the same toric residue (up to ± 1), and hence

$$\text{Res}(\omega) = \pm 1 \iff \text{Res}(\omega_1) = \pm 1. \tag{2.11}$$

We can now prove the proposition. Define $\eta_0 = -\sum_{i=1}^n \eta_i$, where η_1, \dots, η_n are the generators of σ , and let Σ' be the fan whose cones are generated by proper subsets of $\{\eta_0, \dots, \eta_n\}$. This gives a toric variety X' , and note that σ is a cone of Σ' . Now let Σ'' be the fan consisting of all intersections $\sigma_1 \cap \sigma_2$ for $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma'$. Hence Σ'' is a common refinement of Σ and Σ' , and σ is a cone in all three fans. The corresponding toric variety X'' maps to both X and X' . Finally, let $N' \subset N$ be the sublattice generated by η_1, \dots, η_n . Then $\eta_0 \in N'$, and the toric variety determined by N' and Σ' is \mathbf{P}^n . Putting this all together, we get a diagram of toric varieties



where the first two maps come from refinements which preserve σ and the third comes from a change of lattice which preserves the generators η_i . It follows from (2.10) and (2.11) that

$$\text{Res}(\omega) = \pm 1,$$

if and only if the corresponding statement is true for \mathbf{P}^n . The latter is well known; for example, it follows from the Trace Property for \mathbf{P}^n stated in the introduction to [C2]. \square

Remarks 2.12 (i) In Section 4, we will use the relation between toric residues and local residues to give a second proof of Proposition 2.4 in the special case when X is simplicial.

(ii) If we replace the hypothesis that the α_i are \mathbf{Q} -ample with the weaker assumption (2.1), then the conclusion of Theorem 0.2 is still true. In fact, all we need to assume is that $F_i \in B(\Sigma)$ for all i . This will be useful in Section 3.

(iii) The element Δ_σ depends on the choice of simplicial cone σ of dimension n and on the choice of coefficients A_{ij} in (2.2). Once Theorem 0.3 is established, it will follow from Theorem 0.2 that when X is simplicial and the α_i are ample, the class of Δ_σ is unique up to sign modulo the ideal $\langle F_0, \dots, F_n \rangle$. Moreover, if we pick a basis of M such that $\det(\langle m_i, \eta_j \rangle) > 0$, then one can check that $\text{Res}_F(\Delta_\sigma) = 1$.

(iv) Given *any* decomposition

$$F_j = A_{0j} z_1 \dots z_r + \sum_{i=1}^n A_{ij} x_i,$$

the polynomial $\Delta_\sigma = \det(A_{ij})$ satisfies $\text{Res}_F(\Delta_\sigma) = \pm 1$.

(v) The definition of Δ_σ given by (2.3) generalizes a well-known construction in the algebraic setting corresponding to projective space [KK]. Regarding \mathbf{P}^n as a toric variety, we get the usual graded ring $S = \mathbf{C}[x_0, \dots, x_n]$, and the ideal $B(\Sigma)$ is the maximal ideal $\langle x_0, \dots, x_n \rangle$. Given homogeneous polynomials F_0, \dots, F_n , $\deg(F_j) = d_j > 0$, whose only common zero is the origin, let σ be the cone whose generators correspond to the variables x_1, \dots, x_n . Then

$$F_j = \frac{1}{d_j} \frac{\partial F_j}{\partial x_0} x_0 + \sum_{i=1}^n \frac{1}{d_j} \frac{\partial F_j}{\partial x_i} x_i$$

and, consequently, a choice of Δ_σ is given by

$$\Delta_\sigma = \frac{1}{d_0 \dots d_n} \det\left(\frac{\partial F_i}{\partial x_j}\right).$$

3. The codimension one and residue isomorphism theorems

Before we can prove the main results of this section, we need to discuss a toric version of the Cayley trick. Let X be a complete toric variety, and let $\mathcal{L}_0, \dots, \mathcal{L}_n$ be ample line bundles on X . Then consider

$$Y = \mathbf{P}(\mathcal{L}_0 \oplus \dots \oplus \mathcal{L}_n).$$

This variety has a canonical line bundle $\mathcal{O}_Y(1)$, which is ample since the \mathcal{L}_j are ample (see Section 1 of Chapter III of [H]). Note that Y is a \mathbf{P}^n -bundle over X , so that Y has dimension $2n$.

For our purposes, we need to understand Y as a toric variety. We begin with the description given in [BB]. The ample line bundle \mathcal{L}_j is associated to a n -dimensional polytope $\Delta_j \subset M_{\mathbf{R}}$ of the form

$$\Delta_j = \{m \in M_{\mathbf{R}}: \langle m, \eta_i \rangle \geq -a_{ij}, i = 1, \dots, n+r\}. \quad (3.1)$$

Since each \mathcal{L}_j is ample, the *facets* of Δ_j (faces of codimension 1) correspond bijectively to the η_i , where η_i gives the facet $F_{ij} = \{m \in \Delta_j: \langle m, \eta_i \rangle = -a_{ij}\} \subset \Delta_j$.

Now consider $\mathbf{R}^n \oplus M_{\mathbf{R}}$ with the integer lattice $\mathbf{Z}^n \oplus M$. Elements of $\mathbf{R}^n \oplus M_{\mathbf{R}}$ can be uniquely written $\tilde{m} = \lambda_1 e_1 + \dots + \lambda_n e_n + m$, where $\lambda_j \in \mathbf{R}$ and $m \in M_{\mathbf{R}}$. We also have the dual $\mathbf{R}^n \oplus N_{\mathbf{R}}$ with lattice $\mathbf{Z}^n \oplus N$, and elements here are written similarly. Then define $\Delta \subset \mathbf{R}^n \oplus M_{\mathbf{R}}$ to be the convex hull

$$\begin{aligned} \Delta &= \text{Conv}(\{\{0\} \times \Delta_0\} \cup \{\{e_1\} \times \Delta_1\} \cup \dots \cup \{\{e_n\} \times \Delta_n\}) \\ &= \left\{ \lambda_1 e_1 + \dots + \lambda_n e_n + \lambda_0 m_0 + \dots + \lambda_n m_n: \lambda_j \geq 0, \right. \\ &\quad \left. \sum_{j=0}^n \lambda_j = 1, m_j \in \Delta_j \right\}. \end{aligned} \quad (3.2)$$

This is easily seen to be equivalent to the polytope Δ_σ in Section 3 of [BB]. Since $\mathcal{O}_Y(1)$ is ample, Proposition 3.2 of [BB] implies that Y is the toric variety determined by the polytope Δ . The corresponding fan in $\mathbf{R}^n \oplus N_{\mathbf{R}}$ is called the *normal fan* of Δ .

We next show that the generators of the 1-dimensional cones in the normal fan are given by

$$\tilde{\eta}_i = \sum_{j=0}^n (a_{ij} - a_{i0}) e_j + \eta_i, \quad i = 1, \dots, n+r,$$

$$\tilde{\nu}_0 = -e_1 - \dots - e_n,$$

$$\tilde{\nu}_j = e_j, \quad j = 1, \dots, n.$$

The first step is to prove that Δ is defined by the inequalities

$$\begin{aligned} \langle \tilde{m}, \tilde{\eta}_i \rangle &\geq -a_{i0}, \quad i = 1, \dots, n+r, \\ \langle \tilde{m}, \tilde{\nu}_0 \rangle &\geq -1, \\ \langle \tilde{m}, \tilde{\nu}_j \rangle &\geq 0, \quad j = 1, \dots, n. \end{aligned} \quad (3.3)$$

Write $\tilde{m} = \lambda_1 e_1 + \cdots + \lambda_n e_n + m$, where $m \in M_{\mathbf{R}}$, and let $\lambda_0 = 1 - \sum_{j=1}^n \lambda_j$. Then the above inequalities are equivalent to

$$\langle m, \eta_i \rangle \geq - \sum_{j=0}^n \lambda_j a_{ij}, \quad i = 1, \dots, n+r,$$

$$\lambda_j \geq 0, \quad j = 0, \dots, n, \quad \sum_{j=0}^n \lambda_j = 1.$$

If $\tilde{m} \in \Delta$, then (3.2) shows that $m = \sum_{j=0}^n \lambda_j m_j$, where $m_j \in \Delta_j$, and it follows easily from (3.1) that the above conditions are satisfied. Conversely, if $\tilde{m} = \lambda_1 e_1 + \cdots + \lambda_n e_n + m$ satisfies (3.3), consider the linear maps $B(m_0, \dots, m_n) = (-\langle m_j, \eta_i \rangle)$ and $D(m_0, \dots, m_n) = \sum_{j=0}^n \lambda_j m_j$. Then the above inequalities and the Farkas Lemma (as stated in Exercise 1.6 of [Z] with $A = C = 0$) imply that there exists (m_0, \dots, m_n) with $B(m_0, \dots, m_n) \leq (a_{ij})$ and $D(m_0, \dots, m_n) = m$. This shows that $m = \sum_{j=0}^n \lambda_j m_j$, where $m_j \in \Delta_j$ by (3.1), and $\tilde{m} \in \Delta$ follows immediately.

From the inequalities defining Δ , we can read off the facets of Δ as follows. First, one easily shows that

$$\{\tilde{m} \in \Delta : \langle \tilde{m}, \tilde{\eta}_i \rangle = -a_{i0}\}$$

is the convex hull

$$\text{Conv}(\{0\} \times F_{i0}) \cup \{e_1\} \times F_{i1}) \cup \cdots \cup \{e_n\} \times F_{in}),$$

where $F_{ij} \subset \Delta_j$ is the facet defined by η_i . Since this set has dimension $2n-1$, it is a facet of Δ . Similarly, if one looks at the subsets of Δ defined by the equations $\langle \tilde{m}, \tilde{\nu}_0 \rangle = -1$ or $\langle \tilde{m}, \tilde{\nu}_j \rangle = 0$, $1 \leq j \leq n$, then one gets the $(2n-1)$ -dimensional polytopes obtained by taking the convex hulls of n of the $n+1$ sets $\{0\} \times \Delta_0, \{e_1\} \times \Delta_1, \dots, \{e_n\} \times \Delta_n$. Hence these are also facets.

It follows $\tilde{\eta}_i$ and $\tilde{\nu}_j$ define facets of Δ , and these are *all* of the facets since Δ is given by (3.3). This proves that we have found the generators of the 1-dimensional cones of the fan of Y .

We next turn our attention to the coordinate ring of Y , which is the polynomial ring

$$R = \mathbf{C}[x_1, \dots, x_{n+r}, y_0, \dots, y_n],$$

where x_i corresponds to $\tilde{\eta}_i$ and y_j corresponds to $\tilde{\nu}_j$. To determine the grading on R , note that the \mathbf{P}^n -fibration $p: Y \rightarrow X$ gives an exact sequence

$$0 \rightarrow A_{n-1}(X) \xrightarrow{p^*} A_{2n-1}(Y) \rightarrow \mathbf{Z} \rightarrow 0. \quad (3.4)$$

In terms of $p: Y \rightarrow X$, we can think of the x_i as variables coming from the base and the y_j as variables on the fiber. To make this more precise, let the torus invariant divisors on Y corresponding to $\tilde{\eta}_i$ and $\tilde{\nu}_j$ be \tilde{D}_i and \tilde{D}'_j respectively. Then \tilde{D}_i is the pullback of the torus invariant divisor D_i on X corresponding to η_i , and \tilde{D}'_j induces the hyperplane class on each fiber. In particular, $\deg(x_i) = [\tilde{D}_i] \mapsto 0$ and $\deg(y_j) = [\tilde{D}'_j] \mapsto 1$ in (3.4).

We next have the following important lemma.

LEMMA 3.5. *For each $j = 0, \dots, n$, we have $\mathcal{O}_Y(\tilde{D}'_j) \otimes p^*(\mathcal{L}_j) \simeq \mathcal{O}_Y(1)$.*

Proof. The integers a_{ij} in (3.1) mean that $\mathcal{L}_j \simeq \mathcal{O}_X(\sum_{i=1}^{n+r} a_{ij} D_i)$ on X . It follows that on Y , we have

$$\mathcal{O}_Y(\tilde{D}'_j) \otimes p^*(\mathcal{L}_j) \simeq \mathcal{O}_Y\left(\tilde{D}'_j + \sum_{i=1}^{n+r} a_{ij} \tilde{D}_i\right).$$

When $j = 0$, the polytope corresponding to this divisor is *precisely* Δ by (3.3), which proves the lemma in this case. If $j > 0$, we have $e_j \in \mathbf{Z}^n \oplus M$, and the divisor of the corresponding character χ^{e_j} is

$$\begin{aligned} \operatorname{div}(\chi^{e_j}) &= \sum_{i=1}^{n+r} \langle e_j, \tilde{\eta}_i \rangle \tilde{D}_i + \sum_{k=0}^n \langle e_j, \tilde{\nu}_k \rangle \tilde{D}'_k \\ &= \sum_{i=1}^{n+r} (a_{ij} - a_{i0}) \tilde{D}_i - \tilde{D}'_0 + \tilde{D}'_j \\ &= \left(\tilde{D}'_j + \sum_{i=1}^{n+r} a_{ij} \tilde{D}_i \right) - \left(\tilde{D}'_0 + \sum_{i=1}^{n+r} a_{i0} \tilde{D}_i \right), \end{aligned}$$

and the lemma follows immediately. \square

To see what this lemma says about coordinate rings, let $\alpha_j = [\mathcal{L}_j] \in A_{n-1}(X)$ and pick polynomials $F_j \in S_{\alpha_j}$. The F_j may have different degrees in S (since the α_j need not be equal), but Lemma 3.5 implies that the polynomials $y_j F_j$ all have the *same* degree in R . Thus we can form the single homogeneous polynomial $\sum_{j=0}^n y_j F_j \in R$ which contains all the F_j simultaneously. This is the essence of the Cayley trick.

We can now prove the first main result of this section, which gives a sufficient condition for $\langle F_0, \dots, F_n \rangle \subset S$ to have codimension one in the critical degree ρ .

THEOREM 3.6. *Let X be a complete simplicial toric variety of dimension n , and assume $F_j \in S_{\alpha_j}$, for $j = 0, \dots, n$, where α_j is ample and the F_j don't vanish simultaneously on X . If $\rho = \rho(F)$ is the critical degree of the F_j , then*

$$\dim_{\mathbf{C}}(S_{\rho} / \langle F_0, \dots, F_n \rangle_{\rho}) = 1.$$

Proof. If we pick ample line bundles \mathcal{L}_j on X such that $\alpha_j = [\mathcal{L}_j] \in A_{n-1}(X)$, then we get the toric variety $Y = \mathbf{P}(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_n)$ described above. As remarked after the proof of Lemma 3.5, the polynomials $y_j F_j$ all have the same degree in the coordinate ring R of Y . This degree is the ample class $[\mathcal{O}_Y(1)] \in A_{2n-1}(Y)$, which we will denote by γ . Thus we can define the homogeneous polynomial

$$F = y_0 F_0 + \cdots + y_n F_n \in R_\gamma.$$

Let $W \subset Y$ be the hypersurface defined by $F = 0$. The idea of the Cayley trick is that this hypersurface should be closely related to the complete intersection $F_0 = \cdots = F_n = 0$ on X . Since the intersection is empty in our situation, we expect W to be especially simple.

We next check that all of the relevant hypotheses of [BC] are satisfied. We know that γ is ample, and Y is simplicial since it is a \mathbf{P}^n -bundle over the simplicial toric variety X . To show that W is quasi-smooth (as defined in Section 3 of [BC]), note that among the partial derivatives of F , we have

$$\frac{\partial F}{\partial y_j} = F_j. \tag{3.7}$$

Since the F_j don't vanish simultaneously on X , these partials of F can't vanish simultaneously on Y , which proves that W is quasi-smooth.

The *primitive cohomology* of W is defined by the exact sequence

$$H^{2n-1}(Y) \rightarrow H^{2n-1}(W) \rightarrow PH^{2n-1}(W) \rightarrow 0,$$

(with coefficients in \mathbf{C}). To prove Theorem 3.6, we will compute $PH^{2n-1}(W)$ topologically, using $W \hookrightarrow Y \rightarrow X$, and algebraically, using the Jacobian ideal of F .

In the composition $W \hookrightarrow Y \rightarrow X$, the fiber over a point of X with coordinates t_1, \dots, t_{n+r} is the subset of \mathbf{P}^n defined by $\sum_{j=0}^n y_j F_j(t_1, \dots, t_{n+r}) = 0$. Since the F_j don't vanish simultaneously on X , it follows that the fiber is a hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$. Topologically, this means we have a map of fibrations

$$\begin{array}{ccc} \mathbf{P}^{n-1} & \hookrightarrow & \mathbf{P}^n \\ \downarrow & & \downarrow \\ W & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X & = & X. \end{array}$$

For each fibration, we get the usual spectral sequence, and the map between the spectral sequences is surjective at E_2 because $H^q(\mathbf{P}^n) \rightarrow H^q(\mathbf{P}^{n-1})$ is surjective for all q . It follows that $H^{2n-1}(Y) \rightarrow H^{2n-1}(W)$ is surjective, so that $PH^{2n-1}(W)$ vanishes.

We can also compute the Hodge components of $PH^{2n-1}(W)$ using [BC]. In particular, the exact sequence from Theorem 10.13 of [BC] gives an exact sequence

$$\begin{aligned} 0 \rightarrow H^{2n-2}(Y) \rightarrow H^{2n}(Y) \rightarrow (R/J(F))_{(n+1)\gamma-\tilde{\beta}} \\ \rightarrow PH^{n-1,n}(W) (= 0) \rightarrow 0, \end{aligned} \quad (3.8)$$

where $J(F) = \langle \partial F / \partial x_i, \partial F / \partial y_j \rangle$ is the Jacobian ideal of F and $\tilde{\beta} = \sum_{i=1}^{n+r} \deg(x_i) + \sum_{j=0}^n \deg(y_j)$. However, $\gamma = \deg(y_j) + \alpha_j$ for all j by Lemma 3.5, so that

$$\begin{aligned} (n+1)\gamma - \tilde{\beta} &= \sum_{j=0}^n (\deg(y_j) + \alpha_j) - \sum_{i=1}^{n+r} \deg(x_i) - \sum_{j=0}^n \deg(y_j) \\ &= \sum_{j=0}^n \alpha_j - \sum_{i=1}^{n+r} \deg(x_i) = \rho. \end{aligned}$$

In the map $A_{2n-1}(Y) \rightarrow \mathbf{Z}$ of (3.4), we know that $\rho \mapsto 0$ and $\deg(y_j) \mapsto 1$. This implies $R_\rho = S_\rho$. Furthermore, by (3.7), the Jacobian ideal is $J(F) = \langle F_j, \partial F / \partial x_i \rangle$, and $J(F)_\rho = \langle F_0, \dots, F_n \rangle_\rho$ follows since $\partial F / \partial x_i = \sum_{j=0}^n y_j \partial F_j / \partial x_i$. Then (3.8) tells us that

$$\dim_{\mathbf{C}}(S_\rho / \langle F_0, \dots, F_n \rangle_\rho) = h^{2n}(Y) - h^{2n-2}(Y).$$

However, since the spectral sequence for the fibration $\mathbf{P}^n \rightarrow Y \rightarrow X$ degenerates at E_2 (both base and fiber have cohomology only in even degrees), we see that if $q \leq n$, then

$$\begin{aligned} h^{2q}(Y) &= \sum_{k=0}^q h^{2k}(X) h^{2(q-k)}(\mathbf{P}^n) \\ &= h^0(X) + h^2(X) + \dots + h^{2q}(X). \end{aligned}$$

This easily implies $\dim_{\mathbf{C}}(S_\rho / \langle F_0, \dots, F_n \rangle_\rho) = h^{2n}(X) = 1$, and the theorem is proved. \square

We can now prove Theorem 0.3 from the Introduction. The first part of the theorem claims that the toric residue map

$$\text{Res}_F: S_\rho / \langle F_0, \dots, F_n \rangle_\rho \rightarrow \mathbf{C} \quad (3.9)$$

is an isomorphism. Since X is simplicial, every n -dimensional $\sigma \in \Sigma$ is simplicial, so that by Theorem 0.2, we have $\Delta_\sigma \in S_\rho$ such that $\text{Res}_F(\Delta_\sigma) = \pm 1$. Then Theorem 3.6 immediately implies that (3.9) is an isomorphism.

Turning to the second part of Theorem 0.3, we need to show that

$$x_i \cdot S_\rho \subset \langle F_0, \dots, F_n \rangle, \quad i = 1, \dots, n + r. \quad (3.10)$$

To prove this, let σ be a n -dimensional cone of Σ containing η_i , and renumbering as in Section 2, we can assume that $i \leq n$. Then Cramer's Rule, applied to the equations (2.2), shows that $x_i \cdot \Delta_\sigma \subset \langle F_0, \dots, F_n \rangle$. But the previous paragraph implies $S_\rho = \mathbf{C} \cdot \Delta_\sigma + \langle F_0, \dots, F_n \rangle_\rho$, and then (3.10) follows immediately. This completes the proof of Theorem 0.3.

Remark 3.11. As a consequence of these results, we can describe an algorithm for computing toric residues when X is complete and simplicial and $F_j \in S_{\alpha_j}$ for α_j ample. First, pick a Gröbner basis for $\langle F_0, \dots, F_n \rangle$ (using a convenient monomial order on S). Given a polynomial $H \in S$, we can then compute its normal form, denoted $\text{normalform}(H)$. Since $\langle F_0, \dots, F_n \rangle_\rho \subset S_\rho$ has codimension 1, an easy argument shows that the normal forms of elements of S_ρ are multiples of the monomial x^α which is the least (relative to the chosen monomial order) among the monomials of degree ρ not in $\langle F_0, \dots, F_n \rangle_\rho$.

Then choose a n -dimensional cone σ , say with generators η_{i_j} , and pick a basis m_i of M such that $\det(\langle m_i, \eta_{i_j} \rangle) > 0$. If we use this basis to construct the Euler form Ω , then by the remarks made at the end of Section 2, the determinant $\Delta_\sigma \in S_\rho$ satisfies $\text{Res}_F(\Delta_\sigma) = 1$. Finally, let c_σ be the nonzero constant such that $\text{normalform}(\Delta_\sigma) = c_\sigma x^\alpha$.

Given these 'preprocessing' steps, we can now describe the algorithm: given $H \in S_\rho$, its toric residue is given by the quotient

$$\text{Res}_F(H) = \frac{c}{c_\sigma},$$

where $\text{normalform}(H) = c x^\alpha$. This follows because $H \equiv c x^\alpha \pmod{\langle F_0, \dots, F_n \rangle}$ and $\Delta_\sigma \equiv c_\sigma x^\alpha \pmod{\langle F_0, \dots, F_n \rangle}$ imply $H \equiv (c/c_\sigma) \Delta_\sigma \pmod{\langle F_0, \dots, F_n \rangle}$.

We should point out that the use of the Residue Isomorphism Theorem together with the existence of an element such as Δ_σ to compute residues is at the heart of the algebraic approach to the study of residues. In this setting one needs to work with a regular sequence in an appropriate ring. We refer to [SS1] for the classical homogeneous case, and to [KK] and [SS2] for the case when $\text{Pic}(X)$ has rank one. However, as pointed out in [Sta, Sect. 5 of Ch. I], regular sequences of homogeneous elements are not well behaved when the group grading the ring has rank greater than one. Hence, whenever $\text{rank } A_{n-1}(X) > 1$, the algebraic approach doesn't apply directly to the coordinate ring of X . In the case of polynomials of equal ample degree in an arbitrary complete toric variety, one can avoid this difficulty by

reducing to a subring of the coordinate ring which has a rank one grading. This is done in [C2] (based on earlier work of Batyrev [Ba]).

In the final part of this section, we will discuss the hypotheses of the Codimension One Theorem (Theorem 3.6) and the Residue Isomorphism Theorem (Theorem 0.3). In proving both of these results, we assumed that the variety X was simplicial; this appears to be an essential ingredient in the proof of the results of [BC] which we are using in our argument. It is not clear to us whether one should expect the Codimension One Theorem to hold for arbitrary complete toric varieties. For the same reason we also made the assumption that the degrees of F_0, \dots, F_n were ample classes in $A_{n-1}(X)$. Although some assumption of this type is needed (see Example (iv) in (3.14)), we suspect that these theorems should hold under the weaker hypothesis that the degrees are \mathbf{Q} -ample. In fact, there is an even weaker hypothesis which leads to the following conjecture generalizing the Codimension One Theorem.

CONJECTURE 3.12. If X is a complete simplicial toric variety and $F_0, \dots, F_n \in B(\Sigma)$ are homogeneous polynomials which don't vanish simultaneously on X , then

$$\dim_{\mathbf{C}}(S_{\rho}/\langle F_0, \dots, F_n \rangle_{\rho}) = 1,$$

where as usual ρ is the critical degree of F_0, \dots, F_n .

Recall from Section 2 that $B(\Sigma)$ is the ideal generated by the monomials $\hat{x}_{\sigma} = \prod_{\eta_i \notin \sigma} x_i$ for all $\sigma \in \Sigma$ and that $S_{\alpha_j} \subset B(\Sigma)$ when α_j is \mathbf{Q} -ample (see (2.1)). Thus Theorem 3.6 is a special case of Conjecture 3.12.

One useful observation is that Conjecture 3.12 implies the conclusions of the Residue Isomorphism Theorem remain true.

PROPOSITION 3.13. *Let X be a complete simplicial toric variety, and let $F_0, \dots, F_n \in B(\Sigma)$ be homogeneous polynomials which don't vanish simultaneously on X . If Conjecture 3.12 is true for X (i.e., if $\dim_{\mathbf{C}}(S_{\rho}/\langle F_0, \dots, F_n \rangle_{\rho}) = 1$), then*

- (i) *The toric residue map $\text{Res}_F: S_{\rho}/\langle F_0, \dots, F_n \rangle_{\rho} \rightarrow \mathbf{C}$ is an isomorphism.*
- (ii) *For each variable x_i , $0 \leq i \leq n+r$, we have $x_i \cdot S_{\rho} \subset \langle F_0, \dots, F_n \rangle$.*

Proof. The argument is identical to what we used to derive Theorem 0.3 from Theorem 3.6. This is because, as we observed in Remark 2.12 (ii), Theorem 0.2 still applies under the assumption $F_i \in B(\Sigma)$. \square

As evidence for Conjecture 3.12, we present the following examples.

EXAMPLE 3.14. (i) If $X = \mathbf{P}(q_0, \dots, q_n)$ is a weighted projective space with coordinate ring $S = \mathbf{C}[x_0, \dots, x_n]$, then $B(\Sigma)$ is the ideal $\langle x_0, \dots, x_n \rangle$, so that $F_i \in B(\Sigma)$ means that F_i has positive degree. Hence Conjecture 3.12 follows easily

by standard commutative algebra because F_0, \dots, F_n form a regular sequence in S (since they don't vanish simultaneously on X).

For a specific example, consider $X = \mathbf{P}(1, 1, 1, 1, 3, 3, 5)$ and suppose that F_0, \dots, F_6 have degrees 3, 6, 6, 6, 6, 5, 4 respectively. The critical degree is $\rho = 21$, so that $S_{21}/\langle F_0, \dots, F_6 \rangle_{21} \simeq \mathbf{C}$ in this case. Note that the F_i are *not* Cartier, though they are certainly \mathbf{Q} -ample. This is an example from mirror symmetry which arises in connection with certain (0, 2) string theories – see [DK] for more details.

(ii) For another example where the degrees of the F_i are \mathbf{Q} -ample but not Cartier, consider the toric surface X corresponding to the fan in \mathbf{R}^2 determined by the vectors

$$\begin{aligned} \eta_1 &= (1, 0); & \eta_2 &= (0, 1); & \eta_3 &= (-1, 1); \\ \eta_4 &= (-1, -1); & \eta_5 &= (1, -1). \end{aligned}$$

Note that X is singular since η_3, η_4 and η_4, η_5 don't span all of \mathbf{Z}^2 , though X is certainly simplicial.

If we let the variables x, y, z, t, u correspond to η_1, \dots, η_5 , then the exceptional set $Z \subset \mathbf{C}^5$ is defined by the ideal $B(\Sigma) = \langle ztu, xtu, xyu, xyz, yzt \rangle$, that is,

$$\begin{aligned} Z &= \{x = z = 0\} \cup \{z = u = 0\} \cup \{y = u = 0\} \\ &\cup \{x = t = 0\} \cup \{y = t = 0\}. \end{aligned}$$

Thus, $X \simeq (\mathbf{C}^5 - Z)/(\mathbf{C}^*)^3$. Furthermore, one can show that $A_1(X) \simeq \mathbf{Z}^3$ and that we get a grading in the polynomial ring $S = \mathbf{C}[x, y, z, t, u]$ with

$$\begin{aligned} \deg(x) &= (1, 1, -1); & \deg(y) &= (-1, 1, 1); & \deg(z) &= (1, 0, 0); \\ \deg(t) &= (0, 1, 0); & \deg(u) &= (0, 0, 1). \end{aligned}$$

Thus, the sum of the degrees of the variables is $\beta = (1, 3, 1)$.

We next characterize ample divisors on X . First, one checks that a class $(a, b, c) \in \mathbf{Z}^3 \simeq A_1(X)$ lies in $\text{Pic}(X) \subset A_1(X)$ (i.e., the divisor $aD_3 + bD_4 + cD_5$ is Cartier) if and only if $a \equiv b \equiv c \pmod{2}$. Then it is straightforward to verify (using [F, Sect. 3.3-4]) that a Cartier class (a, b, c) is ample if and only if

$$b > a > 0 \quad \text{and} \quad b > c > 0. \tag{3.15}$$

For an arbitrary (a, b, c) , these inequalities tell us when the corresponding class is \mathbf{Q} -ample.

Now consider the polynomials

$$\begin{aligned} F_0 &= xy^2z^3; & F_1 &= x^2yu^3 + yz^2t^2u + xt^2u^3 + y^2z^3t; \\ F_2 &= zt^3u^2 + xt^2u^3 + y^2z^3t. \end{aligned}$$

They are homogeneous and $\deg F_0 = (2, 3, 1)$, $\deg F_1 = \deg F_2 = (1, 3, 2)$. One can check that the common zeros of F_0, F_1 and F_2 in \mathbf{C}^5 are contained in the set Z and therefore the corresponding divisors on X have empty intersection. None of these divisors are Cartier, but they are clearly \mathbf{Q} -ample by (3.15), and their critical degree is given by

$$\rho = (2, 3, 1) + (1, 3, 2) + (1, 3, 2) - (1, 3, 1) = (3, 6, 4).$$

There are 22 monomials of degree ρ , and computing the normalform of each monomial (as in Remark 3.11), we find that the normalforms are all multiples of the same monomial (for example, if we use graded reverse lex with $x > y > z > t > u$, the normalforms are all multiples of $x^3 t^3 u^7$). Thus $\langle F_0, F_1, F_2 \rangle_\rho$ has codimension one in S_ρ .

(iii) We next give an example where $F_i \in B(\Sigma)$ for all i but their degrees are not \mathbf{Q} -ample classes. We use the same singular toric surface X as in (ii), but this time we consider the polynomials

$$F_0 = ztu; \quad F_1 = yzt + xyu; \quad F_2 = xyz + xtu.$$

These are homogeneous with degrees $\deg F_0 = (1, 1, 1)$, $\deg F_1 = (0, 2, 1)$, and $\deg F_2 = (1, 2, 0)$. One can check that F_0, F_1 and F_2 don't vanish simultaneously on X , and by the ampleness criterion (3.15), none of their degrees are \mathbf{Q} -ample, although F_0, F_1, F_2 all lie in $B(\Sigma)$. The critical degree is

$$\rho = (1, 1, 1) + (0, 2, 1) + (1, 2, 0) - (1, 3, 1) = (1, 2, 1).$$

Computing normalforms of the four monomials of degree ρ reveals that $\langle F_0, F_1, F_2 \rangle_\rho$ has codimension one in S_ρ .

(iv) Finally, we give an example to show what can go wrong if not all of the F_i are contained in $B(\Sigma)$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$. Here, it is well known that the homogeneous coordinate ring of X is $S = \mathbf{C}[x, y, z, t]$, with the usual bigrading

$$\deg(x) = \deg(y) = (1, 0); \quad \deg(z) = \deg(t) = (0, 1).$$

Also, $B(\Sigma) = \langle xz, xt, yz, yt \rangle$.

We now let

$$F_0 = (x + y)^2; \quad F_1 = xz; \quad F_3 = yt.$$

Thus $\deg(F_0) = (2, 0)$ and $\deg(F_1) = \deg(F_2) = (1, 1)$. It is easy to check that F_0, F_1, F_2 don't vanish simultaneously on X . Moreover, the divisors defined by F_1, F_2 are ample (a polynomial of degree (a, b) defines an ample divisor if and only if $a > 0$ and $b > 0$), while $F_0 \notin B(\Sigma)$.

The critical degree in this case is $\rho = (2, 0)$ since the sum of the degrees of the variables is $\beta = (2, 2)$. There are three monomials of degree $(2, 0)$: x^2, y^2 and xy ,

and any two of them are linearly independent modulo the ideal $\langle F_0, F_1, F_2 \rangle$. Thus $\langle F_0, F_1, F_2 \rangle_\rho$ does *not* have codimension one in S_ρ . Note also that *no* monomial of degree $(3, 0)$ is in the ideal, which shows that $x \cdot S_\rho \not\subset \langle F_0, F_1, F_2 \rangle$. Hence the second part of Proposition 3.13 fails as well.

Remark 3.16. (i) Notice that if the F_i don't all lie in $B(\Sigma)$, then we can no longer express the F_i as in (2.2), so that the definition of Δ_σ makes no sense. Thus, even if $\langle F_0, \dots, F_n \rangle_\rho$ has codimension one in S_ρ , the second part of Proposition 3.13 could fail. For an example of how this can happen, consider the toric variety X of Example 3.14 (ii), this time using the polynomials

$$F_0 = ztu; \quad F_1 = yzt + xyu; \quad F_2 = xyz + xtu + zt^2.$$

These are very similar to what we used in Example 3.14 (iii)—the only difference is that F_2 has an extra zt^2 term. As in that example, $\deg F_0 = (1, 1, 1)$, $\deg F_1 = (0, 2, 1)$, and $\deg F_2 = (1, 2, 0)$, and they don't vanish simultaneously on X . Note also that $F_0, F_1 \in B(\Sigma)$ but $F_2 \notin B(\Sigma)$ because of the zt^2 term. The critical degree is still $(1, 2, 1)$, and an easy computation shows that $\langle F_0, F_1, F_2 \rangle_\rho$ still has codimension one in S_ρ . However, in this case, one can also compute that

$$x \cdot xyzu \notin \langle F_0, F_1, F_2 \rangle.$$

Since $xyzu \in S_\rho$, we have $x \cdot S_\rho \not\subset \langle F_0, F_1, F_2 \rangle$, so that the second part of Proposition 3.13 fails in this case.

(ii) One question we have not investigated is whether the simplicial hypothesis is needed in Conjecture 3.12 and Proposition 3.13. For example, if X is an arbitrary complete toric variety, then Conjecture 3.12 and the first part of Proposition 3.13 are true when the degrees of the F_i are the same ample class—this is Theorem 5.1 of [C2].

4. Global residues as sums of local residues

In this section we will show that for simplicial toric varieties, the toric residue may be computed as a sum of local Grothendieck residues. The toric setting is not essential here and, in fact, it is convenient to work with the more general notion of a V -manifold or orbifold (see [B], [S]). We begin with a review of the theory of residual currents.

Residual currents on V -manifolds.

We recall that, by results of Prill [P], if an n -dimensional complex variety X is a V -manifold, then for every $x \in X$ there exists a finite subgroup $G \subset \mathrm{GL}(n, \mathbf{C})$ such that for some neighborhood W of $x \in X$, we have $(W, x) \simeq (U/G, 0)$, where U is a G -invariant neighborhood of $0 \in \mathbf{C}^n$. Furthermore, G is *small* (no $g \in G$

has 1 as an eigenvalue of multiplicity $n - 1$) and is unique up to conjugacy. Such a local presentation $(W, x) \simeq (U/G, 0)$ is called a *standard model*.

A simplicial toric variety X is an example of a V -manifold. Indeed, with the notation of Section 2, we may cover X with open sets $X_\sigma \simeq \mathbf{C}^n/G(\sigma)$ and it is easy to verify that $G(\sigma)$ is a small subgroup (see [BC, 3.5]).

It is shown in [Ste, 1.8] that if X is a V -manifold and $(W, x) \simeq (U/G, 0)$ is a standard model, then

$$\Gamma(W, \widehat{\Omega}_X^p) \simeq \Gamma(U, \Omega_{\mathbf{C}^n}^p)^G, \quad (4.1)$$

where, as before, $\widehat{\Omega}_X^p$ denotes the sheaf of Zariski p -forms on X , and the superscript G indicates the subspace of G -invariant forms. Similarly (see [S] and [B]), we consider the sheaves $\mathcal{E}_X^{p,q}$ of C^∞ forms on X of bidegree (p, q) . They are associated with the presheaves which assign to an open set $W \subset X$, which is part of a standard model $(W, x) \simeq (U/G, 0)$, the group $\mathcal{E}_X^{p,q}(W) = \Gamma(U, \mathcal{E}^{p,q})^G$, where $\mathcal{E}^{p,q}$ is the sheaf of C^∞ (p, q) -forms on \mathbf{C}^n . The restriction maps for these presheaves are defined as follows: if $(W', x) \simeq (U'/G', 0)$ is another standard model and $W' \subset W$ then by [P, Theorem 2], there exists a linear map $h \in \text{GL}(n, \mathbf{C})$ such that $h(U') \subset U$ and $G' = h^{-1}Gh$. We then set $r_{W'}^W = h^*: \mathcal{E}_X^{p,q}(W) \rightarrow \mathcal{E}_X^{p,q}(W')$. Note also that any element in $\text{GL}(n, \mathbf{C})$ commutes with the differential operator $\bar{\partial}$ acting on $\Gamma(U, \mathcal{E}^{p,q})$, which means that we can define an operator $\bar{\partial}: \mathcal{E}_X^{p,q} \rightarrow \mathcal{E}_X^{p,q+1}$.

We denote by $\Gamma_c(W, \mathcal{E}_X^{p,q})$ the space of sections of $\mathcal{E}_X^{p,q}$ with compact support in W . For a standard model $(W, x) \simeq (U/G, 0)$, we have $\Gamma_c(W, \mathcal{E}_X^{p,q}) \simeq \Gamma_c(U, \mathcal{E}^{p,q})^G$, and it carries a natural Fréchet topology as a subspace of $\Gamma_c(U, \mathcal{E}^{p,q})$. We will denote by $\mathcal{D}_X^{p,q}$ the sheaf of (p, q) -currents on X , i.e., the sheaf which associates to any open set W of X , the space $\mathcal{D}_X^{p,q}(W)$ of continuous linear functionals on $\Gamma_c(W, \mathcal{E}_X^{n-p, n-q})$.

LEMMA 4.2. *If $(W, x) \simeq (U/G, 0)$ is a standard model, then*

$$\mathcal{D}_X^{p,q}(W) \simeq \mathcal{D}^{p,q}(U)^G,$$

where $\mathcal{D}^{p,q}$ is the sheaf of (p, q) -currents on \mathbf{C}^n and the action of G on $\mathcal{D}^{p,q}(U)$ is the natural one:

$$(gT)(\alpha) = T(g^*\alpha), \quad T \in \mathcal{D}^{p,q}(U) \quad \text{and} \quad \alpha \in \Gamma_c(U, \mathcal{E}^{n-p, n-q}).$$

Proof. The space $\mathcal{D}_X^{p,q}(W)$ is by definition the continuous dual of $\Gamma_c(W, \mathcal{E}_X^{n-p, n-q}) = \Gamma_c(U, \mathcal{E}^{n-p, n-q})^G$, and any continuous linear map

$$T: \Gamma_c(U, \mathcal{E}^{n-p, n-q})^G \rightarrow \mathbf{C},$$

extends to a G -invariant continuous linear map defined on all of $\Gamma_c(U, \mathcal{E}^{n-p, n-q})$ by the formula

$$T(\alpha) = T(\alpha^G), \quad \text{where} \quad \alpha^G = \frac{1}{|G|} \sum_{g \in G} g^*\alpha. \quad (4.3)$$

Conversely, every G -invariant linear functional T on $\Gamma_c(U, \mathcal{E}^{n-p, n-q})$ satisfies

$$T(\alpha) = \frac{1}{|G|} \sum_{g \in G} (gT)(\alpha) = \frac{1}{|G|} \sum_{g \in G} T(g^* \alpha) = T(\alpha^G) \quad (4.4)$$

and, thus, is in the image of (4.3). \square

It will be convenient to assign to $T \in {}^t\mathcal{D}^{p,q}(U)^G$ the element $(1/|G|)T \in {}^t\mathcal{D}_X^{p,q}(W)$. With this convention, the G -invariant continuous linear operator defined by integration $\int_U: \Gamma_c(U, \mathcal{E}^{n,n}) \rightarrow \mathbf{C}$ gives rise, when $(W, x) \simeq (U/G, 0)$ is a standard model, to the usual definition of integration for sections $\alpha \in \Gamma_c(W, \mathcal{E}_X^{n,n})$

$$\int_W \alpha = \frac{1}{|G|} \int_U \alpha.$$

It is clear that this definition is independent of the choice of standard model. Moreover, the existence of C^∞ partitions of unity on X (see [B]) implies that we can define the integral for compactly supported sections of $\mathcal{E}_X^{n,n}$ over any open set of X .

Similarly, given a G -invariant form $\alpha \in \Gamma(U, \mathcal{E}^{p,q})^G$, integration against α defines a G -invariant current $I(\alpha) \in {}^t\mathcal{D}^{p,q}(U)^G$. Thus

$$I(\alpha)(\beta) = \int_U \alpha \wedge \beta \quad \text{for } \beta \in \Gamma_c(U, \mathcal{E}^{n-p, n-q}).$$

The corresponding current in ${}^t\mathcal{D}_X^{p,q}(W)$ will also be denoted by $I(\alpha)$ and we have $I(\alpha)(\beta) = (1/|G|) \int_U \alpha \wedge \beta = \int_W \alpha \wedge \beta$.

We extend the definition of $\bar{\partial}$ to the space of currents by the formula

$$\begin{aligned} (\bar{\partial}T)(\beta) &:= (-1)^{p+q} T(\bar{\partial}\beta), & T \in {}^t\mathcal{D}_X^{p,q}(W) & \text{ and} \\ \beta &\in \Gamma_c(W, \mathcal{E}_X^{n-p, n-q-1}). \end{aligned}$$

PROPOSITION 4.5. *Let X be a compact, connected V -manifold. Then*

(i) *The diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\Omega}_X^p & \longrightarrow & \mathcal{E}_X^{p,0} & \xrightarrow{\bar{\partial}} & \cdots & \xrightarrow{\bar{\partial}} & \mathcal{E}_X^{p,n} & \longrightarrow & 0 \\ & & \parallel & & \downarrow I & & & & \downarrow I & & \\ 0 & \longrightarrow & \widehat{\Omega}_X^p & \xrightarrow{I} & {}^t\mathcal{D}_X^{p,0} & \xrightarrow{\bar{\partial}} & \cdots & \xrightarrow{\bar{\partial}} & {}^t\mathcal{D}_X^{p,n} & \longrightarrow & 0 \end{array}$$

commutes and its rows are exact.

(ii) *The following diagram commutes and all maps are isomorphisms*

$$\begin{array}{ccc}
 & H_{\bar{\partial}}^{n,n}(X) & \\
 \eta \nearrow & \downarrow I & \searrow \int_X \\
 H^n(X, \widehat{\Omega}_X^n) & & \mathbf{C} \\
 \downarrow \prime\eta & & \nearrow Ev_1 \\
 & H_{\bar{\partial}}^n(\Gamma(X, \mathcal{D}_X^{n,\prime})) &
 \end{array}$$

Proof. The commutativity of the first diagram is a consequence of the sign convention in the definition of $\bar{\partial}$. Exactness follows from the corresponding statements in the smooth case. We illustrate this for the bottom row.

Let $\alpha \in \widehat{\Omega}_{X,x}^p$ be such that $I(\alpha) = 0$. We represent α by a G -invariant holomorphic p -form $\tilde{\alpha}$ on U where $(U/G, 0) \simeq (W, x)$ is a standard model. By Lemma 4.2, $I(\tilde{\alpha}) = 0$ as an element in ${}^t\mathcal{D}^{p,0}(U)^G \subset {}^t\mathcal{D}^{p,0}(U)$. Consequently, by exactness in the smooth case, $\tilde{\alpha} = 0$ and, a fortiori, $\alpha = 0$.

Suppose now that $T \in {}^t\mathcal{D}_{X,x}^{p,q}$ is $\bar{\partial}$ -closed. Again, we represent T by a G -invariant current $\tilde{T} \in {}^t\mathcal{D}^{p,q}(U)^G$ satisfying $\bar{\partial}\tilde{T} = 0$. We may replace U by a smaller G -invariant neighborhood U' of $0 \in \mathbf{C}^n$ where $\tilde{T} = \bar{\partial}\tilde{S}$, $\tilde{S} \in {}^t\mathcal{D}^{p,q-1}(U')$. As in (4.4), since $\bar{\partial}$ is a G -invariant operator, $\tilde{T} = \bar{\partial}\tilde{S}^G$, where \tilde{S}^G is the G -invariant current $\tilde{S}^G = (1/|G|)\sum_{g \in G} g\tilde{S}$. Thus $T = \bar{\partial}S^G$, for the induced element in ${}^t\mathcal{D}_{X,x}^{p,q-1}$.

To prove (ii) we note that the sheaves $\mathcal{E}_X^{p,q}$ and ${}^t\mathcal{D}_X^{p,q}$ are fine and, consequently, the rows in the diagram in (i) give fine resolutions of the sheaf $\widehat{\Omega}_X^p$. Now, taking $p = n$, the usual proof of Dolbeault's Theorem gives the isomorphisms η and $\prime\eta$. The isomorphism I is deduced from the map at the level of sheaves and the commutativity follows from (i). Clearly I maps the cohomology class of a $\bar{\partial}$ -closed (n, n) -form α to the (n, n) -current defined by integration of compactly supported C^∞ functions against α .

Stokes' Theorem for V -manifolds [B] implies that integration over X defines an isomorphism $\int_X : H_{\bar{\partial}}^{n,n}(X) \rightarrow \mathbf{C}$ and the map $Ev_1 : H_{\bar{\partial}}^n(\Gamma(X, \mathcal{D}_X^{n,\prime})) \rightarrow \mathbf{C}$ is defined by evaluation of a (global) current on the constant function 1_X . The commutativity of the right triangle then follows from the relation $\int_X \alpha = I([\alpha])(1_X)$, $[\alpha] \in H_{\bar{\partial}}^{n,n}(X)$. \square

We now bring into the picture multiple residue and principal value currents (as in [CH] and [Di]). Let D_1, \dots, D_k be reduced Weil divisors on the V -manifold X . For each D_j , some multiple is a Cartier divisor (since X is a V -manifold), so that D_j may be given locally as the support of the zero set of a holomorphic function.

Let ω be a semimeromorphic (p, q) -form on X with poles on $D = \bigcup_{i=1}^k D_i$. This means that ω can be locally written as ω'/f with ω' a C^∞ (p, q) -form and f a holomorphic function such that $\{f = 0\} \subset D$.

Suppose for a moment that X is smooth and we are given (not necessarily minimal) equations $f_i \in \Gamma(U, \mathcal{O}_X)$ for each hypersurface D_i , $i = 1, \dots, k$, on some open subset U . For any C^∞ form α (resp. β) with compact support contained in U and bidegree $(n - p, n - q - k)$ (resp. $(n - p, n - q - (k - 1))$), we define

$$R_D[\omega](\alpha) = R_{D_1, \dots, D_k}[\omega](\alpha) = \lim_{\delta \rightarrow 0} \int_{T_\delta(f)} \omega \wedge \alpha$$

and

$$RP_D[\omega](\beta) = R_{D_1, \dots, D_{k-1}} P_{D_k}[\omega](\beta) = \lim_{\delta \rightarrow 0} \int_{D_\delta(f)} \omega \wedge \beta,$$

where

$$\begin{aligned} T_\delta(f) &= \{x \in U: |f_i(x)| = \varepsilon_i(\delta), \quad 1 \leq i \leq k\}, \\ D_\delta(f) &= \{x \in U: |f_i(x)| = \varepsilon_i(\delta), \quad 1 \leq i \leq k-1, \quad |f_k(x)| > \varepsilon_k(\delta)\}, \end{aligned}$$

are conveniently oriented semianalytic tubes and the k functions $\varepsilon_i: (0, 1) \rightarrow \mathbf{R}_+$ are analytic and satisfy $\lim_{\delta \rightarrow 0} (\varepsilon_j(\delta)/\varepsilon_{j+1}^q(\delta)) = 0$ for all $1 \leq j \leq k-1$ and all positive integers q . We call $(\varepsilon_1, \dots, \varepsilon_k)$ an *admissible path*.

In [CH], Coleff and Herrera show that the above limits exist for any α, β . Moreover, these limits are independent of the admissible path and the particular equations f_1, \dots, f_k . Thus, on U , we get the *multiple residue* current $R_D[\omega]$ of bidegree $(p, q+k)$ and the *principal value* current $RP_D[\omega]$ of bidegree $(p, q+k-1)$. By means of a C^∞ partition of unity, these local definitions can be collected to obtain global currents on X , also denoted $R_D[\omega]$ and $RP_D[\omega]$, whose supports verify

$$\text{supp}(R_D[\omega]) \subset \left(\bigcap_{i=1}^k D_i \right) \cap \text{supp}(\omega),$$

$$\text{supp}(RP_D[\omega]) \subset \left(\bigcap_{i=1}^{k-1} D_i \right) \cap \text{supp}(\omega).$$

Suppose now that X is a V -manifold, D_1, \dots, D_k reduced Weil divisors as above, and $W \simeq U/G$ is a standard model. We denote by $\tilde{D}_1, \dots, \tilde{D}_k$ the lifted hypersurfaces in U . For any G -invariant, semimeromorphic form $\tilde{\omega}$ on U , with polar set contained in $\tilde{D} = \bigcup_{i=1}^k \tilde{D}_i$, the currents $R_{\tilde{D}}[\tilde{\omega}]$ and $RP_{\tilde{D}}[\tilde{\omega}]$ are also G -invariant. Thus, given a semimeromorphic form ω on W , we denote by $\tilde{\omega}$ its lifting to U and then define:

$$R_D[\omega] = \frac{1}{|G|} R_{\tilde{D}}[\tilde{\omega}] \quad \text{and} \quad RP_D[\omega] = \frac{1}{|G|} RP_{\tilde{D}}[\tilde{\omega}].$$

These definitions may again be globalized using a partition of unity on X . The definition of RP_D and support property stated above imply that

$$\begin{aligned}
 RP_D[\omega] \Big|_{X-D_j} &= 0 \quad \text{for } j < k, \quad \text{and} \\
 RP_D[\omega] \Big|_{X-D_k} &= R_{D_1, \dots, D_{k-1}}[\omega] \Big|_{X-D_k}.
 \end{aligned}
 \tag{4.6}$$

The mappings R_D and RP_D associating to any germ of meromorphic p -form ω with poles contained in D , the germ of the residual currents $R_D[\omega]$ and $RP_D[\omega]$, define sheaf morphisms making the following diagram commutative

$$\begin{array}{ccc}
 \widehat{\Omega}_X^p(*D) & \xrightarrow{RP_D} & \mathcal{D}_X^{p,k-1} \\
 \searrow R_D & & \nearrow \bar{\partial} \\
 & & \mathcal{D}_X^{p,k}
 \end{array}
 \tag{4.7}$$

In particular, $\bar{\partial}R_D[\omega] = 0$ for every meromorphic form $\omega \in \widehat{\Omega}_X^p(*D)$.

We conclude our discussion of residual currents by defining the local Grothendieck residue at a point x on a V -manifold X . Let $(W_1, x) \simeq (U_1/G, 0)$ be a standard model, and let W be a relatively compact neighborhood of x such that $W \subset \overline{W} \subset W_1$. Finally, let U be a G -invariant neighborhood of 0 such that $W \simeq U/G$ and $U \subset \overline{U} \subset U_1$. Suppose that $f_1, \dots, f_n \in \mathcal{O}(\overline{W})$ have x as their only common zero in \overline{W} . Pulling-back to \overline{U} , it follows that the hypersurfaces $\bar{D}_i = \{\tilde{f}_i = 0\}$ intersect only at 0. Given now a meromorphic n -form ω on W with polar set contained in $\cup_{i=1}^n \{f_i = 0\}$, we denote by $\tilde{\omega}$ its pull-back to U and define

$$\text{Res}_x(\omega) = \frac{1}{|G|} \text{Res}_0(\tilde{\omega}),
 \tag{4.8}$$

where, as we recall from [GH] for example, the local Grothendieck residue $\text{Res}_0(\tilde{\omega})$ is defined as

$$\text{Res}_0(\tilde{\omega}) = \left(\frac{1}{2\pi i} \right)^n \int_{\{z \in U: |\tilde{f}_i(z)| = \varepsilon_i, 1 \leq i \leq n\}} \tilde{\omega}.$$

Here $\varepsilon_i > 0$ must be chosen so that $\{w \in \mathbf{C}^n: |w_i|_{1 \leq i \leq n} = \varepsilon_i\}$ is contained in the open set $\tilde{f}(U)$, $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n)$, and $\{z \in \overline{U}: |\tilde{f}_i(z)|_{1 \leq i \leq n} = \varepsilon_i\} \cap \partial U = \emptyset$. Note that the tube $\{z \in U: |\tilde{f}_i(z)| = \varepsilon_i, 1 \leq i \leq n\}$ is compact, of real dimension n , and we orient it with the form $d(\arg \tilde{f}_1) \wedge \dots \wedge d(\arg \tilde{f}_n)$.

If φ is a C^∞ function with compact support in W , which is identically equal to 1 in a neighborhood of x , for its pull-back $\tilde{\varphi}$ we have

$$(2\pi i)^n \text{Res}_0(\tilde{\omega}) = \lim_{\delta \rightarrow 0} \int_{\{|\tilde{f}_i(z)| = \varepsilon_i(\delta), 1 \leq i \leq n\}} \tilde{\varphi} \cdot \tilde{\omega} = R_{\bar{D}}[\tilde{\omega}](\tilde{\varphi}),$$

from which it follows that

$$(2\pi i)^n \operatorname{Res}_x(\omega) = R_D[\omega](\varphi). \quad (4.9)$$

Remark 4.10. Given reduced Weil divisors D_1, \dots, D_n with finite intersection in a compact V -manifold X , and a meromorphic n -form ω whose polar set is contained in the divisor $D_1 \cup \dots \cup D_n$, it follows from (4.9) and (4.7) that

$$\begin{aligned} (2\pi i)^n \sum_{x \in D_1 \cap \dots \cap D_n} \operatorname{Res}_x(\omega) &= R_D[\omega](1_X) = (\bar{\partial} R P_D[\omega])(1_X) \\ &= -R P_D[\omega](\bar{\partial} 1_X) = 0. \end{aligned}$$

This is essentially the proof in [CH, p. 48] of the theorem on the vanishing of the sum of Grothendieck residues due to Griffiths [G].

Global residues.

We will now generalize the notion of toric residue to a *global residue* defined on an arbitrary n -dimensional compact V -manifold X .

Given $n + 1$ reduced Weil divisors D_0, \dots, D_n on X such that

$$D_0 \cap \dots \cap D_n = \emptyset,$$

the open sets $U_i = X - D_i$ constitute an open cover \mathcal{U} of X . A meromorphic n -form $\omega \in \Gamma(X, \widehat{\Omega}_X^n(*D))$, with polar set contained in $D = D_0 \cup \dots \cup D_n$, defines a Čech cocycle in $\mathcal{C}^n(\mathcal{U}, \widehat{\Omega}_X^n)$. After passing to the direct limit we obtain a cohomology class $[\omega] \in H^n(X, \Omega_X^n)$. The Dolbeault isomorphism η from Proposition 4.5 (ii) assigns to $[\omega]$ a Dolbeault cohomology class $\eta(\omega) \in H_{\bar{\partial}}^{n,n}(X)$.

DEFINITION 4.11. The *global residue* of ω relative to the divisors D_0, \dots, D_n is given by

$$\operatorname{Res}(\omega) = \left(\frac{-1}{2\pi i} \right)^n \int_X \eta(\omega).$$

For a simplicial toric variety the global residue agrees with the toric residue. Indeed, we have already noted in (4.1) that for a V -manifold, our notion of holomorphic forms agrees with the Zariski differentials and, as shown in [C2, Proposition A.1]

$$\operatorname{Tr}_X([\omega]) = \left(\frac{-1}{2\pi i} \right)^n \int_X \eta(\omega).$$

Our next goal is to show that under very mild hypotheses, we can write the global residue as a sum of local residues. As above, let D_0, \dots, D_n be $n + 1$ reduced Weil

divisors in X with empty intersection, and assume that for some $k = 0, \dots, n$, the n -fold intersection

$$D_{\hat{k}} = D_0 \cap \dots \cap \widehat{D_k} \cap \dots \cap D_n$$

is finite. If $\omega \in \Gamma(X, \widehat{\Omega}_X^n(*D))$ and $x \in D_{\hat{k}}$ we can write, in a neighborhood of x ,

$$\omega = \frac{\omega'}{f_0 \dots f_n},$$

where, locally, ω' is holomorphic and D_i is the support of $\{f_i = 0\}$, f_i holomorphic. We will denote by $\text{Res}_{k,x}(\omega)$ the local Grothendieck residue

$$\text{Res}_{k,x}(\omega) = \text{Res}_x \left(\frac{\omega' / f_k}{f_0 \dots \widehat{f_k} \dots f_n} \right).$$

Note that $x \in D_{\hat{k}}$ implies that $f_k(x) \neq 0$.

THEOREM 4.12. *If D_0, \dots, D_n are reduced Weil divisors with empty intersection on a n -dimensional compact V -manifold X , then for any $\omega \in \Gamma(X, \widehat{\Omega}_X^n(*D))$, we have*

$$\text{Res}(\omega) = (-1)^k \sum_{x \in D_{\hat{k}}} \text{Res}_{k,x}(\omega),$$

whenever the intersection $D_{\hat{k}} = D_0 \cap \dots \cap \widehat{D_k} \cap \dots \cap D_n$ is finite.

Proof. There is no loss of generality in assuming $k = n$; the sign dependence is a consequence of the fact that the global residue is alternating on the order of the divisors.

The global residue $\text{Res}(\omega) = (-1/(2\pi i))^n \int_X \eta(\omega)$ uses the Dolbeault isomorphism η . However, by Proposition 4.5 (ii), we can also use the Dolbeault isomorphism ${}^l\eta$ for currents. Thus $\text{Res}(\omega)$ equals $(-1/(2\pi i))^n$ times the value on the constant function 1_X of any current representing the image under ${}^l\eta$ of the Čech cohomology class $[\omega]$. Hence, the theorem will follow from the following two assertions

- (i) ${}^l\eta([\omega])$ is the class of the current $RP_D[\omega] = R_{D_0, \dots, D_{n-1}} P_{D_n}[\omega]$.
- (ii) $RP_D[\omega](1_X) = (2\pi i)^n \sum_{x \in D_{\hat{n}}} \text{Res}_{n,x}(\omega)$.

Because of the definition of the Dolbeault isomorphism ${}^l\eta$, to prove (i), it suffices to construct, for each $i = 0, \dots, n-1$, a Čech cochain $\xi^{(i)} \in \mathcal{C}^i(\mathcal{U}, {}^l\mathcal{D}_X^{n, n-i-1})$ satisfying

- (a) $\delta \xi^{(n-1)} = I(\omega)$ (δ is the Čech coboundary).
- (b) $\bar{\partial} \xi^{(i)} = \delta \xi^{(i-1)}$ for all $i = 1, \dots, n-1$.

$$(c) \bar{\partial}\xi^{(0)} = R_{D_0, \dots, D_{n-1}} P_{D_n}[\omega].$$

We define

$$\xi_J^{(n-1)} = \begin{cases} RP_{D_0}[\omega] & \text{if } J = \{1, \dots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and, for any $i = 0, \dots, n-2$ and any $J \subset \{0, \dots, n\}$ with cardinality $i+1$,

$$\xi_J^{(i)} = \begin{cases} R_{D_0, \dots, D_{n-i-2}} P_{D_{n-i-1}}[\omega] & \text{if } J = \{n-i, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is understood that the above currents $RP_{D_0}[\omega]$ and $R_{D_0, \dots, D_{n-i-2}} P_{D_{n-i-1}}[\omega]$ are restricted to the appropriate open sets $U_J = \bigcap_{j \in J} U_j$. We will generally not indicate the restriction when it is irrelevant or clear from the context.

To verify (a), note that $\delta\xi^{(n-1)}$ is the cochain assigning to $U_0 \cap \dots \cap U_n$ the current $RP_{D_0}[\omega]$ restricted to this open set. Since D_0 is disjoint from $U_0 \cap \dots \cap U_n$, the definition of $RP_{D_0}[\omega]$ implies that it must agree with $I(\omega)$.

Suppose now that $1 \leq i \leq n-1$, then it follows from (4.7) that

$$(\bar{\partial}\xi^{(i)})_J = \begin{cases} R_{D_0, \dots, D_{n-i-1}}[\omega] & \text{if } J = \{n-i, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, clearly $(\delta\xi^{(i-1)})_J = 0$ if J is not an index set of the form $J = J_j = \{j, n-i+1, \dots, n\}$ for some $j = 1, \dots, n-i$. But, if $j < n-i$, then $(\delta\xi^{(i-1)})_{J_j}$ also vanishes – as a consequence of (4.6) – since it is the restriction to the open set $U_{J_j} \subset U_j$ of the current $R_{D_0, \dots, D_{n-i-1}} P_{D_{n-i}}[\omega]$ and $j < n-i$.

It remains to consider the case $J = \{n-i, \dots, n\}$. Then, $(\delta\xi^{(i-1)})_J$ is the restriction to U_J of $R_{D_0, \dots, D_{n-i-1}} P_{D_{n-i}}[\omega]$. But, since $U_J \subset U_{n-i}$, we deduce, again from (4.6), that

$$R_{D_0, \dots, D_{n-i-1}} P_{D_{n-i}}[\omega]|_{U_J} = R_{D_0, \dots, D_{n-i-1}}[\omega]|_{U_J}.$$

Thus, (b) is satisfied.

The final assertion (c) is proved in a similar way: the cochain $\bar{\partial}\xi^0$ assigns the zero current to the open sets U_j , $j < n$ and the residue current $R_{D_0, \dots, D_{n-1}}[\omega]$ to U_n . But, then, it follows from (4.6) that $\bar{\partial}\xi^0$ agrees with the global current $RP_D[\omega]$.

The verification of (ii) now reduces to the local formula (4.9). Indeed, since the support of the principal value $RP_D[\omega]$ is contained in the finite set $D_{\hat{n}} = D_0 \cap \dots \cap D_{n-1}$, its value on the constant function 1_X is the same as the value on any function ψ which is equal to one on a neighborhood of each of the points in $D_{\hat{n}}$. We may choose such a function ψ of the form $\psi = \sum_{x \in D_{\hat{n}}} \psi_x$, where ψ_x is equal to 1 in a neighborhood of x and the supports of the ψ_x 's are mutually disjoint and disjoint from D_n as well. Then

$$\begin{aligned}
RP_D[\omega](1_X) &= \sum_{x \in D_{\hat{n}}} RP_D[\omega]|_{U_n}(\psi_x) \\
&= \sum_{x \in D_{\hat{n}}} R_{D_0, \dots, D_{n-1}}[\omega]|_{U_n}(\psi_x),
\end{aligned}$$

where the last equality follows from (4.6). But now, (4.9) yields

$$RP_D[\omega](1_X) = (2\pi i)^n \sum_{x \in D_{\hat{n}}} \text{Res}_x(\omega). \quad \square$$

Remark 4.13. (i) The V -manifolds setting means that in the case of toric varieties we must restrict ourselves to the simplicial case. It is reasonable to expect, however, that Theorem 0.4 will hold for arbitrary complete toric varieties. This is the case, for example, if, with the notation of Theorem 0.4, the (finite) set $D_{\hat{k}}$ is contained in the torus T . It is then possible to reduce to the simplicial case by considering simplicial resolutions (cf. the proof of Theorem 4 in [CD]).

(ii) To understand why we need currents in the proof of Theorem 4.12, we will sketch a proof for the case $n = 2$ using forms rather than currents. The argument will be less than rigorous.

We have $D_0 \cap D_1 \cap D_2 = \emptyset$ in X . Let $T_j(\varepsilon)$ be a fundamental system of (open) tubular neighborhoods of D_j , and let $S_j(\varepsilon) = \partial T_j(\varepsilon)$ and $E_j(\varepsilon) = X - T_j(\varepsilon)$. Also, for i, j, k distinct indices from 0 to 2, consider the intersections $C_{ijk}(\varepsilon) = E_i(\varepsilon) \cap E_j(\varepsilon) \cap E_k(\varepsilon)$, $C_{ij}(\varepsilon) = E_i(\varepsilon) \cap E_j(\varepsilon) \cap S_k(\varepsilon)$ and $C_i(\varepsilon) = E_i(\varepsilon) \cap S_j(\varepsilon) \cap S_k(\varepsilon)$. We will assume that these sets are homology chains of (real) codimension 0, 1, and 2 respectively and that their boundaries behave as one would expect.

Next recall the procedure to define $\eta(\omega)$. Let $\{\sigma_0, \sigma_1, \sigma_2\}$ be a partition of unity subordinated to the covering \mathcal{U} . Then, beginning with $\omega \in \Gamma(U_0 \cap U_1 \cap U_2, \widehat{\Omega}_X^2)$, define $\xi_{ij} = (-1)^k \sigma_k \omega \in \Gamma(U_i \cap U_j, \mathcal{E}_X^{2,0})$, which implies $\omega = \delta(\xi_{ij}) = \xi_{12} - \xi_{02} + \xi_{01}$. Next, define $\xi_i = \pm \sigma_j \bar{\partial} \xi_{ij} \pm \sigma_k \bar{\partial} \xi_{ik}$, with the signs chosen so that $\bar{\partial} \xi_{ij} = \delta(\xi_i) = \xi_j - \xi_i$. Finally, $\eta(\omega)$ is defined to be the global $(2, 2)$ -form $\bar{\partial} \xi_i$ in U_i .

To compute the global residue $(2\pi i)^{-2} \int_X \eta(\omega)$, we first observe

$$\int_X \eta(\omega) = \lim_{\varepsilon \rightarrow 0} \int_{C_{012}(\varepsilon)} \eta(\omega) = \lim_{\varepsilon \rightarrow 0} \int_{C_{012}(\varepsilon)} \bar{\partial} \xi_0.$$

Since ξ_0 has bidegree $(2, 1)$, $d\xi_0 = \bar{\partial} \xi_0$, so we can apply Stokes' Theorem to write

$$\int_{C_{012}(\varepsilon)} \bar{\partial} \xi_0 = \int_{C_{12}(\varepsilon)} \xi_0 + \int_{C_{02}(\varepsilon)} \xi_0 + \int_{C_{01}(\varepsilon)} \xi_0.$$

On the other hand, $\xi_0 = \pm \sigma_1 \bar{\partial} \sigma_2 \wedge \omega \pm \sigma_2 \bar{\partial} \sigma_1 \wedge \omega$, and therefore $\xi_0 = 0$ in $S_1(\varepsilon)$ and $S_2(\varepsilon)$ for sufficiently small ε . Consequently,

$$\int_X \eta(\omega) = \lim_{\varepsilon \rightarrow 0} \int_{C_{12}(\varepsilon)} \xi_0 = \lim_{\varepsilon \rightarrow 0} \int_{C_{12}(\varepsilon)} \xi_1 - \bar{\partial} \xi_{01}.$$

Once again, $\int_{C_{12}(\varepsilon)} \xi_1$ vanishes for ε sufficiently small and, using Stokes' Theorem, we write

$$\int_X \eta(\omega) = \lim_{\varepsilon \rightarrow 0} \int_{C_2(\varepsilon)} \xi_{01} + \lim_{\varepsilon \rightarrow 0} \int_{C_1(\varepsilon)} \xi_{01}.$$

Since $\xi_{01} = \sigma_2 \omega$ vanishes on $S_2(\varepsilon)$ for ε sufficiently small, we have

$$\int_X \eta(\omega) = \lim_{\varepsilon \rightarrow 0} \int_{C_2(\varepsilon)} \sigma_2 \omega.$$

Finally, for ε sufficiently small, σ_2 is identically 1 in $S_0(\varepsilon) \cap S_1(\varepsilon)$, so that

$$\int_X \eta(\omega) = \lim_{\varepsilon \rightarrow 0} \int_{C_2(\varepsilon)} \omega = (2\pi i)^2 \sum_{x \in D_0 \cap D_1} \text{Res}_x(\omega),$$

which gives the desired formula for the global residue.

The use of residual currents in making the above argument rigorous is twofold: first of all, the local nature of the residual currents definition obviates the need to construct global cycles of integration – a step which is not always possible; moreover, the concept of admissible paths explains the passage to the limit necessary for the vanishing of the various integrals.

(iii) For an example of how $D_0, \dots, \widehat{D}_k, \dots, D_n$ can fail to satisfy the finiteness condition in Theorem 4.12, let $X = \mathbf{P}^1 \times \mathbf{P}^1$, and consider the divisors $D_0 = \{0\} \times \mathbf{P}^1$, $D_1 = (\{\infty\} \times \mathbf{P}^1) \cup (\mathbf{P}^1 \times \{\infty\})$ and $D_2 = (\{\infty\} \times \mathbf{P}^1) \cup (\mathbf{P}^1 \times \{0\})$. Then $D_0 \cap D_1 \cap D_2 = \emptyset$, yet $D_1 \cap D_2$ is infinite since it contains $\{\infty\} \times \mathbf{P}^1$.

In light of the last remark, it would be useful to know when the n -fold intersection $D_{\hat{k}} = D_0 \cap \dots \cap \widehat{D}_k \cap \dots \cap D_n$ is finite. Here is one criterion.

LEMMA 4.14. *Let D_0, \dots, D_n be reduced Weil divisors with empty intersection on a n -dimensional projective variety X . If D_k is the support of an ample divisor, then $D_{\hat{k}}$ is finite.*

Proof. This is immediate since $D_{\hat{k}} \cap D_k = \emptyset$ implies that $D_{\hat{k}}$ is a complete subvariety of the affine variety $X - D_k$. □

When applied to toric residues, these results yield Theorem 0.4 which, in turn, may be used to give new proofs of some basic results concerning toric residues. We will conclude this section with three such applications of Theorem 0.4:

Application 1: Res = ±1

The first application is an alternate proof of Proposition 2.4 when the toric variety X is simplicial. We resume the notation of Section 2, where we have a n -dimensional cone σ and the variables are labelled x_1, \dots, x_n (corresponding to the generators of σ) and z_1, \dots, z_r (corresponding to the other generators). Our goal is to prove that

$$\text{Res}\left(\frac{\Omega}{(z_1 \dots z_r) \cdot x_1 \dots x_n}\right) = \pm 1,$$

using Theorem 0.4. Since the divisors

$$D_0 = \{z_1 \dots z_r = 0\}, \quad D_i = \{x_i = 0\}, \quad i = 1, \dots, n,$$

have empty intersection and $D_1 \cap \dots \cap D_n = \{p\}$ is a single point, the hypotheses of Theorem 0.4 are satisfied, so that

$$\text{Res}\left(\frac{\Omega}{(z_1 \dots z_r) \cdot x_1 \dots x_n}\right) = \text{Res}_p(\omega_\sigma),$$

where ω_σ is the restriction of $\Omega/((z_1 \dots z_r) \cdot x_1 \dots x_n)$ to the affine open set $X_\sigma \subset X$. But we have seen that $X_\sigma \simeq \mathbf{C}^n/G(\sigma)$, and since $G(\sigma)$ is a small subgroup, this defines a standard model. Moreover, as noted in (2.9), the pullback to \mathbf{C}^n of ω_σ is given by

$$\tilde{\omega}_\sigma = \pm \frac{|G(\sigma)| \, dx_1 \wedge \dots \wedge dx_n}{x_1 \dots x_n}.$$

Therefore

$$\begin{aligned} \text{Res}_p(\omega_\sigma) &= \frac{1}{|G(\sigma)|} \text{Res}_0(\tilde{\omega}_\sigma) \\ &= \pm \text{Res}_0\left(\frac{dx_1 \wedge \dots \wedge dx_n}{x_1 \dots x_n}\right) = \pm 1, \end{aligned}$$

which proves Proposition 2.4 when X is simplicial.

Application 2: Sums of residues in a torus

Let f_1, \dots, f_n be n -variate Laurent polynomials with a finite set of common zeroes $Z = Z(f_1, \dots, f_n)$ in the torus $T = (\mathbf{C}^*)^n$. Given a Laurent polynomial q , we get the differential form

$$\phi = \frac{q}{f_1 \dots f_n} \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_n}{t_n}.$$

The operator which assigns to q the sum of local residues $\sum_{x \in Z} \text{Res}_x(\phi)$ has interesting applications in a number of different contexts. In certain cases, it is possible to use Theorem 0.4 to give a global interpretation of this sum.

We assume that there exists a simplicial toric compactification X of T such that if D_i is the closure in X of the hypersurface $\{f_i = 0\} \subset (\mathbf{C}^*)^n$ and $D_0 = X - T$ is the ‘divisor at infinity’, then

$$D_0 \cap D_1 \cap \cdots \cap D_n = \emptyset.$$

Such a (smooth) compactification exists, for example, if the polynomials f_i are nondegenerate in the sense of Khovanskii [K1].

In this situation, the meromorphic form ϕ has an extension to X which can be written as

$$\Phi = \frac{Q \Omega}{F_0 \cdots F_n},$$

where Q, F_0, \dots, F_n are homogeneous polynomials in the coordinate ring of X such that $D_i = \{F_i = 0\}$, and Ω is the Euler form of X . Then it follows from Theorem 0.4 that

$$\sum_{x \in Z} \text{Res}_x(\phi) = \text{Res}_F(Q).$$

If we assume, in addition, that the Newton polyhedron of q is contained in the interior of the Minkowski sum of the Newton polyhedra corresponding to f_1, \dots, f_n , then one may show that Q is a multiple of F_0 and hence $\text{Res}_F(Q) = 0$ which gives the classical Euler-Jacobi Theorem in this setting [K2]. In fact, as in this case Φ has poles only on the union of the n divisors $D_1 \cup \cdots \cup D_n$, whose intersection is contained in the torus, the vanishing of the sum of the local residues of ϕ follows directly from the result of Griffiths recalled in Remark 4.10. This global approach to residues in the torus is the subject of [CD].

Application 3: Toric Jacobians.

For our third application, we use Theorem 0.4 to give an alternate proof of Theorem 5.1 (ii) of [C2] for a simplicial toric variety. This result asserts that the *toric Jacobian* J of $F_0, \dots, F_n \in S_\alpha$ (as defined in [C2, Proposition 4.1]) has nonzero toric residue. More precisely, if α is ample and the F_i don’t vanish simultaneously on X , then we will show that the equality

$$\text{Res}_F(J) = (D^n)$$

follows from Theorem 0.4. Here, (D^n) is the n -fold intersection number of any divisor D with $[D] = \alpha$. Note that $J \in S_\rho$, where $\rho = (n + 1)\alpha - \beta$ is the critical degree for the F_i .

To prove this, let $\omega_F(J) = J\Omega/(F_0 \dots F_n)$. Then Theorem 0.4 implies

$$\begin{aligned} \text{Res}_F(J) &= \sum_{x \in D_0} \text{Res}_{0,x}(\omega_F(J)) \\ &= \sum_{x \in D_1 \cap \dots \cap D_n} \text{Res}_x \left(\frac{(J/F_0)\Omega}{F_1 \dots F_n} \right). \end{aligned} \quad (4.15)$$

We will show that each local residue $\text{Res}_{0,x}(\omega_F(J))$ is a local intersection multiplicity of D_1, \dots, D_n at x , which will prove that $\text{Res}_F(J)$ is the intersection number $(D_1 \dots D_n) = (D^n)$.

Given $x \in D_0$, let σ be a n -dimensional cone such that x lies in the affine open set X_σ . Since σ is simplicial, we write the variables as $x_1, \dots, x_n, z_1, \dots, z_r$. Then the form $\omega_F(J)$, restricted to X_σ , may be written in appropriate coordinates as

$$\frac{k(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n}{f_0 \dots f_n},$$

where

$$k(x_1, \dots, x_n) = \det \begin{pmatrix} f_0 & \dots & f_n \\ \partial f_0 / \partial x_1 & \dots & \partial f_n / \partial x_1 \\ \vdots & & \vdots \\ \partial f_0 / \partial x_n & \dots & \partial f_n / \partial x_n \end{pmatrix}$$

and $f_i(x_1, \dots, x_n)$ is the function obtained from $F_i(x_1, \dots, x_n, z_1, \dots, z_r)$ by setting $z_j = 1$ for $j = 1, \dots, r$. It follows that

$$\text{Res}_{0,x}(\omega_F(J)) = \text{Res}_x \left(\frac{(k/f_0) dx_1 \wedge \dots \wedge dx_n}{f_1 \dots f_n} \right).$$

However, expanding the determinant for k along the first row and using $f_0(x) \neq 0$, we see that

$$k/f_0 \equiv \det(\partial f_i / \partial x_j; 1 \leq i, j \leq n) \pmod{\langle f_1, \dots, f_n \rangle},$$

in the local ring $\mathcal{O}_{X,x}$. Consequently,

$$\text{Res}_{0,x}(\omega_F(J)) = \text{Res}_x \left(\frac{\det(\partial f_i / \partial x_j) dx_1 \wedge \dots \wedge dx_n}{f_1 \dots f_n} \right),$$

and this last residue equals the local intersection multiplicity of D_1, \dots, D_n at x (this is well-known in the smooth case and is easy to prove for V -manifolds). By (4.15), it follows that the toric residue of the toric jacobian equals the intersection number $(D_1 \dots D_n)$.

Remark 4.16. (i) Since all the divisors D_i have the same degree, we can write $\text{Res}_F(J) = (D_0 \dots \widehat{D}_k \dots D_n)$ for any $k = 0, \dots, n$.

(ii) The intersection number $(D_1 \dots D_n)$ can also be interpreted as the degree of the map $F = (F_0, \dots, F_n): X \rightarrow \mathbf{P}^n$ (see [C2, Theorem 5.1] for a careful proof). Thus the toric Jacobian has the property that its toric residue is given by $\text{Res}_F(J) = \deg(F)$. This will be useful in Section 5.

5. Toric residues as point residues in the equal degree case

When $X = \mathbf{P}^n$ and F_0, \dots, F_n all have the same degree, the toric residue $\text{Res}_F(H)$ equals the classical Grothendieck residue at $0 \in \mathbf{C}^{n+1}$, i.e.,

$$\text{Res}_F(H) = \frac{1}{(2\pi i)^{n+1}} \int_{|F_i|=\varepsilon} \frac{H dx_0 \wedge \dots \wedge dx_n}{F_0 \dots F_n},$$

(see [PS, 12.10]). Thus, in the projective case, the toric residue is a point residue computed on the related space \mathbf{C}^{n+1} . In Theorem 5.8 below, we will generalize this result to a complete simplicial toric variety X , assuming that the F_i have the *same* degree α in the homogeneous coordinate ring S .

We first describe the space we will use for computing toric residues on X . Given $\alpha \in A_{n-1}(X)$, let $S_{*\alpha} = \bigoplus_{k \geq 0} S_{k\alpha}$ and set

$$X_\alpha = \text{Spec}(S_{*\alpha}).$$

Note that the natural grading of $S_{*\alpha}$ induces a \mathbf{C}^* action on X_α .

PROPOSITION 5.1. *If X is a complete simplicial toric variety and α is ample, then X_α has the natural structure of an affine toric variety. Furthermore, if $0 \in X_\alpha$ is the unique fixed point of the torus action, then $X_\alpha - \{0\}$ is simplicial and \mathbf{C}^* acts on $X_\alpha - \{0\}$ with finite stabilizers and X as geometric quotient.*

Proof. Consider $\mathbf{R} \oplus N_{\mathbf{R}}$ with the lattice $\mathbf{Z} \oplus N$. Elements of $\mathbf{R} \oplus N_{\mathbf{R}}$ will be written $\lambda e_0 + v$, where $\lambda \in \mathbf{R}$ and $v \in N_{\mathbf{R}}$. Now let $D = \sum_i a_i D_i$ (where \sum_i denotes $\sum_{i=1}^{n+r}$) be a divisor on X whose class is α , and let $\psi: N_{\mathbf{R}} \rightarrow \mathbf{R}$ be its support function. This means $\psi(\eta_i) = -a_i$, where the η_i generate the 1-dimensional cones of the fan of X . Given this data, let $\tilde{\sigma} \subset \mathbf{R} \oplus N_{\mathbf{R}}$ be the cone generated by the vectors $\tilde{\eta}_i = a_i e_0 + \eta_i$. Equivalently, $\tilde{\sigma}$ is generated by the graph of $-\psi$ in $\mathbf{R} \oplus N_{\mathbf{R}}$. Since ψ is strictly upper convex (D is ample), we see that $\tilde{\sigma}$ is a strongly convex rational polyhedral cone.

We next observe that the semigroup ring $\mathbf{C}[\tilde{\sigma}^\vee \cap (\mathbf{Z} \oplus M)]$ is naturally isomorphic to $S_{*\alpha}$. To prove this, first note that

$$\begin{aligned} ke_0 + m \in \tilde{\sigma}^\vee \cap (\mathbf{Z} \oplus M) &\iff \langle ke_0 + m, \tilde{\eta}_i \rangle \geq 0 \text{ for all } i \\ &\iff \langle m, \eta_i \rangle + k a_i \geq 0 \text{ for all } i \\ &\iff \prod_i x_i^{\langle m, \eta_i \rangle + k a_i} \in S_{k\alpha}, \end{aligned}$$

(where Π_i denotes $\prod_{i=1}^{n+r}$). Since all monomials in $S_{k\alpha}$ can be described in this form (see Sect. 1 of [C1]), the observation follows easily. Thus X_α is an affine toric variety.

The torus action on X_α has a unique fixed point which we denote by 0 . Furthermore, the complement $X_\alpha - \{0\}$ is the toric variety whose fan is the boundary of $\tilde{\sigma}$. This fan is the graph of $-\psi$, so the strict convexity of ψ implies that under the projection $\pi : \mathbf{R} \oplus N_{\mathbf{R}} \rightarrow N_{\mathbf{R}}$, each cone of the boundary fan maps naturally to the corresponding cone in the fan of X . Thus the projection π induces a map of toric varieties $X_\alpha - \{0\} \rightarrow X$. We leave to the reader the straightforward proof that $X_\alpha - \{0\}$ is simplicial since X is.

Since X is simplicial, we can write

$$X = (\mathbf{C}^{n+r} - Z)/G, \quad (5.2)$$

where $G = \text{Hom}_{\mathbf{Z}}(A_{n-1}(X), \mathbf{C}^*)$ and the exceptional set Z is a union of coordinate subspaces determined by the fan of X (see [BC, Theorem 1.9]). The correspondence $\eta_i \leftrightarrow \tilde{\eta}_i$ implies that X , X_α and $X_\alpha - \{0\}$ have the same homogeneous coordinate ring (though the gradings may differ), and the map $X_\alpha - \{0\} \rightarrow X$ shows that the fans of X and $X_\alpha - \{0\}$ are combinatorially equivalent. Thus X and $X_\alpha - \{0\}$ have the same exceptional set Z . Hence

$$X_\alpha - \{0\} = (\mathbf{C}^{n+r} - Z)/H, \quad (5.3)$$

where $H = \text{Hom}_{\mathbf{Z}}(A_n(X_\alpha - \{0\}), \mathbf{C}^*)$. To compare G and H , we use the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathbf{Z}^{n+r} & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z} \oplus M & \longrightarrow & \mathbf{Z}^{n+r} & \longrightarrow & A_n(X_\alpha - \{0\}) & \longrightarrow & 0 \end{array}$$

to conclude that we have an exact sequence

$$0 \rightarrow \mathbf{Z} \rightarrow A_{n-1}(X) \rightarrow A_n(X_\alpha - \{0\}) \rightarrow 0,$$

where $1 \in \mathbf{Z}$ maps to $\alpha \in A_{n-1}(X)$. Applying $\text{Hom}_{\mathbf{Z}}(-, \mathbf{C}^*)$, we can identify H with the subgroup $\{g \in G : g(\alpha) = 1\} \subset G$, so that $g \mapsto g(\alpha)$ induces an isomorphism $G/H \simeq \mathbf{C}^*$.

Comparing (5.2) and (5.3), X is the quotient of $X_\alpha - \{0\}$ by $G/H \simeq \mathbf{C}^*$. Furthermore, the proof of Theorem 1.9 of [BC] shows that the G -action in (5.2) has finite stabilizers, and it follows that the \mathbf{C}^* -action on X_α must also have finite stabilizers. To describe this action more explicitly, note that G acts on $S_{k\alpha}$ by

$g \cdot F = g(k\alpha) F = g(\alpha)^k F$. Since H acts trivially by definition, the action of $G/H \simeq \mathbf{C}^*$ is exactly the action that gives the grading of $S_{*\alpha}$. This completes the proof of the proposition. \square

Remarks 5.4 (i) When α is very ample (always true when X is smooth), then X_α is the affine cone of X in the projective embedding given by α .

(ii) Besides being a geometric quotient, the map $X_\alpha - \{0\} \rightarrow X$ is a *combinatorial quotient* in the sense of [KSZ, p. 645].

(iii) If we add the 1-dimensional cone generated by e_0 to $\tilde{\sigma}$ and subdivide accordingly, we get a toric variety \tilde{X}_α which maps naturally to X . In [R, Sect. 3], it is proved that $\tilde{X}_\alpha \rightarrow X$ is the total space of the line bundle $\mathcal{O}_X(-\alpha)$. Thus $\tilde{X}_\alpha \rightarrow X_\alpha$ is a blow-up of $0 \in X_\alpha$ with exceptional fiber isomorphic to X . Conversely, we can view X_α as the variety obtained by blowing down the zero section of $\mathcal{O}_X(-\alpha)$.

(iv) Although $X_\alpha - \{0\}$ is simplicial, $0 \in X_\alpha$ can be very singular. For example, let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $\alpha = (1, 1)$. The coordinate ring for X is $S = \mathbf{C}[x, y, z, t]$, where $\deg(x) = \deg(y) = (1, 0)$ and $\deg(z) = \deg(t) = (0, 1)$. Then X_α is the singular affine hypersurface defined by $AD - BC = 0$ in \mathbf{C}^4 since this hypersurface is the affine cone over the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. Note that X_α is not simplicial at the origin.

We next discuss differential forms on X and X_α . As we saw in the proof of Proposition 5.1, X and X_α have the same homogenous coordinate ring (though graded differently). By (2.8), X has the Euler form

$$\Omega = \sum_{|I|=n} \det(\eta_I) \hat{x}_I dx_I.$$

Now let $\sum_i a_i D_i$ be a divisor in the class of α and consider the $(n + 1)$ -form

$$\Omega_\alpha = \left(\sum_i a_i \frac{dx_i}{x_i} \right) \wedge \Omega.$$

LEMMA 5.5. *Let $\beta = \sum_i \deg(x_i) \in A_{n-1}(X)$ and $\rho = (n + 1)\alpha - \beta$. Then*

(i) Ω_α is the Euler form of X_α .

(ii) If θ is any Euler vector field for X (which can be regarded as a map $\theta : A_{n-1}(X) \rightarrow \mathbf{C}$), we have

$$\theta \lrcorner \Omega_\alpha = \theta(\alpha) \Omega.$$

(iii) If $J \in S_\rho$ is the toric Jacobian of $F_0, \dots, F_n \in S_\alpha$ (see [C2, Sect. 4]), then

$$J \Omega_\alpha = dF_0 \wedge \dots \wedge dF_n.$$

Proof. To define Ω , we used a basis m_1, \dots, m_n of M . Then e_0 and m_j for $j > 0$ form a basis of $\mathbf{Z} \oplus M$, and from the proof of Proposition 9.5 of [BC] (which is easily seen to hold in the non-simplicial case), we see that the Euler form of X_α is

$$x_1 \dots x_{n+r} \left(\frac{dt_0}{t_0} \wedge \dots \wedge \frac{dt_n}{t_n} \right), \quad (5.6)$$

where $t_0 = \prod_i x_i^{\langle e_0, \tilde{\eta}_i \rangle} = \prod_i x_i^{a_i}$ and $t_j = \prod_i x_i^{\langle m_j, \tilde{\eta}_i \rangle} = \prod_i x_i^{m_j \cdot \eta_i}$ for $j > 0$. Since $dt_0/t_0 = \sum_i a_i dx_i/x_i$ and $\Omega = x_1 \dots x_{n+r} (dt_1/t_1 \wedge \dots \wedge dt_n/t_n)$ (also by Proposition 9.5 of [BC]), we see that Ω_α is the Euler form of X_α .

For the second part of the lemma, first note that $\theta \lrcorner \Omega = 0$ by Lemma 6.2 of [C2]. Thus

$$\begin{aligned} \theta \lrcorner \Omega_\alpha &= \theta \lrcorner \left(\left(\sum_i a_i \frac{dx_i}{x_i} \right) \wedge \Omega \right) \\ &= \left(\theta \lrcorner \sum_i a_i \frac{dx_i}{x_i} \right) \cdot \Omega. \end{aligned}$$

However, if $\theta = \sum_i b_i x_i \partial / \partial x_i$, then $\theta \lrcorner \sum_i a_i dx_i/x_i = \sum_i a_i b_i = \theta(\alpha)$, which gives the desired formula. (For more background on Euler vector fields, see 3.8–3.10 of [BC].)

Turning to the final part of the lemma, note that each F_i lies in S_α and hence gives a function on $X_\alpha = \text{Spec}(S_{*\alpha})$. Further, the functions t_0, \dots, t_n introduced above are coordinates on the torus $T_{X_\alpha} \subset X_\alpha$. Thus, if we restrict F_i to the torus, we can write $F_i = \tilde{F}_i(t_0, \dots, t_n)$. Then

$$dF_0 \wedge \dots \wedge dF_n = \det(\partial \tilde{F}_i / \partial t_j) dt_0 \wedge \dots \wedge dt_n.$$

Comparing this to the formula (5.6) for Ω_α , we see that

$$dF_0 \wedge \dots \wedge dF_n = \tilde{J} \Omega_\alpha$$

for some rational function \tilde{J} .

It remains to show that \tilde{J} is the toric Jacobian J from [C2]. Pick an Euler form θ such that $\theta(\alpha) \neq 0$. We can find such a θ since α is ample and hence has infinite order in $A_{n-1}(X)$ (see also Lemma 10.5 of [BC]). Then, by (ii) and the above equation for \tilde{J} ,

$$\begin{aligned} \theta(\alpha) \cdot \tilde{J} \Omega &= \theta \lrcorner (\tilde{J} \Omega_\alpha) \\ &= \theta \lrcorner (dF_0 \wedge \dots \wedge dF_n) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^n (-1)^i (\theta \lrcorner dF_i) dF_0 \wedge \cdots \wedge \widehat{dF_i} \wedge \cdots \wedge dF_n \\
&= \theta(\alpha) \cdot \sum_{i=0}^n (-1)^i F_i dF_0 \wedge \cdots \wedge \widehat{dF_i} \wedge \cdots \wedge dF_n
\end{aligned}$$

where the last equality follows because $\theta \lrcorner dF = \theta(\alpha) F$ for all $F \in S_\alpha$. However, on the bottom row, the expression on the right equals $\theta(\alpha) \cdot J \Omega$ by [C2]. Then $\tilde{J} = J$ follows since $\theta(\alpha) \neq 0$, and (iii) is proved. \square

Given $F_0, \dots, F_n \in S_\alpha$, we next consider the integral on X_α

$$\int_{\{|F_i|=\varepsilon, 0 \leq i \leq n\}} \frac{H \Omega_\alpha}{F_0 \cdots F_n}, \tag{5.7}$$

where $\varepsilon > 0$, the cycle $\{|F_i| = \varepsilon, 0 \leq i \leq n\}$ is oriented using $d(\arg F_0) \wedge \cdots \wedge d(\arg F_n)$, and $H \in S_\rho$ for $\rho = (n+1)\alpha - \beta$. To make sense of (5.7), first note that $H \Omega_\alpha / (F_0 \cdots F_n)$ is a meromorphic form on the V -manifold $X_\alpha - \{0\}$. Furthermore, each F_i is a polynomial function on X_α and $\{|F_i| = \varepsilon, 0 \leq i \leq n\} \subset X_\alpha - \{0\}$. It follows that (5.7) exists whenever $(F_0, \dots, F_n): X_\alpha \rightarrow \mathbf{C}^{n+1}$ is finite. We can now state the main result of this Section.

THEOREM 5.8. *Assume that X is complete and simplicial, α is ample, and $F_0, \dots, F_n \in S_\alpha$ don't vanish simultaneously on X . Then*

- (i) *The map $(F_0, \dots, F_n): X_\alpha \rightarrow \mathbf{C}^{n+1}$ is finite.*
- (ii) *If $\rho = (n+1)\alpha - \beta$ is the critical degree of F_0, \dots, F_n , then for every $H \in S_\rho$,*

$$\text{Res}_F(H) = \frac{1}{(2\pi i)^{n+1}} \int_{\{|F_i|=\varepsilon, 0 \leq i \leq n\}} \frac{H \Omega_\alpha}{F_0 \cdots F_n}.$$

Proof. By [C2, Proposition 3.2], we know that $S_{*\alpha} / \langle F_0, \dots, F_n \rangle$ has finite dimension over \mathbf{C} , so that by definition, F_0, \dots, F_n is a homogeneous system of parameters for $S_{*\alpha}$. It follows from [BH, Theorem 1.5.17] that $S_{*\alpha}$ is finitely generated as a module over the subring $\mathbf{C}[F_0, \dots, F_n]$. Thus $\tilde{F} = (F_0, \dots, F_n): X_\alpha \rightarrow \mathbf{C}^{n+1}$ is finite, which proves (i).

To prove (ii), we first observe that each side of the identity in (ii) vanishes when $H \in \langle F_0, \dots, F_n \rangle$. This is obviously true for the toric residue, and for the integral (5.7), one uses the usual argument (see [GH, pp. 650–651]). Since we know $S_\rho / \langle F_0, \dots, F_n \rangle_\rho$ is one dimensional and the toric Jacobian J has nonzero toric residue (see Sect. 4), it suffices to check that (ii) holds for $H = J$.

By Remark 4.16, we know that $\text{Res}_F(J) = \text{deg}(F)$, where $F = (F_0, \dots, F_n)$, regarded as a map $F: X \rightarrow \mathbf{P}^n$. On the other hand, by Lemma 5.5, we have

$$\frac{J \Omega_\alpha}{F_0 \dots F_n} = \frac{dF_0 \wedge \dots \wedge dF_n}{F_0 \dots F_n} = \tilde{F}^* \left(\frac{dz_0 \wedge \dots \wedge dz_n}{z_0 \dots z_n} \right),$$

where z_0, \dots, z_n are coordinates on \mathbf{C}^{n+1} and $\tilde{F} = (F_0, \dots, F_n)$, now regarded as a map $\tilde{F}: X_\alpha \rightarrow \mathbf{C}^{n+1}$. It follows that

$$\begin{aligned} & \frac{1}{(2\pi i)^{n+1}} \int_{\{|F_i|=\varepsilon, 0 \leq i \leq n\}} \frac{J \Omega_\alpha}{F_0 \dots F_n} \\ &= \frac{\text{deg}(\tilde{F})}{(2\pi i)^{n+1}} \int_{\{|z_i|=\varepsilon, 0 \leq i \leq n\}} \frac{dz_0 \wedge \dots \wedge dz_n}{z_0 \dots z_n} = \text{deg}(\tilde{F}), \end{aligned}$$

since \tilde{F} is finite by (i).

Thus, to prove (ii) for J , we must show that $\text{deg}(F) = \text{deg}(\tilde{F})$. However, as noted in the proof of Proposition 5.1, the $\mathbf{C}^* \simeq G/H$ action on X_α satisfies $g \cdot F_i = g(\alpha) F_i$ for $g \in G$. It follows that $\tilde{F}: X_\alpha - \{0\} \rightarrow \mathbf{C}^{n+1} - \{0\}$ is equivariant with respect to \mathbf{C}^* , and since the quotient is $F: X \rightarrow \mathbf{P}^n$, one easily sees that F and \tilde{F} have the same degree. This completes the proof of the theorem. \square

Remarks 5.9 (i) Notice that in general, the integral (5.7) is slightly different from the Grothendieck residue defined in (4.8). This is because X_α need not be simplicial at the point $0 \in X_\alpha$.

(ii) When $X = \mathbf{P}^n$ and F_0, \dots, F_n are homogeneous of degree d , note that the residue of Theorem 5.8 is computed *not* on \mathbf{C}^{n+1} , but rather on $X_d = \text{Spec}(\oplus_{k \geq 0} \mathbf{C}[x_0, \dots, x_n]_k d)$, which is the quotient of \mathbf{C}^{n+1} by the diagonal action of the d th roots of unity μ_d . Furthermore, one can show that the Euler form of X_d is $\Omega_d = d dx_0 \wedge \dots \wedge dx_n$.

Since X_d is simplicial at the origin, the local residue $\text{Res}_{0 \in X_d}(\omega_F(H))$ is defined, and combining Theorem 5.8 and equation (4.8), we see that

$$\begin{aligned} \text{Res}_F(H) &= \text{Res}_{0 \in X_d} \left(\frac{H \Omega_d}{F_0 \dots F_n} \right) \\ &= \text{Res}_{0 \in \mathbf{C}^{n+1}} \left(\frac{H dx_0 \wedge \dots \wedge dx_n}{F_0 \dots F_n} \right). \end{aligned}$$

Thus the toric residue equals both of the local residues that can be defined in this situation, and Theorem 5.8 gives the toric generalization of the first of these equalities.

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