

## Mixed Lefschetz Theorems and Hodge–Riemann Bilinear Relations

**Eduardo Cattani**

Department of Mathematics and Statistics, University of Massachusetts,  
Amherst, MA 01003, USA

*Correspondence to be sent to: [cattani@math.umass.edu](mailto:cattani@math.umass.edu)*

The Hard Lefschetz Theorem (HLT) and the Hodge–Riemann bilinear relations (HRR) hold in various contexts: they impose restrictions on the cohomology algebra of a smooth compact Kähler manifold; they restrict the local monodromy of a polarized variation of Hodge structure; they impose conditions on the  $f$ -vectors of convex polytopes. While the statements of these theorems depend on the choice of a Kähler class, or its analog, there is usually a cone of possible choices. It is then natural to ask whether the HLT and HRR remain true in a mixed context. In this note, we present a unified approach to proving the mixed HLT and HRR, generalizing the known results, and proving it in new cases, such as the intersection cohomology of nonrational polytopes.

### 1 Introduction

The cohomology of a smooth compact Kähler manifold  $X$  is constrained by the existence of a Hodge decomposition in each degree

$$H^d(X, \mathbb{C}) = \bigoplus_{p+q=d} H^{p,q}(X); \quad \overline{H^{p,q}(X)} = H^{q,p}(X), \quad (1.1)$$

Received July 10, 2007; Revised February 17, 2008; Accepted February 18, 2008

© The Author 2008. Published by Oxford University Press. All rights reserved. For permissions, please e-mail: [journals.permissions@oxfordjournals.org](mailto:journals.permissions@oxfordjournals.org).

where, in de Rham terms,  $H^{p,q}(X)$  may be characterized as those cohomology classes with a representative of bidegree  $(p, q)$ , and by the existence of a polarized Lefschetz action on the total cohomology space  $H^*(X, \mathbb{C})$ . The latter structure is encoded in the Hard Lefschetz Theorem (HLT) and the Hodge–Riemann bilinear relations (HRR) (see, for example, [17]).

**Theorem 1.1 (HLT).** Let  $X$  be a smooth, compact,  $k$ -dimensional Kähler manifold and let  $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$  be a Kähler class. Let  $L_\omega \in \text{End}(H^*(X, \mathbb{C}))$  denote multiplication by  $\omega$ . Then, for each  $m$ , such that  $0 \leq m \leq k$ , the map

$$L_\omega^m: H^{k-m}(X, \mathbb{C}) \rightarrow H^{k+m}(X, \mathbb{C}) \quad (1.2)$$

is an isomorphism. □

**Theorem 1.2 (HRR).** Let  $X$  be a smooth, compact,  $k$ -dimensional Kähler manifold and let  $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$  be a Kähler class. Define a real bilinear form  $Q$  on  $H^*(X, \mathbb{C})$  by

$$Q(\alpha, \beta) = (-1)^{\frac{(k-d)(k-d-1)}{2}} \int_X \alpha \cup \beta, \quad (1.3)$$

where  $\deg(\alpha) = d$  and the integral is assumed to be zero if  $\deg(\alpha \cup \beta) \neq 2k$ . Then

$$i^{p-q} Q(\alpha, L_\omega^m \bar{\alpha}) \geq 0 \quad (1.4)$$

for any

$$\alpha \in H^{p,q}(X) \cap \ker(L_\omega^{m+1}); \quad p + q = k - m.$$

Moreover, equality holds if and only if  $\alpha = 0$ . □

Similar statements hold for the action of the local monodromy on the general fiber of a local system underlying a polarized variation of Hodge structure of weight  $k$  [8, 25]. In the context of mirror symmetry, this statement may be viewed as dual to that for the cohomology of a smooth, compact, Kähler manifold [7].

In another direction, the relation between algebraic geometry and the combinatorics of polytopes established by toric geometry, allows us to prove Stanley’s conjecture for simple polytopes as a consequence of HLT and HRR for toric varieties and to deduce

the Alexandrov–Fenchel inequality for the mixed volume of polytopes, as well as other similar properties, from the Hodge index theorem, which is also a consequence of HLT and HRR [20, 26, 28, 29]. Combinatorial proofs of the generalized Stanley conjecture for arbitrary convex polytopes are then obtained through a generalization of the HLT and HRR to convex polytopes and to the intersection cohomology of the associated projective toric varieties. An explicit construction for the simplicial case is due to Timorin [31] (see also [32]), while the general case was first obtained by Karu [19] (see also [1–6]).

The structures described by the Hard Lefschetz Theorem and the Hodge–Riemann bilinear relations have been codified in different settings and with different names appropriate to the various contexts: polarized mixed Hodge structures split over  $\mathbb{R}$  [11], Lefschetz modules [21], Frobenius modules [7], and polarized Hodge structures on cohomology algebras [33]. They involve the choice of a Kähler class in the classical algebro-geometric situation or an appropriate  $\mathfrak{sl}_2$ -action in the variation of Hodge structure or combinatorial settings; this choice takes place in an open cone defined by the Lefschetz property (1.2) and the positivity condition (1.4). The need to consider the action of a family of  $\mathfrak{sl}_2$ 's first arose in connection with the study of the asymptotics of variations of Hodge structure [10] and of  $L^2$  and intersection cohomologies with values in a variation of polarized Hodge structure [11]. In the latter work, it was also pointed out that the Descent Lemma [11, Lemma 1.16] and the Purity Theorem [11, Corollary 1.13] had implications for mixed Lefschetz actions on the cohomology of smooth compact Kähler manifolds. Subsequently, Gromov [18] explicitly stated mixed Hodge–Riemann bilinear relations and proved them in special cases. In the algebro-geometric case, these mixed theorems may be stated as follows.

**Theorem 1.3 (Mixed HLT).** Let  $X$  be a smooth, compact,  $k$ -dimensional Kähler manifold and suppose that  $\omega_1, \dots, \omega_m \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ ,  $0 \leq m \leq k$ , are Kähler classes. For each  $j = 1, \dots, m$ , let  $L_{\omega_j} \in \text{End}(H^*(X, \mathbb{C}))$  denote multiplication by  $\omega_j$ . Then the map

$$L_{\omega_1} \cdots L_{\omega_m}: H^{k-m}(X, \mathbb{C}) \rightarrow H^{k+m}(X, \mathbb{C})$$

is an isomorphism. □

**Theorem 1.4 (Mixed HRR).** Let  $X$  be a smooth, compact,  $k$ -dimensional Kähler manifold. Suppose that  $m \leq k - 2$  and that  $\omega_1, \dots, \omega_m, \omega_{m+1} \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$  are Kähler classes.

As before, let  $Q$  denote the intersection form (1.3). Then if

$$\alpha \in H^{p,q}(X) \cap \ker(L_{\omega_1} \cdots L_{\omega_m} L_{\omega_{m+1}}); \quad p + q = k - m,$$

we have

$$i^{p-q} Q(\alpha, L_{\omega_1} \cdots L_{\omega_m} \bar{\alpha}) \geq 0$$

and equality holds if and only if  $\alpha = 0$ . □

In [18, Section 2.3.A], Gromov proved the above theorem in the case  $p = q = 1$  as a form of the Alexandrov–Fenchel inequality for mixed volumes and stated it in the general  $p = q$  case. Timorin [30] proved the above mixed theorems in the linear algebraic context, i.e. the cohomology algebra of a torus and, more recently, Dinh and Nguyen [13] proved Theorems 1.3 and 1.4 for arbitrary smooth compact Kähler manifolds. In another direction, Timorin [31] indicated how the mixed HLT and HRR could be obtained in the case of simple polytopes.

The purpose of this note is to show that when put in the context of polarized mixed Hodge structures split over the reals, here renamed *polarized Hodge–Lefschetz modules* to conform with more recent nomenclature, the mixed theorems stated above are an easy consequence of the Descent Lemma [11, Lemma 1.16]. It should be noted, however, that the proof of the Descent Lemma makes use of the deep relationship between polarized Hodge–Lefschetz modules and variations of Hodge structure. The advantage of this approach lies, however, in the fact that the notion of polarized Hodge–Lefschetz modules encompasses all cases where the HLT and HRR hold, and consequently one obtains a unified proof of the known mixed versions as well as proofs, in cases such as nonrational polytopes, where the mixed versions had not yet been proved.

This note is organized as follows: in Section 2, we define polarized Hodge–Lefschetz modules and explain how this notion encodes the structure in the cohomology of smooth, compact Kähler manifold and in the combinatorial intersection cohomology of polytopes. In Section 3, we recall the basic results about polarized mixed Hodge structures and variations of Hodge structure, and state the Descent Lemma which is the key result for our purposes. Finally, in Section 4 we show how, in the context of polarized Hodge–Lefschetz modules the mixed HLT and HRR are immediate consequences of the Descent Lemma.

## 2 Polarized Hodge–Lefschetz Modules

In this section, we describe the abstract setting which encodes HLT and HRR. We have chosen as the core, a notion akin to that of Lefschetz modules [21], although other similar objects could have been used.

**Definition 2.1.** Let  $V = V_*$  be a  $\mathbb{Z}$ -graded finite-dimensional real vector space. A linear map  $N \in \text{End}_{-2}(V)$  of pure degree  $-2$  is said to satisfy the Lefschetz property relative to  $V_*$  if and only if

$$N^\ell: V_\ell \rightarrow V_{-\ell}$$

is an isomorphism for all  $\ell \geq 0$ . An abelian subspace  $\mathfrak{a} \subset \text{End}_{-2}(V)$  is said to satisfy the Lefschetz property if some  $N \in \mathfrak{a}$  does. For  $N$  satisfying the Lefschetz property, the primitive subspace  $P_\ell(N) \subset V_\ell$  is the kernel of the map

$$N^{\ell+1}: V_\ell \rightarrow V_{-\ell-2}.$$

□

We shall denote by  $Y \in \text{End}_0(V)$  the semisimple transformation acting by multiplication by  $\ell$  on  $V_\ell$ . It is well known that the pair  $\{Y, N\}$  may be extended to an  $\mathfrak{sl}_2$ -triple  $\{Y, N, N^+\}$ , i.e.  $N^+ \in \text{End}_2(V)$  and the following commutation relations hold:

$$[N_+, N] = Y; \quad [Y, N_+] = 2N_+; \quad [Y, N] = -2N.$$

In other words,  $\{Y, N, N^+\}$  define a representation of the Lie algebra  $\mathfrak{sl}_2$  on  $V$ . It follows from the basic structure theorem of  $\mathfrak{sl}_2$ -representations that the *Lefschetz decomposition* holds.

**Theorem 2.2.** Let  $V = V_*$  be a  $\mathbb{Z}$ -graded finite-dimensional real vector space and  $N \in \text{End}_{-2}(V)$  an endomorphism satisfying the Lefschetz property relative to  $V_*$ . Then, for every  $m \geq 0$ ,

$$V_m = (\ker(N^{m+1}) \cap V_m) \oplus NV_{m+2}. \tag{2.1}$$

□

We recall that a Hodge structure of weight  $d$  on a real vector space  $H$  is a decomposition of its complexification  $H_{\mathbb{C}}$ ,

$$H_{\mathbb{C}} = \bigoplus_{p+q=d} H^{p,q} \quad (2.2)$$

such that  $\overline{H^{p,q}} = H^{q,p}$ . A Hodge structure of weight  $d$  on  $H$  is said to be *polarized* if there exists a real bilinear form  $Q$  of parity  $(-1)^d$ , such that the Hermitian form  $Q^h(\cdot, \cdot) := i^{-d} Q(\cdot, \cdot)$  makes the decomposition (2.2) orthogonal and such that  $(-1)^p Q^h$  is positive definite on  $H^{p,d-p}$ . Given a real vector space  $V$  and a nondegenerate real bilinear form  $S$  on  $V$ , we denote by  $\mathfrak{o}(V, S)$  the Lie algebra of infinitesimal automorphisms of  $(V, S)$ .

**Definition 2.3.** Let  $V_*$  be a  $\mathbb{Z}$ -graded finite-dimensional real vector space,  $k$  a positive integer, and  $S$  a nondegenerate real bilinear form of parity  $(-1)^k$ . Let  $\mathfrak{a} \subset \mathfrak{o}_{-2}(V, S)$  be an abelian subspace and  $N_0 \in \mathfrak{a}$ . Then  $(V_*, S, \mathfrak{a}, N_0)$  is said to be a polarized Hodge–Lefschetz module of weight  $k$  if the following are satisfied:

1. There is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{0 \leq p, q \leq k} V^{p,q}; \quad V^{q,p} = \overline{V^{p,q}},$$

such that

$$(V_{\ell})_{\mathbb{C}} = \bigoplus_{p+q=\ell+k} V^{p,q}.$$

Hence, the bigrading restricts to a Hodge structure of weight  $k + \ell$  on  $V_{\ell}$ .

2.  $T(V^{p,q}) \subset V^{p-1, q-1}$  for all  $T \in \mathfrak{a}$ .
3.  $N_0$  satisfies the Lefschetz property.
4. For  $\ell \geq 0$ , the induced Hodge structure on  $P_{\ell}(N_0) \subset V_{\ell}$  is polarized by the form  $S_{\ell}(\cdot, \cdot) := S(\cdot, N_0^{\ell} \cdot)$ . □

Given a polarized Hodge–Lefschetz module  $(V_*, S, \mathfrak{a}, N_0)$ , we will denote by  $\mathcal{C} = \mathcal{C}(V_*, S, \mathfrak{a}, N_0)$  the largest convex cone in  $\mathfrak{a}$  containing  $N_0$  and such that every element in  $\mathcal{C}$  has the Lefschetz property relative to  $V_*$ . Since it is easy to check that every  $N \in \mathcal{C}$

polarizes the Hodge–Lefschetz module, in the sense that property 4 in Definition 2.3 holds with  $N$  replacing  $N_0$ , we will usually refer to  $\mathcal{C}$  as the *polarizing cone* of  $(V_*, S, \alpha, N_0)$ .

**Remark 2.4.** It is shown in [21, Proposition 1.6] that if  $(V_*, S, \alpha, N_0)$  is a polarized Hodge–Lefschetz module in the sense of the above definition, then the Lie algebra  $\mathfrak{g}(\alpha, V)$  generated by all  $\mathfrak{sl}_2$ -triples  $\{Y, N, N^+\}$ , where  $N$  runs over all  $N \in \mathcal{C}$  satisfying the Lefschetz property, is semisimple. Hence, a Hodge–Lefschetz module is a Lefschetz module in the sense of [21].  $\square$

**Remark 2.5.** Given a polarized Hodge–Lefschetz module  $(V_*, S, \alpha, N_0)$ , we can construct two filtrations in  $V_{\mathbb{C}}$ ,

$$W_\ell := \bigoplus_{a \leq \ell} (V_a)_{\mathbb{C}}; \quad F^p := \bigoplus_{a \geq p} V^{a,b}.$$

The filtration  $W_*$  is increasing and defined over  $\mathbb{R}$ , while the filtration  $F^*$  is decreasing.  $V_*$  is a grading of  $W_*[-k]$ . The pair  $(W_*, F^*)$  defines a mixed Hodge structure split over  $\mathbb{R}$ . Moreover, it is polarized, in the sense of [8], by  $(N, S)$ , where  $N$  is any element in the polarizing cone  $\mathcal{C}$  (cf. [11, Section 2]).  $\square$

**Example (Cohomology of compact Kähler manifolds.)** Let  $X$  be a  $k$ -dimensional smooth compact Kähler manifold and  $V := H^*(X, \mathbb{R})$ . We let  $V_\ell := H^{k-\ell}(X, \mathbb{R})$  and  $V^{p,q} := H^{k-q, k-p}(X)$ . The Hodge decomposition (1.1) implies that property (2) in Definition 2.3 is satisfied. Every cohomology class  $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$  defines, by cup product, an element  $L_\omega \in \text{End}_{-2}(V_*)$  which is pure of bidegree  $(-1, -1)$  relative to the bigrading  $V^{*,*}$ . The Hard Lefschetz Theorem (cf. Theorem 1.1) asserts that if  $\omega$  is a Kähler class, then  $L_\omega$  satisfies the Lefschetz property. On the other hand, given  $Q$  as in equation (1.3), Theorem 1.2 implies that the polarization condition (5) in Definition 2.3 is satisfied. Hence, if we let  $\alpha$  denote  $H^{1,1}(X) \cap H^2(X, \mathbb{R})$  acting by multiplication on  $V$ , then for any Kähler class  $\omega$ ,  $(V_*, Q, \alpha, L_\omega)$  is a polarized Hodge–Lefschetz module of weight  $k$ .

**Example (Combinatorial intersection cohomology of polytopes.)** Given a  $k$ -dimensional polytope  $\Delta$ , one may construct a combinatorial intersection cohomology. This is a real vector space with an even grading

$$\text{IH}(\Delta) = \bigoplus_{\ell=0}^k \text{IH}^{2\ell}(\Delta)$$

and a perfect intersection pairing

$$S: \mathrm{IH}^q(\Delta) \times \mathrm{IH}^{2k-q}(\Delta) \rightarrow \mathbb{R}.$$

Setting  $V_\ell = \mathrm{IH}^{k-\ell}(\Delta)$ , we have  $V_\ell = \{0\}$  for  $\ell$  odd. If we set  $V^{p,p} := (V_{2p-k})_{\mathbb{C}} = (\mathrm{IH}^{2k-2p}(\Delta))_{\mathbb{C}}$ , we obtain a mixed Hodge structure of Hodge–Tate type on  $V$ . There is a natural action of the space  $\mathfrak{a}$  of maps which are conewise linear on the normal fan of  $\Delta$  and, for a strictly convex map  $\psi$ , the Lefschetz property is satisfied (see [19, Theorem 0.1]). Moreover, HRR is satisfied relative to the intersection form. Hence,  $(V_*, S, \mathfrak{a}, \psi)$  is a polarized Hodge–Lefschetz module of weight  $k$  whose polarizing cone consists of strictly convex conewise linear maps. We refer the reader to [1, 2, 4, 5, 19] for the details of this general case and describe, instead, Timorin’s construction for the case of simple polytopes which is based on a beautiful description of the cohomology algebra due to Pukhlikov and Khovanskiĭ [23, 24]. We point out that, in the case of simple polytopes, the Lefschetz package was first obtained by McMullen [22].

Let us then assume that  $\Delta$  is a simple  $k$ -dimensional polytope, i.e each vertex of  $\Delta$  is incident to exactly  $k$  facets. Let  $r$  be the total number of facets of  $\Delta$ . A polytope  $P$  is said to be *analogous* to  $\Delta$  if  $P$  and  $\Delta$  have the same outward normal directions and if their facets are analogous when considered in a common hyperplane. Any two segments on a line are analogous. The space  $\mathcal{C}(\Delta)$  of polytopes analogous to  $\Delta$  has a natural  $\mathbb{R}$ -cone structure under Minkowski sum. It may be extended to a real vector space of virtual polytopes  $\mathcal{A}(\Delta)$  in the usual way. The space  $\mathcal{A}(\Delta)$  comes equipped with a natural polynomial function  $\nu$  of degree  $k$  which restricts to the usual volume on  $\mathcal{C}(\Delta)$ . There is also a distinguished set of linear coordinates  $x_1, \dots, x_r$  on  $\mathcal{A}(\Delta)$  defined in the following way: let  $\xi_1, \dots, \xi_r$  be a choice of outward normal vectors for  $\Delta$ , we then set for each  $P \in \mathcal{C}(\Delta)$ ,

$$x_i(P) := \max_{m \in P} \langle \xi_i, m \rangle.$$

Let  $\partial_i$  denote the operator  $\partial/\partial x_i$  and consider the graded algebra  $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \dots, \partial_r]$  of partial differential operators on  $\mathcal{A}(\Delta)$  with constant coefficients. Let  $I \subset \mathbb{C}[\partial]$  denote the ideal

$$I := \{D \in \mathbb{C}[\partial] : D \cdot \nu = 0\}.$$



Since  $\nu$  is a homogeneous polynomial,  $I$  is a homogeneous ideal and the quotient algebra

$$H(\Delta) := \mathbb{C}[\partial]/I$$

is naturally graded. We set  $V = \mathbb{R}[\partial]/(I \cap \mathbb{R}[\partial])$  with the grading

$$V = \bigoplus_{j=-k}^k V_j; \quad (V_{k-2\ell})_{\mathbb{C}} := (H(\Delta))_{\ell}, \quad \ell = 0, \dots, k.$$

The set  $\mathfrak{a}$  of linear operators acts on  $V$  as linear transformations degree  $-2$ . Moreover, we have the following HLT [31, Theorem 5.3.1].

**Theorem 2.6.** Let  $P \in \mathcal{C}(\Delta)$  and  $L_P = \sum x_i(P)\partial_i$ . Then  $L_P$  satisfies the Lefschetz property.  $\square$

The vector space  $V$  comes equipped with a nondegenerate pairing, defined by  $S([D_1], [D_2]) = D_1 D_2 \cdot \nu$  if the degrees of  $D_1$  and  $D_2$  are complementary and  $S([D_1], [D_2]) = 0$  otherwise. It then follows from [31, Theorem 5.1.1] that HRR holds. Hence, if we define  $(V_{k-2\ell})_{\mathbb{C}} := V^{k-\ell, k-\ell}$  we have that  $(V_*, S, \mathfrak{a}, L_P)$  is a Hodge–Lefschetz module for every  $P$  in the cone  $\mathcal{C}(\Delta)$ . We note that  $\mathcal{C}(\Delta)$  is the polarizing cone of this Hodge–Lefschetz module.

### 3 Period Mappings and Variations of Hodge Structure

In this section, we recall the basic definitions and main properties of polarized variations of Hodge structure (PVHS) and their period mappings. We refer to [9, 14–16, 25] for details. We will show, in particular, that period mappings, more particularly nilpotent orbits [8, 10, 25], may be viewed as the universal example of polarized Hodge–Lefschetz modules.

Let  $B$  be a connected complex manifold. A *real variation of Hodge structure* of weight  $k$  (VHS) over  $B$  is given by the data  $(\mathcal{V}, \nabla, \mathcal{V}_{\mathbb{R}}, \mathcal{F})$ , where  $\mathcal{V} \rightarrow B$  is a holomorphic vector bundle,  $\nabla$  a flat connection on  $\mathcal{V}$ ,  $\mathcal{V}_{\mathbb{R}}$  a flat real form, and  $\mathcal{F}$  a finite decreasing filtration of  $\mathcal{V}$  by holomorphic subbundles—the *Hodge filtration*—satisfying

1.  $\nabla \mathcal{F}^p \subset \Omega_B^1 \otimes \mathcal{F}^{p-1}$  (Griffiths' transversality)
2.  $\mathcal{V} = \mathcal{F}^p \oplus \bar{\mathcal{F}}^{k-p+1}$  ( $\bar{\mathcal{F}}$  = conjugate of  $\mathcal{F}$  relative to  $\mathcal{V}_{\mathbb{R}}$ )

As a  $C^\infty$ -bundle,  $\mathcal{V}$  may then be written as a direct sum

$$\mathcal{V} = \bigoplus_{p+q=k} \mathcal{V}^{p,q}, \quad \mathcal{V}^{p,q} = \mathcal{F}^p \cap \bar{\mathcal{F}}^q; \quad (3.1)$$

the integers  $h^{p,q} = \dim \mathcal{V}^{p,q}$  are the *Hodge numbers*. A *polarization* of the VHS is a flat nondegenerate bilinear form  $Q$  on  $\mathcal{V}$ , whose specialization at each fiber of  $\mathcal{V}$  polarizes the Hodge structure induced by equation (3.1) on the fiber.

Fixing a fiber  $V_{\mathbb{C}}$  together with the real structure  $V_{\mathbb{R}}$ , the polarizing form  $Q$ , the weight  $k$ , and the Hodge numbers  $\{h^{p,q}\}$  and allowing the Hodge filtration  $F$  to vary, we define the *classifying space*  $D := D(V, k, Q, \{h^{p,q}\})$  of polarized Hodge structures. Its Zariski closure  $\check{D}$  in the appropriate variety of flags consists of all filtrations  $F$  in  $V$  with  $\dim F^p = \sum_{r \geq p} h^{r, k-r}$  satisfying

$$Q(F^p, F^{k-p+1}) = 0.$$

The complex Lie group  $G_{\mathbb{C}}$  of all automorphisms of  $(V, Q)$  acts transitively on  $\check{D}$ —therefore  $\check{D}$  is smooth—and the group of real points  $G_{\mathbb{R}}$  has  $D$  as an open dense orbit. Let  $\mathfrak{g} \subset \mathfrak{gl}(V)$  denote the Lie algebra of  $G_{\mathbb{C}}$ ,  $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$  that of  $G_{\mathbb{R}}$ . The choice of a base point  $F \in \check{D}$  defines a filtration in  $\mathfrak{g}$

$$F^a \mathfrak{g} = \{ T \in \mathfrak{g} : T F^p \subset F^{p+a} \}. \quad (3.2)$$

The Lie algebra of the isotropy subgroup  $U \subset G_{\mathbb{C}}$  at  $F$  is  $F^0 \mathfrak{g}$  and  $F^{-1} \mathfrak{g} / F^0 \mathfrak{g}$  is an  $\text{Ad}(U)$ -invariant subspace of  $\mathfrak{g} / F^0 \mathfrak{g}$ . The corresponding  $G_{\mathbb{C}}$ -invariant subbundle of the holomorphic tangent bundle of  $\check{D}$  is the *horizontal tangent bundle*, denoted by  $T_h(\check{D})$ . A polarized VHS over a manifold  $B$  determines—via parallel translation to a typical fiber—a holomorphic map  $\Phi: B \rightarrow D/\Gamma$ , where  $\Gamma$  is the monodromy group (Griffiths' period map). By definition, it has local liftings into  $D$  whose differentials take values on the horizontal tangent bundle.

In order to understand the local situation at infinity, we suppose now that  $B = (\Delta^*)^r$  is a product of punctured disks,  $\mathcal{U}^r$  its universal cover, i.e.  $\mathcal{U} := \{w \in \mathbb{C} : \text{Im}(w) > 0\}$ , and

$$\Phi: \mathcal{U}^r \rightarrow D$$

a horizontal map such that

$$\Phi(z + e_j) := \gamma_j \Phi(z),$$

for some  $\gamma_j \in G_{\mathbb{R}}$ , where  $z = (z_1, \dots, z_r) \in \mathcal{U}^r$ , and  $e_j$  represents the  $j$ th standard vector. We assume that the transformations  $\gamma_j$  are unipotent. If the period map arises from a polarized variation of Hodge structure defined over  $\mathbb{Z}$ , the Picard–Lefschetz transformations  $\gamma_j$  are automatically quasi-unipotent and passing, if necessary, to a finite cover of  $B$  we may assume that they are unipotent. We will abuse notation and refer to such a map  $\Phi$  as a local period map with unipotent monodromy.

We set

$$N_j := \log \gamma_j \in F^{-1} \mathfrak{g} \cap \mathfrak{g}_{\mathbb{R}}, \quad (3.3)$$

and denote by  $\mathfrak{a}$  the abelian subalgebra of  $\mathfrak{g}_{\mathbb{R}}$  generated by  $N_1, \dots, N_r$ . The following theorem follows from results in [8, 10, 12, 25].

**Theorem 3.1.** Let  $\Phi: \mathcal{U}^r \rightarrow D$  be a local period mapping with unipotent monodromy and values in the classifying space  $D(V, k, Q, \{h^{p,q}\})$ . Let  $\mathfrak{a}$  be the abelian Lie algebra generated by the logarithmic monodromies  $N_1, \dots, N_r$  and let  $N_0 = N_1 + \dots + N_r$ . Then there exists a grading of  $V$ , such that  $(V_*, Q, \mathfrak{a}, N_0)$  is a polarized Hodge–Lefschetz module of weight  $k$ . Moreover, every polarized Hodge–Lefschetz module  $(V_*, Q, \mathfrak{a}, N_0)$  of weight  $k$  arises, in this manner, from a local period mapping with unipotent monodromy.  $\square$

**Proof.** It follows from Schmid’s Nilpotent Orbit Theorem [25] that if  $\Phi: \mathcal{U}^r \rightarrow D$  is a local period mapping with unipotent monodromy, then there exists a filtration  $F_{\text{lim}} \in \check{D}$  such that the map

$$(z_1, \dots, z_r) \in \mathcal{U}^r \mapsto \exp \left( \sum_{j=1}^r z_j N_j \right) \cdot F_{\text{lim}}$$

takes values in  $D$  for  $\text{Im}(z_j) \gg 0$ . We then know from [8] that every element in the positive cone  $\mathcal{C}$  spanned by  $N_1, \dots, N_r$  defines the same weight filtration  $W_*$  and as a consequence of Schmid’s  $SL_2$ -orbit theorem, it follows that  $(W, F_{\text{lim}}, Q, N)$  is a polarized mixed Hodge structure for every  $N \in \mathcal{C}$ . Finally, it follows from [10, 12] that there exists a canonical

splitting of this mixed Hodge structure over  $\mathbb{R}$ . We thus obtain a Hodge–Lefschetz module structure of weight  $k$ .

Conversely, suppose  $(V_*, Q, \mathfrak{a}, N_0)$  is a polarized Hodge–Lefschetz module of weight  $k$ . Let  $N_1, \dots, N_r$  be the elements in the polarizing cone  $\mathcal{C}$ , which are a basis of  $\mathfrak{a}$  and such that  $N_0 = N_1 + \dots + N_r$ . Then as noted in Remark 2.5, there exist filtrations  $(W, F)$  defining a mixed Hodge structure of weight  $k$  split over  $\mathbb{R}$  and polarized by  $(N_j, Q)$  for each  $j = 1, \dots, k$ . It then follows from [8, Proposition 2.18] that

$$\exp \left( \sum_{j=1}^k z_j N_j \right) \cdot F \in D(V, k, Q, \{h^{p,q}\}),$$

where

$$h^{p,q} = \sum_{a=p} \dim V^{a,b}.$$

By property (3) in Definition 2.3, the map

$$\Phi(z) = \exp \left( \sum_{j=1}^k z_j N_j \right) \cdot F$$

is horizontal and consequently, it defines a local period mapping with unipotent monodromy, in fact, a nilpotent orbit in the sense of [25].  $\blacksquare$

The following result is a restatement of the Descent Lemma [11, Lemma 1.16].

**Theorem 3.2.** Let  $(V_*, Q, \mathfrak{a}, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$ . Let  $T \in \mathfrak{a}$  be such that  $T + \lambda N_0$  has the Lefschetz property, relative to  $V_*$ , for all  $\lambda > 0$ . Let  $\tilde{V}$  denote the image  $T \cdot V$  graded by  $\tilde{V}_\ell = T \cdot V_{\ell+1}$ . Set

$$\tilde{Q}(T u, T v) := Q(u, T v); \quad u, v \in V.$$

The commutative subspace  $\mathfrak{a} \subset \mathfrak{o}_{-2}(V, Q)$  acts on  $\tilde{V}$  and we denote by  $\tilde{\mathfrak{a}} \subset \mathfrak{o}_{-2}(\tilde{V}, \tilde{Q})$  the induced space of endomorphisms. Then  $(\tilde{V}_*, \tilde{Q}, \tilde{\mathfrak{a}}, \tilde{N}_0)$  is a polarized Hodge–Lefschetz module of weight  $k - 1$ .  $\square$

The proof of Theorem 3.2 is the content of Section 2 of [11]. The argument, which makes extensive use of the equivalence between variations of Hodge structure and polarized Hodge–Lefschetz structures described by Theorem 3.1, is quite involved and we will not attempt to summarize it here. We may illustrate the result by considering the case  $T = N_0$ . Although this only applies to the unmixed case, the discussion makes clear the connection between the Descent Lemma and the mixed HLT and HRR.

Since  $N_0 \cdot V^{0,q} = N_0 \cdot V^{p,0} = 0$ , we have

$$(\tilde{V})_{\mathbb{C}} = N_0 \cdot V_{\mathbb{C}} = \bigoplus_{1 \leq p, q \leq k} N_0 \cdot V^{p,q}.$$

Hence, if we set  $\tilde{V}^{a,b} := N_0 \cdot V^{a+1, b+1}$ , we have a bigrading

$$\tilde{V}_{\mathbb{C}} = \bigoplus_{0 \leq a, b \leq k-1} \tilde{V}^{a,b}.$$

Moreover, setting  $\tilde{V}_{\ell} := N_0 \cdot V_{\ell+1}$  we have

$$(\tilde{V}_{\ell})_{\mathbb{C}} = \bigoplus_{p+q=\ell+1+k} N_0 \cdot V^{p,q} = \bigoplus_{a+b=\ell+(k-1)} \tilde{V}^{a,b}$$

and, since clearly  $\overline{\tilde{V}^{a,b}} = \tilde{V}^{b,a}$ , the bigrading  $\tilde{V}^{*,*}$  defines a Hodge structure of weight  $(k-1) + \ell$  on  $\tilde{V}_{\ell}$ . We note that so far we have only used the fact that  $N_0$  is of pure bidegree  $(-1, -1)$  relative to the bigrading  $V^{*,*}$ .

In order to verify that  $\tilde{N}_0$  satisfies the Lefschetz property, we note that it follows from the Lefschetz decomposition (2.1) that

$$V_{-\ell+1} = N_0^{\ell-1} \cdot V_{\ell-1} = N_0^{\ell-1} \cdot P_{\ell-1}(N_0) \oplus N_0^{\ell} \cdot V_{\ell+1},$$

and consequently

$$\tilde{V}_{-\ell} = N_0 \cdot V_{-\ell+1} = N_0^{\ell+1} \cdot V_{\ell+1} = N_0^{\ell} \cdot \tilde{V}_{\ell}.$$

Since clearly  $N_0^{\ell}$  is injective in  $\tilde{V}_{\ell} = N_0 \cdot V_{\ell+1}$ , it follows that  $\tilde{N}_0^{\ell}: \tilde{V}_{\ell} \rightarrow \tilde{V}_{-\ell}$  is an isomorphism.

It remains to check that  $N_0$  polarizes the Hodge–Lefschetz structure on  $\tilde{V}$  relative to  $\tilde{Q}$ . We note, first of all, that it is easy to check that if  $Q$  is of parity  $(-1)^k$ , then  $\tilde{Q}$

has parity  $(-1)^{k-1}$ . Finally, we verify the polarization condition (4) in Definition 2.3. Let  $u \in P_\ell(\tilde{N}_0) \cap \tilde{V}^{a,b}$  with  $a + b = (k - 1) + \ell$ . Then we can write  $u = N_0 v$ , where

$$v \in P_{\ell+1}(N_0) \cap V^{a+1,b+1}; \quad (a + 1) + (b + 1) = k + (\ell + 1).$$

Hence

$$i^{a-b} \tilde{Q}(u, \tilde{N}_0^\ell \bar{u}) = i^{(a+1)-(b+1)} Q(v, \tilde{N}_0^{\ell+1} \bar{v}) \geq 0$$

and equality holds if and only if  $v = 0$  or, equivalently, if  $u = 0$ .

**Remark 3.3.** Repeated application of Theorem 3.2 allows us to replace  $\tilde{V} = T \cdot V$  by  $\tilde{V} = T^m \cdot V$ ,  $m \leq k$ , graded by  $\tilde{V}_\ell = T \cdot V_{\ell+m}$ . Similarly, we may replace  $\tilde{V} = T^m \cdot V$  by  $\tilde{V} = V/\ker(T^m)$ .  $\square$

**Corollary 3.4.** Let  $(V_*, Q, \alpha, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$  and  $\mathcal{C}$  its polarizing cone. Let  $T_1, \dots, T_m \in \mathcal{C}$ . Set  $\tilde{V} := T_1 \cdots T_m \cdot V$  graded by  $\tilde{V}_\ell = T_1 \cdots T_m \cdot V_{\ell+m}$ . Define  $\tilde{Q}$  by

$$\tilde{Q}(T_1 \cdots T_m u, T_1 \cdots T_m v) := Q(u, T_1 \cdots T_m v); \quad u, v \in V.$$

Let  $\tilde{\alpha}$  denote  $\alpha$  viewed as endomorphisms of  $\tilde{V}$ . Then  $(\tilde{V}_*, \tilde{Q}, \tilde{\alpha}, \tilde{N}_0)$  is a polarized Hodge–Lefschetz module of weight  $k - m$ .  $\square$

**Proof.** This corollary follows from repeated application of Theorem 3.2.

In [11], the Descent Lemma appears as a step toward the proof of a subtler result on the mixed  $\mathfrak{sl}_2$  action on a polarized Hodge–Lefschetz module, namely the Purity Theorem [11, (1.13)]. Although we do not yet have a geometric or combinatorial interpretation of this result, we include its statement for the sake of completeness.

Let  $(V_*, Q, \alpha, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$  and  $T_1, \dots, T_m$  elements in the polarizing cone  $\mathcal{C}$ . Consider the Koszul complex  $K^*$  whose terms are defined by

$$K^p := \bigoplus_{1 \leq j_1 \leq \dots \leq j_p \leq m} T_{j_1} \cdots T_{j_p} \cdot V$$

and whose differentials are given by the maps

$$(-1)^{s-1} T_{j_s} : T_{j_1} \cdots \widehat{T_{j_s}} \cdots T_{j_p} \cdot V \rightarrow T_{j_1} \cdots T_{j_p} \cdot V$$

between the summands of  $K^{p-1}$  and those of  $K^p$ . Letting  $W_*(V)$  be the natural filtration of  $V$  defined by the grading  $V_*$ , we filter  $K^p$  by

$$W_\ell(T_{j_1} \cdots T_{j_p} \cdot V) := T_{j_1} \cdots T_{j_p} \cdot W_{\ell+p}(V). \quad \blacksquare$$

**Theorem 3.5.** The cohomology of the filtered complex  $K^*$  occurs entirely in weight zero or less.  $\square$

**Remark 3.6.** In the context of variations of Hodge structure, the complex  $K^*$  arises as an intersection cohomology complex and Theorem 3.5 was conjectured by Deligne as an analog of Gabber’s Purity Theorem in the  $\ell$ -adic case.  $\square$

#### 4 Mixed Hard Lefschetz Theorem and Hodge–Riemann Bilinear Relations

In this section we show how, in the context of polarized Hodge–Lefschetz modules, the mixed HLT and HRR follow from the Descent Lemma. We begin with the following key lemma.

**Lemma 4.1.** Let  $(V_*, Q, \alpha, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$  and  $\mathcal{C}$  its polarizing cone. Let  $W_*$  denote the filtration defined by the grading  $V_*$ . Let  $m \leq k$  and  $T_1, \dots, T_m \in \mathcal{C}$ . Then

$$\ker(T_1 \cdots T_m) \subset W_{m-1} = \bigoplus_{\ell \leq m-1} V_\ell. \quad (4.1)$$

$\square$

**Proof.** We prove the statement by induction on  $m$ . For  $m = 1$ , the result follows from the assumption that  $T_1$  satisfies the Lefschetz property relative to  $V_*$ . Let now

$$\tilde{V} = V / \ker(T_2 \cdots T_m) \cong T_2 \cdots T_m \cdot V.$$

Since  $(V_*, Q, \alpha, T_1)$  is also a polarized Hodge–Lefschetz module with the same polarizing cone, it follows from Corollary 3.4 that  $(\tilde{V}_*, \tilde{Q}, \tilde{\alpha}, \tilde{T}_1)$  is a polarized Hodge–Lefschetz module

of weight  $k - m + 1$ . Hence,

$$\ker(\tilde{T}_1) \subset \tilde{W}_0$$

which implies that

$$\ker(T_1 \cdots T_m) \subset W_{m-1} + \ker(T_2 \cdots T_m) \subset W_{m-1},$$

since  $\ker(T_2 \cdots T_m) \subset W_{m-2} \subset W_{m-1}$  by inductive hypothesis.  $\blacksquare$

We can now state and prove the mixed HLT for polarized Hodge–Lefschetz modules. When applied to the cohomology algebra of a smooth, compact Kähler manifold, it becomes Theorem 1.3.

**Theorem 4.2.** Let  $(V_*, Q, \alpha, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$  and  $\mathcal{C}$  its polarizing cone. Let  $T_1, \dots, T_m \in \mathcal{C}$ ,  $m \leq k$ . Then, the map

$$T_1 \cdots T_m: V_m \rightarrow V_{-m}$$

is an isomorphism.  $\square$

**Proof.** By Lemma 4.1, the map

$$T_1 \cdots T_m: V_m \rightarrow V_{-m}$$

is 1:1. Since  $\dim V_m = \dim V_{-m}$ , the result follows.  $\blacksquare$

The following result is the mixed Lefschetz decomposition for polarized Hodge–Lefschetz modules. It reduces to Theorem 2.2 when all the transformations  $T_j$  agree.

**Theorem 4.3.** Let  $(V_*, Q, \alpha, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$  and  $\mathcal{C}$  its polarizing cone. Let  $T_1, \dots, T_m, T_{m+1} \in \mathcal{C}$ ,  $m + 2 \leq k$ . Then

$$V_m = (\ker(T_1 \cdots T_{m+1}) \cap V_m) \oplus T_{m+1} \cdot V_{m+2}. \quad (4.2)$$

$\square$



**Proof.** By Lemma 4.1,

$$\ker(T_1 \cdots T_m T_{m+1}^2) \subset W_{m+1} = \bigoplus_{\ell \leq m+1} V_\ell.$$

Hence

$$\ker(T_1 \cdots T_m T_{m+1}) \cap T_{m+1} \cdot V_{m+2} = \{0\}.$$

Thus, it suffices to prove that

$$V_m \subset (\ker(T_1 \cdots T_{m+1}) \cap V_m) + T_{m+1} \cdot V_{m+2}.$$

Let  $\tilde{V} = V/\ker(T_1 \cdots T_m) \cong T_1 \cdots T_m \cdot V$ . Then, since  $(\tilde{V}_*, \tilde{Q}, \tilde{\alpha}, \tilde{T}_{m+1})$  is a polarized Hodge–Lefschetz module of weight  $k - m$ , we have

$$\tilde{V}_0 = (\ker(\tilde{T}_{m+1}) \cap \tilde{V}_0) + \tilde{T}_{m+1} \cdot \tilde{V}_2.$$

Hence

$$V_m \subset (\ker(T_1 \cdots T_{m+1}) \cap V_m) + T_{m+1} \cdot V_{m+2} + \ker(T_1 \cdots T_m).$$

Since, by Lemma 4.1

$$\ker(T_1 \cdots T_m) \cap V_m = \{0\},$$

the theorem follows. ■

The following is the mixed version of the Hodge–Riemann bilinear relations for polarized Hodge–Lefschetz modules. In the geometric case, it is Theorem 1.4.

**Theorem 4.4.** Let  $(V_*, Q, \alpha, N_0)$  be a polarized Hodge–Lefschetz module of weight  $k$  and  $\mathcal{C}$  its polarizing cone. Let  $T_1, \dots, T_m, T_{m+1} \in \mathcal{C}$ ,  $m + 2 \leq k$ . Then if

$$v \in V^{p,q} \cap \ker(T_1 \cdots T_{m+1}); \quad p + q = k + m,$$

we have

$$i^{p-q} Q(v, T_1 \cdots T_m \bar{v}) \geq 0$$

with equality if and only if  $v = 0$ . □

**Proof.** Let  $\tilde{V} = T_1 \cdots T_m V$ . Then by Corollary 3.4 applied to the polarized Hodge-Lefschetz module  $(V_*, Q, \alpha, T_{m+1})$ , we have that  $(\tilde{V}_*, \tilde{Q}, \tilde{\alpha}, \tilde{T}_{m+1})$  is a polarized Hodge-Lefschetz module of weight  $k - m$ , where

$$\tilde{Q}(T_1 \cdots T_m u, T_1 \cdots T_m v) := Q(u, T_1 \cdots T_m v); \quad u, v \in V.$$

Let now  $v \in V^{p,q} \cap \ker(T_1 \cdots T_{m+1})$ . Since  $0 \leq p, q \leq k$  and  $p + q = k + m$ , we must have  $p, q \geq m$  and the image

$$T_1 \cdots T_m v \in \tilde{V}^{p-m, q-m} \cap \ker(\tilde{T}_{m+1}); \quad (p - m) + (q - m) = k - m.$$

Hence,

$$i^{p-q} \tilde{Q}(T_1 \cdots T_m v, T_1 \cdots T_m \bar{v}) = i^{p-q} Q(v, T_1 \cdots T_m \bar{v}) \geq 0.$$

Moreover, equality holds if and only if  $T_1 \cdots T_m \cdot v = 0$  but, by Lemma 4.1,

$$V_m \cap \ker(T_1 \cdots T_m) = \{0\}$$

and the theorem is proved. ■

## References

- [1] Barthel, G., J.-P. Brasselet, K.-H. Fieseler, and L. Kaup. "Combinatorial intersection cohomology for fans." *The Tohoku Mathematical Journal (2)* 54, no. 1 (2002): 1–41.
- [2] Barthel, G., J.-P. Brasselet, K.-H. Fieseler, and L. Kaup. "Combinatorial duality and intersection product: A direct approach." *The Tohoku Mathematical Journal (2)* 57, no. 2 (2005): 273–92.
- [3] Braden, T. "Remarks on the combinatorial intersection cohomology of fans." *Pure and Applied Mathematics Quarterly* 2, no. 4 (2006): 1149–86.
- [4] Bressler, P., and V. A. Lunts. "Intersection cohomology on nonrational polytopes." *Compositio Mathematica* 135, no. 3 (2003): 245–78.

- [5] Bressler, P., and V. A. Lunts. “Hard Lefschetz theorem and Hodge–Riemann relations for intersection cohomology of nonrational polytopes.” *Indiana University Mathematics Journal* 54, no. 1 (2005): 263–307.
- [6] Brion, M. “The structure of the polytope algebra.” *The Tohoku Mathematical Journal (2)* 49, no. 1 (1997): 1–32.
- [7] Cattani, E., and J. Fernandez. “Frobenius modules and Hodge asymptotics.” *Communications in Mathematical Physics* 238, no. 3 (2003): 489–504.
- [8] Cattani, E., and A. Kaplan. “Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure.” *Inventiones Mathematicae* 67, no. 1 (1982): 101–15.
- [9] Cattani, E., and A. Kaplan. “Degenerating variations of Hodge structure.” Actes du Colloque de Théorie de Hodge (Luminy, 1987). *Astérisque* 9, no. 179–180 (1989): 67–96.
- [10] Cattani, E., A. Kaplan, and W. Schmid. “Degeneration of Hodge structures.” *Annals of Mathematics (2)* 123, no. 3 (1986): 457–535.
- [11] Cattani, E., A. Kaplan, and W. Schmid. “ $L^2$  and intersection cohomologies for a polarizable variation of Hodge structure.” *Inventiones Mathematicae* 87, no. 2 (1987): 217–52.
- [12] Deligne, P. “Structures de Hodge mixtes réelles.” In *Motives* (Seattle, WA, 1991), 509–14. Proceedings of Symposia in Pure Mathematics 55. Providence, RI: American Mathematical Society, 1994.
- [13] Dinh, T.-C., and V.-A. Nguyễn. “The mixed Hodge–Riemann bilinear relations for compact Kähler manifolds.” *Geometric and Functional Analysis* 16, no. 4 (2006): 838–49.
- [14] Griffiths, P. A. “Periods of integrals on algebraic manifolds I. Construction and properties of the modular varieties.” *American Journal of Mathematics* 90 (1968): 568–626.
- [15] Griffiths, P. A. “Periods of integrals on algebraic manifolds II. Local study of the period mapping.” *American Journal of Mathematics* 90 (1968): 805–65.
- [16] Griffiths, P. A. “Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems.” *American Mathematical Society Bulletin* 76 (1970): 228–96.
- [17] Griffiths, P. A., and J. Harris. *Principles of Algebraic Geometry*. New York: John Wiley & Sons, 1994.
- [18] Gromov, M. “Convex sets and Kähler manifolds.” In *Advances in Differential Geometry and Topology*, 1–38. Teaneck, NJ: World Scientific Publishing, 1990.
- [19] Karu, K. “Hard Lefschetz theorem for nonrational polytopes.” *Inventiones Mathematicae* 157, no. 2 (2004): 419–47.
- [20] Khovanskiĭ, A. G. “Algebra and mixed volumes.” In *Geometric Inequalities*, edited by D. Yu. Burago and V. A. Zalgaller. Grundlehren der Mathematischen Wissenschaften 285. Berlin: Springer, 1988.
- [21] Looijenga, E., and V. A. Lunts. “A Lie algebra attached to a projective variety.” *Inventiones Mathematicae* 129, no. 2 (1997): 361–412.
- [22] McMullen, P. “On simple polytopes.” *Inventiones Mathematicae* 113, no. 2 (1993): 419–44.
- [23] Pukhlikov, A. V., and A. G. Khovanskiĭ. “Finitely additive measures of virtual polyhedra.” *Algebra i Analiz* 4, no. 2 (1992): 161–85.

- [24] Pukhlikov, A. V., and A. G. Khovanskiĭ. "The Riemann–Roch theorem for integrals and sums of quasipolynomials on virtual polytopes." *Algebra i Analiz* 4, no. 4 (1992): 188–216.
- [25] Schmid, W. "Variation of Hodge structure: The singularities of the period mapping." *Inventiones Mathematicae* 22 (1973): 211–319.
- [26] Stanley, R. P. "The number of faces of a simplicial convex polytope." *Advances in Mathematics* 35, no. 3 (1980): 236–38.
- [27] Stanley, R. P. "Generalized  $H$ -vectors, intersection cohomology of toric varieties, and related results." In *Commutative Algebra and Combinatorics*, 187–213. Advanced Studies in Pure Mathematics 11. Amsterdam, The Netherlands: North-Holland, 1987.
- [28] Teissier, B. "Du théorème de l'index de Hodge aux inégalités isopérimétriques." *Comptes Rendus Academie des Sciences Paris Séries A–B* 288, no. 4 (1979): A287–9.
- [29] Teissier, B. "Variétés toriques et polytopes." In *Bourbaki Seminar*, 71–84. Lecture Notes in Mathematics 901. Berlin: Springer, 1981.
- [30] Timorin, V. A. "Mixed Hodge–Riemann bilinear relations in a linear context." *Funktsional. Anal i Prilozhen* 32, no. 4 (1998): 63–8, 96.
- [31] Timorin, V. A. "An analogue of the Hodge–Riemann relations for simple convex polyhedra." *Russian Mathematical Surveys* 54, no. 2 (1999): 381–426.
- [32] Timorin, V. A. "On polytopes that are simple at edges." *Funktsional Anal i Prilozhen* 35, no. 3 (2001): 36–47, 95.
- [33] C. Voisin. Hodge structures on cohomology algebras and geometry. *Mathematische Annalen* 341, no. 1 (2008): 39–70.