[Chapter 6. Functions of Random Variables]

6.1 Introduction

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6.1 Introduction

Objective of statistics is to make inferences about a population based on information contained in a sample taken from that population. All quantities used to make inferences about a population are functions of the n random observations that appear in a sample.

Consider the problem of estimating a population mean μ. One draw a random sample of n observations, $y_1, y_2, \ldots, y_n$, from the population and employ the sample mean

$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

for μ.
How good is this sample mean for μ?
The answer depends on the behavior of the random variables $Y_1, Y_2, \ldots, Y_n$ and their effect on the distributions of a random variable $\bar{Y} = (1/n) \sum_{i=1}^{n} Y_i$.

To determine the probability distribution for a function of n random variables, $Y_1, Y_2, \ldots, Y_n$ (say $\bar{Y}$), one must find the joint probability functions for the random variable themselves $P(Y_1, \ldots, Y_n)$ or $f(Y_1, \ldots, Y_n)$.

The assumption that we will make is that

"$Y_1, Y_2, \ldots, Y_n$ is a random sample from a population with probability function $p(y)$ or probability density function $f(y)$"

: the random variables $Y_1, Y_2, \ldots, Y_n$ are independent with common probability function $p(y)$ or common density function $f(y)$

: $Y_1, \ldots, Y_n \overset{iid}{\sim} p(y)$ or $f(y)$

6.2 Finding the probability distribution of a function of random variables

We will study two methods for finding the probability distribution for a function of r.v.'s.

Consider r.v. $Y_1, Y_2, \ldots, Y_n$ and a function $U(Y_1, Y_2, \ldots, Y_n)$, denoted simply as U, e.g. $U = (Y_1 + Y_2 + \ldots + Y_n)/n$. Then three methods for finding the probability distribution of U are as follows:

- The method of distribution functions($\surd$)
- The method of transformations.
- The method of moment-generating functions($\surd$)

6.3 Method of distribution functions

Suppose that we have r.v. $Y_1, \ldots, Y_n$ with joint pdf $f(y_1, \ldots, y_n)$. Let $U = U(Y_1, \ldots, Y_n)$ be a function of the r.v.'s $Y_1, Y_2, \ldots, Y_n$.

1. Draw the region over which $f(y_1, \ldots, y_n)$ is positive in $(y_1, y_2, \ldots, y_n)$, and find the region in the $(y_1, y_2, \ldots, y_n)$ space for which $U = u$.
2. Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, \ldots, y_n)$ over the region for which $U \leq u$.
3. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus, $f_U(u) = dF_U(u)/du$.

(Example 6.1)

(Example 6.2)

(Example 6.3)

(Example 6.4)

6.5 The method of Moment Generating Functions

This method is based on a *uniqueness theorem* of M.G.F., which states that, *if two r.v. have identical moment-generating functions, the two r.v.'s possess the same probability distributions.*

Let U be a function of the r.v.'s $Y_1, Y_2, \ldots, Y_n$.

1. Find the moment generating function for U, $m_U(t)$.
2. compare $m_U(t)$ with other well-known moment generating functions. If $m_U(t) = m_V(t)$ for all values of t, then U and V have identical distributions (by uniqueness theorem)

(Theorem 6.1)[**Uniqueness Theorem**]
Let $m_X(t)$ and $m_Y(t)$ denote the moment generating functions of r.v.'s X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

(Example 6.10)

(Example 6.11)

The moment generating function method is often very useful for finding the distributions of sums of independent r.v.'s.

(Theorem 6.2(p.304))

Let $Y_1, Y_2, \ldots, Y_n$ be independent r.v.'s with moment generating functions $m_{Y1}(t), m_{Y2}(t), \ldots,$ $m_{Yn}(t)$, respectively. If $U = Y_1 + Y_2 + \ldots + Y_n$ then

$$m_U(t) = m_{Y1}(t) \times m_{Y2}(t) \times \cdots \times m_{Yn}(t).$$

(Proof) in class

(Example 6.12)

The m.g.f method can be used to establish some interesting and useful results about the distributions of some functions of normally distributed r.v.'s.

(Theorem 6.3)
Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed r.v.'s with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$ and let $a_1, a_2, \ldots, a_n$ be constants. If

$$U = \sum_{i=1}^{n} a_i Y_i,$$

then U is a **normally distributed** random variable with $E(U) = \sum_{i=1}^{n} a_i \mu_i$ and $V(U) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$.

(Proof)

(Exercise 6.35)

(Theorem 6.4)
Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed r.v.'s with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$ and define Z_i by

$$Z_i = \frac{Y_i - \mu_i}{\sigma_i}, \ i = 1, 2, \ldots, n.$$

Then $\sum_{i=1}^{n} Z_i^2$ has a χ^2-distribution with n degrees of freedom.

(Proof)

(Exercise 6.34)

(Exercise 6.43)

(Exercise 6.44)