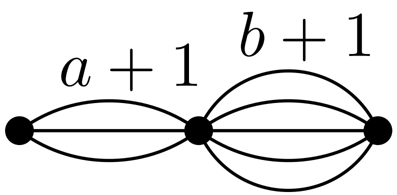
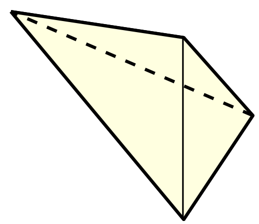
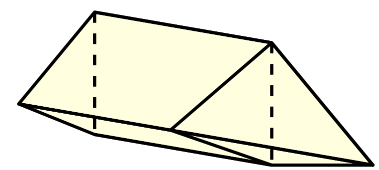
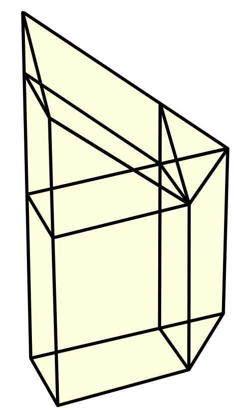
Volumes and triangulations of flow polytopes of graphs

Alejandro H. Morales
UMass Amherst









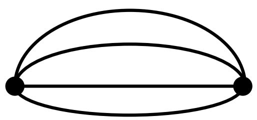
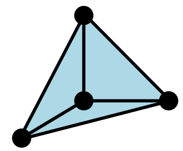
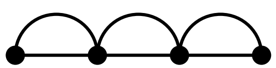
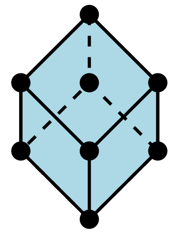
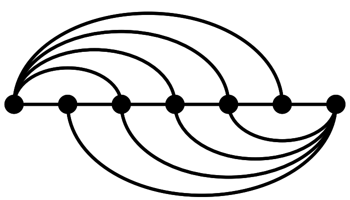
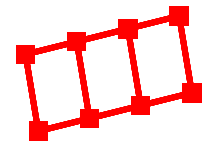
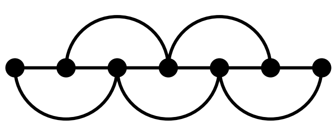
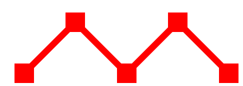
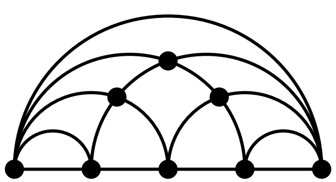
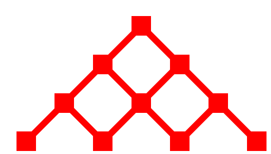
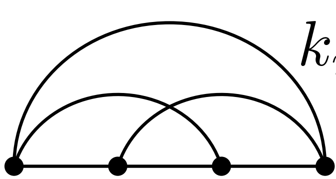
Discrete Math Day of the NorthEast

April 25, 2020

based on joint work with Karola Mészáros (Cornell)
and Jessica Striker (NDSU)

slides available at people.math.umass.edu/∼ahmorales/talks/DMD.pdf

Flow polytopes of graphs

graph G	flow polytope $\mathcal{F}_G$	poset	volume
			1
			$n!$
			$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$
			Euler numbers E_n
			# standard tableaux staircase shape
 k_{n+1}		?	?

Integral polytopes

P a polytope in $\mathbb{R}^N$ with integral vertices:

P is the **convex hull** of finitely many vertices $\mathbf{v}$ in $\mathbb{Z}^N$

OR

P is the intersection of finitely many **half spaces**

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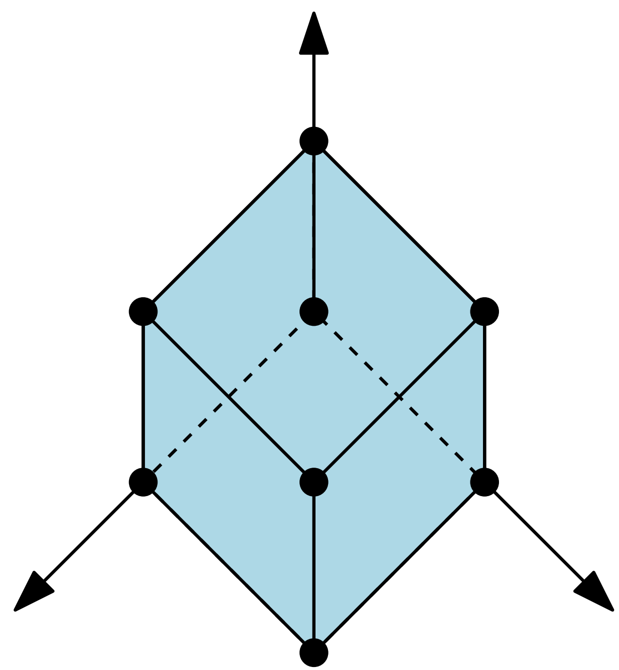


d-cube: convex hull of $\{0,1\}^d$

$$C_d = \left\{ (x_1, \ldots, x_d) \;\middle|\; 0 \le x_i \le 1, \quad i = 1, \ldots, d \right\}$$

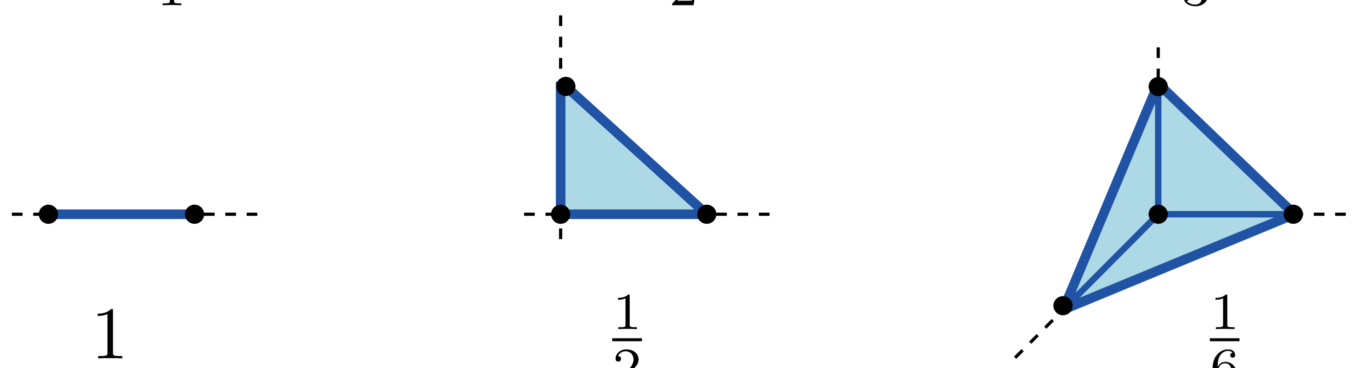
Volume of polytopes

normalized volume of $P := \dim(P)! \cdot$ (euclidean volume of P)

Example:

standard simplex $\Delta_n = \{(x_1, \ldots, x_n) \mid \sum x_i \leq 1,\ x_i \geq 0\}$

Δ_1 $\quad$ Δ_2 $\quad$ Δ_3



euclidean volume $\quad 1 \quad \frac{1}{2} \quad \frac{1}{6}$

(normalized) volume 1

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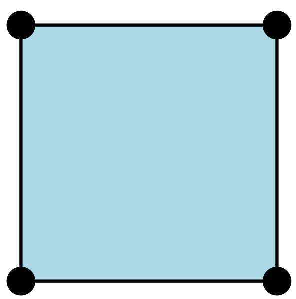

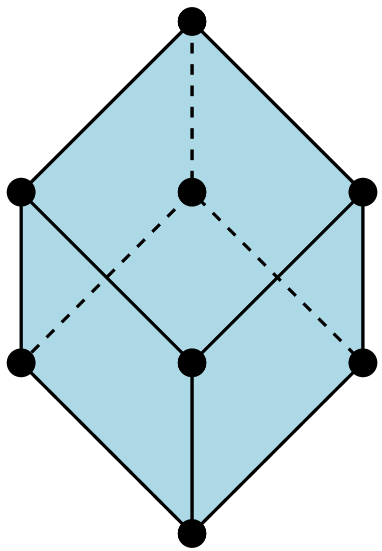

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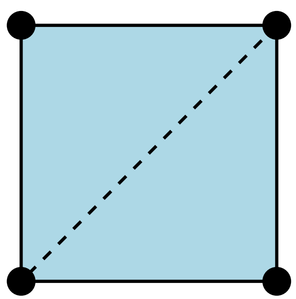

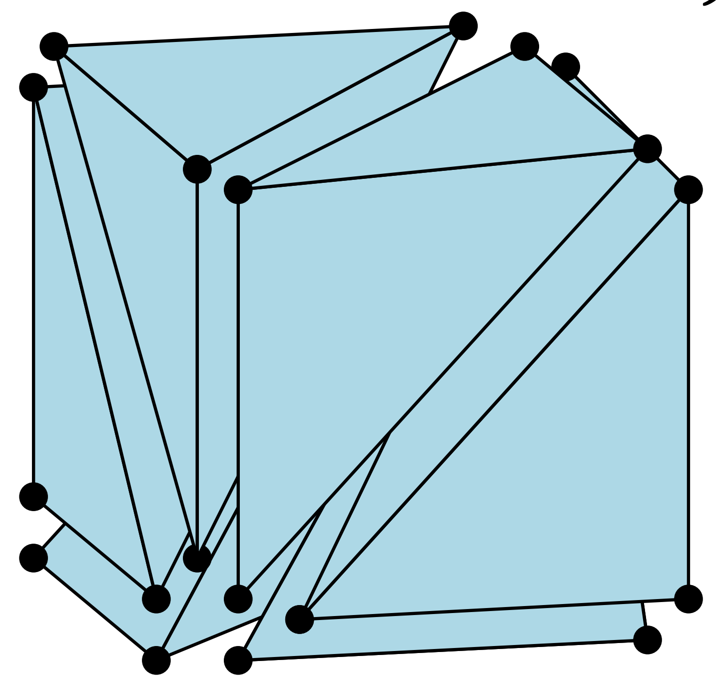

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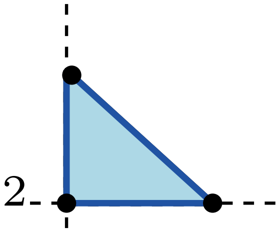
Lattice points of polytopes

- $\#P \cap \mathbb{Z}^N$ number of lattice points (discrete volume)

Example:

Δ_2



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$L_P(t) := \#(tP \cap \mathbb{Z}^N)$ counts lattice points in t-**dilation** of P.

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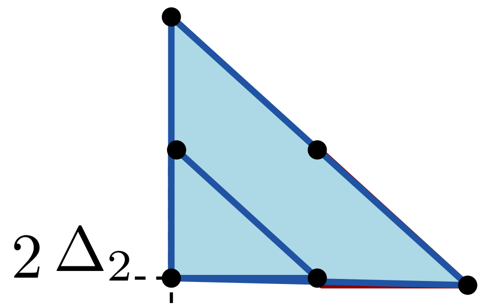

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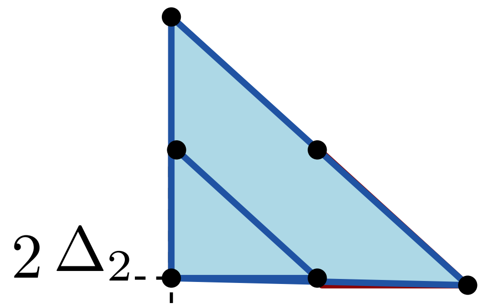

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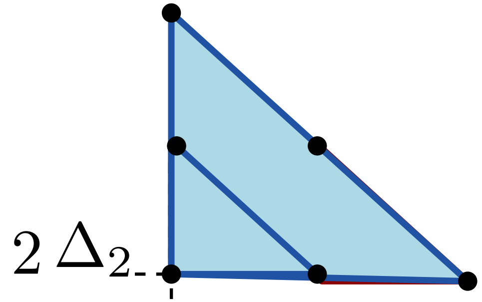

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$\#(t\Delta_d \cap \mathbb{Z}^d) = \binom{t+d}{d}$

Flow polytopes

G graph $n+1$ vertices $|E|$ edges

$\mathbf{a} = (a_1, a_2, \ldots, a_n, -\sum a_i)$, $\quad a_i \in \mathbb{Z}^n_{\geq 0}$

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G) \mid$ netflow vertex $i = a_i\}$

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$$x_{12} + x_{13} + x_{14} = a_1$$
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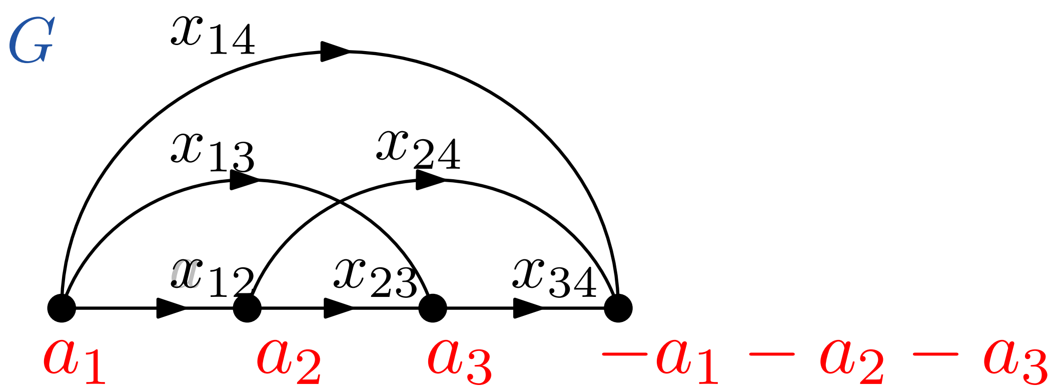

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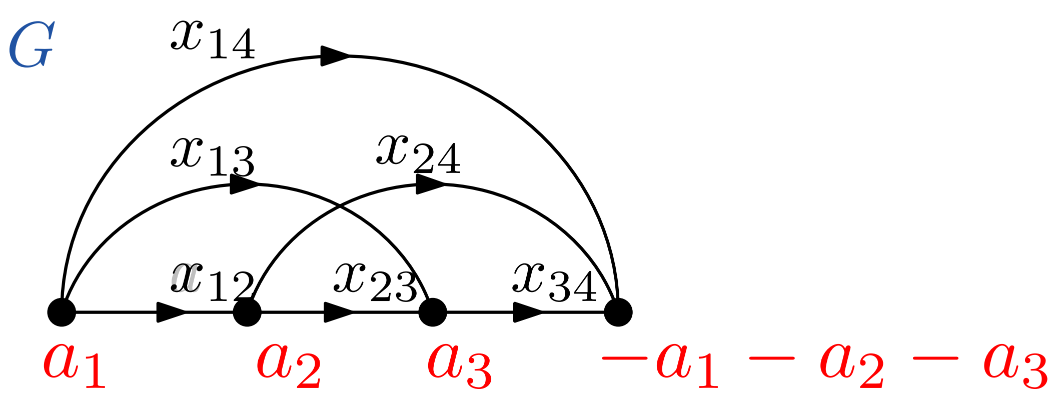

dimension of $\mathcal{F}_G(\mathbf{a})$ is $|E| - n$

Flow polytopes

G graph $n+1$ vertices m edges

$\mathbf{a} = (1, 0, \ldots, 0, -1)$

$\mathcal{F}_G(1, 0, \ldots, 0, -1)$ is flows on G: netflow first vertex is 1, netflow last vertex -1, netflow other vertices is 0.

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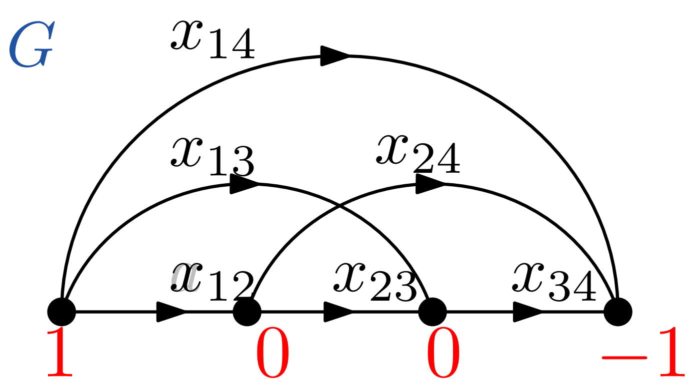
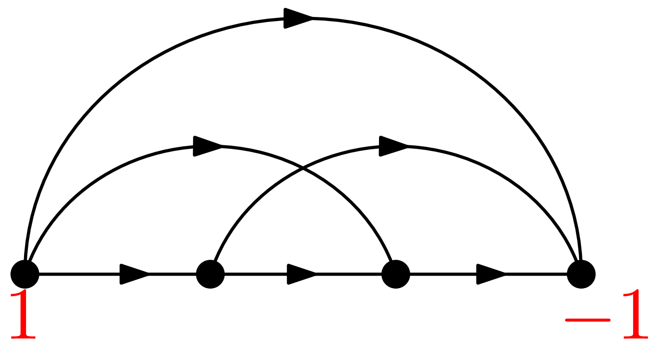
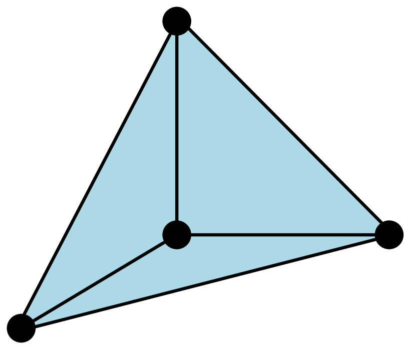
$$-x_{14} - x_{24} - x_{34} = -1$$



Vertices of flow polytopes

- vertices of flow polytopes of $\mathcal{F}_G(1, 0, \ldots, 0, -1)$ can be viewed as unit flows on directed paths from vertex 1 to $n+1$ called **routes**.





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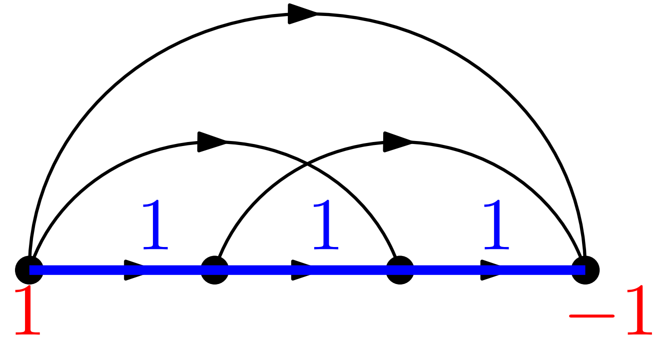

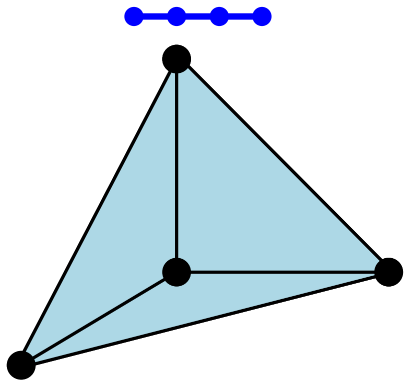

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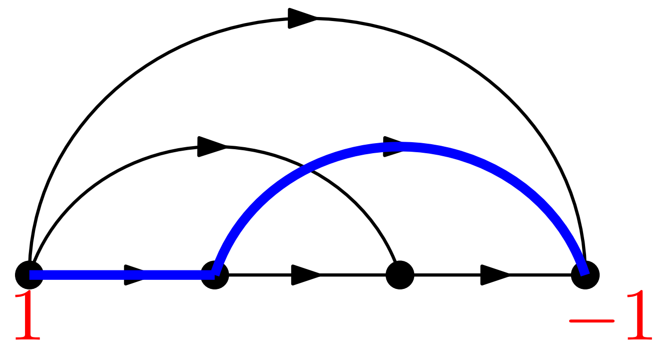

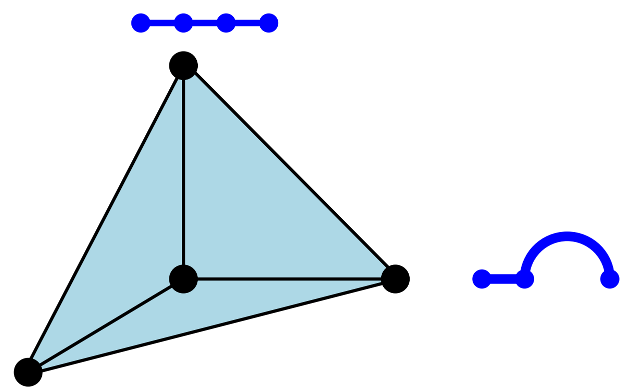

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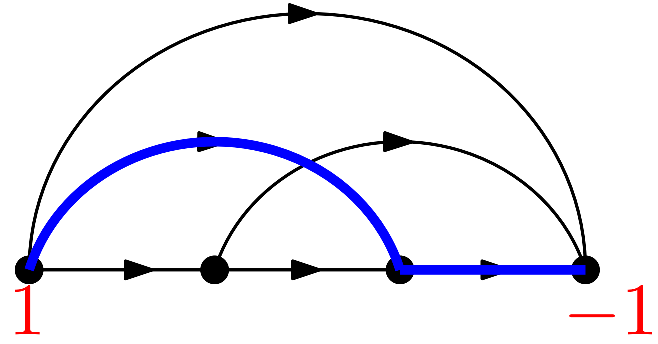

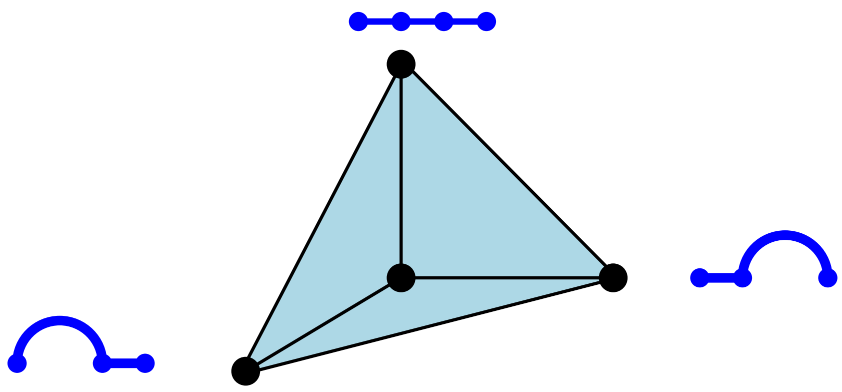

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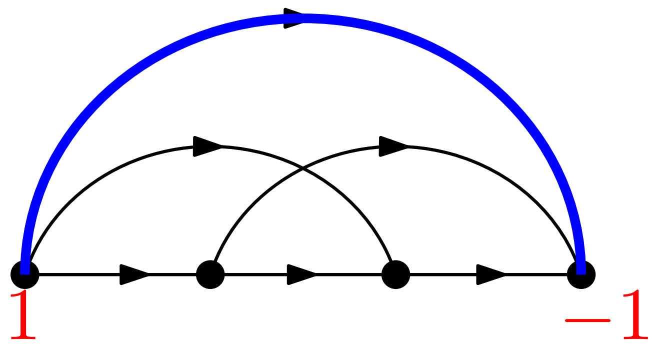

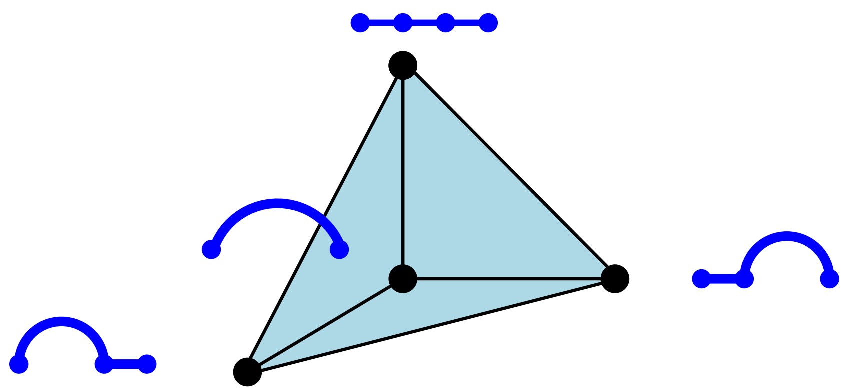

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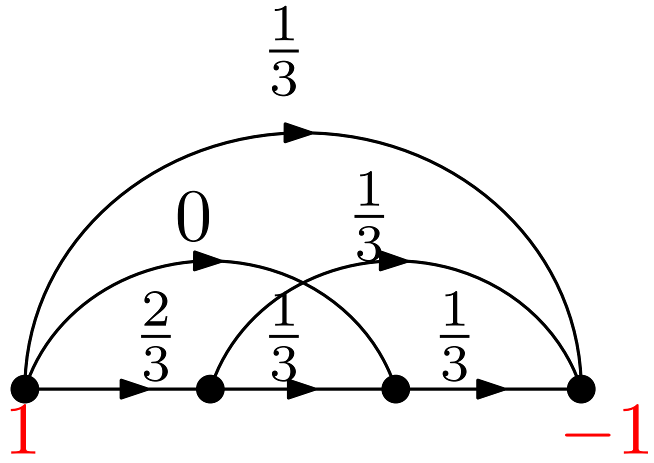
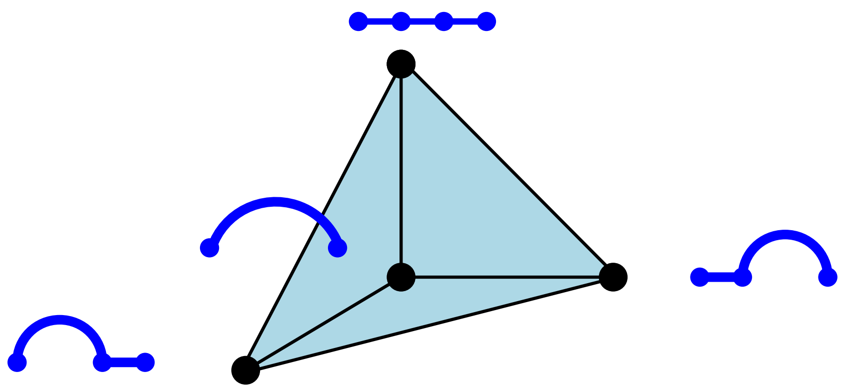




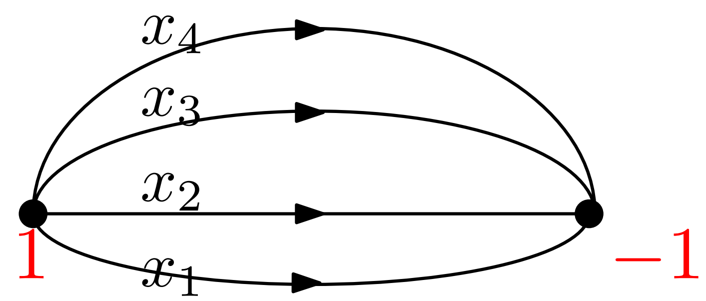
Examples of flow polytopes

$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow vertex } i = a_i\}$

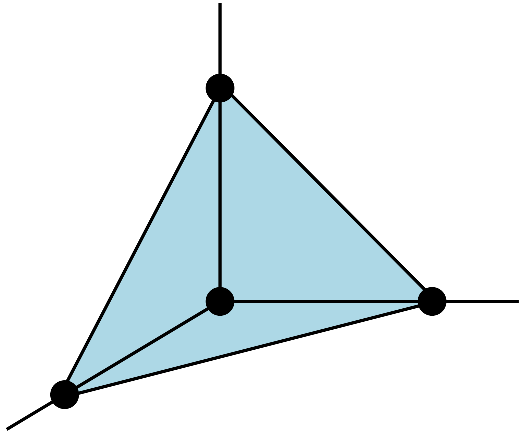
Example

$x_1 + x_2 + x_3 + x_4 = 1$

G



$\mathcal{F}_G(1,-1)$ is a simplex



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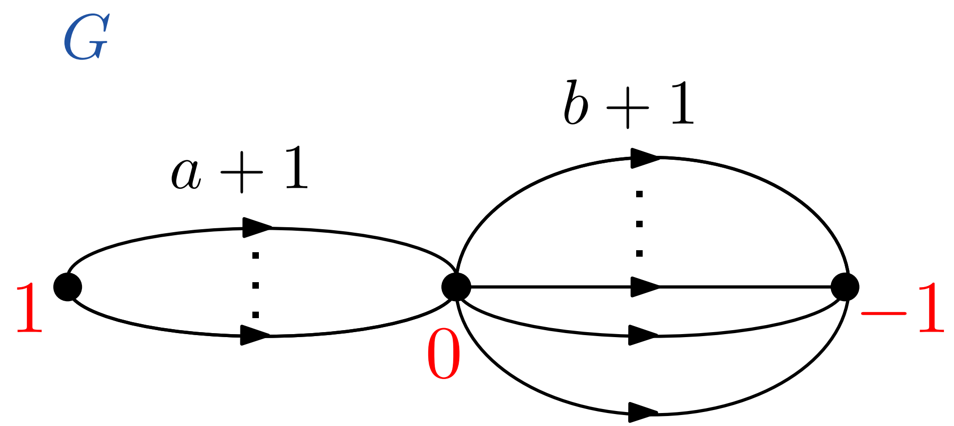

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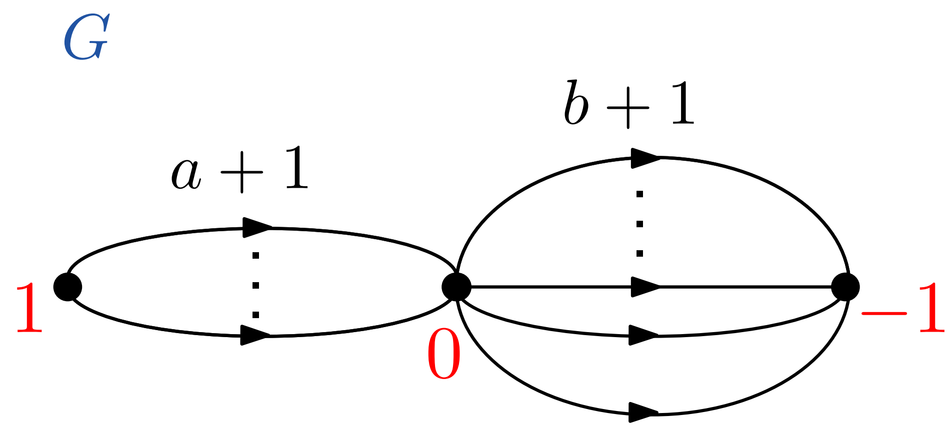
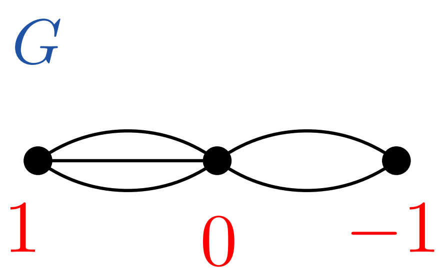
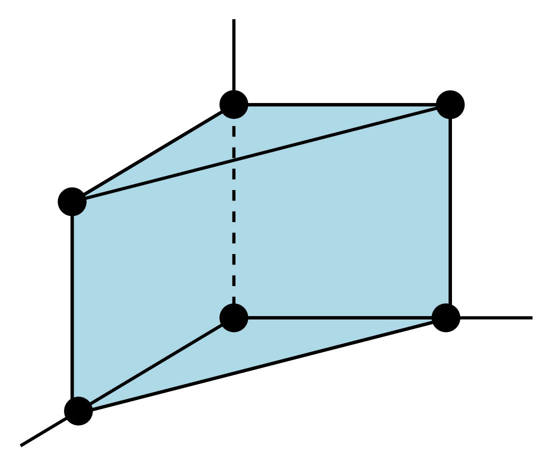


$\mathcal{F}_G(1, 0 - 1)$ is a product of simplicies $\Delta_a \times \Delta_b$



$\Delta_2 \times \Delta_1$



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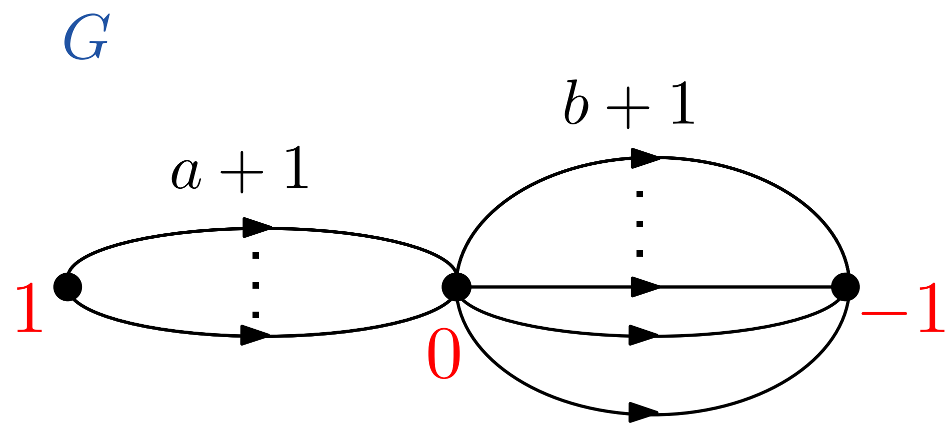

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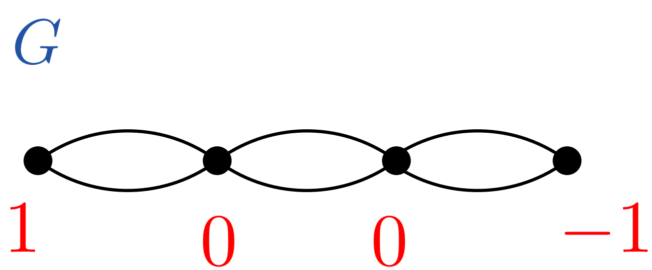
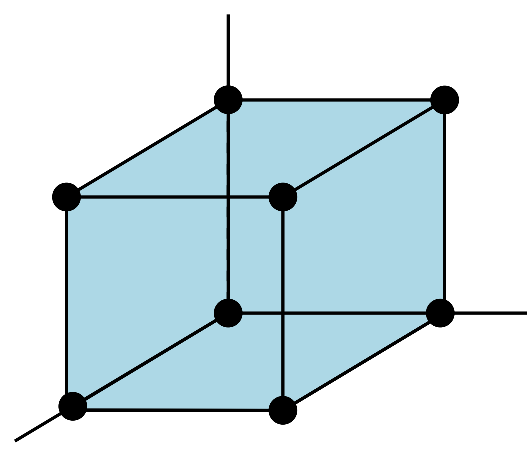


$\Delta_1 \times \Delta_1 \times \Delta_1$



Flow polytopes are "transcendental" too

flow polytopes have been related to:

- *Toric geometry* (Hille 2003)
- *Jeffrey–Kirwan residues* (Baldoni–Vergne 2009)
- *cluster algebras* (Danilov–Karzanov–Koshevoy 2012)

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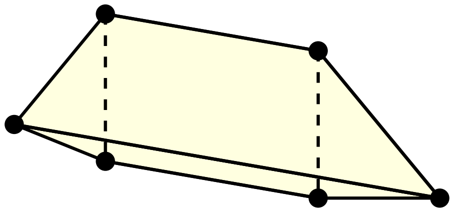

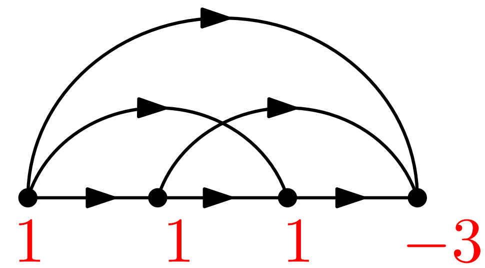

- *diagonal harmonics* (Mészáros-M-Rhoades 17, Liu-Mészáros-M 18)
- *generalized permutahedra* (Mészáros-St. Dizier 2017)
- *Schubert polynomials* (Escobar-Mészáros 2018)
  (Fink-Mészáros-St. Dizier 2018)
- *Gelfand-Tsetlin polytopes* (Liu-Mészáros-St. Dizier 2019)

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- *Brändén-Huh's Lorentzian polynomials* (Mészáros-Setiabatra 2019)
- *juggling sequences* (Harris-Insko-Omar 2015, B-H-H-M-S 2020)
- *rational Catalan combinatorics* (B-G-H-H-K-M-Y 2018, Yip 2019, Jang-Kim 2019)
- *Alternating sign matrices* (Mészáros-M-Striker 2019)

Flow polytopes of planar graphs

Theorem (Postnikov 13, Mészáros-M-Striker 19)
If G is a planar graph then $\mathcal{F}_G(1, 0, \ldots, 0, -1)$ is equivalent to an **order polytope** of a certain poset P.

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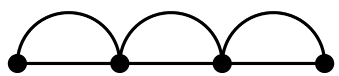

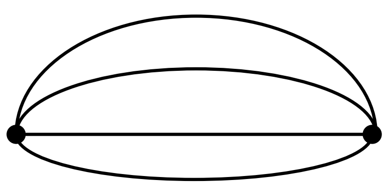

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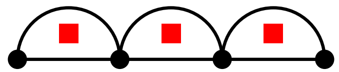

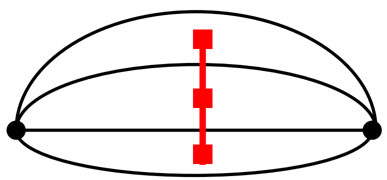

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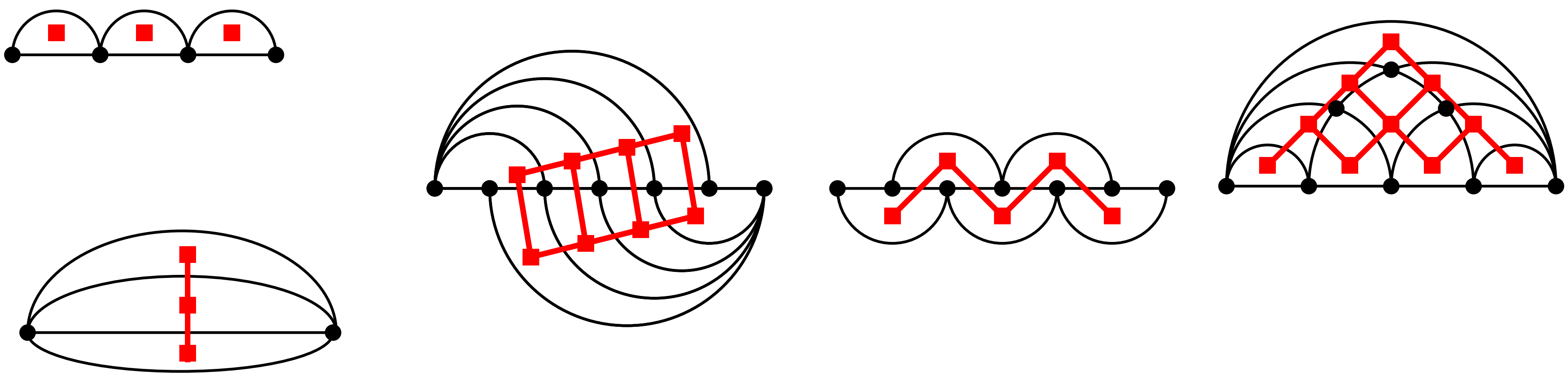

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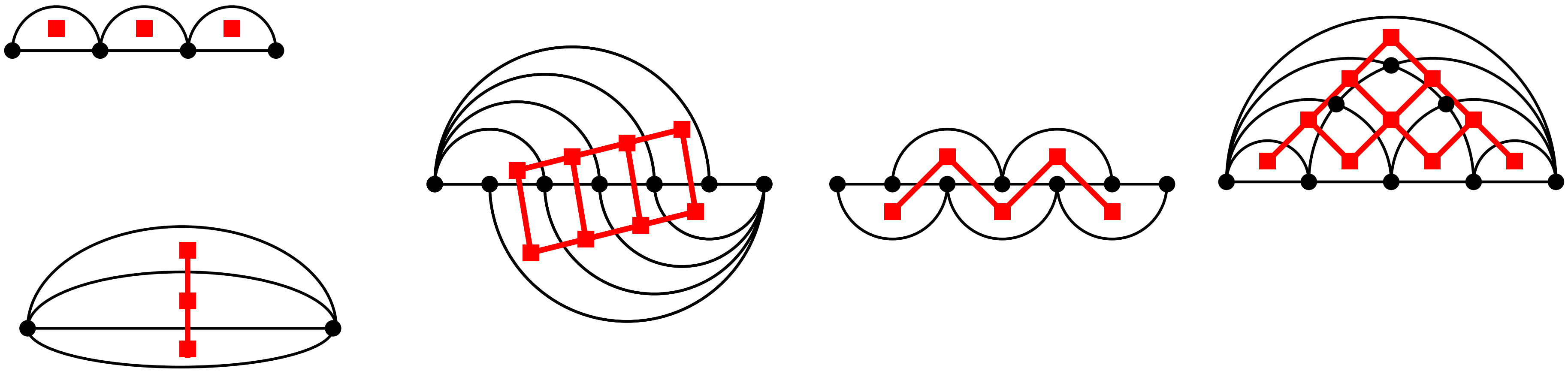

By Stanley's theory of order polytopes:

Corollary If G is a planar graph then

$$\mathrm{volume}\mathcal{F}_G(1,0,\ldots,0,-1) = \#\text{ linear extensions } P.$$

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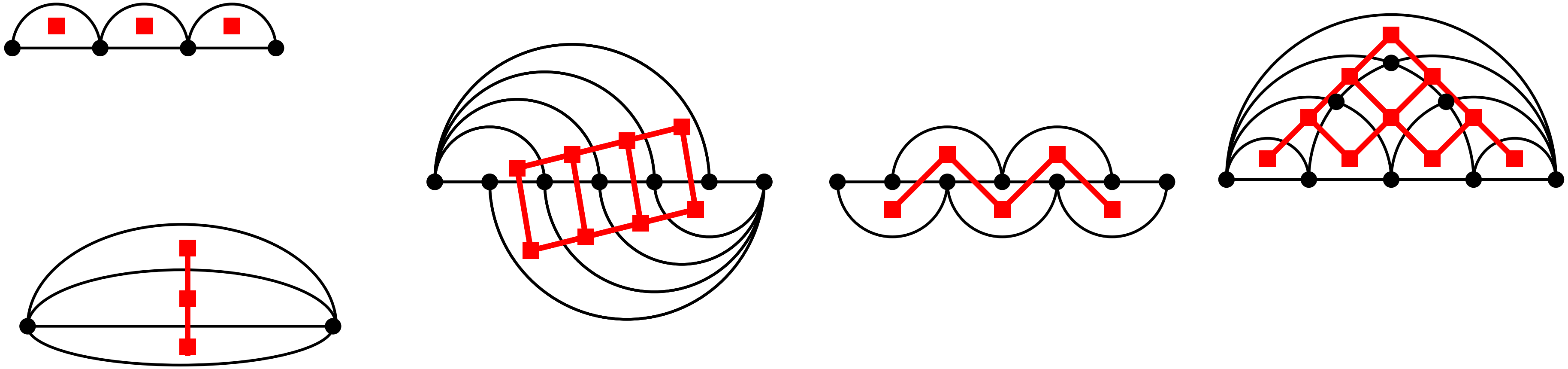

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A *linear extension* of a poset P is an ordering of the poset elements compatible with the partial order.

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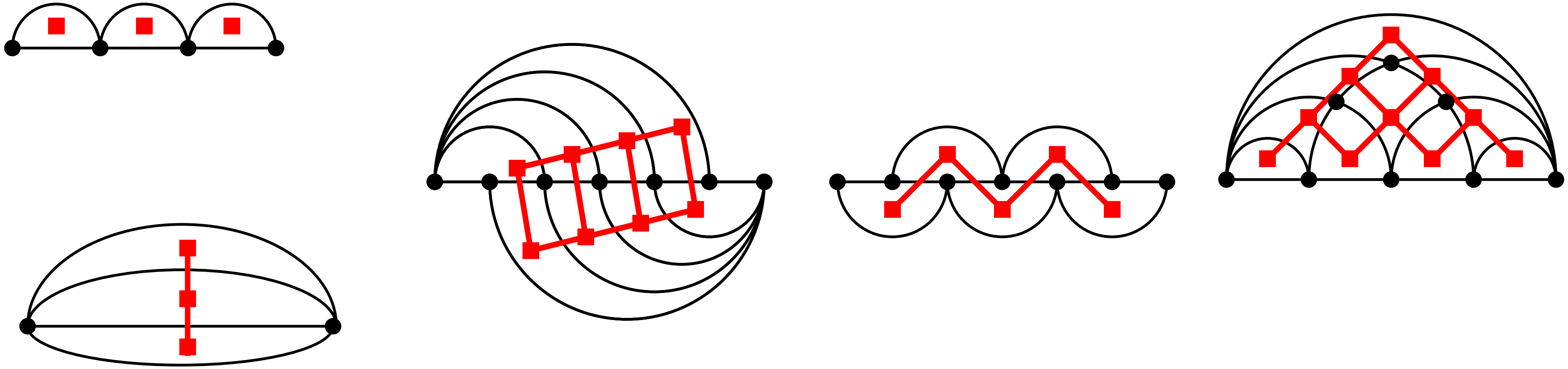

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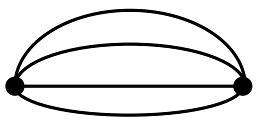
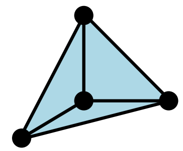
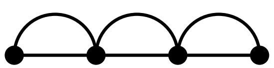
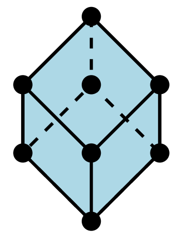
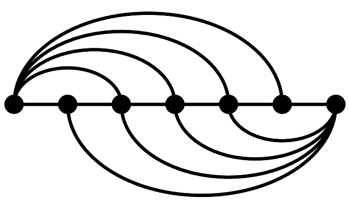
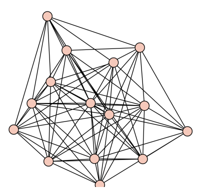
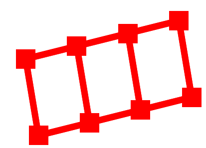
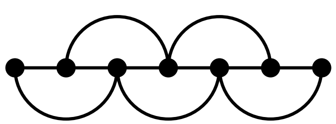
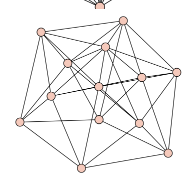
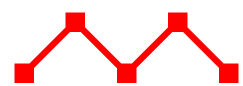
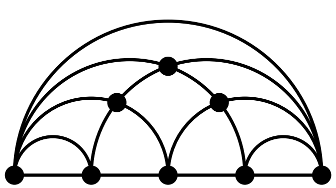
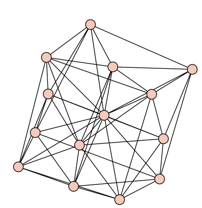
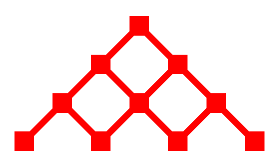
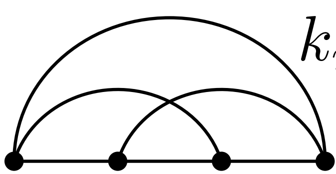
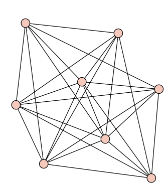
- $\mathcal{F}_{k_7}(1,0,0,0,0,0,-1)$ is not an order polytope. (Behrend-M-Panova 20+)

Flow polytopes of graphs

graph G	flow polytope $\mathcal{F}_G$	poset	volume
			1
			$n!$
			$\text{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$
			Euler numbers E_n
			# standard tableaux staircase shape
 k_{n+1}		?	?

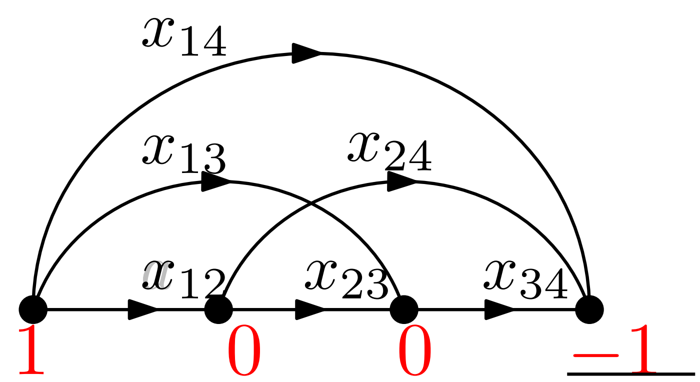
Examples flow polytopes

$$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow vertex } i = a_i\}$$

Example

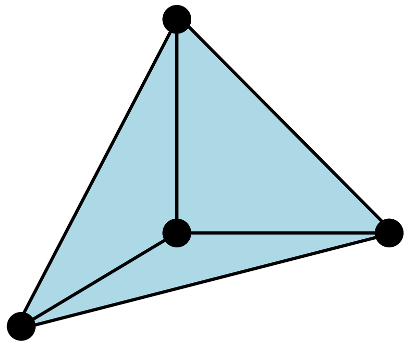
G is the complete graph k_{n+1}

$$\mathbf{a} = (1, 0, \ldots, 0, -1)$$



$\mathcal{F}_{k_{n+1}}(1, 0, \ldots, 0, -1)$ is called the **Chan-Robbins-Yuen** (CRY_n) polytope

has 2^{n-1} vertices, dimension $\binom{n}{2}$



Volume of the CRY_n polytope

$v_n := \mathrm{volume}(CRY_n)$

n	2	3	4	5	6	7
v_n	1	1	2	10	140	5880

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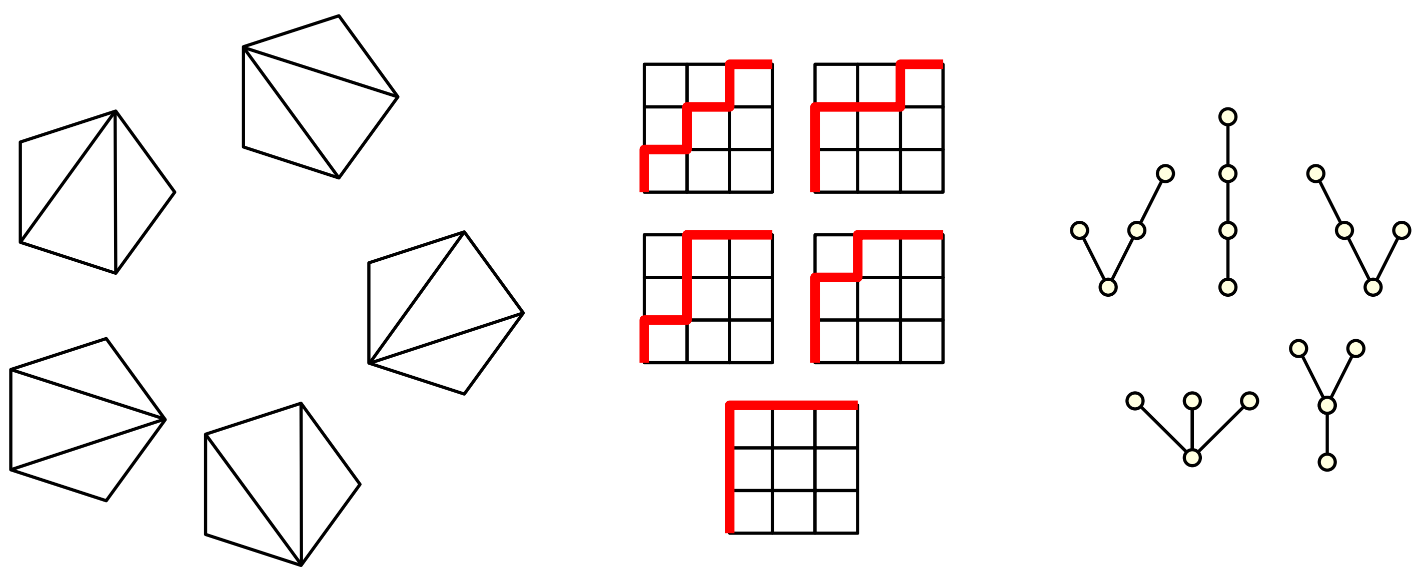
Catalan numbers $(1, 1, 2, 5, 14, 42, \ldots)$ count more than **200 different objects**



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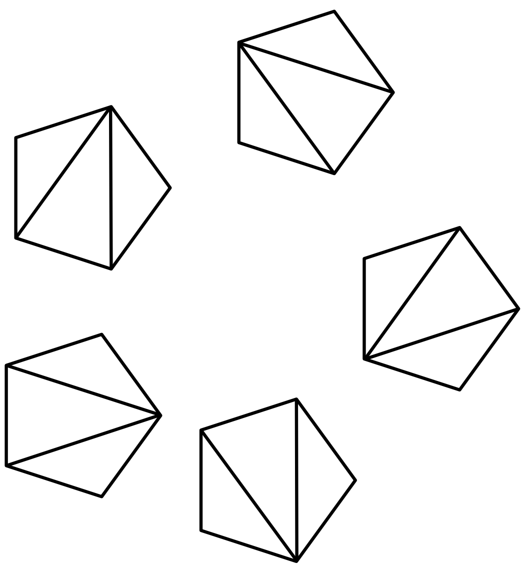
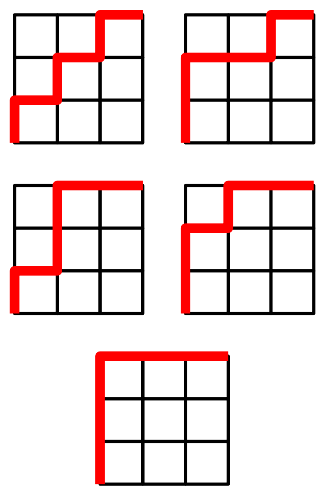
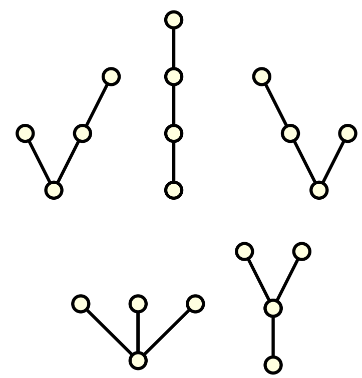
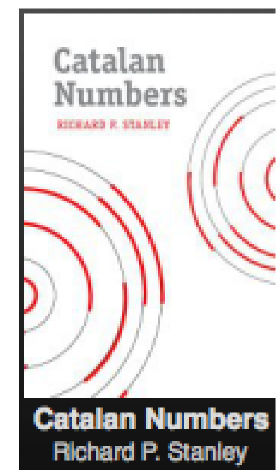

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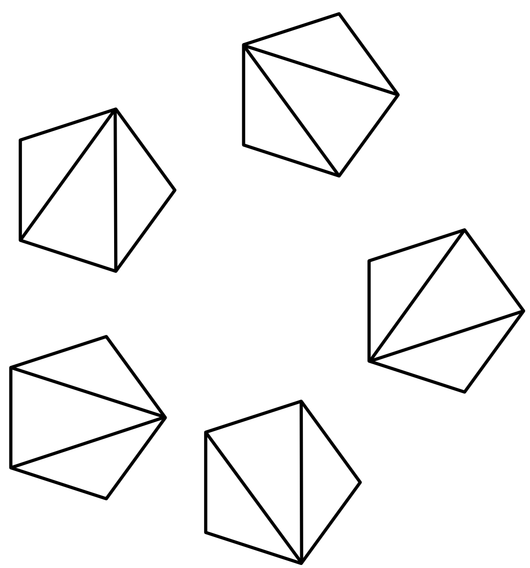

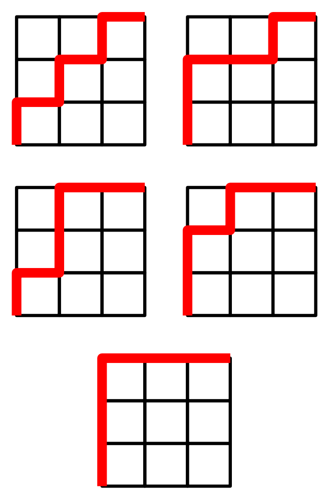

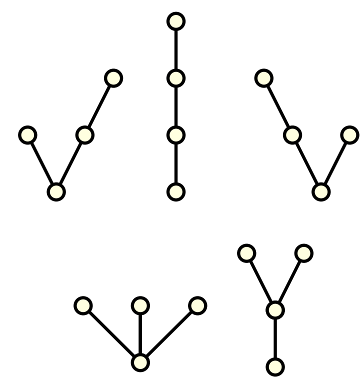

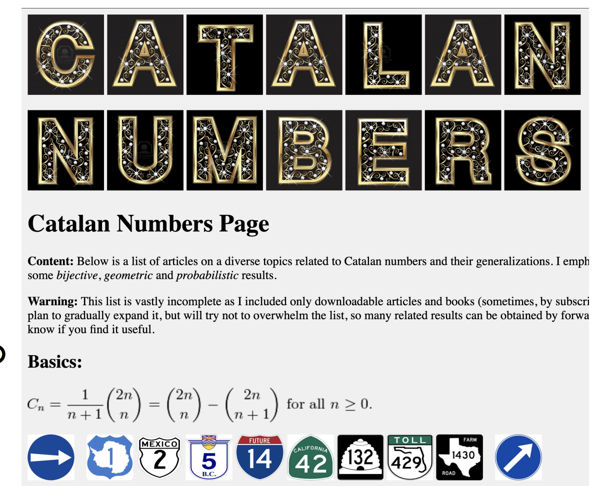

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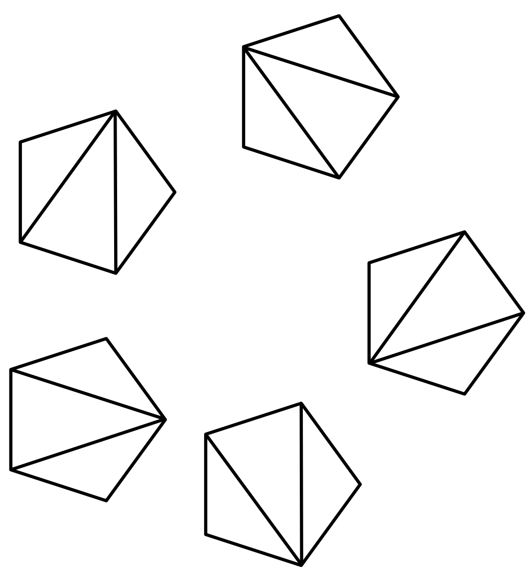
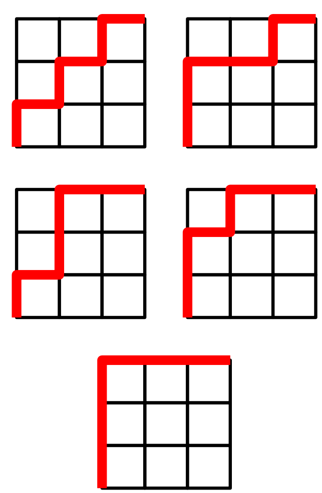
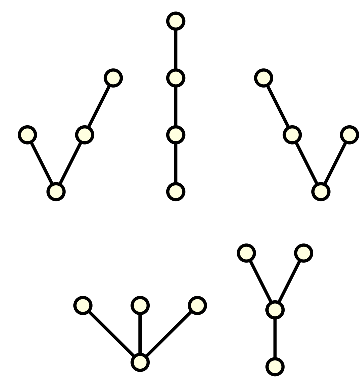
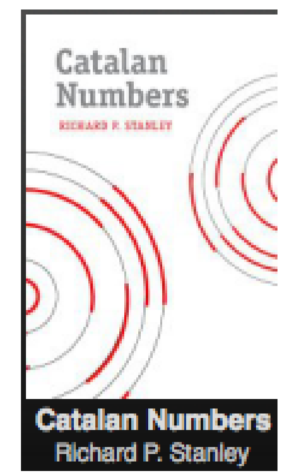


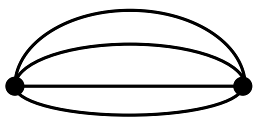
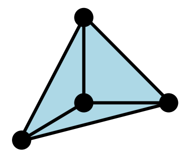
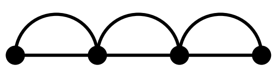
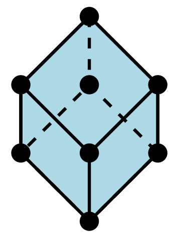
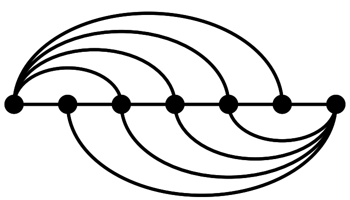
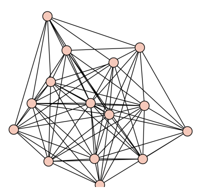
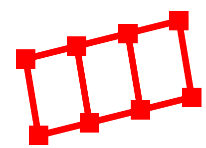
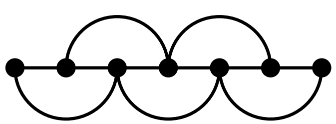
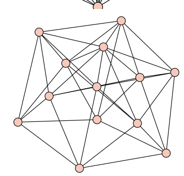
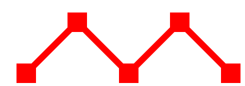
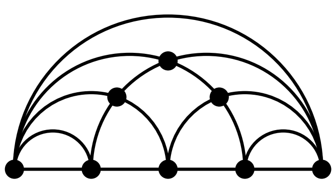
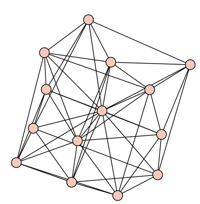
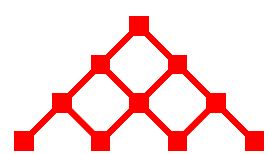
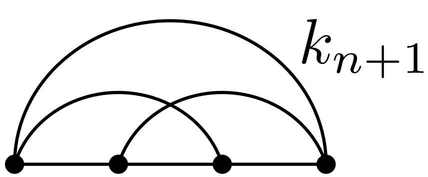
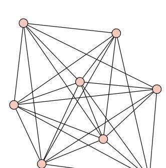






. . . however, there is no combinatorial proof of formula for v_n

Flow polytopes of graphs

graph G	flow polytope $\mathcal{F}_G$	poset	volume
			1
			$n!$
			$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$
			Euler numbers E_n
			# standard tableaux staircase shape
		no poset	$\mathrm{Cat}_1\mathrm{Cat}_2\cdots\mathrm{Cat}_{n-2}$?

Lattice points: Kostant's partition function

lattice points of $\mathcal{F}_G(\mathbf{a})$ are integral flows on G with netflow $\mathbf{a}$

let $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m) = L_{\mathcal{F}_G(\mathbf{a})}(1)$

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$K_{k_{n+1}}(\mathbf{a}) = \#$ of ways of writing $\mathbf{a}$ as an $\mathbb{N}$-combination of vectors
$e_i - e_j$ for $1 \leq i < j \leq n+1$

Lattice points: Kostant's partition function

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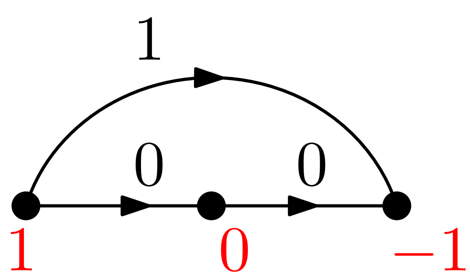
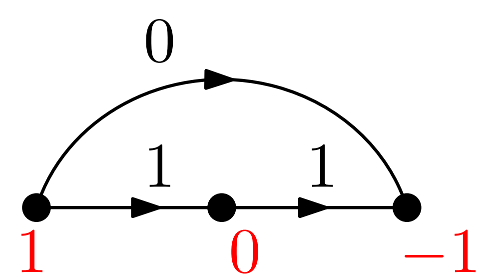


$(1, 0, -1) = e_1 - e_3$



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Generating function for $K_{k_{n+1}}(\mathbf{a})$:

$$\sum_{\mathbf{a}} K_G(\mathbf{a})\mathbf{x}^{\mathbf{a}} = \prod_{1\leq i<j\leq n+1} (1 - x_i x_j^{-1})^{-1}.$$

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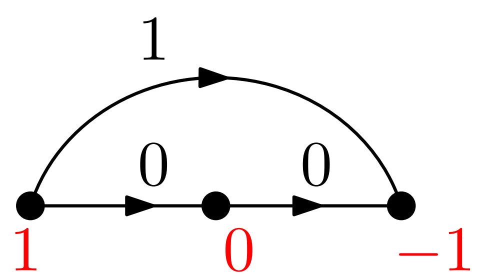

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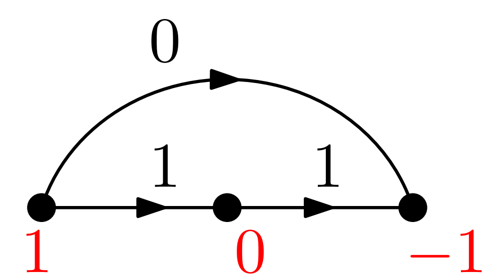

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Formulas for **weight multiplicities** and **tensor product multiplicities** of *type A semisimple Lie algebras* in terms of $K_{k_{n+1}}(\mathbf{a})$.

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"...he said to me that in any good mathematical theory there should be at least one "transcendental" element ... should account for many of the subtleties of the theory. In the Cartan-Weyl theory, he said that my partition function was the transcendental element."

Bertram Kostant on profile of I. M. Gelfand (Notices of the AMS, Jan. 2013)

Fundamental theorem volume of flow polytopes

let $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m)$

Theorem (Stanley-Postnikov 09, Baldoni-Vergne 09)

$$\text{volume}\mathcal{F}_G(1,0,\ldots,0,-1) = K_G(0,i_2,i_3,\ldots,-\textstyle\sum_k i_k),$$

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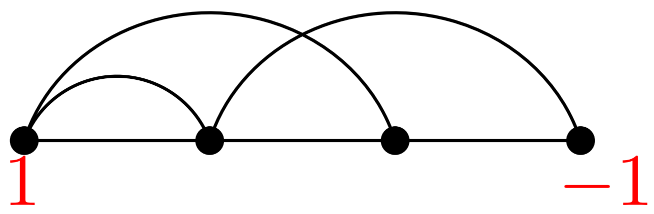
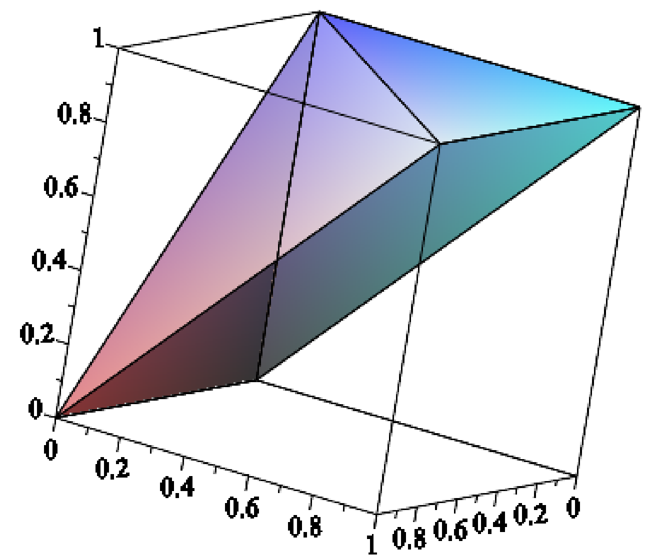
Example





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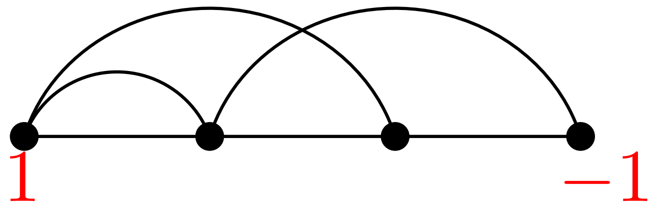

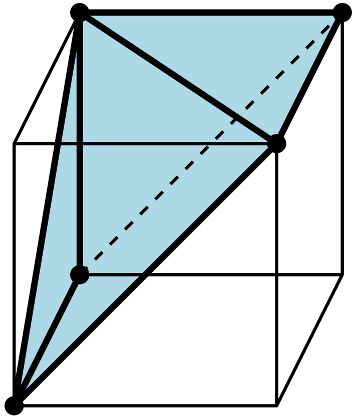

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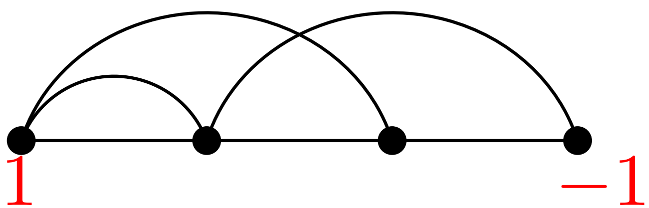

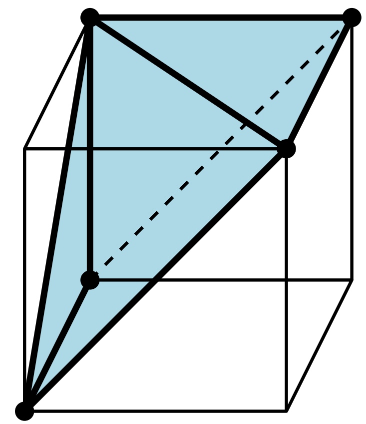

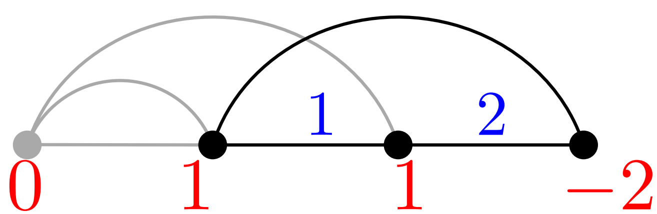

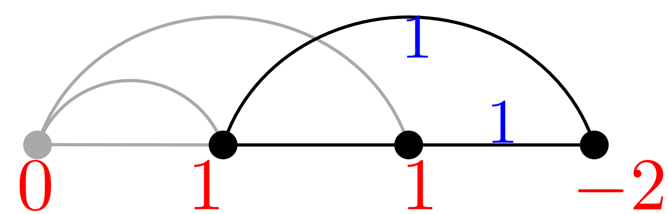

volume $= 2$.

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at $a=b=c=1$ gives $\text{Cat}_1 \cdots \text{Cat}_{n-2}$.

Zeilberger's entire paper

PROOF OF A CONJECTURE OF CHAN, ROBBINS, AND YUEN

Doron ZEILBERGER [1]

Abstract: Using the celebrated Morris Constant Term Identity, we deduce a recent conjecture of Chan, Robbins, and Yuen (math.CO/9810154), that asserts that the volume of a certain $n(n-1)/2$-dimensional polytope is given in terms of the product of the first $n-1$ Catalan numbers.

Chan, Robbins, and Yuen[CRY] conjectured that the cardinality of a certain set of triangular arrays $\mathcal{A}_n$ defined in pp. 6-7 of [CRY] equals the product of the first $n-1$ Catalan numbers. It is easy to see that their conjecture is equivalent to the following *constant term identity* (for any rational function $f(z)$ of a variable z, $CT_z f(z)$ is the coeff. of z^0 in the formal Laurent expansion of $f(z)$ (that always exists)):

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-2} \prod_{1 \le i < j \le n} (x_j - x_i)^{-1} = \prod_{i=1}^{n} \frac{1}{i+1} \binom{2i}{i} \quad . \qquad (CRY)$$

But this is just the special case $a=2$, $b=0$, $c=1/2$, of the *Morris Identity*[M] (where we made some trivial changes of discrete variables, and 'shadowed' it)

$$CT_{x_n} \dots CT_{x_1} \prod_{i=1}^{n} (1-x_i)^{-a} \prod_{i=1}^{n} x_i^{-b} \prod_{1 \le i < j \le n} (x_j - x_i)^{-2c} = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a+b+(n-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)} \, . \qquad (Chip)$$

To show that the right side of $(Chip)$ reduces to the right side of (CRY) upon the specialization $a=2, b=0, c=1/2$, do the plugging in the former and call it M_n. Then manipulate the products to simplify M_n/M_{n-1}, and *then* use *Legendre's duplication formula* $\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)\Gamma(1/2)/2^{2z-1}$ three times, and *voilà*, up pops the Catalan number $\binom{2n}{n}/(n+1)$. $\square$

Remarks: 1. By converting the left side of $(Chip)$ into a contour integral, we get the same integrand as in the Selberg integral (with $a \to -a$, $b \to -b-1$, $c \to -c$). Aomoto's proof of the Selberg integral (SIAM J. Math. Anal. **18**(1987), 545-549) goes verbatim. **2.** Conjecture 2 in [CRY] follows in the same way, from (the obvious contour-integral analog of) Aomoto's extension of Selberg's integral. Introduce a new variable t, stick $CT_t t^{-k}$ in front of (CRY), and replace $(1-x_i)^{-2}$ by $(1-x_i)^{-1}(t + x_i/(1-x_i))$. **3.** Conjecture 3 follows in the same way from another specialization of $(Chip)$.

References

[CRY] Clara S. Chan, David P. Robbins, and David S. Yuen, *On the volume of a certain polytope*, `math.CO/9810154`.

[M] Walter Morris, *"Constant term identities for finite and affine root systems, conjectures and theorems"*, Ph.D. thesis, University of Wisconsin, Madison, Wisconsin, 1982.

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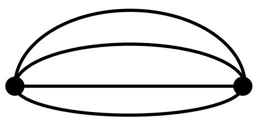
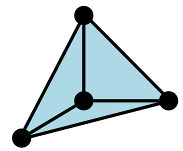
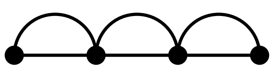
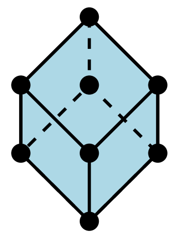
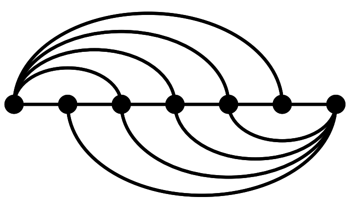
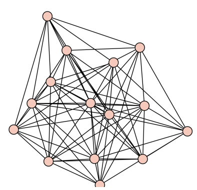
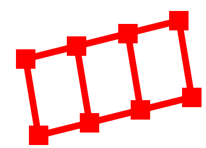
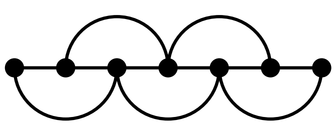
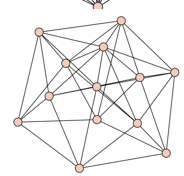
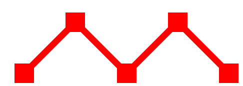
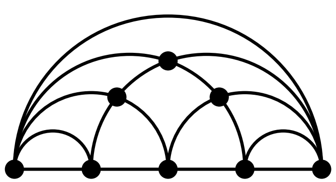
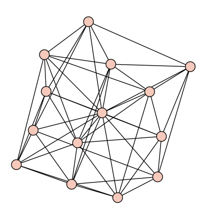
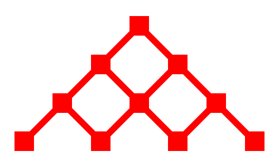
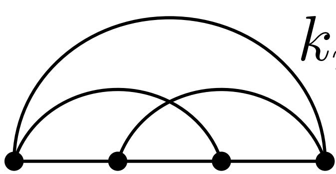
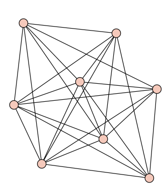
[1] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA. `zeilberg@math.temple.edu` `http://www.math.temple.edu/~ zeilberg/` . Nov. 17, 1998. Supported in part by the NSF.

Flow polytopes of graphs

graph G	flow polytope $\mathcal{F}_G$	poset	volume
			1
			$n!$
			$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$
			Euler numbers E_n
			# standard tableaux staircase shape
		no poset	$\mathrm{Cat}_1\mathrm{Cat}_2\cdots\mathrm{Cat}_{n-2}$?

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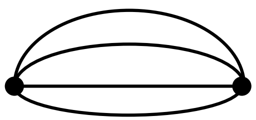
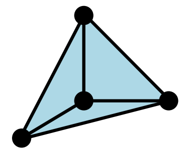
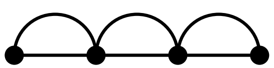
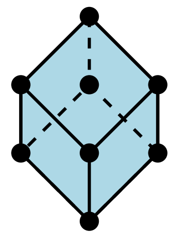
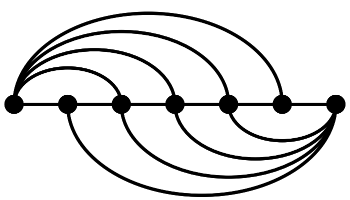
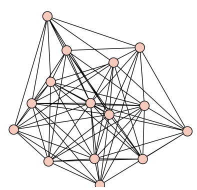
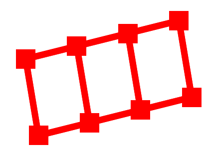
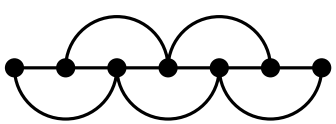
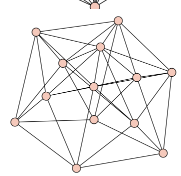
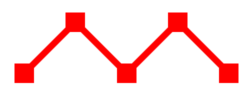
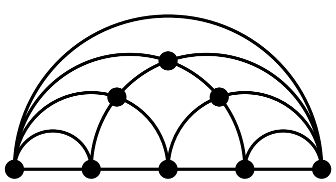
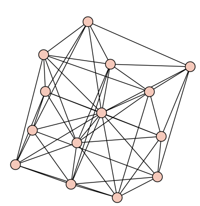
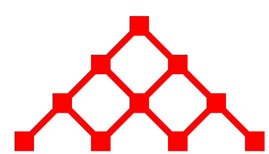
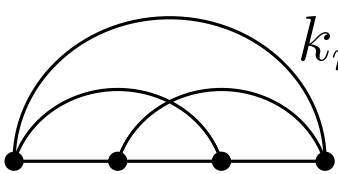
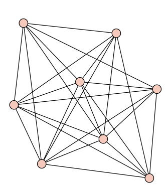
			1
			$n!$
			$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}$
			Euler numbers E_n
			# standard tableaux staircase shape
 k_{n+1}		no poset	$\mathrm{Cat}_1\mathrm{Cat}_2\cdots\mathrm{Cat}_{n-2}$??

Fundamental theorem + symmetry

Theorem (Stanley-Postnikov 09, Baldoni-Vergne 09)

$$\mathrm{volume}\mathcal{F}_G(1,0,\ldots,0,-1) = K_G(0,i_2,i_3,\ldots,-\textstyle\sum_k i_k),$$

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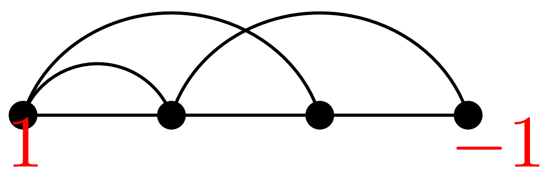

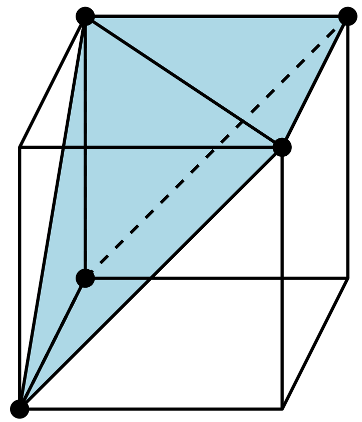

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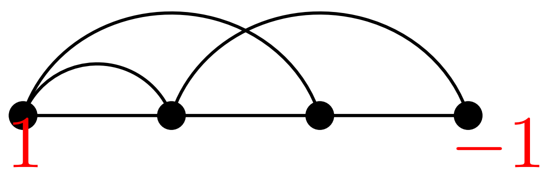

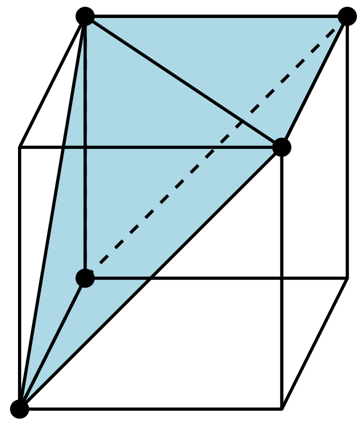

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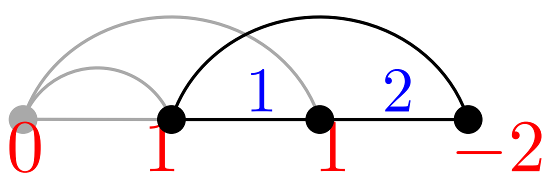

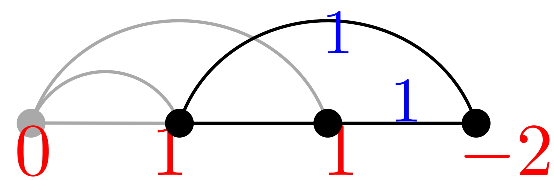

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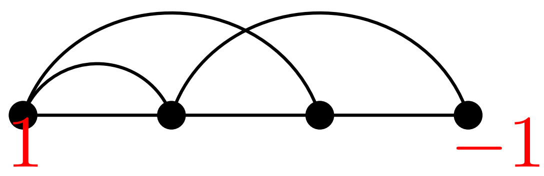

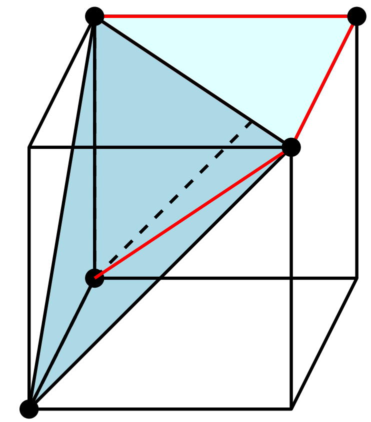

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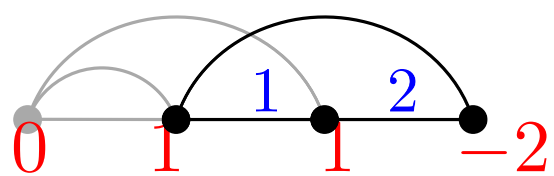

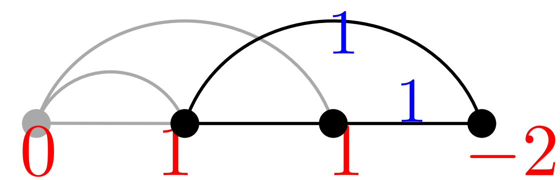

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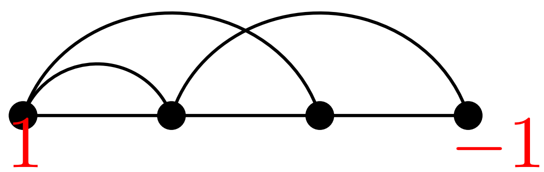

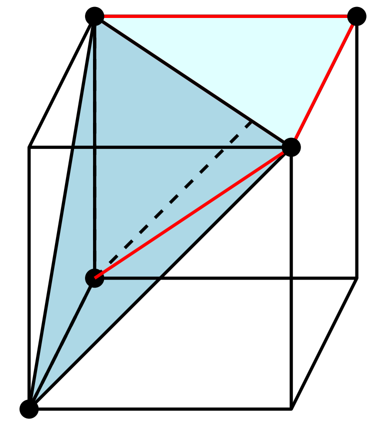

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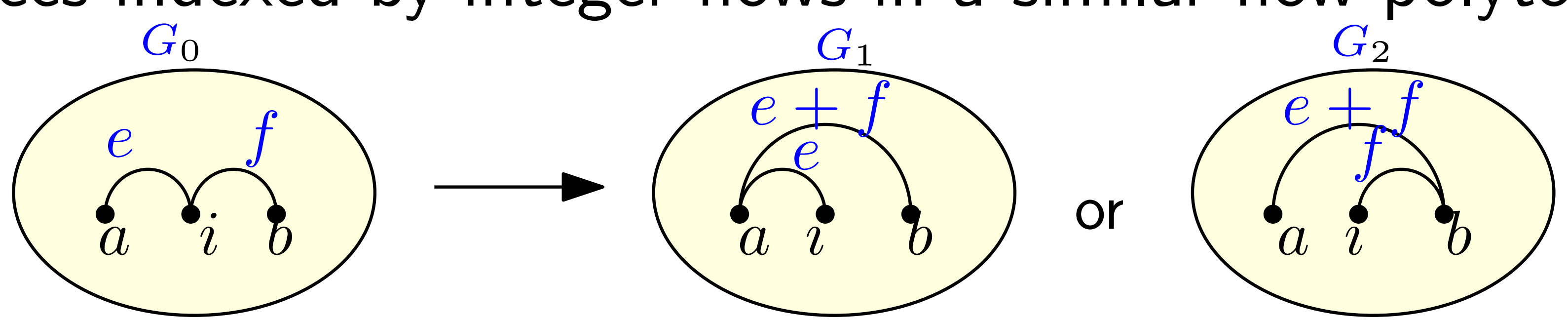

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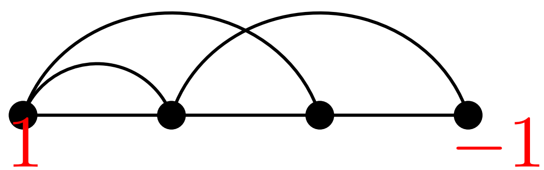

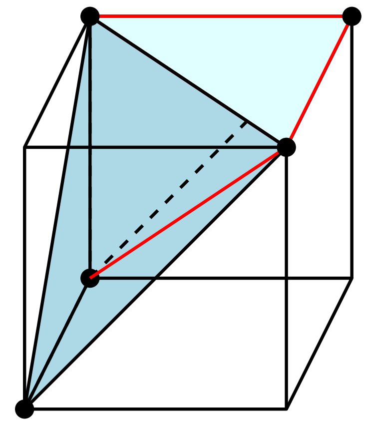

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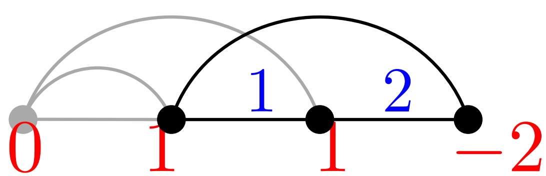

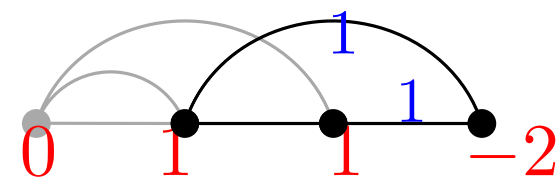

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Proof

Both sides are the volume of $\mathcal{F}_G(10\cdots 0-1) \equiv \mathcal{F}_{G^r}(10\cdots 0-1)$. $\square$

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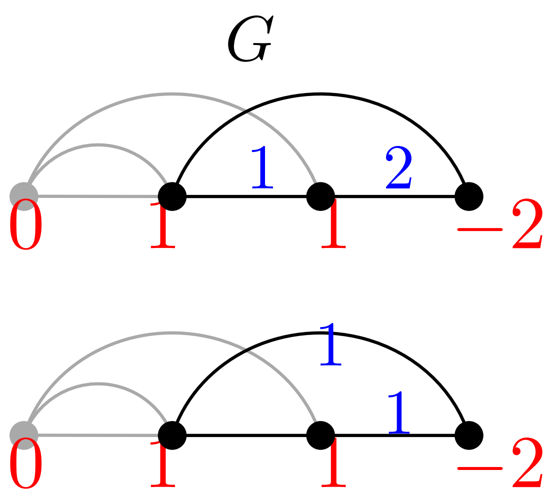

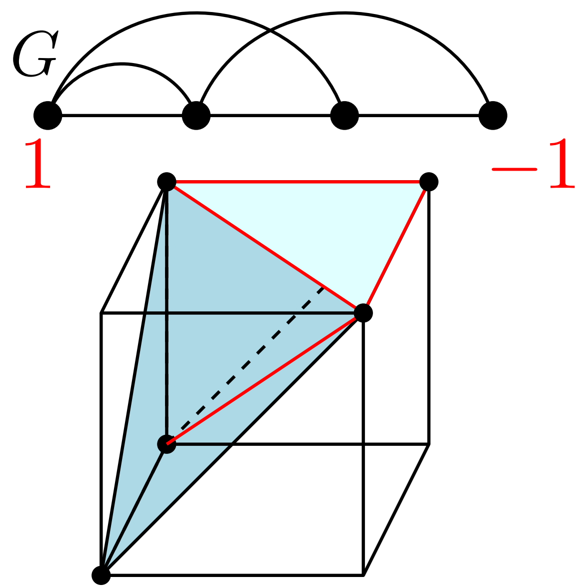

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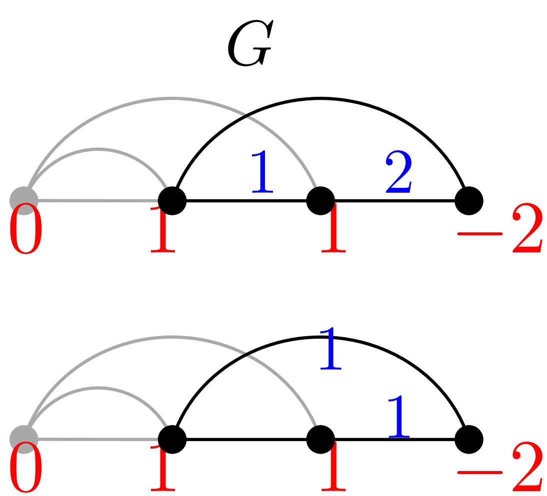

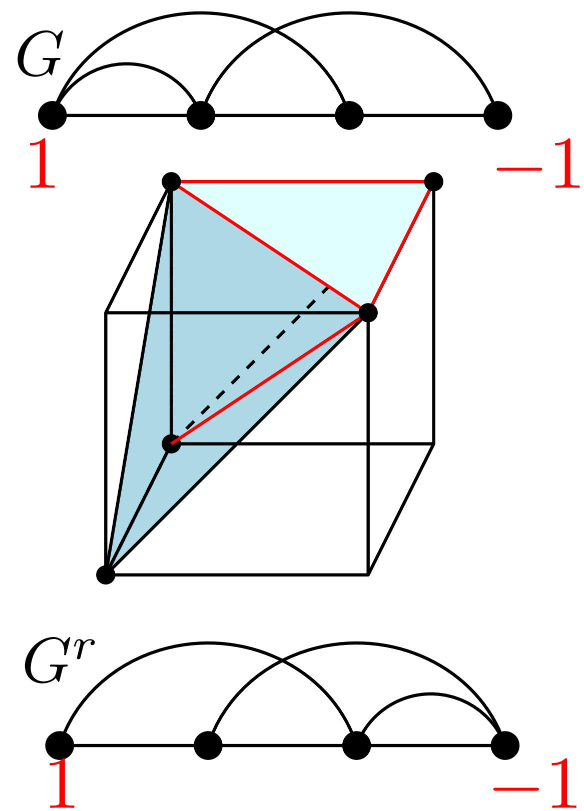

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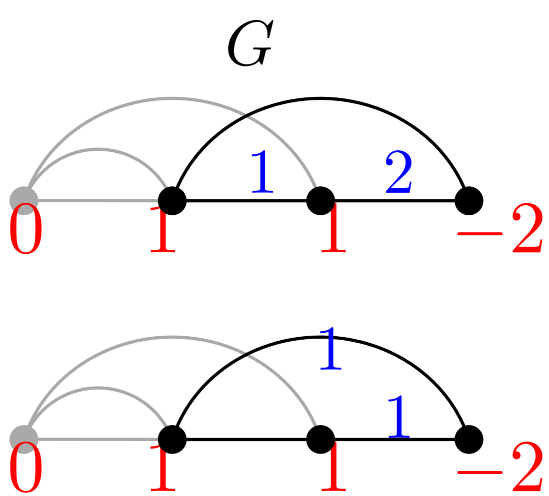

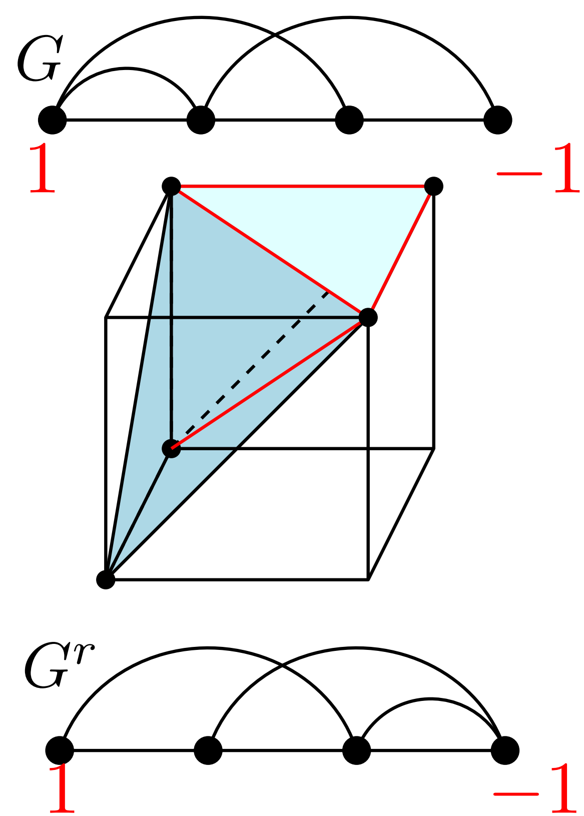

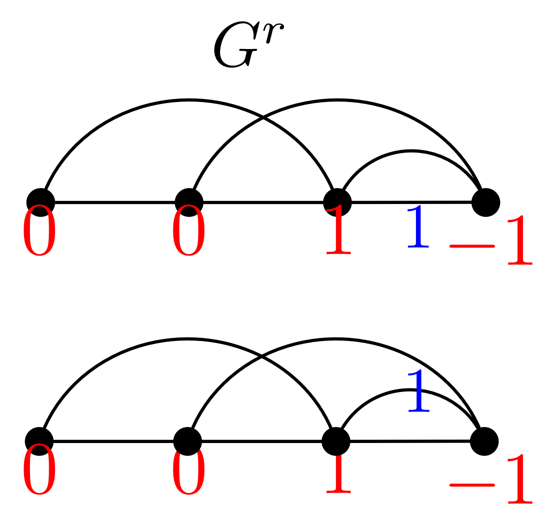

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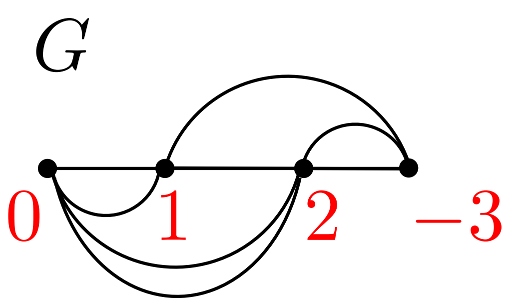

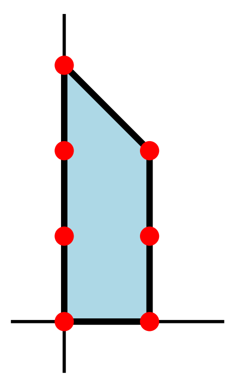

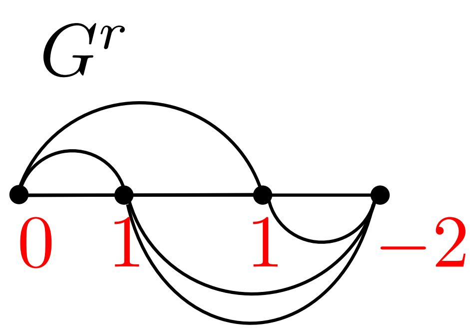

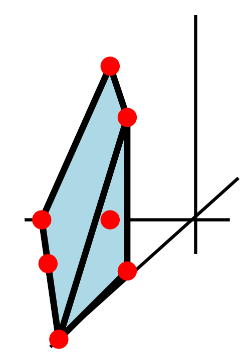

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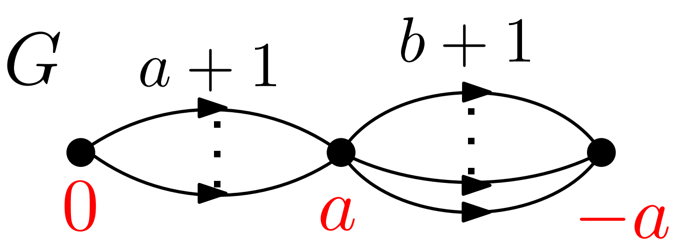

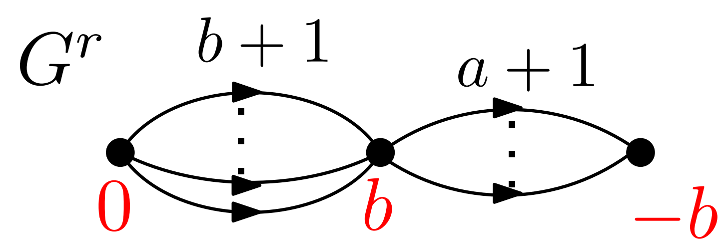

$$\binom{a+b}{b} = \#(a\Delta_b \cap \mathbb{Z}^b) = \#(b\Delta_a \cap \mathbb{Z}^a) = \binom{a+b}{a}$$

Bijection between lattice points K_G and K_{G^r}

- $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m)$

Corollary

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Summary

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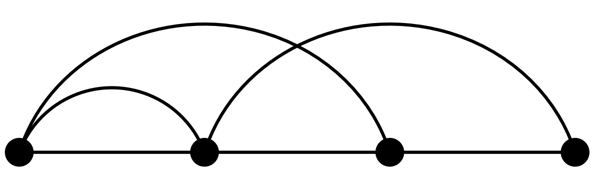

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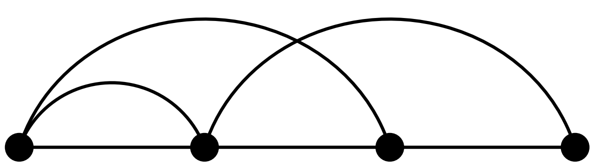
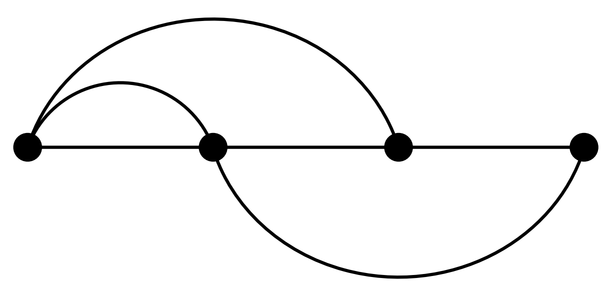
Example: two framings of G





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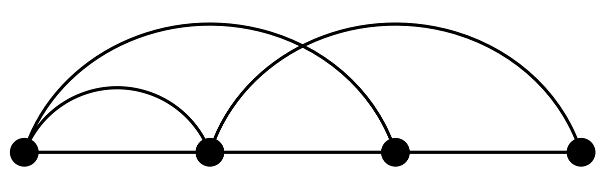

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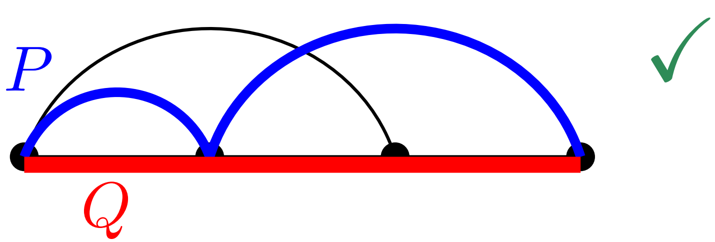

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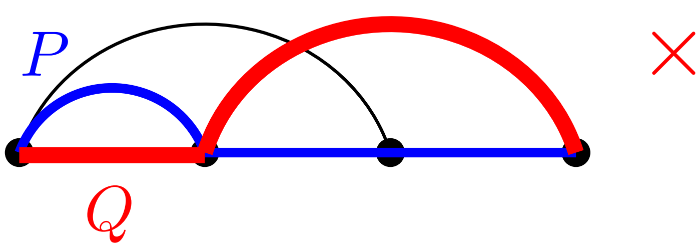

DKK triangulation of flow polytopes

In 2012 Danilov-Karzanov-Koshevoy gave another triangulation of $\mathcal{F}_G$ characterizing the vertices of each simplex

setup

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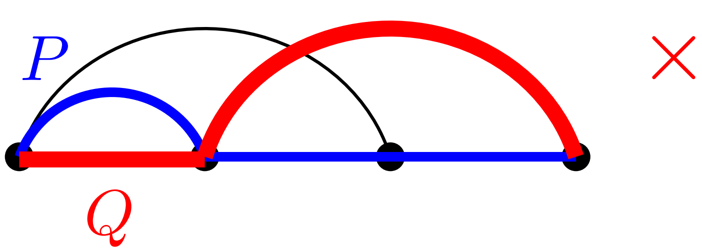
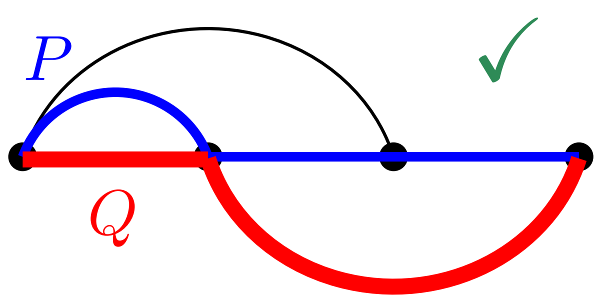
Example: two framings of G





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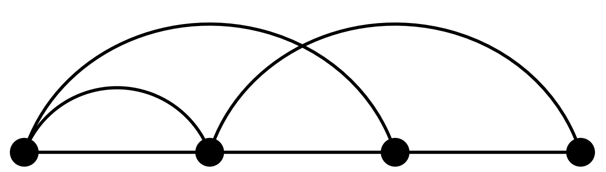

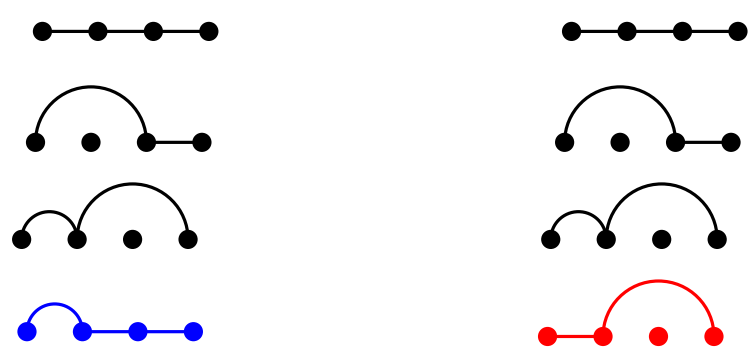

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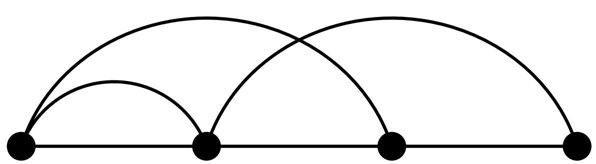

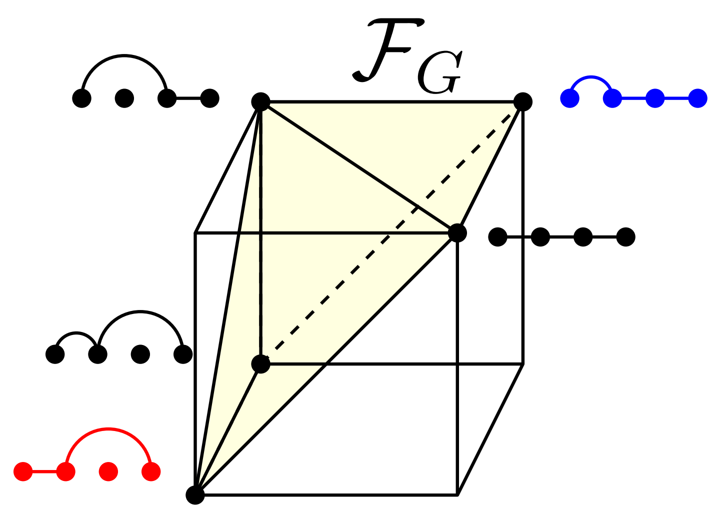

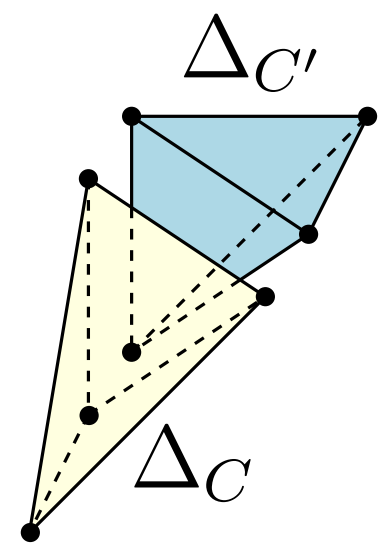

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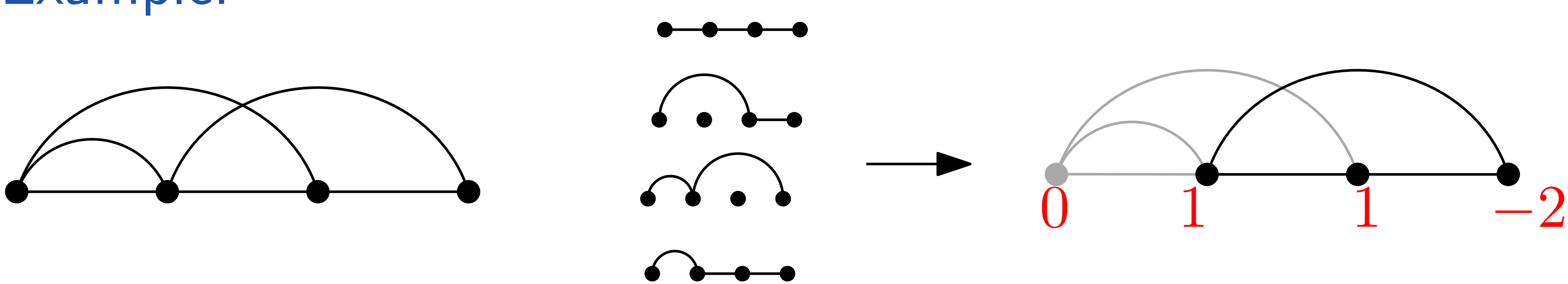

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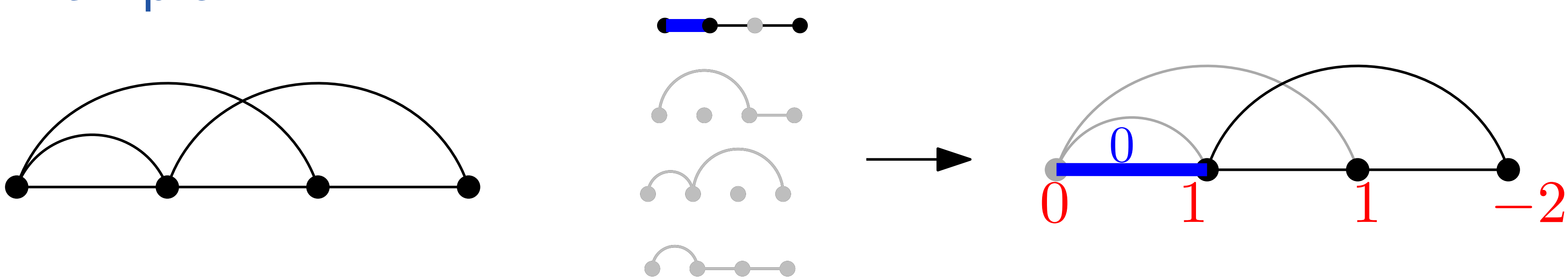

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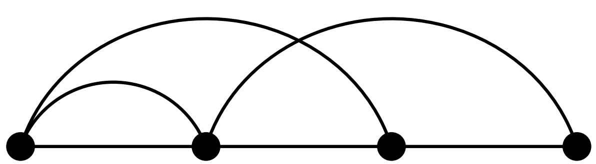

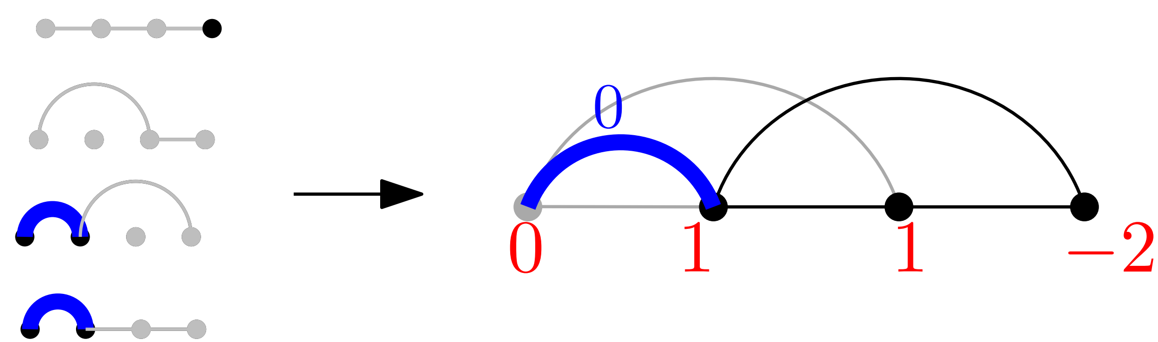

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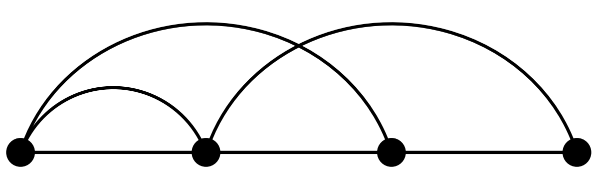

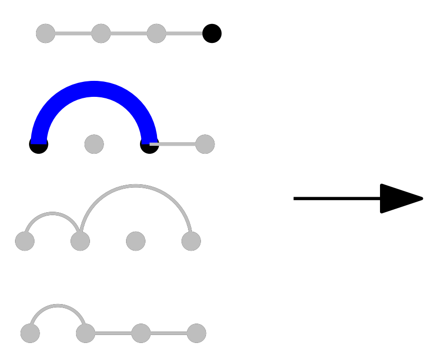

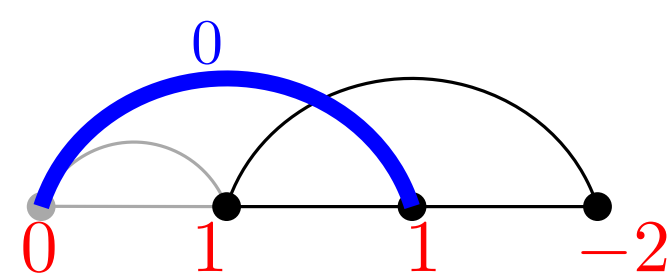

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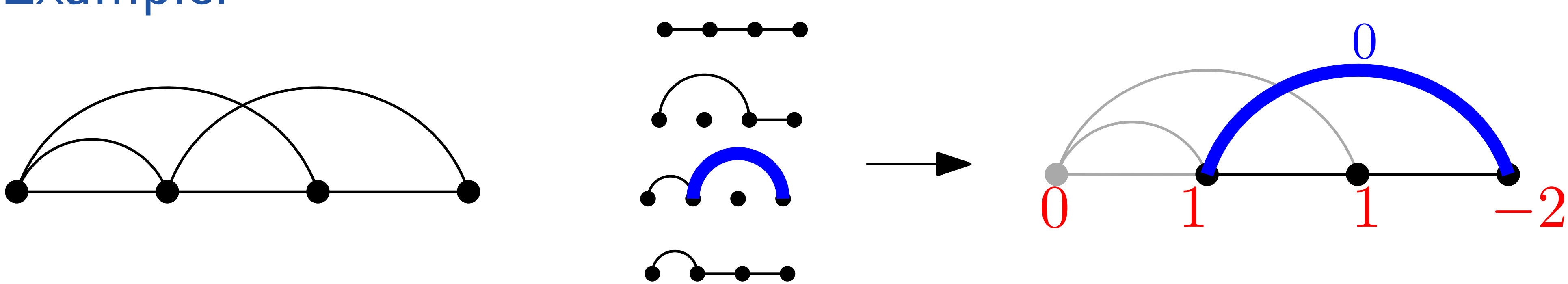

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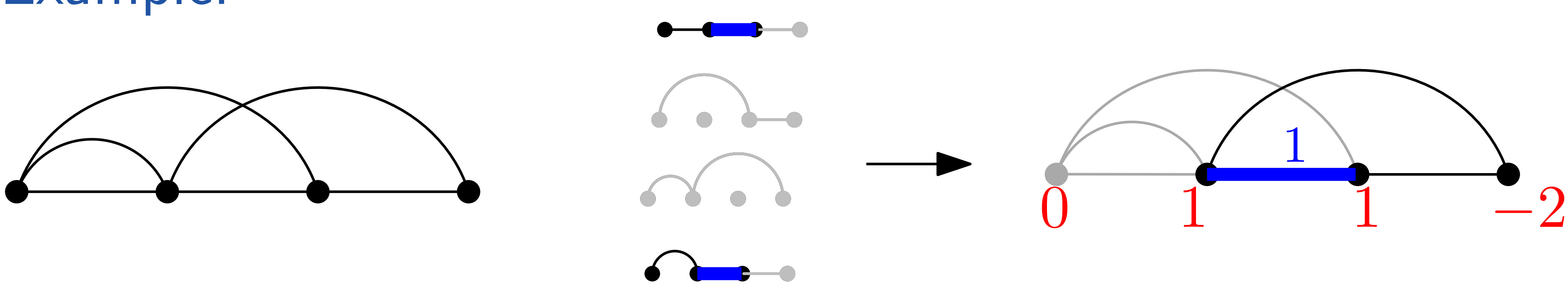

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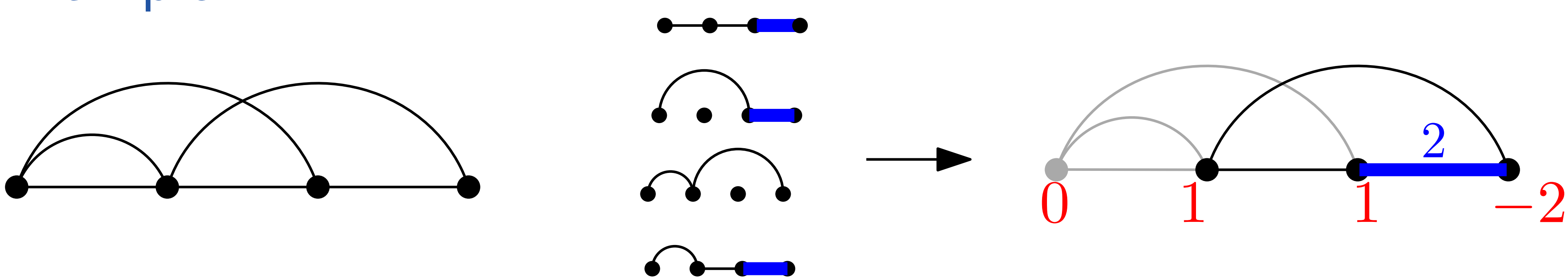

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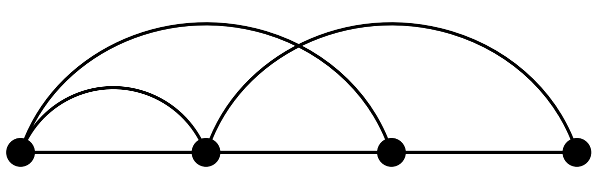

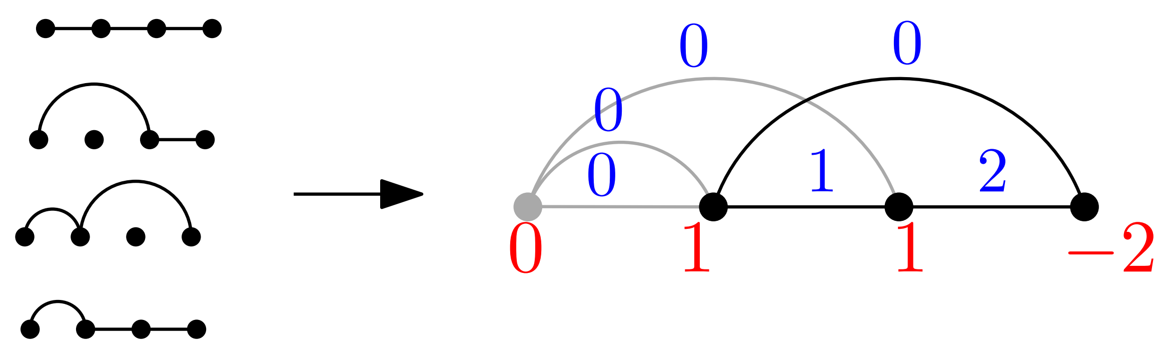

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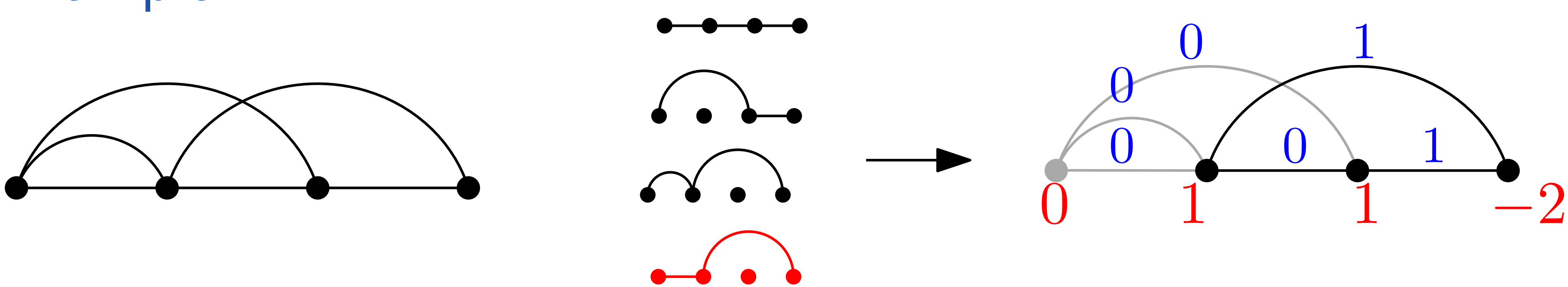

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Corollary

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G^r reverse of G, i_k (i'_k) is indegree -1 vertex k in G (G^r).

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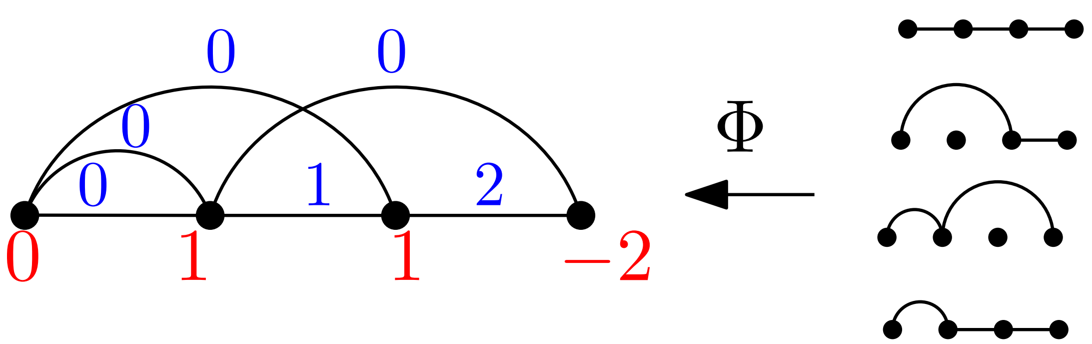

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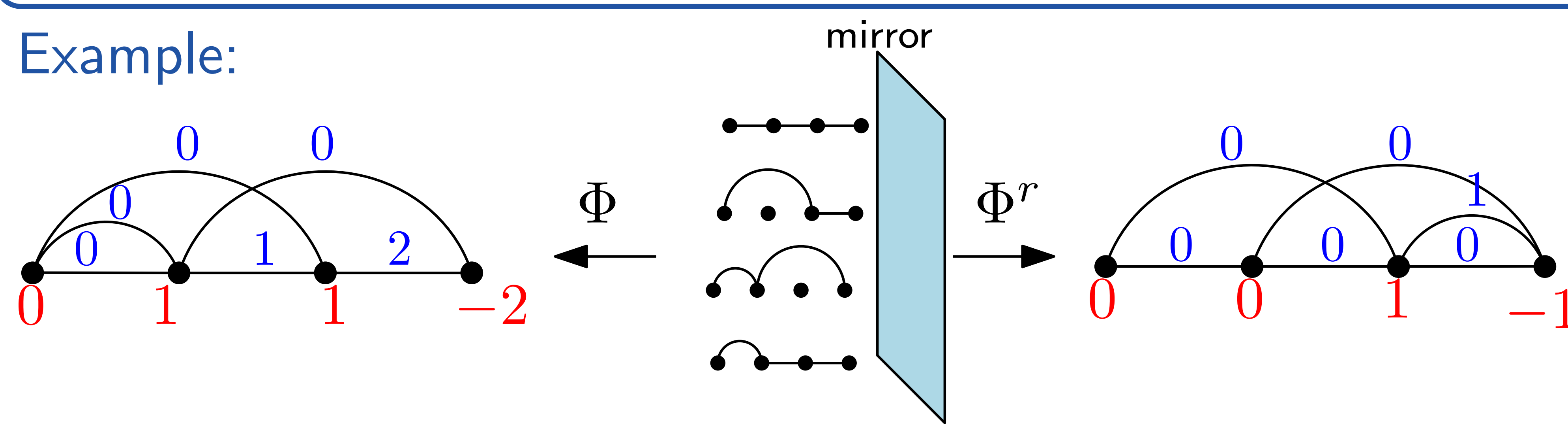

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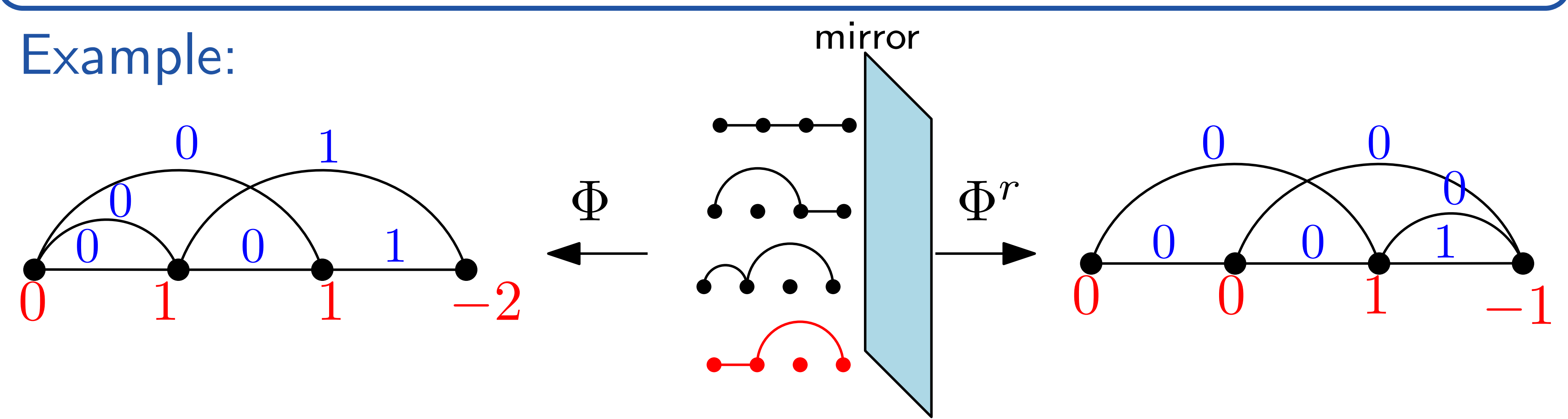

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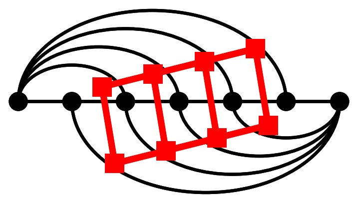

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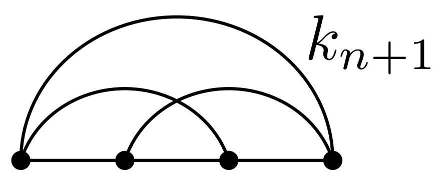

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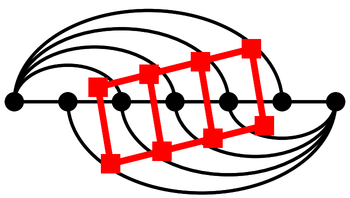

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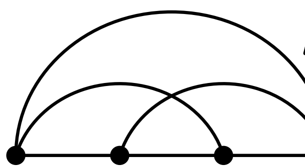


- We give an explicit bijection between the integer flows of $\mathcal{F}_G(0,i_2,i_3,\ldots)$ and the simplices of a DKK triangulation.

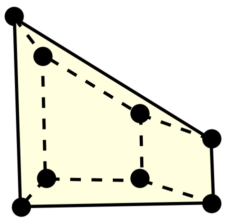
- The bijection depends on a framing of G and has interesting symmetry properties.

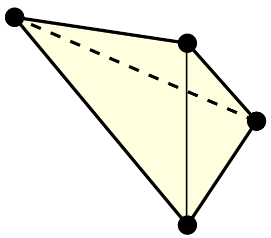
- (with K. Mészáros, J. Striker) On flow polytopes, order polytopes, and certain faces of the alternating sign matrix polytope, **arxiv:1510.03357v2**, *Discrete and Computational Geometry*, Volume **62** (2019) 128–163

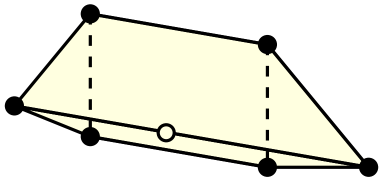
- Triangulations of flow polytopes, in preparation

Panta Rhei = everything flows (Heraclitus)

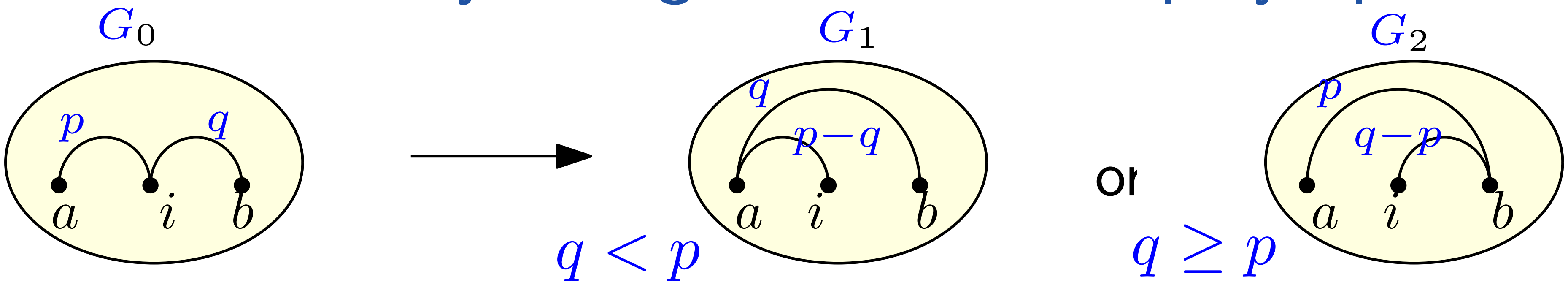
Gracias







Postnikov-Stanley triangulation of flow polytopes



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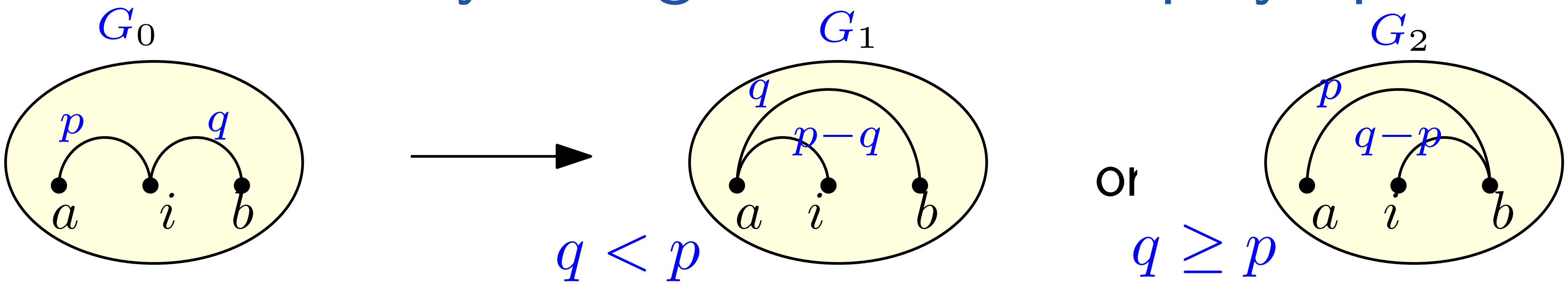

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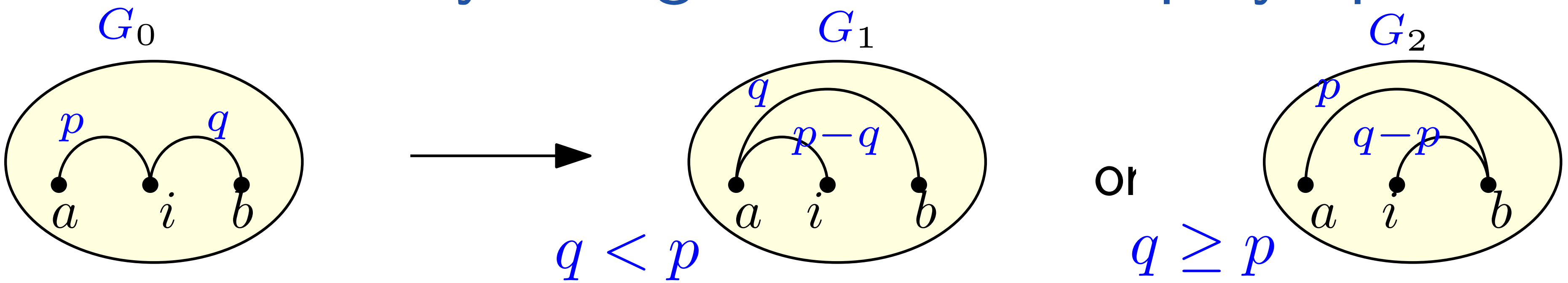

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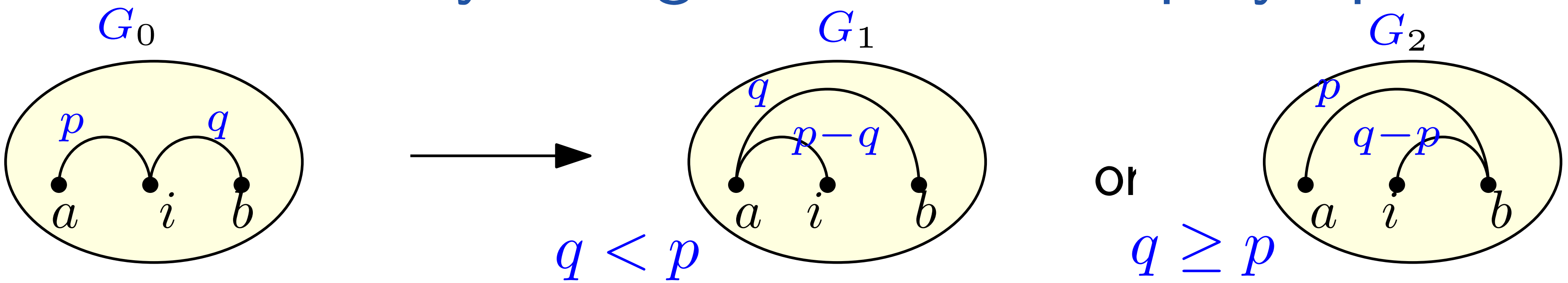

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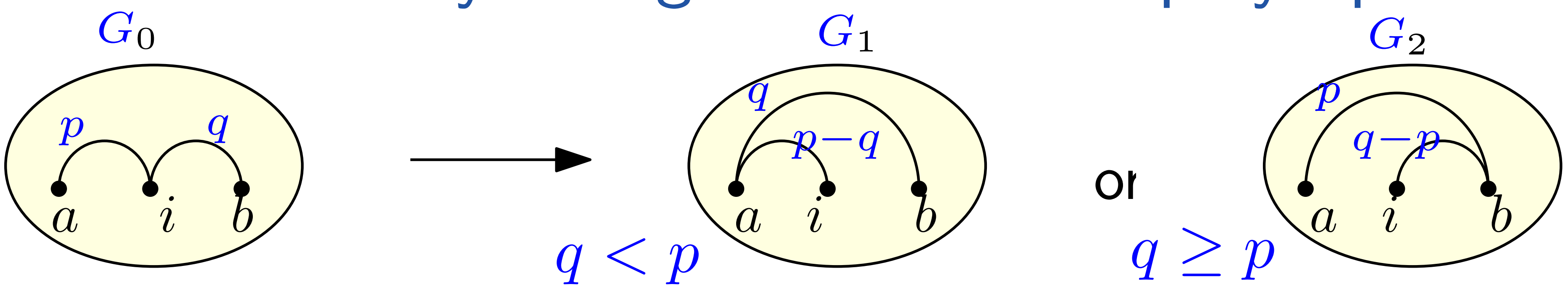

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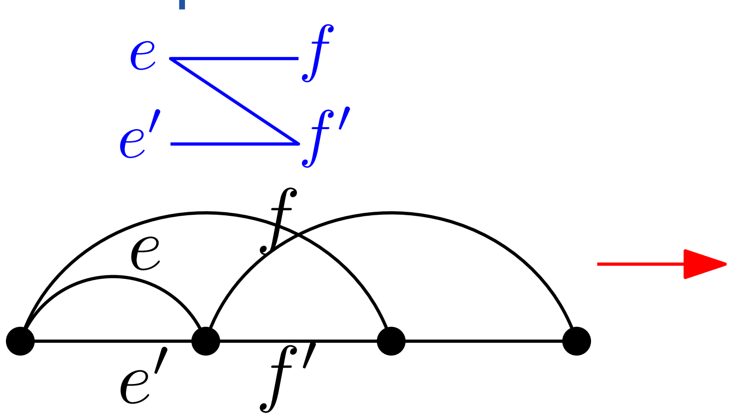

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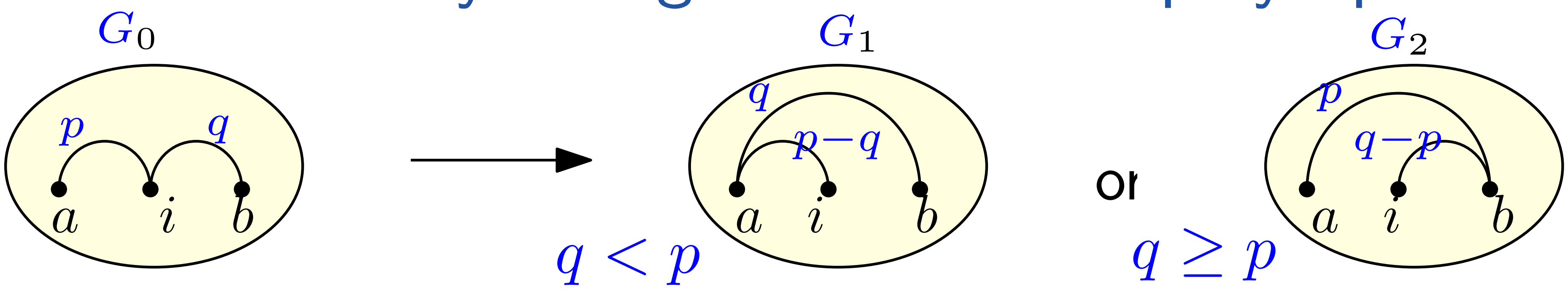

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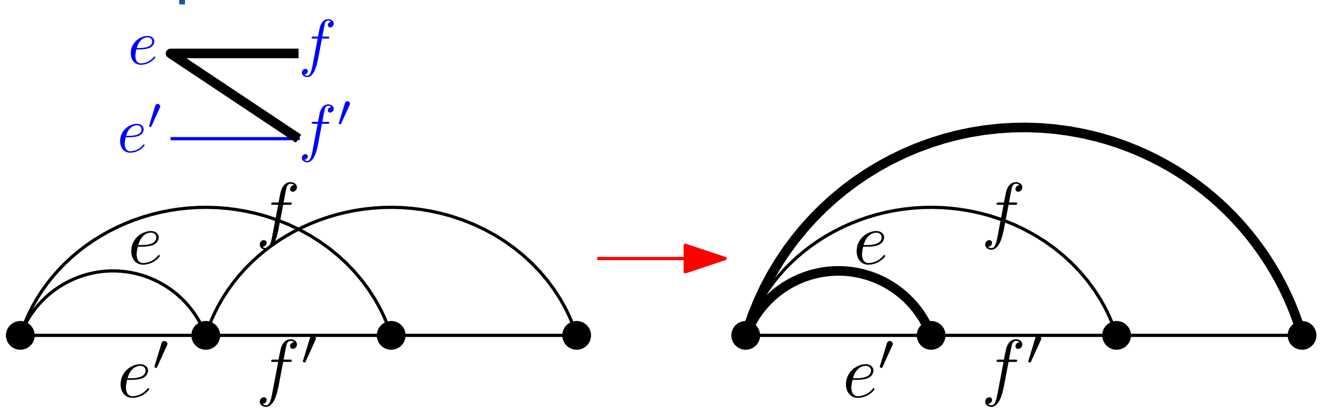

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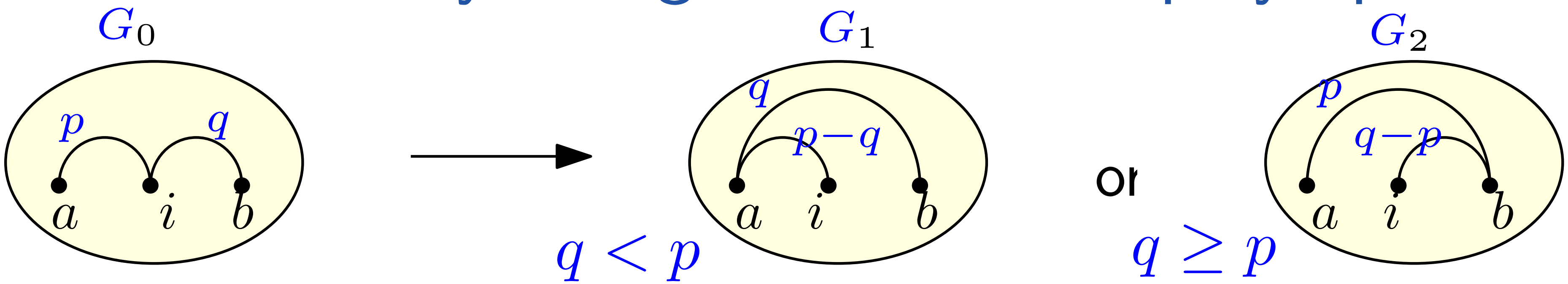

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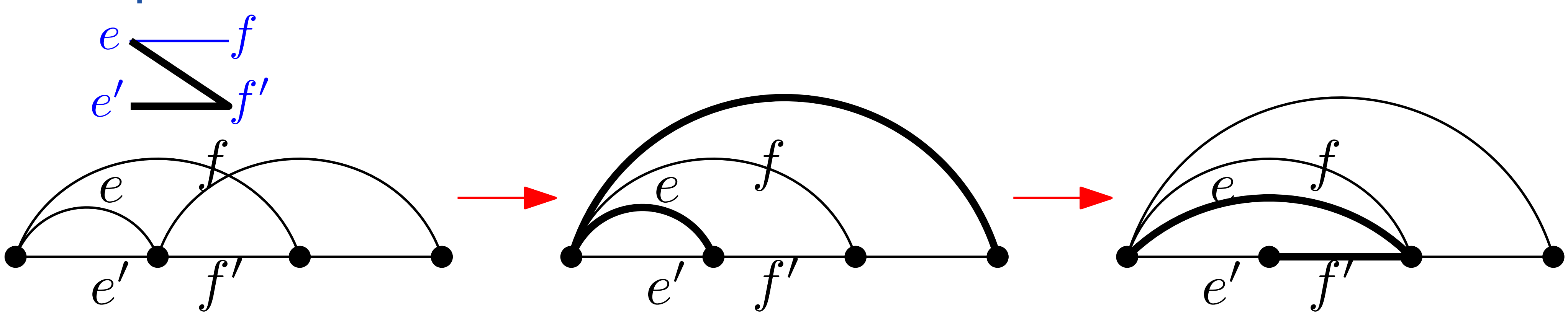

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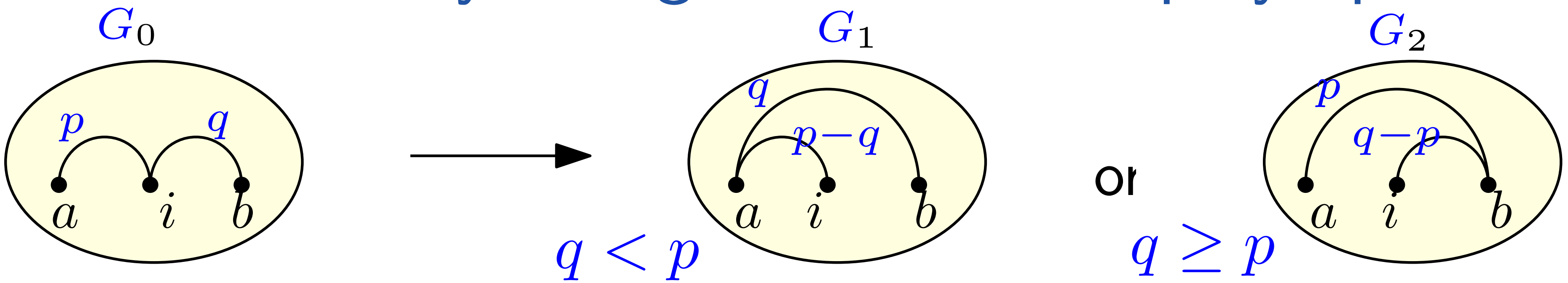

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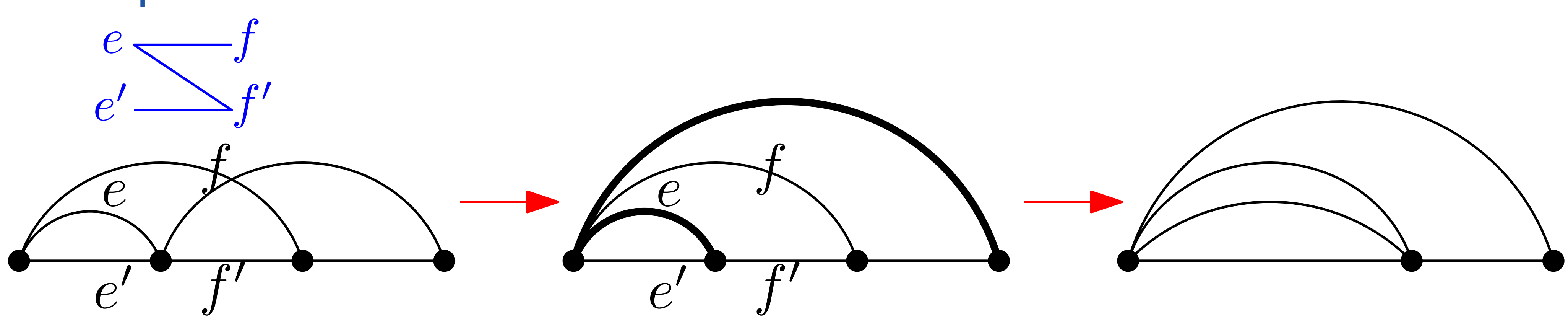

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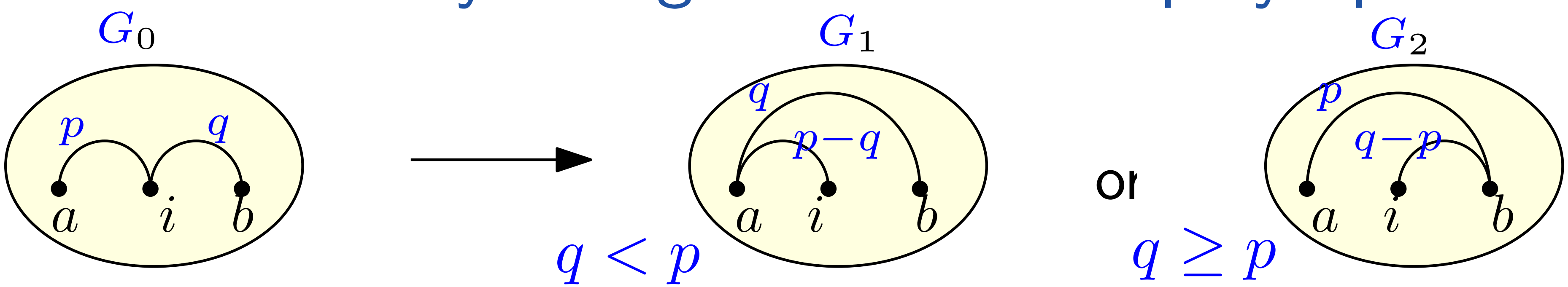

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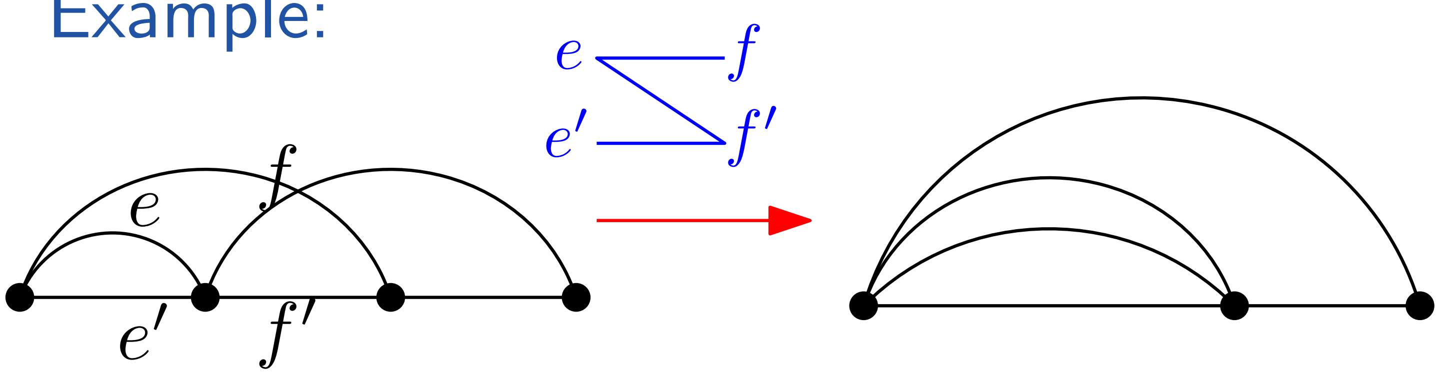

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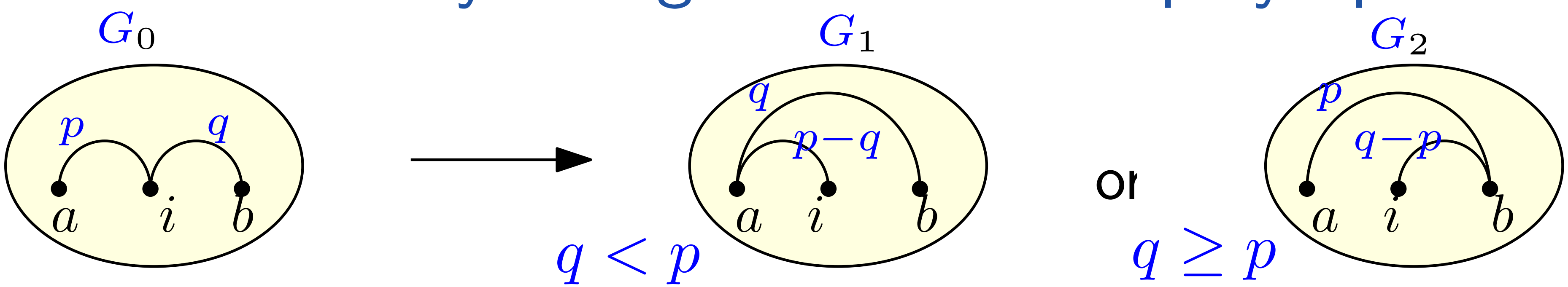

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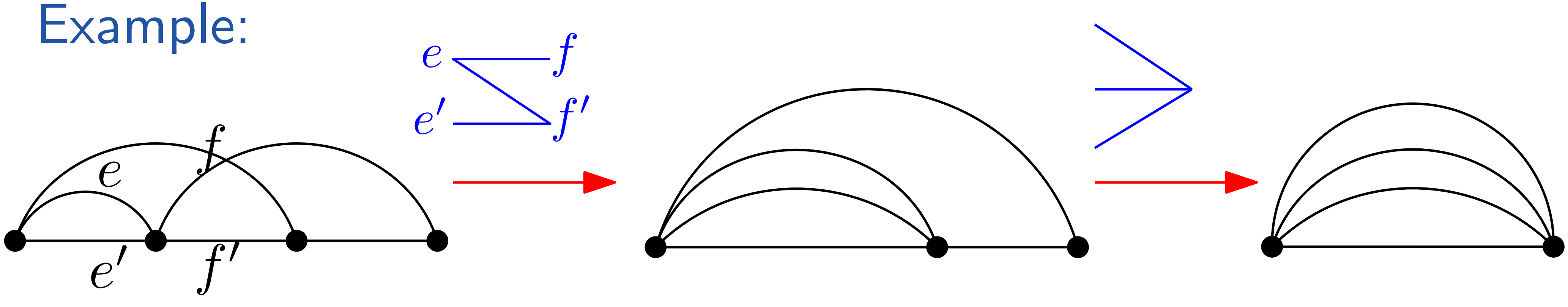

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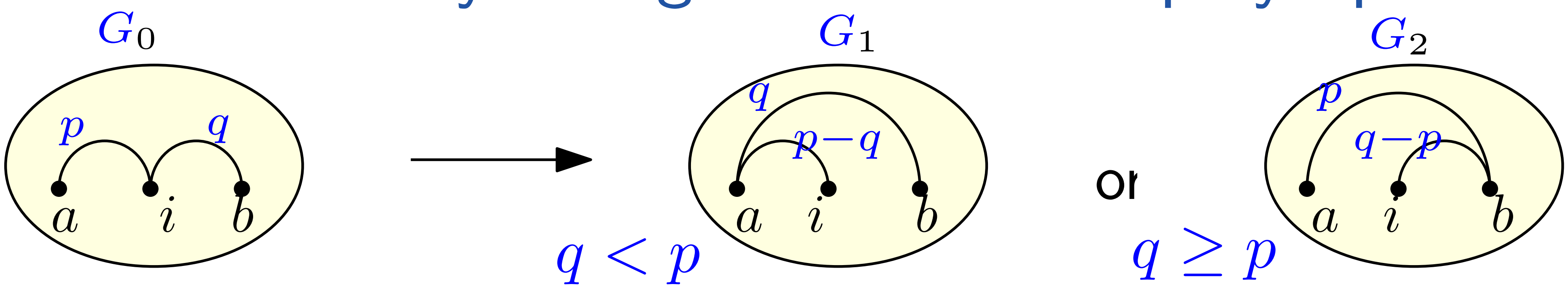

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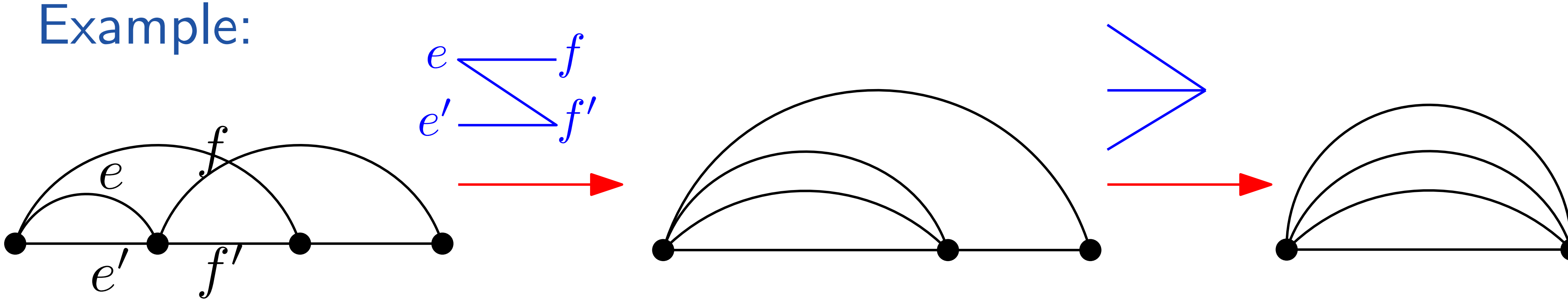

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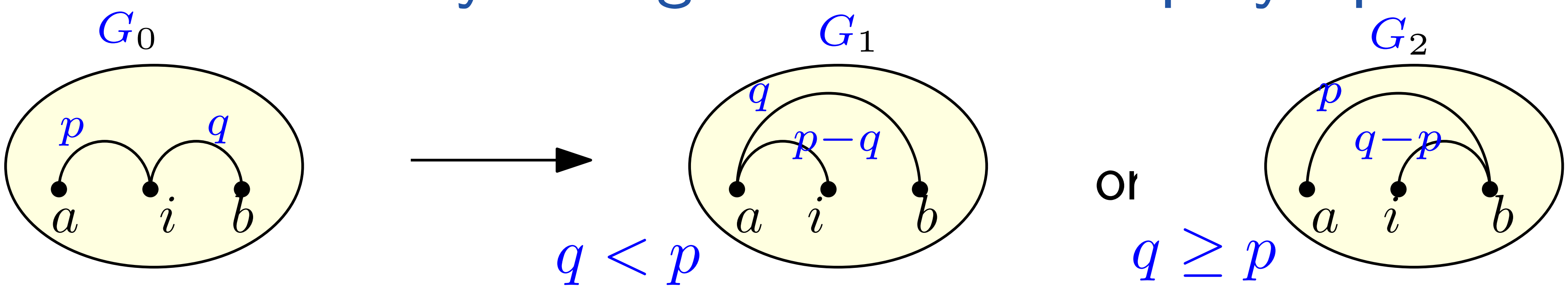

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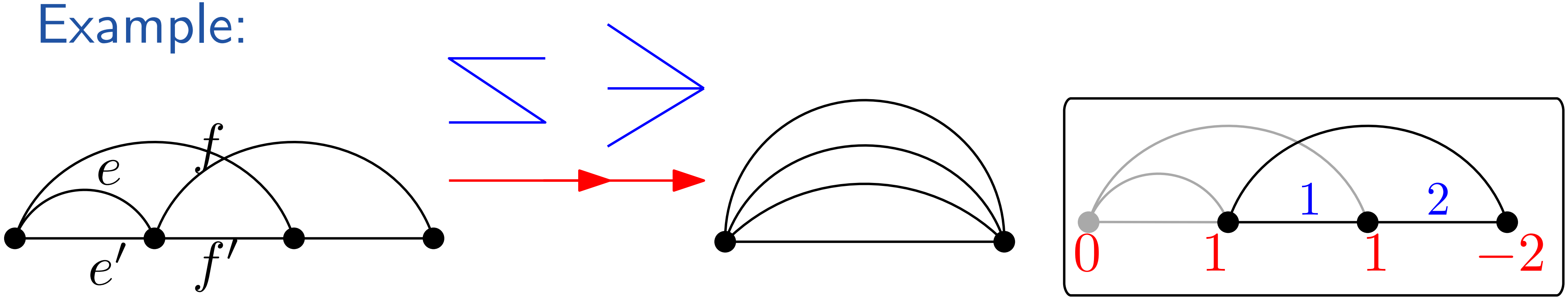

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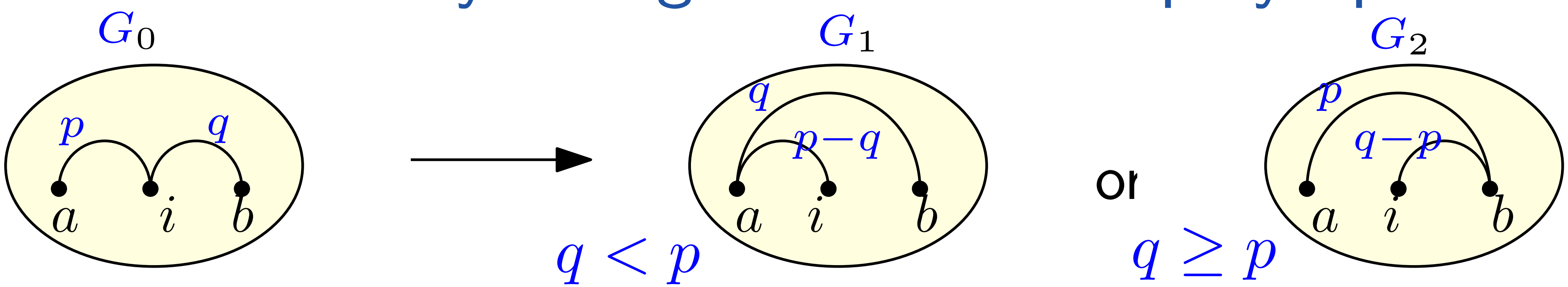

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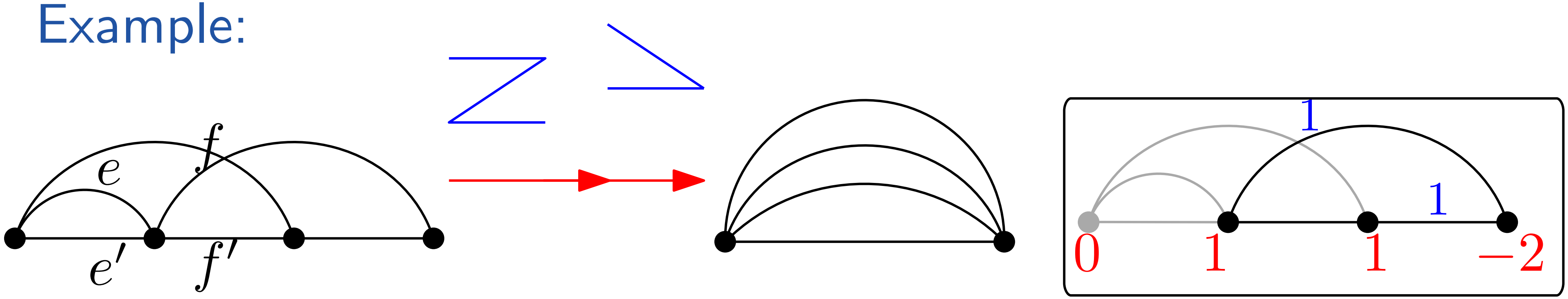
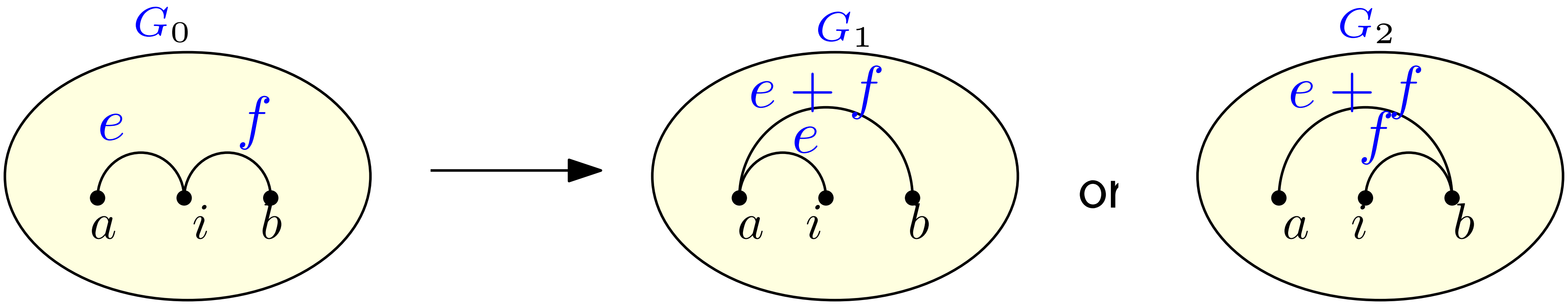


Proof sketch: correspondence

In subdivision view new edges as sum/path of original edges



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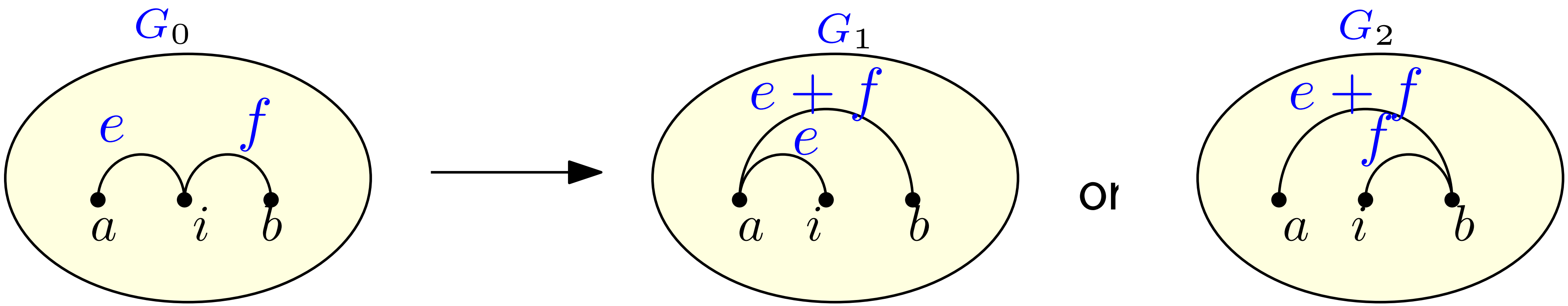

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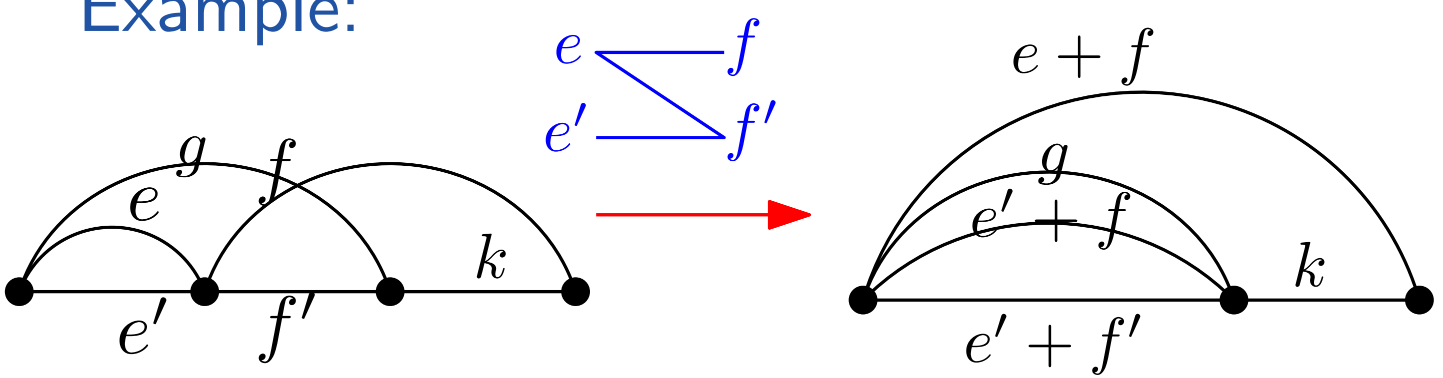

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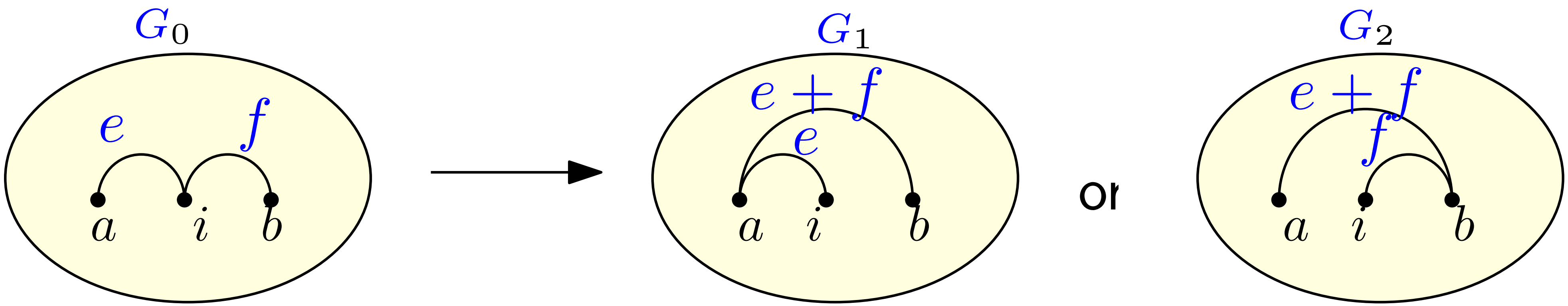

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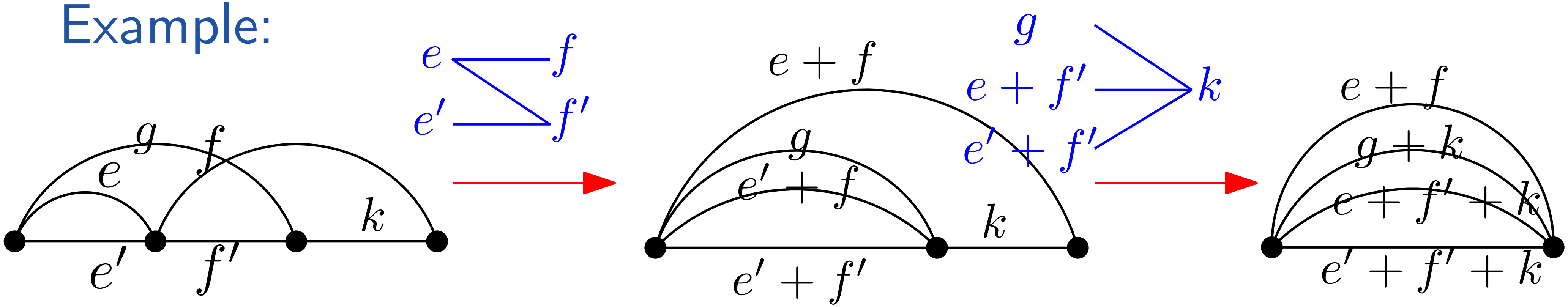

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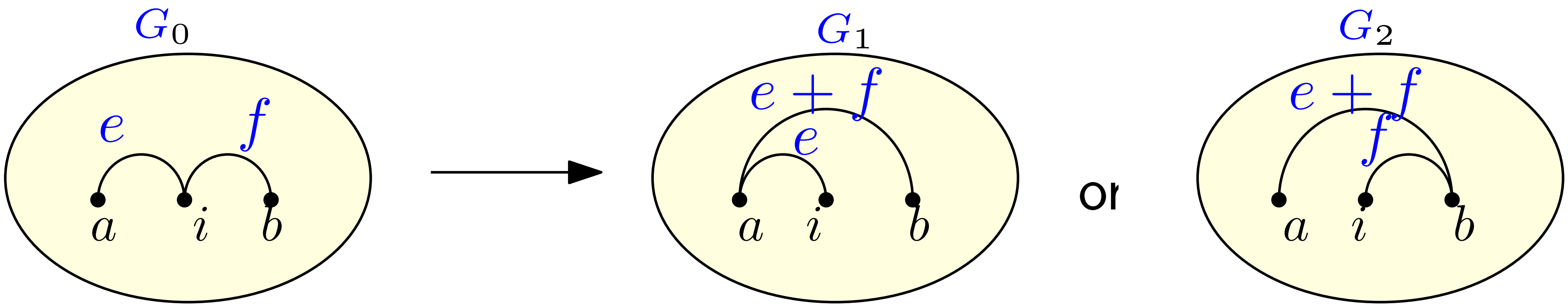

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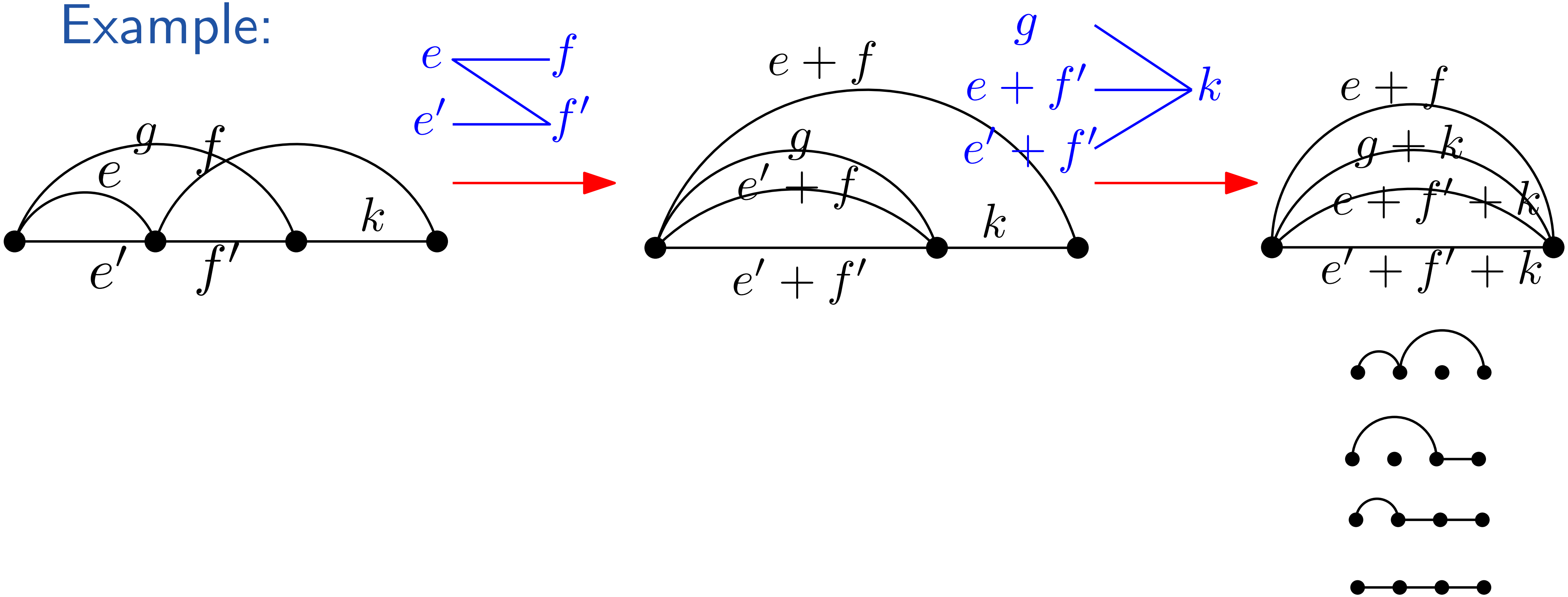
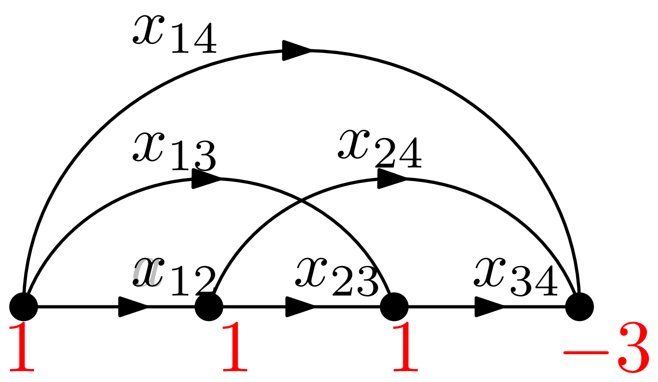


Connection to diagonal harmonics

G is the complete graph k_{n+1}

$\mathbf{a} = (1, 1, \ldots, 1, -n)$



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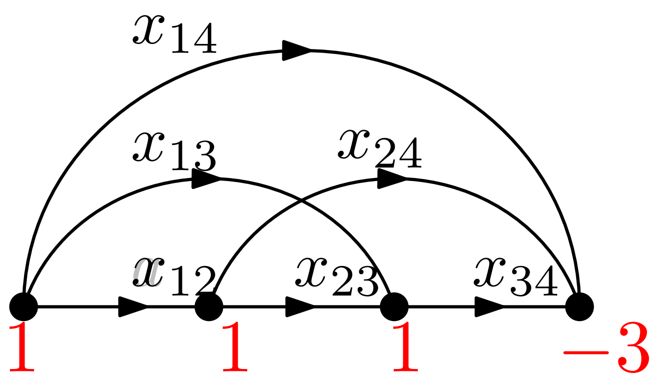

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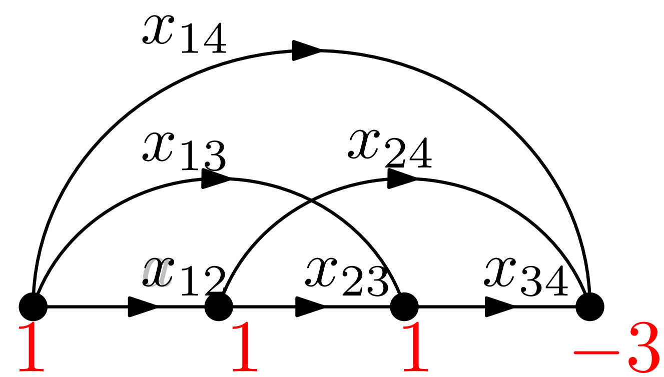

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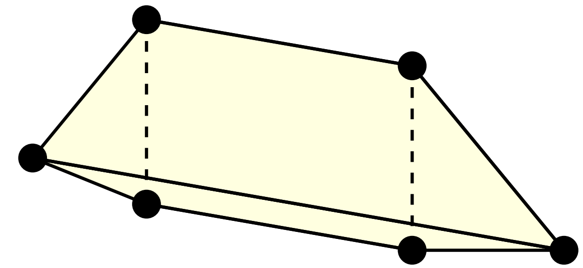

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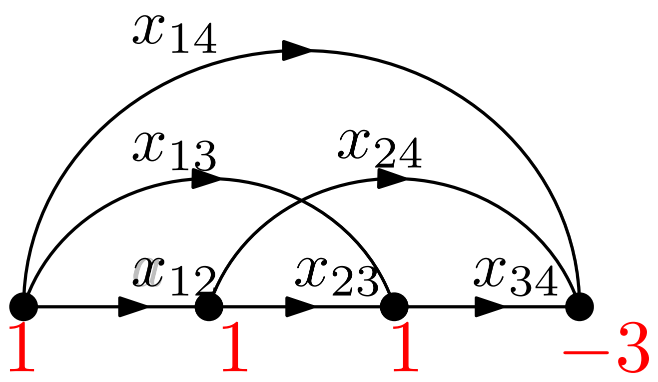

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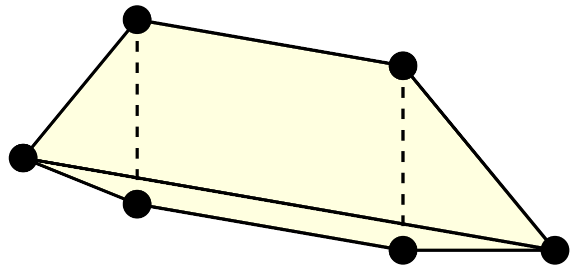

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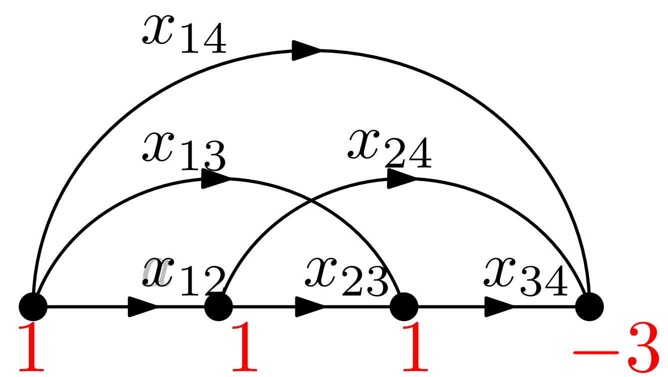

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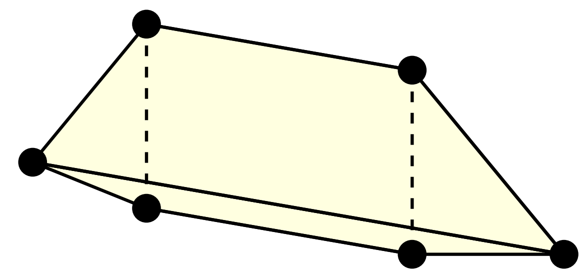

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- Connection was suggested by François Bergeron

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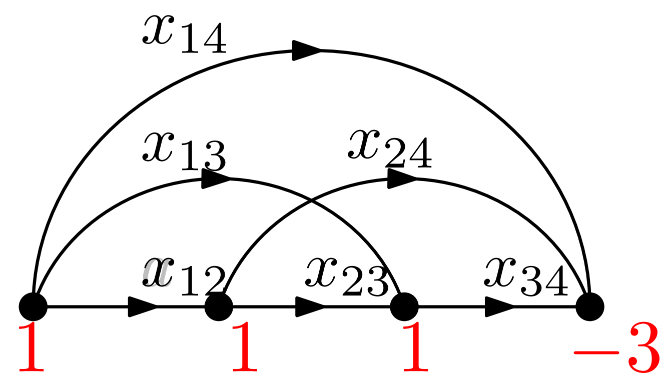

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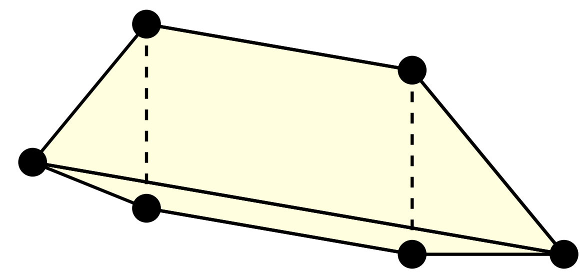

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Theorem (Mészáros, M, Rhoades 2014)

volume equals $\frac{\binom{n}{2}!}{\prod_{i=1}^{n-2}(2i+1)^{n-i-1}} \cdot \mathrm{Cat}_1 \mathrm{Cat}_2 \cdots \mathrm{Cat}_{n-1}$

Other results: Lidskii volume formula

Theorem (Baldoni-Vergne 08, Postnikov-Stanley 08)

graph G with m edges, $n+1$ vertices, $a_i \geq 0$

$$\mathrm{volume}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

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Corollary:

$K_G(j_1-o_1,\ldots,j_n-o_n,0)$ are **mixed volumes** and so are **log-concave** in $j_1,\ldots,j_n$.

Other results: Lidskii volume formula

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- give a proof using polytope subdivisions (Mészáros-M 2019)
- define combinatorial objects like **parking functions** that index the volume of $\mathcal{F}_G(\mathbf{a})$ (B-G-H-H-K-M-Y 2019)