Factorization algebras and free field theories

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ABSTRACT. We study the Batalin-Vilkovisky (BV) formalism for quantization of field theories in several contexts. First, we extract the essential homological procedure and study it from the perspective of derived algebraic geometry. Our main result here is that the BV formalism provides a natural determinant functor we call "cotangent quantization," sending a perfect R-module to an invertible R-module and quasi-isomorphisms to quasi-isomorphisms, where R is an artinian commutative differential graded algebra over a field of characteristic zero. Second, we introduce the formalism of factorization algebras, a local-to-global object much like a sheaf, and describe several perspectives on how the BV formalism makes the observables of a free quantum field theory into a factorization algebra. We study in detail the free $\beta\gamma$ system, a holomorphic field theory living on any Riemann surface, and we recover the $\beta\gamma$ vertex algebra from the factorization algebra of quantum observables. We also construct the factorization algebras on a Riemann surface that recover the vertex algebras arising from affine Kac-Moody Lie algebras. Finally, we study quantization of families of elliptic complexes. Our main result here is an index theorem relating the associated family of factorization algebras to the determinant line of the family of elliptic complexes. At the heart of our work is the formalism for perturbative quantum field theory developed by Costello [**Cos11**] and for the associated observables by Costello-Gwilliam [**CG**], and this thesis provides an exposition of the ideas and techniques in an accessible context.

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CHAPTER 1

Introduction

An ongoing endeavor of mathematics is to provide a language adequate for expressing rigorously the ideas of physics, and this thesis is a product of that endeavor. Before discussing the contents of this thesis, we explain the general context and some mathematical questions it raises.

Our starting point is the path integral approach to quantum field theory. In this formalism a physical system consists of a bundle $P \to M$ over a smooth manifold, whose space of smooth sections $\mathcal{M} := \Gamma(M, P)$ we call the fields, equipped with a local[1] functional $S : \mathcal{M} \to \mathbb{R}$ called the action. An observable of the system is a function $\mathcal{O} : \mathcal{M} \to \mathbb{R}$, and its expected value is computed as

$$\langle \mathcal{O} \rangle := \frac{1}{Z_S} \int_{\phi \in \mathcal{M}} \mathcal{O}(\phi) e^{-S(\phi)/\hbar} \, D\phi,$$

where $e^{-S(\phi)/\hbar} \, D\phi$ is a putative measure on $\mathcal{M}$ and the partition function

$$Z_S := \int_{\phi \in \mathcal{M}} e^{-S(\phi)/\hbar} \, D\phi$$

makes this measure into a probability measure.[2] This perspective on field theory, as a kind of probabilistic system, leads to beautiful insights into many areas of mathematics and physics, but it is often merely a heuristic because measure theory on infinite-dimensional spaces rarely has the properties we desire.

Nonetheless, physicists have provided algorithms for computing expectation values of observables, rooted in this perspective, that are wildly successful. It is a challenge for mathematicians to find explanations and formalisms that justify mathematically these algorithms. In [**Cos11**], Costello has developed a theory that provides a rigorous approach to the algorithms that constitute perturbative quantum field theory (i.e., viewing $\hbar$ as a formal parameter). In [**CG**], we have studied the mathematical structure of the observables of such a perturbative quantum field theory, organized around the idea of a factorization algebra. The basic concept is simple. In a classical field theory, we study the solutions to the Euler-Lagrange equations of S, which pick out the critical points of S. As the Euler-Lagrange equations are partial differential equations, there is a sheaf $\mathcal{EL}$ of solutions on the manifold M, so the functions $\mathscr{O}(\mathcal{EL})$ on these solutions form a cosheaf of commutative algebras on M. For each open set $U \subset M$, the algebra $\mathscr{O}(\mathcal{EL}(U))$ consists of the

[1] "Local" means that S is given by integrating a pointwise function of the jets of a section against a measure on the manifold M.

[2] We are discussing here Euclidean field theories, since we weight S by -1 rather than i.

observables for the classical theory with support in U. In a quantum field theory, one can still talk about the support of observables, but the expected value of a product of observables (with disjoint support) includes quantum corrections, depending on $\hbar$, to the classical expected value. Indeed, these quantum corrections satisfy algebraic relations arising from the Feynman diagram expansion used to compute them. For the precosheaf $\mathrm{Obs}^{\mathrm{q}}$ of quantum observables, these algebraic relations modify the structure maps

$$\mathrm{Obs}^{\mathrm{q}}(U) \otimes \mathrm{Obs}^{\mathrm{q}}(V) \to \mathrm{Obs}^{\mathrm{q}}(W),$$

where U and V are disjoint opens contained in the open W, by adding $\hbar$-dependent terms to the structure maps of the cosheaf of classical observables $\mathrm{Obs}^{\mathrm{cl}} = \mathscr{O}(\mathcal{EL})$. In particular, the precosheaf $\mathrm{Obs}^{\mathrm{q}}$ is no longer a cosheaf of commutative algebras. Instead, it is a factorization algebra, a notion introduced by Beilinson and Drinfeld [BD04] in their work on conformal field theory.

Perturbative quantum field theories are rich and subtle objects, and the constructions in [CG], while explicit, can be very involved because they mix analysis, homological algebra, and category theory in complicated ways. The central aim of this thesis is to study a special class of theories where the constructions are much simpler. We focus on free field theories, in which the action functional S is a quadratic function of the fields. This restriction might seem to limit the possibility of interesting results, but the framework of [Cos11] allows any elliptic complex on a manifold to provide a free theory. Thus, there is a plethora of examples and the possibility that one might obtain new insights into geometry, where elliptic complexes are ubiquitous. Moreover, the factorization algebras arising from free field theories are a small step away from familiar constructions with elliptic complexes, and thus they are more amenable to human understanding.

At the heart of Costello's approach to quantum field theory is the Batalin-Vilkovisky (BV) formalism, which is a homological approach to defining the path integral. It forms the basic mechanism by which we obtain the quantum observables $\mathrm{Obs}^{\mathrm{q}}$ from the classical observables $\mathrm{Obs}^{\mathrm{cl}}$. Unfortunately, it is notoriously difficult to learn and hard to motivate. Thus a secondary aim of this thesis is to provide an introduction to the BV formalism where its virtues are apparent. Again, free theories provide such a context. In fact, we show how the homological algebra of the BV formalism can be deployed outside field theory and apply it to (well-behaved, i.e., perfect) modules over any commutative dg algebra.

Our main results in this thesis are the following.

(1) BV quantization defines a determinant functor from perfect R-modules to invertible R-modules, for R an artinian commutative dg algebra.
(2) One can recover rigorously a vertex algebra from an action functional. In particular, we start with the free $\beta\gamma$ system on $\mathbb{C}$ and show that its factorization algebra of quantum observables recovers the $\beta\gamma$ vertex algebra.
(3) We prove an index theorem arising from the study of quantization of free field theories in families (i.e., families of elliptic complexes). In particular, the global observables on a

closed manifold are given by the determinant of the underlying elliptic complex, so that the factorization algebra provides a local avatar of this determinant. The index theorem describes how this "local determinant" varies in families.

The first result provides mathematical insight into the somewhat-mysterious power of the BV formalism: it is a homological approach to defining volume forms (recall that the volume forms on a vector space live in the determinant of the dual vector space). The second result verifies that our formalism gives the "right answer" when we apply it to a well-known example. Physicists view vertex algebras as capturing the relations between the observables in the chiral sector of a conformal field theory, so it is gratifying that our procedure recovers the vertex algebra — moreover, the computations are easy and explicit and arise directly from the action functional. The third result is much deeper and relies on the full power of Costello's formalism (in fact, it uses nearly every structural theorem in [**Cos11**]). Even to state the theorem precisely requires the language of factorization algebras and field theory we develop in this thesis.

1.1. An overview of the chapters

Chapter 2 is an introduction to the BV formalism for "0-dimensional field theories," namely when the space of fields $\mathcal{M}$ is in fact a finite-dimensional manifold. We begin by extracting the axiomatics from familiar constructions in geometry. We then explain how to recover Wick's lemma and Feynman diagrams directly from the homological algebra of the BV formalism. In the final section, we move beyond 0-dimensional field theories, define "free BV theories," and explain how the Hodge theorem allows one to use exactly these same techniques to compute expectation values of global observables for such theories.

The next chapter extends the BV formalism into a general setting: we define the BV quantization of a perfect R-module for R a commutative dg algebra. We then show that for R an artinian k-algebra, where k is a characteristic zero field, the BV quantization of every perfect module is an invertible module. For instance, for $R = k$ and V an ordinary finite-dimensional vector space, the BV quantization is (a cohomological shift) of $\det V = \Lambda^{\dim V} V$.

The next two chapters introduce the central objects of the thesis: factorization algebras and the observables of free BV theories. In chapter 4, we define factorization algebras, provide general methods for constructing them, and show that factorization algebras on the real line have an intimate relationship to associative algebras and their modules. For instance, we use a BV quantization process to recover the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ as a factorization algebra living on $\mathbb{R}$. In chapter 5, we explain what the quantum observables of a free BV theory are and describe several approaches to their construction.

Chapter 6 applies this formalism in the context of Riemann surfaces. We examine the free $\beta\gamma$ system in detail and show how to recover the vertex operation of the $\beta\gamma$ vertex algebra from the structure maps of the factorization algebra. These arguments apply almost verbatim to a large class of BV theories on Riemann surfaces and so we obtain a method for constructing vertex algebras from action functionals. We also write down explicitly the factorization algebras that recover the vertex algebras associated to affine Kac-Moody Lie algebras, although in this case we do not derive the factorization algebra from an action functional. Instead, we construct the factorization algebra directly, using ideas from the deformation theory of holomorphic G-bundles, for G an algebraic group.

The final chapter, chapter 7, studies deformations of free BV theories. Given a sheaf $\mathfrak{g}$ of dg Lie algebras that acts locally on our fields (for instance, the sheaf of holomorphic vector fields acting on a holomorphic field theory on a Riemann surface), we ask whether we can $\mathfrak{g}$-equivariantly BV quantize. The obstruction to this quantization is a section of the sheaf $C^*\mathfrak{g}$, but to describe it requires the full machinery of [Cos11].

1.2. Notations

Our base field is $\mathbb{C}$, although most arguments work fine with $\mathbb{R}$ as well.

We use *dg vector space* to mean a $\mathbb{Z}$-graded vector space $V = \oplus_n V^n$ with a degree 1 differential d; equivalently, we will speak of cochain complexes in vector spaces. There is a category *dgVect* whose objects are dg vector spaces and whose morphisms are cochain maps (so they are cohomological degree 0 and commute with the differentials).

When we refer to elements of a dg vector space $V = \bigoplus_{n\in\mathbb{Z}} V^n$, we always mean *homogeneous* elements (i.e., they have pure cohomological degree). We denote the cohomological degree of x by $|x|$, so $|x| = n$ for $x \in V^n$.

Shifts of complexes are denoted as follows: $V[k]$ is the complex with $V[k]^n := V^{k+n}$.

We denote the *dual* of a vector space V by $V^\vee$. For a dg vector space (V, d), the dual is the dg vector space $(V^\vee, d)$ where $(V^\vee)^n = \mathrm{Hom}_{\mathbb{C}}(V^{-n}, \mathbb{C})$, the $\mathbb{C}$-linear maps as ungraded vector spaces and d on $V^\vee$ abusively denotes the obvious induced differential.

Because we always use cohomological conventions (i.e., the differential has degree 1), we regrade *chain* complexes by swapping the signs: $V_k \mapsto V^{-k}$. For example, given a Lie algebra $\mathfrak{g}$, we define the *Chevalley-Eilenberg* chain complex for Lie algebra homology as

$$C_*\mathfrak{g} = \left(\bigoplus_{n\in\mathbb{N}} \Lambda^n \mathfrak{g}[n], d_{CE} \right) = (\mathrm{Sym}(\mathfrak{g}[1]), d_{CE}),$$

where $d_{CE}(X \wedge Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

For $\pi : E \to M$ a smooth vector bundle, we use the following notations:

- $\mathscr{E} := C^\infty(M, E)$ is the smooth sections;
- $\mathscr{E}_c := C_c^\infty(M, E)$ is the compactly supported smooth sections;
- $\overline{\mathscr{E}} := C^{-\infty}(M, E)$ is the distributional sections;
- $\overline{\mathscr{E}}_c := C_c^{-\infty}(M, E)$ is the compactly supported distributional sections.

We will abusively denote the sheaf of smooth (respectively, distributional) sections by $\mathscr{E}$ ($\overline{\mathscr{E}}$) and the cosheaf of compactly supported sections by $\mathscr{E}_c$.

Let $E^! = E^\vee \otimes \mathrm{Dens}_M$ denote the vector bundle on M whose fiber is the linear dual of the fiber of E tensored with the density line. Then $\overline{\mathscr{E}^!}$ is the continuous linear dual of $\mathscr{E}_c$.

CHAPTER 2

Motivation and algebraic techniques

The Batalin-Vilkovisky (BV) formalism is a body of ideas and techniques for constructing and studying gauge theories using homological algebra. The essential ideas, however, can be demonstrated in a geometric context where other issues from field theory, like renormalization, do not appear. In the first part of this chapter, sections 2.1 to 2.3, we distinguish the two stages of the BV formalism,[1]

(1) the *classical* BV formalism, which applies derived geometry to describe the critical locus of a function, and
(2) the *quantum* BV formalism, which provides a homological version of integration theory amenable to generalization to infinite-dimensional manifolds.

Finally, we show how Feynman diagrams appear naturally when you apply the quantum BV formalism to compute Gaussian integrals. These sections are purely expository in character and aim to provide simple models for the homological techniques we use throughout the text. In other words, we try to explain "where the BV formalism comes from" by providing a story for its introduction that guides the audience along current research trajectories.[2]

Sections 2.4 and 2.5 provide definitions and techniques that systematize the viewpoint introduced earlier. We introduce the notion of -1-symplectic vector spaces and construct a canonical BV quantization functor on these spaces.[3] (In chapter 3, we provide an interpretation of this quantization as a determinant functor.) We then introduce homological perturbation theory, a tool that clarifies the origins of Feynman diagrams (at least in the BV formalism) and renormalization group flow. We apply it to reprove the results of section 2.3.

In the final section, section 2.6, we introduce the notion of a free field theory on a closed manifold in the sense of [**Cos11**] and explain how the techniques developed in this chapter allow a purely homological approach to computing the expectation value of global observables. Moreover, it illuminates how, in the BV formalism, the Feynman diagrams really used to compute

[1]We always mean the Lagrangian BV formalism, not the Hamiltonian version sometimes known as the BFV formalism.

[2]Ignoring the actual origin story and instead offering an alternative history that motivates one's own approach is a narrative device beloved by mathematicians.

[3]These are analogs of systems with quadratic Lagrangians and hence have canonical quantizations. BV quantizing a nonlinear space is far more subtle.

correlation functions are simply a convenient graphical description of the homological perturbation lemma. With enough control on the underlying elliptic complex (e.g., on tori, where Fourier analysis makes the spectral theory of the Laplacian explicit), this method is effective in computations.

NOTE 2.0.1. *The material in sections 2.3 and 2.5 was developed in collaboration with Theo Johnson-Freyd, although it was undoubtedly well-known to experts in the BV formalism. The viewpoint on the BV formalism articulated here is due in large part to Kevin Costello, who introduced me to it.*

2.1. Classical BV formalism: the derived critical locus

In the Lagrangian approach to physics, a physical system is a space of fields $\mathcal{M}$ (often an infinite-dimensional manifold) with an action functional $S : \mathcal{M} \to \mathbb{R}$. The classical physics is described by the critical locus of S, namely

$$\mathrm{Crit}(S) = \{\phi \in \mathcal{M} \ : \ dS(\phi) = 0\},$$

which, by the calculus of variations, is the space of solutions to the Euler-Lagrange equations for S. We introduce the classical BV formalism — the BV formalism as its applies to classical field theory — in a simplified, finite-dimensional context. A more extensive development of this viewpoint can be found in [**Cos11**], [**CG**], and [**Vez**].

Let M be a finite-dimensional smooth manifold or affine variety (our substitute for the fields $\mathcal{M}$) and let $S : M \to \mathbb{C}$ be a smooth function.We want to study a better-behaved, derived version of $\mathrm{Crit}(S)$. First, observe that

$$\mathrm{Crit}(S) = \mathrm{graph}(dS) \underset{T^*M}{\times} M,$$

the intersection of the graph of dS and the zero section inside the cotangent bundle T^*M. For generic S, this intersection is well-behaved, but we want a construction that behaves well even when $\mathrm{graph}(dS)$ and the zero section M are not transverse. In particular, we want a construction that captures how the intersection fails to be transverse.

The perspective of derived geometry suggests that we take the derived intersection $\mathrm{dCrit}(S)$, which is the dg manifold[4] whose sheaf of functions is the commutative dg algebra

$$\mathscr{O}(\mathrm{dCrit}(S)) := \mathscr{O}(\mathrm{graph}(dS)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

This construction simply enacts the idea that functions on a fiber product are the relative tensor product, but it takes the "homologically correct" tensor product. Not only does it detect the naive

[4] A dg manifold is a ringed space $(X, \mathscr{O})$ such that X is a smooth manifold and $\mathscr{O}$ is a sheaf of commutative dg algebras whose underlying graded algebra is locally of the form $\mathrm{Sym}_{C^\infty_X} \mathscr{E}$, where $\mathscr{E}$ is the sheaf of smooth sections of a $\mathbb{Z}$-graded vector bundle.

intersection — notice that this sheaf on T^*M has support precisely on the topological subspace $\mathrm{Crit}(S)$ — but the rest of the complex detects refined, syzygial information.[5]

It is helpful to give an explicit presentation of $\mathscr{O}(\mathrm{dCrit}(S))$ by picking an explicit resolution for $\mathscr{O}(\mathrm{graph}(dS))$ over $\mathscr{O}(T^*M) = \mathrm{Sym}_{\mathscr{O}(M)}(T_M)$. Let $n = \dim M$. There is a natural Koszul complex K^* providing such a resolution:

$$0 \longrightarrow \mathscr{O}(T^*M) \otimes_{\mathscr{O}(M)} \Lambda^n T_M \longrightarrow \mathscr{O}(T^*M) \otimes_{\mathscr{O}(M)} \Lambda^{n-1} T_M \longrightarrow \cdots$$

$$\longrightarrow \mathscr{O}(T^*M) \otimes_{\mathscr{O}(M)} \Lambda^2 T_M \longrightarrow \mathscr{O}(T^*M) \otimes_{\mathscr{O}(M)} T_M \longrightarrow \mathscr{O}(T^*M)$$

where the differential is

$$\begin{array}{ccc} \mathscr{O}(T^*M) \otimes_{\mathscr{O}(M)} T_M & \to & \mathscr{O}(T^*M) \\ 1 \otimes X & \mapsto & X - dS(X) \end{array}$$

on vector fields and we extend to the left as a Koszul complex. Thus, we obtain an explicit commutative dg algebra describing functions on $\mathrm{dCrit}(S)$:

$$K^* \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M) = \ \Lambda^{\dim M} T_M \longrightarrow \cdots \longrightarrow \Lambda^2 T_M \longrightarrow T_M \longrightarrow \mathscr{O}(M)$$

with differential $-\iota_{dS}$, which sends X to $-dS(X) = -X(S)$. This complex $(\mathrm{Sym}_{\mathscr{O}(M)}(T_M[1]), -\iota_{dS})$ can be viewed as functions on the *shifted* cotangent bundle $T^*[-1]M$ with a nontrivial differential. We call the underlying graded space the *polyvector fields*.

This explicit description of the derived critical locus also showcases another property. Namely, polyvector fields come equipped with a natural bracket: extend the Lie bracket on vector fields (which has degree 1 here) and the Lie derivative on functions (also degree 1) in the natural, graded-symmetric way to all polyvector fields. Thus, for instance, given X, Y, Z vector fields,

$$[X, Y \wedge Z] := [X, Y] \wedge Z + Y \wedge [X, Z],$$

where $\wedge$ is to indicate the product of vector fields. This bracket is known as the *Schouten bracket*. It is, in fact, a Poisson bracket of cohomological degree 1 and so we denote it by $\{-,-\}$.

REMARK 2.1.1. For this choice of resolution, the Poisson structure is strict. If we use a different resolution, we still have a *homotopy* Poisson bracket, although it need not be strict. In other words, it only makes sense to talk about such a Poisson structure in the homotopical sense when working in derived geometry. Throughout this thesis, however, we will restrict our attention to examples where it suffices to use the strict versions of these notions.

The Schouten bracket yields another description of the differential.

LEMMA 2.1.2. *The operator $-\iota_{dS}$ on polyvector fields is equal to the operator $\{S,-\}$, the derivation given by bracketing with S.*

[5] It is beyond my scope here to explain why this derived intersection is better than the usual intersection. The standard story in algebraic geometry grows out of Serre's *Tor* formula for intersection multiplicities [**Ser00**]. For a beautiful motivation of the derived perspective on intersections, see the introduction to Lurie's thesis [**Lura**]. Spivak has developed a version appropriate for manifolds in [**Spi10**].

PROOF. Let X be a vector field and hence have cohomological degree -1 in the polyvector fields. Then, by definition,

$$\{S, X\} = -\{X, S\} = -\mathcal{L}_X S = -X(S) = -\iota_{dS} X.$$

We extend the bracket as a derivation, just as we do the contraction. □

2.1.1. Axiomatizing this structure. We now axiomatize the structure we've uncovered on the derived critical locus.

DEFINITION 2.1.3. A *Pois$_0$ algebra* $(A, d, \{-,-\})$ is a commutative dg algebra (A, d) equipped with a Poisson bracket $\{-,-\}$ of cohomological degree 1. Explicitly, the bracket is a degree 1 map $\{-,-\} : A \otimes A \to A$ such that

- (skew-symmetry) $\{x, y\} = -(-1)^{(|x|+1)(|y|+1)}\{y, x\}$ for all $x, y \in A$;
- (compatibility with d) $d\{x, y\} = \{dx, y\} + (-1)^{|x|}\{x, y\}$ for all $x, y \in A$;
- (biderivation) $\{x, yz\} = \{x, y\}z + (-1)^{(|x|+1)|y|}y\{x, z\}$ for all $x, y, z \in A$.

Our prime example of a Pois$_0$ algebra is $(\mathrm{Sym}_{\mathscr{O}_M}(T_M[1]), -\iota_{dS})$

REMARK 2.1.4. Just to clarify, we emphasize here that the classical BV formalism (the introduction of antifields) is a distinct procedure from BRST (the introduction of ghosts). The BV process allows us to construct the derived critical locus of a function, whereas the BRST process allows us to construct the derived quotient of a space by a Lie algebra. In gauge theory, one must do both, and so these constructions are typically learned almost simultaneously. Since we make no claims about knowing the real history of the subject, we simply state that in this text, BV will mean the use of antifields *aka* taking the "shifted cotangent bundle" of the fields.

2.2. Quantum BV formalism: the twisted de Rham complex

Just as the classical BV formalism put a homological twist on the usual heuristic picture of classical field theory (take the derived critical locus rather than just the critical locus), the quantum BV formalism takes a homological approach to the heuristic picture of quantum field theory. Again, let $\mathcal{M}$ denote the space of fields and $S : \mathcal{M} \to \mathbb{R}$ denote the action functional. In the path integral approach to QFT (the quantum version of the Lagrangian approach), we use $\mathcal{M}$ and S to define a kind of probabilistic system. An *observable* is a measurement we could take of the system, and hence defines a function $\mathcal{O} : \mathcal{M} \to \mathbb{R}$. In classical physics, our system would correspond to some point $\phi \in \mathrm{Crit}(S) \subset \mathcal{M}$ and the measurement takes the value $\mathcal{O}(\phi)$. In the quantum setting, we use S to define a probability measure on $\mathcal{M}$ where the expectation value of an observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle := \frac{1}{Z_S} \int_{\phi \in \mathcal{M}} \mathcal{O}(\phi) e^{-S(\phi)/\hbar} \mathcal{D}\phi,$$

where the quantity $e^{-S(\phi)/\hbar}\mathcal{D}\phi$ is supposed to be some kind of measure on $\mathcal{M}$ and we've normalized by a constant

$$Z_S := \int_{\phi\in\mathcal{M}} e^{-S(\phi)/\hbar}\mathcal{D}\phi$$

known as the partition function of the theory. There are some obvious challenges, not yet surmounted in many cases, to making this picture mathematically rigorous.

For our purposes, however, it suffices to note that the BV approach to quantum systems needs to do two things:

(1) provide a homological approach to integration or, more accurately, to defining such expectation values;
(2) provide a procedure for relating this homological integration to the classical BV formalism already introduced.

These two steps have different flavors, so we undertake them in order.

2.2.1. The de Rham complex as a homological approach to integration. Although this point of view is well-known, we briefly review the set-up to emphasize the aspects relevant to the BV formalism. For simplicity, let M be a closed, oriented, smooth, finite-dimensional n-manifold (i.e., compact and without boundary). Then the top forms $\Omega^n(M)$ are smooth measures, and there is the linear map known as integration $\int_M : \Omega^n(M) \to \mathbb{R}$. By Stokes' theorem, we know

$$\int_M \mu = 0 \Leftrightarrow \mu \in d\Omega^{n-1}(M),$$

so that the integration map descends to a map $\int_M : \Omega^n(M)/d\Omega^{n-1}(M) = H^n_{dR}(M) \to \mathbb{R}$.

In the homological spirit, we might view $H^n_{dR}(M)$ as the space of "integrals" and ask for a resolution. Place $H^n_{dR}(M)$ in degree zero. Then we have a resolution by $\Omega^*(M)[n]$ by shifting the de Rham complex down by n. The cosheaf $\Omega^*_c[n]$ given by the compactly supported de Rham complex, also shifted, naturally provides a local-to-global object that locally resolves the integrals (thanks to the Poincaré lemma) and globally recovers the correct notion in H^0 (thanks to our shift).

REMARK 2.2.1. There is another way to write the de Rham complex that emphasizes the central role of the top forms (or the densities more generally). The exterior derivative

$$\Omega^{n-1}_M \xrightarrow{d} \Omega^n_M$$

can be rewritten as

$$T_M \otimes_{\mathscr{O}_M} \Omega^n_M \xrightarrow{\mathcal{L}} \Omega^n_M,$$

where T_M denotes vector fields, contraction provides the isomorphism $T_M \otimes_{\mathscr{O}_M} \Omega^n_M \cong \Omega^{n-1}_M$, and

$$\mathcal{L}(X \otimes \mu) := \mathcal{L}_X\mu = d\iota_X\mu.$$

We can extend the identification $\Lambda^k T_M \otimes \Omega^n_M \cong \Omega^{n-k}_M$ all the way to the left and re-express the de Rham complex as

$$\Lambda^n T_M \otimes \Omega^n_M \to \cdots \to \Lambda^2 T_M \otimes \Omega^n_M \xrightarrow{\mathcal{L}} T_M \otimes \Omega^n_M \xrightarrow{\mathcal{L}} \Omega^n_M.$$

In other words, the de Rham complex corresponds to describing a natural action of polyvector fields $\mathrm{Sym}_{\mathscr{O}_M} T_M[1]$ on top forms.

2.2.2. BV quantization and the twisted de Rham complex. This rephrasing of integration theory suggests the following maneuver, which lies at the heart of the quantum BV formalism. Again, for simplicity, we work with a closed, oriented manifold M. Suppose we fix a top form μ, which we view as defining a kind of probability density on M (it's the analog of $e^{-S/\hbar}\mathcal{D}\phi$ from above). We thus obtain a map $C^\infty_M \to \Omega^n_M$ by $f \mapsto f\mu$. Observe a simple but compelling consequence of this choice. Let $[\mu]$ denote the image of μ in $H^n_{dR}(M)$, and let $\langle f \rangle_\mu$ denote the expectation value of f relative to the probability measure induced by μ. By construction, we see

$$\langle f \rangle_\mu := \frac{\int_M f\mu}{\int_M \mu} = \frac{[f\mu]}{[\mu]}.$$

Thus we have a purely cohomological way to compute the expectation value of any function f with respect to the probability measure defined by a volume form μ. The basic goal of the quantum BV formalism is to find an abstract, axiomatic version of this process. (To our knowledge, the first reference that emphasizes this point of view is [Wit90].)

Note that a choice of μ gives us a map

$$\begin{array}{ccc} \Lambda^k T_M & \xrightarrow{m_\mu} & \Omega^{n-k}_M \\ \mathcal{X} & \mapsto & \iota_{\mathcal{X}}\mu \end{array}$$

and so we might hope to transfer the exterior derivative d from the de Rham complex to the polyvector fields. *We now assume that μ is nowhere vanishing.* This assumption allows us to invert the "contract with μ" map m_μ and hence to define an operator $\Delta_\mu = m_\mu^{-1} \circ d \circ m_\mu$ on polyvector fields. We call Δ_μ a *BV Laplacian* and we call $(\mathrm{Sym}_{\mathscr{O}_M} T_M[1], \Delta_\mu)$ the *quantum BV complex for μ*. It is isomorphic to the de Rham complex. In other words, the quantum BV complex is simply an obfuscated version of the de Rham complex. Thus we obtain the following.

LEMMA 2.2.2. *Given f a function on M, the cohomology class $[f]_{BV}$ in $H^0(\mathrm{Sym}\, T_M[1], \Delta_\mu)$ satisfies*

$$[f]_{BV} = \langle f \rangle_\mu [1]_{BV}.$$

Other descriptions may provide some intuition for what Δ_μ means. For instance, on T_M it is just divergence with respect to μ,

$$\Delta_\mu X = \mathrm{div}_\mu X \text{ where } (\mathrm{div}_\mu X)\mu = \mathcal{L}_X \mu,$$

and we then extend it to polyvector fields in the natural way. (This interpretation is helpful in reading the standard literature on BV formalism.) A description in local coordinates provides further insight. In particular, we will see that Δ_μ is a second-order differential operator and that the quantum BV complex can be viewed as a twisted de Rham complex.

CONSTRUCTION 2.2.3 (BV complex in local coordinates). Let $M = \mathbb{R}^n$.[6] We study the problem in two stages. Denote the basic vector fields by $\partial_i = \partial/\partial x_i$.

First, suppose μ_{Leb} is the Lebesgue measure $dx_1 \wedge \cdots \wedge dx_n$ and let Δ_{Leb} denote its BV Laplacian. Then $m_{\mu_{Leb}}$ is the following correspondence:

$$\partial_{i_1} \wedge \cdots \wedge \partial_{i_k} \leftrightarrow \pm dx_1 \wedge \cdots \widehat{dx_{i_1}} \cdots \widehat{dx_{i_k}} \cdots \wedge dx_n,$$

where $1 \leq i_1 \leq \cdots \leq i_k \leq n$ and the sign is given by the usual sign for the Hodge star. Hence

$$\Delta_{Leb}(f\partial_1 \wedge \cdots \wedge \partial_n) = \sum_i (-1)^{i-1}(\partial_i f)\partial_1 \wedge \cdots \widehat{\partial_i} \cdots \wedge \partial_n.$$

In fact, a concise form of Δ_{Leb} is

$$\Delta_{Leb} = \sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial(\partial_i)},$$

where as usual we use the Koszul rule of signs.

Second, write an arbitrary density μ in the form $e^{-S(x)}dx_1 \wedge \cdots \wedge dx_n$, as we can express any positive function in the form $e^{-S(x)}$ for some function S. An explicit computation shows that

$$\begin{aligned}\Delta_\mu &= \Delta_{Leb} - \sum_i \frac{\partial S}{\partial x_i}\frac{\partial}{\partial(\partial_i)} \\ &= \Delta_{Leb} - \iota_{dS} \\ &= \Delta_{Leb} + \{S, -\}.\end{aligned}$$

In other words, the quantum BV complex for $\mu = e^{-S}\mu_{Leb}$ is given by modifying the differential of the quantum BV complex for the Lebesgue measure. Using the correspondence between de Rham forms and polyvector fields given by the *Lebesgue* measure, this BV complex for S corresponds to the *twisted de Rham complex* $(\Omega^*_M, d + dS\wedge)$.

With this construction in hand, we now show that the construction of the quantum BV complex for $\mu = e^{-S/\hbar}dx_1 \wedge \cdots \wedge dx_n$ is very close to the complex of functions on the derived critical locus of S. (Notice that we included $\hbar$ into μ to adhere to the path integral story at the beginning of the section.) Then

$$\Delta_\mu = \Delta_{Leb} - \frac{1}{\hbar}\iota_{dS}.$$

We suppose here that $\hbar$ is some nonzero value so we can multiply by $\hbar$. We now have two complexes:

$$\underbrace{(\mathrm{Sym}\, T_M[1], -\iota_{dS})}_{\text{the classical BV complex}} \quad \text{vs.} \quad \underbrace{(\mathrm{Sym}\, T_M[1], -\iota_{dS} + \hbar\Delta)}_{\text{the quantum BV complex}}.$$

[6] We only need M to be compact to get cohomology in the correct degrees. The map between polyvector fields and forms is local in nature, so much of the rest of construction works in general. We are free to use compactly-supported differential forms or polyvector fields to obtain the integration interpretation from above.

By changing $\hbar$, we move from describing functions on the derived critical locus ($\hbar = 0$ is the classical problem) to describing integration of functions against the correct probability measure ($\hbar \neq 0$ is the quantum problem). This example is the model of *BV quantization* that we wish to codify.

REMARK 2.2.4. Note that ι_{dS} is a first-order differential operator on polyvector fields, as the first line of Δ_μ (in the construction) makes apparent.

2.2.3. Axiomatizing this structure. We now look for structural properties of Δ_μ that we can use to make a definition. Notice (in local coordinates is easiest) that

(1) Δ_μ is a second-order differential operator on $\mathrm{Sym}\, T_M[1]$;
(2) $\Delta_\mu^2 = 0$;
(3) we have the following relationship between Δ_μ and the Poisson bracket:

$$\Delta_\mu(\mathcal{X}\mathcal{Y}) = (\Delta_\mu \mathcal{X})\mathcal{Y} + (-1)^{|\mathcal{X}|}\mathcal{X}(\Delta_\mu \mathcal{Y}) + \{\mathcal{X}, \mathcal{Y}\}$$

for any polyvector fields $\mathcal{X}, \mathcal{Y}$.

DEFINITION 2.2.5. A *Beilinson-Drinfeld (BD) algebra*[7] $(A, d, \{-,-\})$ is a graded commutative algebra A, flat as a module over $\mathbb{R}[[\hbar]]$, equipped with a degree 1 Poisson bracket such that

$$d(ab) = (da)b + (-1)^{|a|}a(db) + \hbar\{a, b\}. \tag{1}$$

Observe that given a BD algebra A^q, we can restrict to "$\hbar = 0$" by setting

$$A_{\hbar=0} := A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/(\hbar).$$

Note that the induced differential on $A_{\hbar=0}$ is a derivation, so that $A_{\hbar=0}$ is a Pois_0 algebra! Likewise, when we restrict to "$\hbar \neq 0$" by setting

$$A_{\hbar\neq 0} := A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)),$$

we obtain just a cochain complex. In particular, the cohomology does not inherit an algebra structure, unlike $H^* A_{\hbar=0}$.

DEFINITION 2.2.6. A *BV quantization* of a Pois_0 algebra A is a BD algebra A^q such that $A_{\hbar=0} = A$.

Our typical approach to constructing a BV quantization of a Pois_0 algebra (A, d) is to search for BV Laplacians Δ such that $d + \hbar\Delta$ makes $A[[\hbar]]$ a BD algebra. Sometimes one needs to add $\hbar$-dependent terms to d.

[7] We would prefer to call these *BV* algebras, but that name has come to refer to a different but very similar class of objects.

2.2.4. Projective volume forms. The construction above of a BV Laplacian on polyvector fields of a smooth manifold has two important features:

(1) The construction is local and hence does not depend on global properties of the volume form μ (e.g., integrability), and
(2) The construction only depends on μ up to a scalar. If we multiply μ by a nonzero constant C, then $\Delta_{C\mu} = \Delta_\mu$.

The second feature is clear because $\Delta_\mu = m_\mu^{-1} \circ d \circ m_\mu$, so the constant C cancels itself.

One corollary of these features is that we do not need μ to be globally well-defined to do the construction! For instance, if we have a covering $\{U_i\}$ of M and a nowhere-vanishing top form μ_i for each open U_j such that μ_i and μ_j differ by a constant on $U_i \cap U_j$ for every i and j, then we still get a well-defined BV Laplacian $\Delta_{\{\mu_i\}}$ on the polyvector fields. Such a collection $\{\mu_i\}$ is called a *projective volume form* in [**Cosb**] and [**Cosa**]. It is clearly equivalent to putting a flat connection on top forms. Thus every projective volume form yields a BV quantization of polyvector fields.

For M a general dg manifold, not every BV quantization of polyvector fields $\mathscr{O}(T^*[-1]M)$ comes from a projective volume form on M. But the property that characterizes such quantizations is very simple: these BV Laplacians are equivariant under scaling of the cotangent fiber, in particular they must have weight one under this $\mathbb{G}_m$ action. In [**Cosb**], Costello proves there is equivalence between the simplicial set of projective volume forms and the simplicial set of $\mathbb{G}_m$-equivariant quantizations.

REMARK 2.2.7. This story about the quantum BV formalism suggests that BV quantization can often be interpreted as choosing a projective volume form. This perspective can be quite useful, especially in searching for quantum field theories of mathematical interest. A good discussion can be found in [**Cosa**].

2.2.5. Berezin integration. The usual motivation for Berezin integration falls naturally out of the BV approach to constructing "homological integration." We quickly overview it as a pleasant digression.

Let V be a purely odd vector space of dimension $0|n$. We want a BV Laplacian Δ on polyvector fields $\mathrm{Sym}(V^\vee \oplus V[1])$ that is the analogue of the "Lebesgue" BV Laplacian. We will see that it recovers the Berezin integral.

LEMMA 2.2.8. *There is a unique, translation-invariant BV quantization of $T^*[-1]V$.*

PROOF. Pick a basis $\{x_1, \dots, x_n\}$ for V and let $\mathbb{C}[\xi_1, \dots, \xi_n]$ denote $\mathscr{O}(V)$ with respect to the corresponding linear coordinate functions. The polyvector fields are then the graded commutative algebra $\mathbb{C}[\xi_1, \dots, \xi_n, x_1, \dots, x_n]$, where we view the ξ_j as cohomological degree 0 and the x_j as cohomological degree -1. Our Poisson bracket — here, the Schouten bracket — is

$$\{\xi_i, \xi_j\} = 0 = \{x_i, x_j\} \quad \text{and} \quad \{\xi_i, x_j\} = \delta_{ij}$$

with respect to this basis.

Observe that any second-order differential operator P of cohomological degree 1 on polyvector fields is of the form

$$\sum_{i,j} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x_j} + \sum_k b_k(\xi) \frac{\partial}{\partial x_k} + \sum_{l,m,n} c_{lmn}(\xi) x_l \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_n}.$$

Translation-invariance forces the coefficients a_{ij}, b_k, and c_{mn} to be constants.

Now we show P is unique by using equation (1). We must have the equality

$$\begin{aligned} P(\xi_i x_j) &= P(\xi_i) x_j \pm \xi_i P(x_j) + \{\xi_i, x_j\} \\ \pm a_{ij} &= 0 \pm \xi_i b_j + \delta_{ij}. \end{aligned}$$

Thus $b_j = 0$ for all j and $a_{ij} = \pm\delta_{ij}$. We also require the equality

$$\begin{aligned} P(x_i x_j) &= P(x_i) x_j \pm x_i P(x_j) + \{x_i, x_j\} \\ \pm c_{kij} x_k &= b_i x_j \pm x_i b_j + 0 \\ \pm c_{kij} x_k &= 0. \end{aligned}$$

Hence the BV Laplacian P is completely determined.

Although we used a choice of basis, it does not affect P. A change of basis $\{x\} \to \{x'\}$ leads to a compensating change of linear coordinates $\{\xi\} \to \{\xi'\}$, and the Poisson bracket is defined through the evaluation pairing, so it looks exactly the same. Thus, the BV Laplacian "looks the same" for any basis, much like an identity matrix. □

LEMMA 2.2.9. *For the unique, translation-invariant BV Laplacian*

$$\Delta = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial \xi_i},$$

using the basis as in the proof, the cohomology of the BV complex

$$(\mathbb{C}[\xi_1, \dots, \xi_n, x_1, \dots, x_n], \Delta)$$

is one-dimensional and concentrated in degree 0. In particular, H^0 is generated by the monomial $\xi_1 \xi_2 \cdots \xi_n$.

PROOF. We verify the claim for $n = 1$. The case for n is a corollary by taking the n-fold tensor product of the BV complex for the one-dimensional case.

Observe that $\Delta(x^m) = 0$ and $\Delta(\xi x^m) = m x^{m-1}$. In cohomological degree $-m \leq 0$, the BV complex is spanned by these two elements. For $-m < 0$, this shows that the cohomology is zero, as x^m is a boundary and ξx^m is not a cycle. For $m = 0$, both elements are cycles but only $1 = x^0$ is a boundary. □

COROLLARY 2.2.10. *A linear map*

$$\int : \mathscr{O}(V) = \mathbb{C}[\xi_1, \ldots, \xi_n] \to \mathbb{C}$$

that vanishes on divergences of vector fields factors through the zeroth cohomology of the BV complex and hence is determined by assigning a number to the monomial $\xi_1 \xi_2 \cdots \xi_n$.

Such a linear map is an *integration map* and corresponds to the usual Berezin integral. Although this example is somewhat silly — after all, the BV formalism arose in part by applying systematically a viewpoint originating in the theory of supermanifolds — it gives a feel for how to use the quantum BV formalism.

REMARK 2.2.11. This argument essentially rests on finding a translation-invariant projective volume form for V. As a projective volume form is equivalent to putting a *right* D-module structure on $\mathscr{O}(V)$, we are rediscovering an appealing approach to super-integration due to Rothstein [**Rot87**], who showed how to properly extend super-integration to non-compact super-manifolds.

2.3. Wick's lemma and Feynman diagrams, homologically

In the previous section, we introduced the quantum BV formalism as a version of integration. Our goal in this section is to extend this relationship by directly recovering, with the BV formalism, the Feynman diagrams that appear in computing asymptotic integrals over finite-dimensional spaces. The "usual story" behind Feynman diagrams (see [**Man99**],[**Pol05**], or [**Cos11**]) has two parts:

(1) one proves Wick's lemma (Lemma 2.3.2 below), which gives a formula for the moments of a Gaussian integral
$$\langle x^n \rangle = \int_{\mathbb{R}} x^n e^{-x^2/\hbar} dx;$$
(2) for an "interaction term" I a polynomial with only cubic and higher terms, one gives an expression (often formal) for Gaussian integrals like
$$\int_{\mathbb{R}} f(x) e^{-x^2/\hbar - I(x)/\hbar} dx,$$
by using Feynman diagrams to encode the combinatorics that express this integral in terms of Wick's lemma.

Our approach proceeds in parallel to the "usual story" but proves the main results purely homologically. Although these results are quite simple, we show in section 2.6 that these techniques do apply essentially verbatim to computing expectation values in free quantum field theories on closed manifolds.

REMARK 2.3.1. This section is a minor rewriting of [**GJF**], a joint paper with Theo Johnson-Freyd, that expounds these ideas in more detail.

2.3.1. Translating the problem into homological algebra. The basic problem is as follows. Let $V = \mathbb{R}^N$ denote Euclidean space and equip it with the Gaussian probability measure

$$\mu_{Gauss} := \frac{(2\pi\hbar)^{N/2}}{\sqrt{\det A}} e^{-\langle x, Ax\rangle/2\hbar} dx^1 \cdots dx^N,$$

with $A = (a_{ij})$ a positive-definite, symmetric, real, $N \times N$ matrix and $\hbar > 0$. We want to compute expectation values

$$\langle f \rangle_{Gauss} := \int_V f \mu_{Gauss},$$

or, more accurately, have explicit descriptions at least for polynomials. With these formulas in hand, we can treat $\hbar$ as a formal parameter and give a nice expression for the expectation value $\langle f \rangle_{Gauss}$ of any formal power series f in $\hbar^{N/2}\mathbb{C}[[\hbar]]$. This expression provides the simplest appearance of Feynman diagrams.

Following the discussion in section 2.2, we rephrase this problem homologically. Naively, we want to work with the de Rham complex of V and identify the cohomology class $[f\mu_{Gauss}]$ in $H^N_{dR}(V)$. Of course, we know this naive idea fails because $H^N_{dR}(V) = 0$ and so the cohomology class is always zero. This failure is related to the fact that most smooth top forms are not integrable on a vector space.[8] One might attempt to fix this problem by working with *compactly-supported* de Rham cohomology, since compactly-supported top forms are honestly integrable, but our main example μ_{Gauss} is not compactly-supported. As a first step, let $\mathcal{S} = \mathcal{S}(V)$ denote the Schwartz functions on V and consider the Schwartz-de Rham complex

$$\Omega^*_{\mathcal{S}}(V) := \mathcal{S} \xrightarrow{d} \bigoplus_{i=1}^N \mathcal{S}\, dx_i \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}\, dx_1 \wedge \cdots \wedge dx_N,$$

where we work with de Rham forms whose coefficients live in $\mathcal{S}$. Like the compactly-supported de Rham complex, this complex has cohomology concentrated in degree N. For $f \in \mathcal{S}$, we define $\langle f \rangle_{Gauss}$ as the number such that

$$[f\mu_{Gauss}] = \langle f \rangle_{Gauss}[\mu_{Gauss}] \in H^N_{\mathcal{S}}(V),$$

where $H^*_{\mathcal{S}}(V)$ denotes the cohomology of the Schwartz-de Rham complex. The translation between differential forms and polyvector fields described in section 2.2 (see Construction 2.2.3) applies in this context, so that we can work with the Schwartz polyvector fields and BV Laplacian

$$\Delta_{Gauss} = \Delta_{Leb} - \frac{1}{\hbar}\sum_{ij} a_{ij} x_i \frac{\partial}{\partial \xi_j}.$$

Thus we can use the BV formalism to study the expectation values. In particular, once we work with the BV complex, we know that $\langle f \rangle_{Gauss}$ is given by the cohomology class $[f]_{BV}$ in the zeroth cohomology of this BV complex (recall lemma 2.2.2).

[8] Dealing with this sort of disconnect between "homological integration" and usual integration is one of the minor challenges in this formalism.

A further algebraic idealization is possible and it makes the comparison to the usual story clearer. We shift $\hbar$ onto the Lebesgue BV Laplacian, view $\hbar$ as a formal parameter, and replace $\mathcal{S}$ by formal power series on V. The problem is then as follows.

Consider the algebra of formal power series

$$\mathscr{A}(V) := \widehat{\mathrm{Sym}}_{\mathbb{C}[[\hbar]]}(V^\vee \oplus V[1]) = \mathbb{C}[[\hbar, x_1, \ldots, x_N, \xi_1, \ldots, \xi_N]]$$

where the x_i have cohomological degree 0 (these are the coordinate functions on V) and the ξ_i have cohomological degree -1 (these correspond to the vector fields $\partial/\partial x_i$). We equip it with the BV Laplacian

$$\begin{aligned}\Delta &:= -\sum_{i,j=1}^N a_{ij} x_i \frac{\partial}{\partial \xi_i} + \hbar \sum_{i=1}^N \frac{\partial}{\partial x_i}\frac{\partial}{\partial \xi_i} \\ &= -\sum_{i,j=1}^N a_{ij} x_i \frac{\partial}{\partial \xi_i} + \hbar \Delta_{Leb},\end{aligned}$$

where $A = (a_{ij})$ is our original matrix. There is the associated degree 1 Poisson bracket $\{-,-\}$ satisfying

$$\{x_i, x_j\} = 0 = \{\xi_i, \xi_i\} \text{ and } \{x_i, \xi_j\} = \delta_{ij}.$$

Then, to mimic the usual story (2.3) about Feynman diagrams and integration, we need

(1) to compute the cohomology of $(\mathscr{A}(V), \Delta)$, and
(2) to compute the cohomology of $(\mathscr{A}(V)), \Delta + \{I, -\})$ for an "interaction" $I \in \mathbb{C}[[x_1, \ldots, x_N]]$ having only cubic and higher terms in the x's.[9]

For us, "computing the cohomology" of $(\mathscr{A}(V)), \Delta + \{I, -\})$ means that we have an explicit expression for the cohomology class $[f]_I$ of any f in $\mathbb{C}[[\hbar, x_1, \ldots, x_N]]$. We define the *expectation value* $\langle f \rangle_I$ to be the element of $\mathbb{C}[[\hbar]]$ such that

$$[f]_I = \langle f \rangle_I [1]_I \in H^0(\mathscr{A}(V), \Delta + \{I, -\}).$$

Clearly, this number depends on I. For step one, we use $I = 0$. We now attack these problems in order.

2.3.2. Step one: Wick's lemma. We want to compute the cohomology without an interaction. We begin by considering the simplest case $V = \mathbb{R}$. Our complex is then

$$\mathbb{C}[[x, \hbar]]\, \xi \xrightarrow{\Delta} \mathbb{C}[[x, \hbar]]$$

where

$$\Delta = -ax \frac{\partial}{\partial \xi} + \hbar \frac{\partial^2}{\partial x \partial \xi}.$$

[9]The term $\{I, -\}$ is exactly the contraction $-\iota_{dI}$ from section 2.2. We are viewing the function S as the sum $\frac{1}{2}\langle x, Ax \rangle + I(x)$. The first term is quadratic and embodies the "free theory" while I is the "interaction" term for the theory.

Given an element in degree -1, namely $f(x)\xi$, we see $\Delta(f\xi) = -axf(x) + \hbar f'(x)$. A little formal calculus tells us that $\Delta(f\xi) = 0$ only when $f(x) = \exp(ax^2/2\hbar)$, but as this f is not in $\mathbb{C}[[x, \hbar]]$, the cohomology in degree -1 vanishes.

We now want to know $[x^n]$ for all n. Observe that since

$$\Delta(x^{n-1}\xi) = -ax^n + \hbar(n-1)x^{n-2},$$

we know

$$[x^n] = \frac{\hbar}{a}(n-1)[x^{n-2}].$$

Applying this relation recursively, we see

$$[x^n] = \begin{cases} 0, & \text{if } n \text{ odd} \\ \left(\frac{\hbar}{a}\right)^k (2k-1)!![1] & \text{if } n = 2k \end{cases}$$

where $(2k-1)!!$ denotes the "double factorial" $(2k-1)(2k-3)\cdots 5 \cdot 3$. We now have an explicit combinatorial formula for the expectation values $\langle x^n \rangle$. Those familiar with the usual story will recognize this result.

LEMMA 2.3.2 (Wick's lemma). *The cohomology of* $(\mathscr{A}(V), \Delta)$ *is* $\mathbb{C}[[\hbar]]$ *concentrated in degree 0. Moreover, for any monomial*

$$x^\nu = x_1^{n_1} \cdots x_N^{n_N},$$

the expectation value is

$$\langle x^\nu \rangle_0 = \hbar^{|\nu|/2} \sum_{\substack{\text{pairings } P \text{ of} \\ \text{the multiset } \nu}} \prod_{(i,j) \in P} a^{ij},$$

where the multiset ν *is*

$$\{\underbrace{1, \ldots, 1}_{n_1 \text{ times}}, \underbrace{2, \ldots, 2}_{n_2 \text{ times}}, \ldots, \underbrace{N, \ldots, N}_{n_N \text{ times}}\},$$

$|\nu| = \sum n_i$, $A^{-1} = (a^{ij})$, *and a pairing* P *is a partition of a multiset into a union of two-element multisets. If* $|\nu|$ *is odd, this expectation value is zero.*

PROOF. The assertion about cohomology follows from a spectral sequence argument. Consider the filtration by powers of the ideal $(\hbar)$. The first page of the spectral sequence is the cohomology of the complex

$$(\mathbb{C}[[x_1, \ldots, x_N, \xi_1, \ldots, \xi_N]], \sum_{i,j=1}^{N} a^{ij} x_i \frac{\partial}{\partial \xi_i}),$$

which is just the Koszul complex for the regular sequence given by the hyperplanes

$$\sum_j a^{0j} x_j, \sum_j a^{1j} x_j, \ldots.$$

The intersection of these hyperplanes is just the origin of V. Thus the spectral sequence collapses here.

The assertion about the expectation value follows directly along the lines of the one-dimensional case discussed before the lemma. For instance, one can diagonalize the matrix A by the spectral theorem. This reduces the problem to an N-fold tensor product of the one-dimensional case. $\square$

2.3.3. Step two: Feynman diagrams. When we have a nontrivial interaction I, the computation of $H^*(\mathscr{A}(V), \Delta - \{I, -\})$ is more complicated. The language of Feynman diagrams provides a succinct, combinatorial description of the expectation values, which we introduce below. (A more thorough, chatty discussion of these constructions can be found in [GJF].)

Before delving into diagrams, we fix some notation. In this section, we fix $V = \mathbb{R}^N$. The interaction term I is an element of $\mathscr{O}(V) = \mathbb{C}[[x_1, \dots, x_N]]$ that has only cubic and higher order terms. To describe elements of $\mathbb{C}[[x_1, \dots, x_N]]$, we use the following notation. View $\mathscr{O}(V)$ as the subspace of symmetric tensors inside the tensor algebra $T(V^\vee)$. For $\vec{\imath} \in \{1, \dots, N\}^m$, we write $x_{\vec{\imath}}$ for the symmetric m-tensor $x_{i_1} \cdots x_{i_m}$. For example,

$$(x_1)^m = x_{1,\dots,1}.$$

We define the Taylor coefficients of our interaction term I via

$$I_{\vec{\imath}}^{(m)} = \left.\frac{\partial^m I}{\partial x_{i_1} \cdots \partial x_{i_m}}\right|_{(x)=0}.$$

In particular, each $I^{(m)}$ is a symmetric m-tensor, and

$$\frac{\partial I(x)}{\partial x_i} = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{\vec{\jmath} \in \{1,\dots,N\}^m} I_{i,\vec{\jmath}}^{(m+1)} x_{\vec{\jmath}}.$$

This term describes the coefficient of $\partial/\partial x_i$ in ι_{dI}, which appears in the differential of the BV complex.

Our goal can be stated as follows: compute

$$\langle f \rangle_I = \left\langle \sum_{\vec{\imath}} f_{\vec{\imath}} x_{\vec{\imath}} \right\rangle_I$$

for I our interaction term and $f = \sum_{\vec{\imath}} f_{\vec{\imath}} x_{\vec{\imath}}$ an arbitrary power series.

We now introduce the version of Feynman diagram appropriate for our purposes.

DEFINITION 2.3.3. A *Feynman diagram* is a finite connected graph (self-loops and parallel edges are allowed) built from the following pieces:

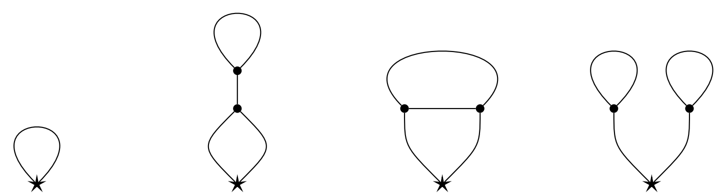
- Precisely one *marked* vertex, with valence n, which is labeled by an n-tensor $f \in (\mathbb{C}^N)^{\otimes n}$, and whose incident half-edges are totally ordered; we will draw the marked vertex with a star $\star$, and leave the tensor and the total ordering implicit.
- Some number of *internal* vertices, which are required to have valence 3 or more; we will draw internal vertices as solid bullets $\bullet$.

- Some number of univalent *external* vertices; we will draw external vertices as open circles $\circ$.

REMARK 2.3.4. The marked vertex will be labelled by f, the function whose expectation value we wish to compute.

An *automorphism* of a Feynman diagram is a permutation of its half-edges that does not change the combinatorial type of the diagram — it may separately permute both the internal and external vertices, but it should not permute the half-edges incident to the marked vertex. Given a Feynman diagram Γ, its *first Betti number* $b_1(\Gamma)$ is its total number of edges minus its number of un-marked vertices. We say that an edge is *internal* if it connects internal and marked vertices and *external* if one of its ends is an external vertex.

Below are Feynman diagrams whose marked vertex has valence 2, whose internal vertices have valence 3, whose Betti number are 1 or 2, and which possess no external vertices. We indicate the numbers of automorphisms beneath each diagram.



$|\mathrm{Aut}| = 1$ $\quad$ $|\mathrm{Aut}| = 2$ $\quad$ $|\mathrm{Aut}| = 2$ $\quad$ $|\mathrm{Aut}| = 4$

Finally, we introduce the basic operation on Feynman diagrams, which we use to compute expectation values. We fix an element $f \in \mathbb{C}[[x_1, \ldots, x_N]]$ whose expectation value we wish to compute.

DEFINITION 2.3.5. The *evaluation* $\mathrm{ev}_I(\Gamma, f)$ of a Feynman diagram Γ on f is as follows. First, suppose we are given a labeling of the half-edges by numbers $\{1, \ldots, N\}$. To such a labeled Feynman diagram we associate a product of matrix coefficients:

- The marked vertex contributes $f_{\vec{\imath}}$, where $\vec{\imath}$ is the vector of labels formed by reading the labels on the incident half-edges in the prescribed order (recall that part of the data of Γ was a total ordering of these vertices).
- Each internal vertex with valence m contributes $-I^{(m)}_{\vec{\imath}}$, where $\vec{\imath}$ is the vector of labels formed by reading the incident half-edges in any order (recall that the tensors $I^{(m)}$ are symmetric).
- Each external vertex with incident half-edge labeled by $i \in \{1, \ldots, N\}$ contributes the variable $x_i \in \mathbb{C}[[x_1, \ldots, x_N, \hbar]]$.
- Each internal edge with half-edges labeled i, j contributes $a^{ij} = a^{ji}$, where $A^{-1} = (a^{ij})$.
- Each external edge with half-edges labeled i, j contributes $\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

Thus a labeled Feynman diagram evaluates to some monomial in $\mathbb{C}[[x_1, \dots, x_N, \hbar]]$. The evaluation $\mathrm{ev}_I(\Gamma, f)$ of an unlabeled Feynman diagram Γ is defined to be the sum over all possible labelings of its evaluation as a labeled Feynman diagram.

Thus, we have a map

$$\begin{array}{ccc} \{\text{Feynman diagrams}\} & \to & \mathbb{C}[[x_1, \dots, x_N, \hbar]] \\ \Gamma & \mapsto & \frac{\mathrm{ev}_I(\Gamma, f)\hbar^{b_1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \end{array}$$

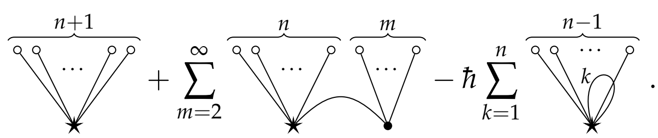
that relates Feynman diagrams to the power series we care about.

The utility of diagrammatic notation is showcased by the question of "recognizing a boundary" $\Delta g + \{I, g\}$ in $\mathbb{C}[[x_1, \dots, x_N, \hbar]]$. We first examine the problem using the algebraic notation from above. Just as in the proof of Wick's lemma, we start with simple monomials and see that

$$(\Delta - \{I, -\}) \left(x_{\vec\imath}\, \xi_j \right) \;=\; -a_{ij} x_i x_{\vec\imath} - \sum_{m=2}^{\infty} \sum_{\vec\jmath} \frac{1}{m!}\, I^{(m+1)}_{j,\vec\jmath}\, x_{\vec\jmath}\, x_{\vec\imath} + \hbar \sum_{k=1}^{n} \delta_{j,\imath_k} x_{i_1,\dots,\hat\imath_k,\dots,i_n}\,.$$

By "$\hat\imath_k$" we mean "remove this term from the list." Thus we can write the class $[x_i x_{\vec\imath}]$ as a sum of various other terms, each of which has either more xs or more $\hbar$s.

In the diagrammatic notation, we have a "picture" of a boundary:



In the final diagram, the self-loop connects the kth and $(n+1)$th half-edges on the marked vertex.

This picture suggests how to evaluate any $\langle \sum_{\vec\imath} f_{\vec\imath}\, x_{\vec\imath} \rangle$. In Johnson-Freyd's words, we play "Hercules' game of the many-headed Hydra." Pick some external vertex of the graph (a "head of the Hydra") corresponding to $f_{\vec\imath}\, x_{\vec\imath}$. Up to boundaries in $\mathscr{A}$, we can

- either attach this vertex to some other external vertex, thus making a loop and increasing the Betti number of the graph (this is the Δ term),
- or try to "chop this head off," at which point our Hydra grows at least two new external vertices (this is the $\{I, -\}$ term).

In the profinite topology on $\mathscr{A}$, any sequence of Hydra with strictly-increasing head number converges to 0, and for any given nonnegative integer β the game only produces finitely many graphs with Betti number $b_1 \le \beta$. Thus the whole game converges in the profinite topology.

What does our sequence of Hydra converge to? The only Feynman diagrams left at the end of the game are those with no external vertices at all: these are the only Hydra that do not have a head that Hercules can chop off. All together, we have proved the following.

PROPOSITION 2.3.6.

$$\left\langle \sum_{\vec{\imath}} f_{\vec{\imath}}\, x_{\vec{\imath}} \right\rangle_I = \sum_{\substack{\text{Feynman diagrams } \Gamma \\ \text{with no external vertices} \\ \text{and marked vertex labeled by } f}} \frac{\mathrm{ev}_I(\Gamma, f)\, \hbar^{b_1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \in \mathbb{C}[[\hbar]].$$

2.4. A compendium of essential definitions and constructions

We begin to extend these ideas to a more general setting.

DEFINITION 2.4.1. A -1-*symplectic vector space* is a dg vector space (V, d) with a degree -1 bilinear pairing $\langle -, - \rangle$ such that

- (skew-symmetry) $\langle x, y\rangle = -(-1)^{(|x|+1)(|y|+1)}\langle y, x\rangle$ for all $x, y \in V$;
- (nondegeneracy) for any nonzero $x \in V^k$, the linear functional $\langle x, -\rangle : V^{1-k} \to \mathbb{C}$ is nonzero;
- (compatibility with d) for all $x, y \in V$, $\langle dx, y\rangle = -(-1)^{|x|}\langle x, dy\rangle$.

REMARK 2.4.2. Because we want to work with infinite-dimensional vector spaces, we do not require that the symplectic pairing induces an isomorphism $V \to V^\vee$.

The compatibility with d has a crucial consequence.

LEMMA 2.4.3. *For $(V, d, \langle -, -\rangle)$ a -1-symplectic vector space, the cohomology $(H^*(V), 0)$ is canonically a -1-symplectic vector space with pairing $\langle -, -\rangle_{H^*V}$ defined by*

$$\langle [x], [y]\rangle_{H^*V} := \langle x, y\rangle$$

for any closed elements $x, y \in V$. In particular, the subspace of boundaries $B \subset V$ is isotropic in V.

Just as with ordinary symplectic vector spaces, maps are tricky. We will only need (for now) the analog of isomorphism.

DEFINITION 2.4.4. A *symplectomorphism* $\phi : V \to W$ of -1-symplectic vector spaces is a quasi-isomorphism such that

$$\langle \phi(x), \phi(y)\rangle_W = \langle x, y\rangle_V$$

for all $x, y \in V$.

REMARK 2.4.5. As we are working with vector spaces, a symplectomorphism always has an inverse symplectomorphism (just by picking intelligent splittings). This aspect does not extend well to arbitrary dg commutative algebras.

Note that this implies $H^*\phi$ is an isomorphism of graded vector spaces preserving the induced symplectic pairing. We denote by -1-*SympVect* the category whose objects are -1-symplectic vector spaces and whose morphisms are the symplectomorphisms.

REMARK 2.4.6. It would be interesting to develop a Weinsteinian category of -1-symplectic vector spaces with Lagrangian correspondences for morphisms. In general, a further exploration of derived symplectic geometry beckons (current work can be seen in [PTVV] and [CS]).

We want to view a -1-symplectic vector space V as a *derived* space, so we define its ring of functions as the commutative dg algebra

$$\mathscr{O}(V) := (\mathrm{Sym}(V^\vee), d),$$

where d is the differential on the dual $V^\vee$ extended as a derivation. By analogy with ordinary symplectic geometry, we might expect that $\mathscr{O}(V)$ has some kind of Poisson structure, as we now see.

We say a graded vector space V is *locally finite* if each graded component V^k is finite-dimensional. For a locally finite V, the symplectic pairing yields an isomorphism $V \xrightarrow{\cong} V^\vee$ and so we obtain a degree 1 pairing $\{-,-\}$ on $V^\vee$ dual to $\langle -,- \rangle$.

LEMMA 2.4.7. *For V a locally finite -1-symplectic vector space, $\mathscr{O}(V)$ has a natural* Pois_0 *structure, which arises by extending the pairing $\{-,-\}$ on $V^\vee$ to a Poisson bracket.*

REMARK 2.4.8. In the infinite-dimensional setting, it becomes more challenging to equip $\mathscr{O}(V)$ with a natural Pois_0 structure. We do this in section 2.6 for V an elliptic complex on a closed manifold.

CONSTRUCTION 2.4.9. There is a natural BV Laplacian Δ induced on $\mathscr{O}(V) = \mathrm{Sym}(V^\vee)$ when the Poisson pairing $\{-,-\}$ sends $V \otimes V$ to $\mathbb{C}$ (as is the case for functions on a -1-symplectic vector space). We set $\Delta \equiv 0$ on $\mathrm{Sym}^{\leq 1}(V^\vee)$ and

$$\Delta(xy) = \{x, y\}$$

for $x, y \in V^\vee$. Knowing Δ on $\mathrm{Sym}^{\leq 2}(V^\vee)$, we can extend to all of $\mathscr{O}(V)$ by recursively applying the equation

$$\Delta(xy) = (\Delta x)y + (-1)^{|x|}x(\Delta y) + \{x, y\}.$$

Equivalently, the pairing on $V^\vee$ has a kernel $P \in V \otimes V$ and we use the second-order differential operator $\partial_P = \iota_P$ on $\mathrm{Sym}\, V^\vee$. This construction implies the following proposition/definition.

PROPOSITION 2.4.10. *There is a functor*

$$\begin{array}{rccc} \mathcal{BVQ}: & -1\text{-}SympVect^{lf} & \to & dgVect \\ & (V, d, \langle -,- \rangle) & \mapsto & (\mathrm{Sym}(V^\vee), d + \Delta) \end{array}$$

where the -1-$SympVect^{lf}$ denotes the subcategory of locally finite spaces.

This functor has two appealing properties:

- $\mathcal{BVQ}(0) = \mathbb{C}$, and
- $\mathcal{BVQ}(V \oplus W) \cong \mathcal{BVQ}(V) \otimes \mathcal{BVQ}(W)$.

This functor also possesses a remarkable property on a natural subcategory with even stronger finiteness condition.

PROPOSITION 2.4.11. *Let (V, d) be a locally finite -1-symplectic vector space with finite-dimensional cohomology (i.e., $\sum_k \dim H^k V < \infty$). Then*

$$H^* \mathcal{B}VQ(V) \simeq \mathbb{C}[d(V)]$$

where $d(V) = -\sum_k (2k+1) \dim H^{2k+1}(V)$.

This proposition says that $\mathcal{B}VQ$ is a kind of *Pfaffian functor* on the cohomologically finite-dimensional -1-symplectic vector spaces. Just as the Pfaffian eats a skew-symmetric form on a vector space (finite-dimensional with orientation) and returns a number, $\mathcal{B}VQ$ eats a -1-symplectic space (cohomologically finite-dimensional) and returns a graded line. Just as the Pfaffian sends direct sum of matrices to products, $\mathcal{B}VQ$ sends direct sums of -1-symplectic spaces to tensor products. In chapter 3, we develop this point of view to obtain a a very precise interpretation of BV quantization for shifted cotangent bundles as providing a *determinant functor* (again building on the analogy that the Pfaffian is a square-root of the determinant).[10]

PROOF. Consider the filtration $F^k \mathcal{B}VQ(V) = \mathrm{Sym}^{\leq k}(V^\vee)$. In the induced spectral sequence, the first page is given by $\mathrm{Sym}(H^*(V^\vee))$ and the differential is the BV Laplacian coming from the -1-symplectic structure on $H^*(V)$. The following lemma shows the cohomology of this first page is one-dimensional, so that the spectral sequence collapses. $\square$

LEMMA 2.4.12. *Consider the complex*

$$(\mathbb{C}[x_1, \ldots, x_N, \xi_1, \ldots, \xi_N], \Delta),$$

where — without loss of generality — all the x_i are even, $|\xi_i| = 1 - |x_i|$, and

$$\Delta = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \frac{\partial}{\partial \xi_j}.$$

Its cohomology is one-dimensional and generated by the pure odd element $\xi_1 \xi_2 \cdots \xi_N$.

PROOF. We've picked a basis for $V^\vee$ (or $H^* V^\vee$, in the case of the preceding proposition) simply to make the proof as explicit as possible,. We use induction on N. For $N = 1$, it is clear that $\Delta(x^m) = 0$ and $\Delta(\xi) = 0$ but $\Delta(x^m \xi) = m x^{n-1}$. Hence ξ generates the cohomology.

For the induction step, suppose the lemma holds for $N = k$. Given an element f, we decompose it as a finite sum

$$f = \sum_{m \in \mathbb{N}} x_{k+1}^m f_{m0} + \sum_{n \in \mathbb{N}} x_{k+1}^n \xi_{k+1} f_{n1}$$

[10] In a sense, none of these assertions should be surprising. As we've seen, the BV formalism in finite dimensions provides a homological encoding of Wick's lemma and Berezinian integration. As such, we've obtained a way to discuss these structures at the level of vector spaces rather than sets.

where the terms f_{ij} depend only on the first k variables of x and ξ, i.e.,

$$f_{ij} \in \mathbb{C}[x_1, \ldots, x_k, \xi_1, \ldots, \xi_k] \subset \mathbb{C}[x_1, \ldots, x_{k+1}, \xi_1, \ldots, \xi_{k+1}].$$

We write

$$\Delta = \Delta_k + \frac{\partial}{\partial x_{k+1}} \frac{\partial}{\partial \xi_{k+1}},$$

where Δ_k is the BV Laplacian for the first k variables. Observe that

$$\Delta f = \sum_{m \in \mathbb{N}} x_{k+1}^m (\Delta_k f_{m0}) - \sum_{n \in \mathbb{N}} x_{k+1}^n \xi_{k+1} (\Delta_k f_{n1}) + \sum_{n \in \mathbb{N}} n x_{k+1}^{n-1} f_{n1}.$$

If $\Delta f = 0$, we find

- only the second term depends on ξ_{k+1}, so $\Delta_k f_{n1} = 0$ for all n;
- consequently, the first and third terms cancel, so $n f_{n1} = -\Delta_k f_{(n-1)0}$.

Thus, *for* $n > 1$, the f_{n1} terms are determined by the f_{m0} terms. Conversely, if $\Delta f = 0$ and $f_{01} = 0$, then f is a boundary:

$$\Delta : \tilde{f} = \sum_n \frac{1}{n+1} x_{k+1}^{n+1} \xi_{k+1} f_{n0} \mapsto \sum_n x_{k+1}^n f_{n0} - \sum_n \frac{1}{n+1} x_{k+1}^{n+1} \xi_{k+1} \Delta_k f_{n0} = f.$$

When $f_{01} \neq 0$, however, f is not a boundary. Since we have $\Delta_k f_{01} = 0$, the induction hypothesis tells us that, up to boundaries, there is only a one-dimensional space of choices and that $\xi_1 \cdots \xi_k$ is a generator for cohomology in the $N = k$ case. Hence $\xi_1 \cdots \xi_k \xi_{k+1}$ generates cohomology for $N = k + 1$. □

2.5. The homological perturbation lemma in the BV formalism

There is a natural toolkit from homological algebra that makes the manipulations in section 2.3 appear more systematic and less *ad hoc*. In particular, we want a technique to solve the basic problem:

> If we know the cohomology $H^*(V, d)$ of some complex (V, d), is there a method for computing the cohomology $H^*(V, d + \delta)$ where δ is some "small" perturbation of the original differential?

This problem appears frequently in mathematics; the spectral sequence of a double complex can be viewed as a tool for relating the horizontal cohomology to its "perturbation," the full double complex. For us, we have the classical BV complex and its deformation, the quantum BV complex.

2.5.1. Reminder on homological perturbation theory. The homological perturbation lemma provides an answer to the problem but requires some extra control over the original complex.

DEFINITION 2.5.1. A *contraction* (or *strong deformation retract*) consists of the following data:

(i) a pair of complexes (V, d_V) and (W, d_W);
(ii) a pair of cochain maps $\pi : V \to W$ and $\iota : W \to V$;
(iii) a degree -1 map of graded vector spaces $\eta : V \to V$.

This data must satisfy:

(a) W is a retract of V, so $\pi \circ \iota = \mathbb{1}_W$;
(b) η is a chain homotopy between $\mathbb{1}_V$ and $\iota \circ \pi$, so

$$\iota \circ \pi - \mathbb{1}_V = d_V \eta + \eta d_V = [d_V, \eta];$$

(c) the *side conditions*

$$\eta^2 = 0, \ \eta \circ \iota = 0, \text{ and } \pi \circ \eta = 0.$$

We draw this data as

$$(*) \qquad \begin{tikzcd} (W, d_W) \arrow[r, shift right, "\iota"'] & (V, d_V) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\eta"] \end{tikzcd}$$

and use it as a visual shorthand throughout the text.

DEFINITION 2.5.2. A *perturbation* of a complex (V, d_V) is a degree 1 map $\delta : V \to V$ such that $(d_V + \delta)^2 = 0$. A *small perturbation of a contraction* (using the notation in ($*$)) is a perturbation of V such that $\mathbb{1}_V - \delta\eta$ is invertible or, equivalently, if $\mathbb{1}_V - \eta\delta$ is invertible.

With these definitions in hand, we now introduce the useful trick.

THEOREM 2.5.3 (Homotopy Perturbation Lemma). *Given a small perturbation δ of a contraction, there is a new contraction*

$$\begin{tikzcd} (W, d_W + \delta_W) \arrow[r, shift right, "\widetilde{\iota}"'] & (V, d_V + \delta) \arrow[l, shift right, "\widetilde{\pi}"'] \arrow[loop right, "\widetilde{\eta}"] \end{tikzcd}$$

where

$$\begin{aligned} \delta_W &= \pi \circ (\mathbb{1}_V - \delta\eta)^{-1} \circ \delta \circ \iota, \\ \widetilde{\iota} &= \iota + \eta \circ (\mathbb{1}_V - \delta\eta)^{-1} \circ \delta \circ \iota, \\ \widetilde{\pi} &= \pi + \pi \circ (\mathbb{1}_V - \delta\eta)^{-1} \circ \delta \circ \eta, \\ \widetilde{\eta} &= \eta + \eta \circ (\mathbb{1}_V - \delta\eta)^{-1} \circ \delta \circ \eta. \end{aligned}$$

In short, there is a perturbed contraction.

We recommend [Cra] as a succinct reference for this lemma and some generalizations. In particular, one can work with general homotopy equivalences rather than contractions, but we do not need that level of generality.

2.5.2. Applying perturbation theory in the BV formalism. In the general situation of the BV formalism, we have two complexes, the classical observables

$$\mathrm{Obs}^{\mathrm{cl}} = (V, d)$$

and the quantum observables

$$\mathrm{Obs}^{\mathrm{q}} = (V[[\hbar]], d_q := d + \hbar d_1 + \hbar^2 d_2 + \cdots).$$

If we have a contraction of the classical observables, e.g.,

$$\begin{tikzcd} (H^*V, 0) \arrow[r, shift left, "\iota"'] & (V, d) \arrow[l, shift left, "\pi"'] \arrow[loop right, "\eta"] \end{tikzcd} ,$$

we might hope to obtain a contraction of the quantum observables

$$\begin{tikzcd} (H^*V[[\hbar]], D) \arrow[r, shift left, "\tilde{\iota}"'] & (V[[\hbar]], d_q) \arrow[l, shift left, "\tilde{\pi}"'] \arrow[loop right, "\tilde{\eta}"] \end{tikzcd} ,$$

with D something relatively simple. In particular, the projection operator $\tilde{\pi}$ provides a method for computing the expectation value of an element $f \in V[[\hbar]]$. Namely, $\tilde{\pi}(f)$ lives in a much smaller complex than f, so that its cohomology class is easy to compute explicitly.

Indeed, the main results of section 2.3 can be phrased as "Feynman diagrams are an implementation of the homological perturbation lemma." We now indicate how to justify that assertion. First, we will outline the proof without writing a formal proof, as it involves combinatorics and algebra that is straightforward but involved. Second, we do an explicit example that demonstrates the requisite computations.

As in section 2.3, let $V = \mathbb{R}^N$ and $A = (a_{ij})$ a positive-definite, symmetric, real $N \times N$ matrix. We define

$$\mathrm{Obs}^{\mathrm{cl}}{}_I := (\mathbb{C}[[x_1, \ldots, x_N, \xi_1, \ldots, \xi_N]], d_I)$$

where

$$d_I = -\sum_{i,j=1}^{N} a_{ij} x_i \frac{\partial}{\partial \xi_i} + \{I, -\}.$$

We wish to apply the homological perturbation lemma by adding

$$\hbar \Delta_{Leb} = \hbar \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \frac{\partial}{\partial \xi_i}$$

to d_I.

As shown in the proof of Wick's lemma (lemma 2.3.2), the complex for $I = 0$ is precisely a Koszul complex describing the origin inside V (i.e., arising from the regular sequence given by the rows of A). Hence, there is a simple contraction

$$(\mathbb{C}, 0) \underset{\iota_0}{\overset{\pi_0}{\leftrightarrows}} (\mathrm{Obs^{cl}}_0, d_0) \circlearrowleft \eta_0 .$$

Here ι_0 is the inclusion of $\mathbb{C} \cong \mathrm{Sym}^0$ into $\mathrm{Obs^{cl}}$ and π_0 is projection onto $\mathrm{Sym}^0 \cong \mathbb{C}$. *Note that π_0 is given by "evaluation at 0," where 0 is the critical point of the quadratic function $\langle x, Ax\rangle$.* The homotopy η_0 can be constructed quite explicitly (see proposition 2.5.5 below).

Wick's lemma is now a direct consequence of the homological perturbation lemma. If we deform the differential on $\mathrm{Obs^{cl}}[[\hbar]]$ by adding $\hbar\Delta_{Leb}$, we get a new projection map

$$\begin{aligned}\widetilde{\pi}_0 &= \pi_0 + \pi_0 \circ (\mathbb{1} - \hbar\Delta_{Leb}\eta_0)^{-1} \circ \hbar\Delta_{Leb} \circ \eta_0 \\ &= \sum_{n\geq 0} \hbar^n \pi_0(\Delta_{Leb}\eta_0)^n.\end{aligned}$$

Direct calculation verifies that $\widetilde{\pi}_0(x^\nu)$ recovers precisely the formula given in lemma 2.3.2.[11] That is, $\widetilde{\pi}_0$ *computes the expectation value.* In this homological setting, the fact that Δ_{Leb} is a constant-coefficient second-order differential operator makes the origins of Wick's lemma a bit clearer: in the perturbation, we lower the polynomial degree by two for every power of $\hbar$. In terms of Feynman diagrams, we are iteratively attaching both ends of a loop.

The differential $\{I, -\}$ is also a small perturbation of $\mathrm{Obs^{cl}}_0$, so we can construct a homotopy equivalence

$$(\mathrm{Obs^{cl}}_0, d_0) \underset{\iota'}{\overset{\pi'}{\leftrightarrows}} (\mathrm{Obs^{cl}}_I, d_I) \circlearrowleft \eta' ,$$

which composes with the earlier homotopy equivalence (π_0, ι_0, η_0) to yield

$$(\mathbb{C}, 0) \underset{\iota_I}{\overset{\pi_I}{\leftrightarrows}} (\mathrm{Obs^{cl}}_I, d_I) \circlearrowleft \eta_I .$$

The projection π_I is given by iteratively applying an operation that depends on I. In terms of Feynman diagrams, this is the tree-level expansion. Again, π_I amounts to "evaluation at the origin, the critical point."

We now perturb by $\hbar\Delta_{Leb}$. Diagrammatically, the new projection is given by iteratively attaching a loop, just as in the case of Wick's lemma. If one draws the diagrams, one sees the Herculean game of Hydra reappear. *The projection operator $\widetilde{\pi}_I$ computes the expectation value of any $f \in \mathrm{Obs^q}_I$.* In other words, the Feynman diagram expansion is just a combinatorial description of this projection operator.

[11]As earlier, calculation is quite direct in the case of $V = \mathbb{R}^1$. For the higher dimensional case, it's easiest to diagonalize A and then take tensor products of the one-dimensional case.

These techniques apply *mutatis mutandis* to more general BD algebras, and thus we will obtain a point of view on how Feynman diagrams enter into the study of general QFT. In the next subsection, we discuss the essential algebra before introducing the analytic issues that appear in QFT proper.

EXAMPLE 2.5.4. Let $N = 1$. We start with the contraction of classical observables

$$\begin{tikzcd} (\mathbb{C}, 0) \arrow[r, shift right, "\iota_0"'] & (\mathbb{C}[[x,\xi]], d_0) \arrow[l, shift right, "\pi_0"'] \arrow[loop right, "\eta_0"] \end{tikzcd} .$$

Here ξ has degree -1 and $d_0 = -ax\partial_\xi$. The homotopy is

$$\eta_0(x^n) = \begin{cases} 0, & n = 0 \\ \frac{1}{a}x^{n-1}, & n > 0 \end{cases}$$

as one can quickly check.

Consider the perturbation by $\hbar\Delta_{Leb}$. We compute that

$$\hbar\Delta_{Leb}\eta_0(x^n) = \begin{cases} 0, & n = 0, 1 \\ \frac{\hbar}{a}(n-1)x^{n-2}, & n > 1 \end{cases}$$

and hence

$$(\hbar\Delta_{Leb}\eta_0)^m(x^n) = \begin{cases} 0, & n < 2m \\ \left(\frac{\hbar}{a}\right)^m (n-1)(n-3)\cdots(n-(2m-1))x^{n-2m}, & n \geq 2m \end{cases} .$$

We see that

$$\widetilde{\pi}_0(x^n) = \sum_{m \geq 0} \hbar^n \pi_0(\Delta_{Leb}\eta_0)^m(x^n) = \begin{cases} 0, & n = 2k \\ \left(\frac{\hbar}{a}\right)^k (2k-1)!!, & n = 2k \end{cases} .$$

This is precisely Wick's lemma.

2.5.3. The "free field" case. Much of the rest of this thesis will focus on a situation roughly of the following form, which is a caricature of the "free field theories" we will study. Let (V, d) be a bounded cochain complex of vector spaces (the "fields") and set

$$\mathscr{O}(T^*[-1]V) := (\mathrm{Sym}(V^\vee \oplus V[1]), d_V),$$

the commutative dg algebra where d denotes the obvious differentials on $V^\vee$ and $V[1]$ extended to a derivation. This algebra is "functions on the shifted cotangent bundle of V." The evaluation pairing between $V^\vee$ and V induces a canonical 1-symplectic pairing that we then extend to a Poisson bracket $\{-,-\}$ by the Leibniz rule. This bracket makes $\mathscr{O}(T^*[-1]V)$ into a Pois_0 algebra. There is then a canonical BV Laplacian we use to quantize. (We are using the construction from section 2.4.)

Now suppose we had a contraction of the linear observables $V^\vee \oplus V[1]$ onto their cohomology $\mathcal{H}$. We want to obtain a contraction of $\mathscr{O}(T^*[-1]V)$ onto $\mathscr{O}(\mathcal{H})$ and then use the homotopy perturbation lemma to contract $\mathrm{Obs}^{\mathrm{q}}$ onto $\mathscr{O}(\mathcal{H})[[\hbar]]$. Thus, our first order of business is to construct a contraction on the symmetric algebras coming from a contraction.

PROPOSITION 2.5.5. *Given a contraction*

$$
\begin{tikzcd}
(W, d_W) \arrow[r, shift right, "\iota"'] & (V, d_V) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\eta"]
\end{tikzcd} ,
$$

there is a natural contraction on the associated symmetric algebras

$$
\begin{tikzcd}
(\mathrm{Sym}\, W, d_W) \arrow[r, shift right, "\mathrm{Sym}\,\iota"'] & (\mathrm{Sym}\, V, d_V) \arrow[l, shift right, "\mathrm{Sym}\,\pi"'] \arrow[loop right, "\mathrm{Sym}\,\eta"]
\end{tikzcd}
$$

where $\mathrm{Sym}\,\eta$ *is constructed explicitly in the proof below.*

PROOF. We elaborate on some simple properties of a contraction that clarify the idea of the proof. First, observe that $\iota \circ \pi$ is a projection operator on the graded module V, so we will view W as a submodule of V and denote the projection $P : V \to W \subset V$. The operator $P^\perp = \mathbb{1}_V - P$ is thus also a projection operator, and we denote its image by $W^\perp \subset V$. By construction, W and $W^\perp$ are subcomplexes, i.e., d_V respects the decomposition $V = W \oplus W^\perp$. Second, observe that the side conditions on η imply that η also respects the decomposition:

$$P \circ \eta \circ P^\perp = 0 = P^\perp \circ \eta \circ P.$$

Finally, note that this decomposition implies that $\mathrm{Sym}\, V \cong \mathrm{Sym}\, W \otimes \mathrm{Sym}\, W^\perp$. Moreover, $P^\perp$, extended to a derivation on $\mathrm{Sym}\, V$, decomposes it into a direct sum of eigen-complexes

$$\mathrm{Sym}\, V = \bigoplus_{n \geq 0} M_n \text{ where } M_n = (\mathrm{Sym}\, W) \otimes \mathrm{Sym}^n\, W^\perp$$

and $P^\perp$ has eigenvalue n on M_n. The map $\mathbb{1}_{\mathrm{Sym}\, V} - \mathrm{Sym}\,\iota \circ \mathrm{Sym}\,\pi$ is then simply the projection of $\mathrm{Sym}\, V$ onto $\oplus_{n \geq 1} M_n$.

Let η denote the operator on $\mathrm{Sym}\, V$ obtained by extending η on V as a derivation. The side conditions imply that η respects the eigendecomposition. We now define a map $\mathrm{Sym}\,\eta$:

$$\mathrm{Sym}\,\eta\big|_{M_n} := \begin{cases} \frac{1}{n}\eta, & \text{for } n > 0 \\ 0, & n = 0 \end{cases} .$$

It remains to verify $\mathrm{Sym}\,\eta$ is the desired homotopy.

Observe that the maps d, η, $P^\perp$ all preserve the subcomplexes $\mathrm{Sym}^m\, W \otimes \mathrm{Sym}^n\, W^\perp$. It is straightforward to verify that, for $n > 0$,

$$[d, \eta]\Big|_{\mathrm{Sym}^m\, W \otimes \mathrm{Sym}^n\, W^\perp} = P^\perp\Big|_{\mathrm{Sym}^m\, W \otimes \mathrm{Sym}^n\, W^\perp} = n\mathbb{1}_{\mathrm{Sym}^m\, W \otimes \mathrm{Sym}^n\, W^\perp}$$

and hence that

$$[d, \operatorname{Sym}\eta]\Big|_{\operatorname{Sym}^m W \otimes \operatorname{Sym}^n W^\perp} = \mathbb{1}_{\operatorname{Sym}^m W \otimes \operatorname{Sym}^n W^\perp}.$$

Thus we see $\operatorname{Sym}\eta$ is the required homotopy. $\square$

Using the same notation as above, suppose we have a complex (V, d) and a contraction for the linear elements of $\mathscr{O}(T^*[-1]V)$ onto their cohomology $\mathcal{H}$:

$$\begin{tikzcd} (\mathcal{H}, 0) \arrow[r, shift left, leftarrow, "\pi"] \arrow[r, shift right, "\iota"'] & (V^\vee \oplus V[1], d) \arrow[loop right, "\eta"] \end{tikzcd}.$$

Let Δ denote the BV Laplacian on $\operatorname{Sym}(V^\vee \oplus V[1])$. There are two, closely-related perturbations of the classical observables that interest us:

(a) the parameter $\hbar$ is not formal (i.e., can take nonzero values)

$$\mathrm{Obs}^{\mathrm{q}} := (\operatorname{Sym}(V^\vee \oplus V[1])[\hbar], d + \hbar\Delta),$$

(b) the parameter $\hbar$ is formal (i.e., can take "infinitesimal" nonzero values)

$$\mathrm{Obs}^{\mathrm{q}} := (\operatorname{Sym}(V^\vee \oplus V[1])[[\hbar]], d + \hbar\Delta).$$

In either case, $\hbar\Delta$ is a small perturbation. For $\hbar$ formal, it is clear that the geometric series

$$(\mathbb{1} - \hbar\Delta \operatorname{Sym}\eta)^{-1} = 1 + \hbar\Delta \operatorname{Sym}\eta + \hbar^2(\Delta \operatorname{Sym}\eta)^2 + \cdots = \sum_{n\geq 0} \hbar^n (\Delta \operatorname{Sym}\eta)^n$$

is well-defined. When $\hbar$ is not formal (case (a) above), the same geometric series is well-defined because Δ is locally nilpotent: if $f \in \operatorname{Sym}^{\leq k}(V^\vee \oplus V[1])$, then $\Delta^m f = 0$ for $2m > k$.

By the perturbation lemma, we then obtain a perturbation of $\operatorname{Sym}\mathcal{H}[[\hbar]]$ whose differential is now

$$\begin{aligned} D &= \operatorname{Sym}\pi \circ (\mathbb{1} - \hbar\Delta \operatorname{Sym}\eta)^{-1} \circ (\hbar\Delta) \circ \operatorname{Sym}\iota \\ &= \hbar \sum_{n\geq 0} \hbar^n \operatorname{Sym}\pi \circ \underbrace{\Delta \operatorname{Sym}\eta \circ \cdots \Delta \operatorname{Sym}\eta}_{\text{with } n\ \Delta \operatorname{Sym}\eta\text{'s}} \circ \Delta \circ \operatorname{Sym}\iota. \end{aligned}$$

The projection map is

$$\begin{aligned} \widetilde{\operatorname{Sym}\pi} &= \operatorname{Sym}\pi + \operatorname{Sym}\pi \circ (\mathbb{1} - \hbar\Delta \operatorname{Sym}\eta)^{-1} \circ \Delta \circ \operatorname{Sym}\eta \\ &= \sum_{n\geq 0} \hbar^n \operatorname{Sym}\pi \circ \underbrace{\Delta \operatorname{Sym}\eta \circ \cdots \Delta \operatorname{Sym}\eta}_{\text{with } n\ \Delta \operatorname{Sym}\eta\text{'s}} \\ &= \operatorname{Sym}\pi \circ (\mathbb{1} - \hbar\Delta \operatorname{Sym}\eta)^{-1}. \end{aligned}$$

As in the previous section, *this projection map* $\widetilde{\operatorname{Sym}\pi}$ *is useful because it gives an explicit procedure for computing the expectation value of any* f *in* $\mathrm{Obs}^{\mathrm{q}}$. More precisely, it characterizes how to find an element in a smaller complex that is hopefully more manageable.

2.6. Global observables and formal Hodge theory

The algebra of the BV formalism can be applied directly to the global observables of free quantum field theories on *closed* manifolds. (Interactions introduce serious challenges that are overcome with the machinery of renormalization. See [**CG**] for a discussion of these issues.) In essence, because we work with elliptic complexes (more accurately, the classical theories are given by elliptic complexes), the global solutions — given by the cohomology of the elliptic complex — are finite-dimensional. Formal Hodge theory in this context, plus the homological perturbation lemma, then allow us to translate the problem of computing expectation values of global observables into the corresponding problem on a finite-dimensional problem. As we've seen, we can visualize the transfer maps using Feynman diagrams, and in this context, the computations of a physicist exactly match up with those of the homological algebraist.

2.6.1. Reminder on formal Hodge theory. In this section, we will work in a context well-known in differential geometry. We will work with a smooth, closed manifold M equipped with a strictly positive smooth measure μ. This measure is rarely part of the data of the theory but is necessary for many of the analytic results we use. The dependence on choice of measure is easy to deal with: the space of strictly positive smooth measures is convex, so the space of choices is contractible. Our standard reference here is [**Wel08**], particularly chapter 4.

DEFINITION 2.6.1. An *elliptic complex with inner product* is an elliptic complex $(\mathscr{E}, Q)$ where each vector bundle E^i has a hermitian inner product $(-,-)_{E^i}$, inducing a pre-Hilbert structure on $\mathscr{E}^i$:

$$(f,g)_{\mathscr{E}^i} := \int_{x\in M} (f(x), g(x))_{E^i}\ \mu.$$

The inner product allows us to introduce *adjoint operators* Q^* on $\mathscr{E}$ of degree -1 and thus to obtain the "Laplacian" $D = [Q, Q^*]$. [12] Let $\mathcal{H} = \ker D$ denote the *harmonic sections*. There is an orthogonal projection map

$$\pi : \mathscr{E} \to \mathcal{H}$$

onto this closed subspace. There is a parametrix for D, which we will denote by G (for Green's function), and satisfying

$$\mathbb{1}_{\mathscr{E}} = \pi + GD = \pi + DG.$$

(This result is Theorem 4.12 in [**Wel08**].) The following is Theorem 5.2 in [**Wel08**].

THEOREM 2.6.2 (Formal Hodge theorem). *For $(\mathscr{E}, Q)$ an elliptic complex with inner products,*

(a) there is an orthogonal decomposition

$$\mathscr{E} = \mathcal{H} \oplus QQ^*(\mathscr{E}) \oplus Q^*Q(\mathscr{E});$$

(b) the following commutation relations hold:

[12] We use D rather than Δ because want to reserve Δ for the BV Laplacian.

(a) $\mathbb{1}_{\mathscr{E}} = \pi + DG = \pi + GD$,
(b) $\pi G = G\pi = \pi D = D\pi = 0$,
(c) $QD = DQ$ *and* $Q^* D = DQ^*$,
(d) $QG = GQ$ *and* $Q^* G = GQ^*$;

(c) $\dim \mathcal{H}$ *is finite and there is a canonical isomorphism*

$$H^i(\mathscr{E}) \cong \mathcal{H} \cap \mathscr{E}^i.$$

In short, we obtain a homotopy retraction

$$\begin{tikzcd} (\mathcal{H}, 0) \arrow[r, shift left, leftarrow, "\pi"] \arrow[r, shift right, "\iota"'] & (\mathscr{E}, Q) \arrow[loop right, "Q^* G"] \end{tikzcd} .$$

using formal Hodge theory. (Note $[Q, Q^* G] = DG = \mathbb{1} - \pi$.) Our goal is to use this result in field theory.

2.6.2. Field theories and their global observables. Our notion of field theory is borrowed from [**Cos11**] and our definition of observables is borrowed from [**CG**].

2.6.2.1. *Free theories.*

DEFINITION 2.6.3. A *free BV theory* on a manifold M consists of the following data:

- a finite rank, $\mathbb{Z}$-graded vector bundle (or super vector bundle) E on M;
- a vector bundle map $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}_M$ that is fiberwise nondegenerate, antisymmetric, and of cohomological degree -1; this local pairing induces a pairing on compactly-supported sections
$$\langle -, - \rangle : \mathscr{E}_c \otimes \mathscr{E}_c \to \mathbb{C},$$
$$\langle s_0, s_1 \rangle = \int_{x \in M} \langle s_0(x), s_1(x) \rangle_{loc};$$
- a differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 such that
  (1) $(\mathscr{E}, Q)$ is an elliptic complex;
  (2) Q is skew-self-adjoint with respect to the pairing, i.e., $\langle s_0, Q s_1 \rangle = -(-1)^{|s_0|} \langle Q s_0, s_1 \rangle$.

In practice we want access to theorem 2.6.2, so we are forced to introduce an auxiliary object. Pick a hermitian metric $h = (-, -)$ on the vector bundles E_j so that we obtain an h-adjoint Q^* and h-Laplacian $D = [Q, Q^*]$. Whenever we speak of the free theory $(\mathscr{E}, Q, \langle -, - \rangle)$, it should be understood that we have made a choice of hermitian metric once and for all. Thankfully, the space of metrics is convex and hence contractible, and it is straightforward to relate different choices. (For a sophisticated treatment of these issues, see section 10, chapter 5 of [**Cos11**]. We also give a precise definition that picks out theories well-suited to Hodge theory in section 7.6 in chapter 7.)

REMARK 2.6.4. This definition is a slight modification of the definition from [**Cos11**]. We allow theories for which D need not be a generalized Laplacian (in contrast to [**Cos11**], where D needs to be a second-order differential operator and so on) because our constructions apply to this more general situation. The generalized Laplacian condition is necessary for the renormalization techniques in [**Cos11**], in particular the construction of the counterterms that obtain effective field theories from local Lagrangians.

The action functional associated to this theory is

$$S(\phi) := \int_M \langle \phi, Q\phi \rangle$$

where ϕ is a compactly-supported section of E. To be more accurate, we should say that the classical field theory given by S looks for fields ϕ (not necessarily compactly supported) that are critical points of S with respect to perturbations by compactly supported sections. In other words, there is a 1-form dS with respect to the foliation of $\mathscr{E}$ by $\mathscr{E}_c$ and we are looking for ϕ such that $dS_\phi = 0$. Note that S is quadratic so that dS is linear. This relationship is why quadratic actions correspond to free field theories.

Notice that (the global sections of) a free field theory provides a -1-symplectic vector space, and hence we can try to apply the framework we've developed to it. Our first step is to obtain a well-behaved Pois_0 algebra of functions on the fields, which is somewhat subtle for analytic reasons.

Recall that $E^!$ denotes the vector bundle $E^\vee \otimes \mathrm{Dens}_M$ on M. Then global distributional sections $\overline{\mathscr{E}^!}$ of this bundle are precisely the distributions dual to global smooth sections of E. The differential Q on $\mathscr{E}$ naturally induces a differential Q on $\mathscr{E}^!$ and makes it into an elliptic complex.

DEFINITION 2.6.5. The *global classical observables* of the free theory $(M, \mathscr{E}, Q, \langle -, - \rangle)$ are the commutative dg algebra $\mathrm{Obs}^{\mathrm{cl}} := (\mathrm{Sym}\, \mathscr{E}^!, Q)$.

There is a natural degree 1 Poisson bracket defined as follows. Recall that our free theory has a pairing $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}_M$ on vector bundles. As it is fiberwise nondegenerate, we use it to define a pairing

$$\langle -, - \rangle_{loc}^! : E^! \otimes E^! \to \mathrm{Dens}_M$$

that is also fiberwise nondegenerate and skew-symmetric. It has cohomological degree -1. We thus obtain a skew-symmetric pairing

$$\{-,-\} : \mathscr{E}_c^! \otimes \mathscr{E}_c^! \to \mathbb{C} \quad \text{where} \quad \lambda \otimes \mu \mapsto \int_{x \in M} \langle \lambda(x), \mu(x) \rangle_{loc}^!.$$

Extend this pairing as a biderivation to $\mathrm{Obs}^{\mathrm{cl}}$ by using the Leibniz rule. Thus, $\mathrm{Obs}^{\mathrm{cl}}$ is a a Pois_0 algebra.

REMARK 2.6.6. It probably seems more natural to use $\mathrm{Sym}\, \mathscr{E}^\vee$ as observables, since $\mathscr{E}^!$ is not the continuous dual of $\mathscr{E}$. Unfortunately, this choice does not have a Pois_0 algebra because one cannot pair distributions (the usual analytic problem arising in QFT). A lemma of Atiyah and Bott [AB67] (see 5.2.13 for further discussion) implies that there is nonetheless a homotopy equivalence between $\mathrm{Sym}\,\mathscr{E}^!$ and $\mathrm{Sym}\,\mathscr{E}^\vee$. We discuss these issues in more depth when we construct the factorization algebra of observables.

Theorem 2.6.2 applies to $\mathscr{E}^!$ so, letting $\mathcal{H}$ denote the cohomology of the linear observables $(\mathscr{E}^!, Q)$, we have a contraction

$$(\mathcal{H}, 0) \underset{\iota}{\overset{\pi}{\leftrightarrows}} (\mathscr{E}^!, Q) \circlearrowright \eta\ .$$

By proposition 2.5.5, we then obtain a very small replacement of the classical observables:

$$(\mathrm{Sym}\,\mathcal{H}, 0) \underset{\mathrm{Sym}\,\iota}{\overset{\mathrm{Sym}\,\pi}{\leftrightarrows}} (\mathrm{Sym}\,\mathscr{E}^!, Q) \circlearrowright \mathrm{Sym}\,\eta$$

The right-hand side has a natural interpretation as "functions on the derived space of solutions for the complex $(\mathscr{E}, Q)$," and the contraction allows us to describe those functions in a more tractable form. We have contracted information about this PDE problem onto a discussion just of its finite-dimensional space of solutions.

REMARK 2.6.7. To justify the use of proposition 2.5.5, we note that the space $\mathscr{E}^{\otimes n}$ is precisely sections of $E^{\boxtimes n}$ on M^n. We can then apply the Hodge theorem to $\mathscr{E}^{\otimes n}$ on M^n because Q naturally induces an elliptic complex on this product space.

We now construct the BV quantization. With the Poisson bracket in hand, we obtain a canonical BV Laplacian Δ, defined via Construction 2.4.9. We use Δ to perturb $\mathrm{Obs}^{\mathrm{cl}}$.

DEFINITION 2.6.8. The *global quantum observables* of the free theory $(M, \mathscr{E}, Q, \langle -,- \rangle)$ are the dg vector space $\mathrm{Obs}^{\mathrm{q}} := (\mathrm{Sym}(\mathscr{E}^!)[\hbar], Q + \hbar\Delta)$.

The homological perturbation lemma now allows us to obtain a very small replacement of the quantum observables:

$$(\mathrm{Sym}(\mathcal{H})[\hbar], D) \underset{\widetilde{\iota}}{\overset{\widetilde{\pi}}{\leftrightarrows}} \mathrm{Obs}^{\mathrm{q}} \circlearrowright \widetilde{\mathrm{Sym}\,\eta}$$

Again, we have contracted the essential data onto the finite-dimensional space of solutions. We say that for an observable $\mathcal{O}$ in $\mathrm{Obs}^{\mathrm{q}}$, its *expectation value* $\langle \mathcal{O} \rangle$ is its image in $H^* \mathrm{Obs}^{\mathrm{q}}$.

With our contraction in hand, we have an algorithm to compute $\langle \mathcal{O} \rangle$ order by order in $\hbar$. This algorithm amounts to a Feynman diagram expansion, as we saw in section 2.5.

2.6.2.2. *Digression on the relationship with renormalization group flow à la Costello.* In [**Cos11**], there is an operator $W(P(\ell, L), -)$, called renormalization group (RG) flow, that relates "a field theory at length scale ℓ" to "a field theory at length scale L." It has an explicit description in terms of Feynman diagrams where the edges are labelled by the kernel $P(\ell, L)$ of the heat flow operator $e^{-(L-\ell)D}$. Given an observable $\mathcal{O}$, its expectation value is given by its image $W(P(0,\infty), \epsilon\mathcal{O})$, with ϵ a square-zero element with cohomological degree $-|\mathcal{O}|$. The interpretation given in [**CG**] is that we have integrated out all the nonzero modes of the field theory so that we get a function just on the zero modes.

We have described a different approach to computing the same expectation value, based on the homological perturbation lemma. A direct, albeit involved, combinatorial argument leads to the following.

LEMMA 2.6.9. *The perturbed projection operator* $\widetilde{\mathrm{Sym}\,\pi}$ *is the RG flow operator* $\exp(\partial_P)$*, where* P *is the kernel of the operator* η*.*

REMARK 2.6.10. This lemma provides one perspective on the choices made in [**Cos11**]. Notice, for instance, that Costello requires his free field theories to possess a "gauge-fix" operator Q^* which allows the construction of a contracting homotopy satisfying the side conditions of the homological perturbation lemma. In consequence, the RG flow in Costello's formalism, even between finite length scales, provides a means of transferring the BD algebra structure.

2.6.2.3. *The question of locality.* We've now explained what observables are and how Feynman diagrams show up in their computation. But there's a further aspect to explore: a measurement measures something about a field in some region of the manifold M, and it's natural to organize the observables by their *support*. In other words, we could ask about $\mathrm{Obs}^{\mathrm{q}}(U)$ for each open $U \subset M$ — the observables with support in U — and this local structure of observables leads naturally to the notion of *factorization algebra*, which is the main concern of chapter 4.

CHAPTER 3

BV formalism as a determinant functor

Our main result in this chapter can be glossed as "BV quantization provides a kind of determinant functor." The relationship with quantization is likely not transparent, so we sketch it now (for more discussion, see chapter 2 or [CG]). Recall that the path integral is supposed to be an oscillatory integral over a (typically infinite dimensional) manifold, and that the BV formalism is a homological approach to defining such an integration theory. More precisely, the BV formalism is a procedure for defining an "integration" map $\int : \mathscr{O}(X) \to \mathbb{C}((\hbar))$, where X is a derived space.[1] In this chapter, we restrict our attention to the BV formalism on linear spaces: for example, the "space" given by a perfect complex $X \in \mathrm{Perf}(R)$ over a commutative dg algebra R. It should be no surprise that

(1) for such linear spaces, the BV formalism is much simpler, and
(2) the BV formalism, being homological in nature, constructs a "determinant" for each linear space, since the determinant is the natural home of "volume forms" on a linear space.

We find it compelling that the heuristic picture motivating the BV formalism — integration and volume forms — can be made mathematically precise in this context of linear spaces. In his work on the Witten genus [Cosa], Costello explains how this viewpoint extends to "nonlinear" spaces (e.g., quasi-smooth derived schemes) and how it provides an integration map in the context of quantum field theory.

REMARK 3.0.11. Although the results in this chapter are not needed for any constructions in this thesis, they provide a natural context for the index theorem discussed in chapter 7.

Recall that Knudsen and Mumford [KM76] introduced, for X a scheme, a functor

$$\det : \mathrm{Perf}(X) \to \mathbb{Z} \times \mathrm{Pic}(X)$$

that sends a cochain complex of locally free $\mathscr{O}_X$-modules to a $\mathbb{Z}$-graded invertible $\mathscr{O}_X$-module. This determinant functor is an enhancement of the Euler characteristic, with $\mathbb{Z} \times \mathrm{Pic}(X)$ a categorical refinement of the integers. More recently, Schürg, Toën, and Vezzosi [STV] have developed a generalization of this construction to derived algebraic geometry, namely a map of derived stacks

$$\det : \mathbb{R}\,\mathrm{Perf} \to \mathbb{R}\,\mathrm{Pic},$$

[1] In perturbative QFT, the integrals take values in $\mathbb{C}((\hbar))$ because we need $\hbar$ to be infinitesimal (i.e., perturbative) and invertible (i.e., $\hbar$ is nonzero and hence we're in the quantum regime).

where $\mathbb{R}\,\mathrm{Perf}$ denotes the derived stack of perfect complexes and $\mathbb{R}\,\mathrm{Pic}$ denotes the derived stack of graded line bundles. This map has the property that on a quasiprojective scheme X, the derivative of det at a perfect complex E is precisely the trace map tr_E of Illusie.

We will show that the Batalin-Vilkovisky formalism leads to a natural construction in the setting of *formal* derived geometry that is similar to the work of Schürg, Toën, and Vezzosi. In the "functor of points" approach to geometry, one views a space as a well-behaved functor on the category of commutative rings (equivalently, affine schemes). Formal geometry then means the study of well-behaved functors on artinian rings (equivalently, fattened points). In the derived setting, we enhance both the source and target categories as follows. Fix a field k of characteristic zero. There is a natural simplicially-enriched category of artinian local commutative dg algebras $dgArt_{/k}$ (defined below in section 3.4), which is the source category. Our main objects of interest are two functors

$$\mathrm{Perf}^{iso}, \mathrm{Pic} : dgArt_{/k} \to Cat_{sSets}$$

that assign to each artinian dg algebra R the simplicially-enriched category $\mathrm{Perf}(R)^{iso}$ of perfect R-modules and quasi-isomorphisms or, respectively, the simplicially-enriched category $\mathrm{Pic}(R)$ of invertible R-modules (we define both these categories carefully in section 3.2). These functors are simply restrictions of derived stacks to artinian k-algebras. Our main theorem uses the simplest version of BV quantization, which we call *cotangent quantization*, to provide a relationship between these spaces.

THEOREM 3.0.12. *There is a natural transformation $\mathcal{CQ} : \mathrm{Perf}^{iso} \to \mathrm{Pic}$ given by the cotangent quantization of each perfect R-module. Over k, $\mathcal{CQ}(k) : \mathrm{Perf}^{iso}(k) \to \mathrm{Pic}(k)$ is a determinant functor.*

By a *determinant functor* $F : \mathrm{Perf}^{iso}(R) \to \mathrm{Pic}(R)$ on the commutative dg algebra R, we mean that F satisfies the following two properties:

(1) $F(0) = R$, the free dg R-module of rank 1 (and the unit for $\otimes_R$), and
(2) we have an isomorphism between $F(M \oplus N)$ and $F(M) \otimes F(N)$.

These properties are categorical analogues of properties of the matrix determinant: it is 1 on an identity matrix and the determinant sends a direct sum of matrices to the product of their determinants.

Although our result is more limited than that of Schürg, Toën, and Vezzosi, it has several appealing features:

(1) Our morphism is given by an explicit construction.
(2) This construction applies to more general R-modules, although it ceases to satisfy determinantal properties (we use this feature in the context of factorization algebras).
(3) It unveils a new aspect of the BV formalism that provides greater conceptual justification for its use (and hints at possible further elaborations).

In future work, we hope to deepen our understanding of this construction.

3.0.3. Overview of the chapter. In the first section, we construct the cotangent quantization functor $\mathcal{C}Q(k) : \mathrm{Perf}^{iso}(k) \to \mathrm{Pic}(k)$ and demonstrate that it is determinantal. In the next section, we introduce the necessary language for extending this construction over other commutative dg algebras: what it means for a module to be perfect, dualizable, or invertible. With that language available, we then extend our construction of cotangent quantization to arbitrary commutative dg algebras. In the final section, we demonstrate that $\mathcal{C}Q(R)$ returns invertible modules when R is artinian and discuss some issues surrounding the notion of "artinian" in the derived setting.

3.1. Cotangent quantization of k-vector spaces

In this section, we introduce cotangent quantization of k-vector spaces and show that it provides a determinant functor. The homological and categorical issues here are minimal and hence we can focus on the main idea.

3.1.1. The arena. Let $\mathrm{Ch}(k)_{ab}$ denote the category (not simplicial) whose objects are cochain complexes of k-vector spaces and whose morphisms $\mathrm{Hom}_k(M, N)$ are cochain maps (i.e., degree 0 maps commuting with the differentials). There is a simplicial enhancement $\mathrm{Ch}(k)$ where the morphism space $\mathrm{Ch}(k)(M, N)$ has n-simplices given by $\mathrm{Hom}_k(M, N \otimes \Omega^*(\Delta^n))$ (using, say, polynomial de Rham forms over k) and the obvious face and degeneracy maps.

A *strictly perfect k-module* is simply a bounded complex of finite-dimensional k-vector spaces. A perfect k-module is a complex quasi-isomorphic to a strictly perfect k-module. For concreteness, we will always work directly with strictly perfect complexes. Given a perfect k-module M, let $M^\vee$ denote a *dual module*, for which a good choice is $\mathrm{Hom}_k(M, k)$ (this is precisely the complex whose degree n component is the k-linear dual of the degree $-n$ component of M and whose differential is the dual to d_M). We now define the simplicial categories $\mathrm{Perf}(k)$ and $\mathrm{Pic}(k)$.

DEFINITION 3.1.1. Let $\mathrm{Perf}(k)$ denote the simplicially-enriched category in which an object is a bounded complex of finite-dimensional k-vector spaces and the space of morphisms $\mathrm{Perf}(k)(M, N)$ has n-simplices given by $\mathrm{Hom}_k(M, N \otimes \Omega^*(\Delta^n))$ and the obvious face and degeneracy maps.

DEFINITION 3.1.2. Let $\mathrm{Pic}(k)$ denote the full simplicial subcategory of $\mathrm{Ch}(k)$ in which an object is a cochain complex $L \in Ch(k)$ with cohomology given by a one-dimensional k-vector space.

3.1.2. The construction. In a nutshell, BV quantization is a procedure for deforming a commutative dg algebra to a plain cochain complex. There is no algorithm that does this for all commutative algebras. Here our focus is on a class of algebras where such a process does exist. For comparison, recall that the theory of deformation quantization provides a standard way to quantize a cotangent bundle, although quantizing arbitrary symplectic and Poisson manifolds is substantially more subtle. (A more accurate comparison for our case would be cotangent bundles of vector spaces.)

DEFINITION 3.1.3. A Pois_0 *algebra* $(A, d, \{-,-\})$ is a commutative dg algebra (A, d) equipped with a closed, degree 1 map $\{-,-\} : A \otimes A \to A$ such that

- the pairing is skew-symmetric;
- the pairing is a biderivation, so that $\{a, bc\} = \{a, b\}c + (-1)^{(|a|+1)(|b|+1)} b\{a, c\}$ for all $a, b, c \in A$.

At the heart of our work is the following construction.

CONSTRUCTION 3.1.4. For any $(M, d) \in \mathrm{Perf}(k)$, consider the commutative dg algebra $(\mathrm{Sym}(M^\vee \oplus M[1]), d)$ where d is the extension as a derivation of the differential d on $M^\vee \oplus M[1]$. This algebra has a natural Poisson bracket arising from the evaluation pairing. We define the bracket explicitly on Sym^1 and extend to the whole algebra as a biderivation (i.e., use the Leibniz rule repeatedly till one obtains linear elements). We set

$$\{\alpha + a, \beta + b\} = \alpha(b) - (-1)^{(|a|+1)(|\beta|+1)} \beta(a)$$

where $\alpha, \beta \in M^\vee$ and $a, b \in M[1]$. This algebra can be interpreted as $\mathscr{O}(T^*[-1]M)$, namely "functions on the shifted cotangent bundle of the derived space M," $T^*[-1]M$, which is naturally equipped with a shifted symplectic structure.

This Pois_0 structure indicates how the algebra "wants to deform," just as an ordinary Poisson algebra "wants to deform" to an associative algebra. In general, it is a subtle problem to construct the deformation, but for this example, there is a straightforward, general process. The basic idea is to find a way to deform the derivation on the algebra to a mere differential, and the BV formalism offers a controlled way to do this, by adding to the derivation an operator known as a BV Laplacian.

DEFINITION 3.1.5. A *BV Laplacian* on a Pois_0 algebra $(A, d, \{-,-\})$ is a closed, degree 1 map $\Delta : A \otimes A \to A$ such that

- $\Delta^2 = 0$;
- Δ is a second-order differential operator;
- it has the following compatibility with the Poisson bracket:

$$\Delta(ab) = (\Delta a)b + (-1)^{|a|} a(\Delta b) + \{a, b\} \tag{2}$$

  for all $a, b \in A$.

CONSTRUCTION 3.1.6 (construction 3.1.4 continued). We equip $\mathrm{Sym}(M^\vee \oplus M[1])$ with a canonical BV Laplacian Δ by setting $\Delta = 0$ on Sym^0 and Sym^1 and by defining

$$\Delta(ab) = \{a, b\}$$

for $ab \in \mathrm{Sym}^2$ a pure tensor. We compute Δ on the higher Sym^k by applying equation (2) recursively.[2]

DEFINITION 3.1.7. The *cotangent quantization* of $(M, d) \in \mathrm{Perf}(k)$ is the cochain complex $\mathcal{CQ}(M) := (\mathrm{Sym}(M^\vee \oplus M[1]), d + \Delta)$.

PROPOSITION 3.1.8. *This construction has the following properties.*

(1) For perfect modules M, N, we have a natural isomorphism $\mathcal{CQ}(M \oplus N) \cong \mathcal{CQ}(M) \otimes \mathcal{CQ}(N)$.
(2) For a perfect module M, the cohomology $H^*\mathcal{CQ}(M)$ is isomorphic to the one-dimensional graded vector space $k[d(M)]$, where

$$d(M) = -\sum_n (2n+1)(\dim_k H^{2n}(M) + \dim_k H^{2n+1}(M)).$$

(3) For an acyclic perfect module M, there is a natural quasi-isomorphism $i : k \xrightarrow{\simeq} \mathcal{CQ}(M)$ sending k to $\mathrm{Sym}^0(M^\vee \oplus M[1])$.

Since we work with vector spaces, we can always obtain a decomposition of a complex M into a direct sum $H \oplus A$, with A an acyclic complex and H a complex with zero differential and isomorphic to H^*M. Thus, by combining (1) and (3), we obtain

(4) For each choice of such decomposition $M = H \oplus A$, we obtain a natural quasi-isomorphism $i : \mathcal{CQ}(H) \xrightarrow{\simeq} \mathcal{CQ}(M)$.

PROOF. Property (1) is manifest from our construction.

Property (2) follows in two steps. We begin by restricting to modules M with zero differential so that

$$\mathcal{CQ}(M) = (\mathrm{Sym}(M^\vee \oplus M[1]), \Delta).$$

By picking a basis $\{x_1, \ldots, x_N, \xi_1, \ldots, \xi_N\}$ for $M^\vee \oplus M[1]$, with the x_i even, the ξ_i odd, and $\langle x_i, \xi_j \rangle = \delta_{ij}$, we have

$$\mathcal{CQ}(M) = \left(k[x_1, \ldots, \xi_N], \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i \partial \xi_i} \right).$$

When $N = 1$ (i.e., $\dim_k M = 1$), one sees directly that ξ_1 generates the cohomology of $\mathcal{CQ}(M)$. If M is concentrated in degree $2n$, then $\xi_1 \in M[1]$ and has degree $-2n-1$. If M is concentrated in degree $2n+1$, then $\xi_1 \in M^\vee$ and has degree $-2n-1$. By induction or invoking property (1), we obtain property (2) for arbitrary modules with zero differential.

Now consider M with nontrivial differential. Consider the filtration on $\mathcal{CQ}(M)$ given by $F^k := \mathrm{Sym}^{\leq k}(M^\vee \oplus M[1])$. Note that the first page of the associated spectral sequence is $\mathrm{Sym}(H^*(M^\vee) \oplus H^*(M)[1])$ with differential Δ_H arising from the evaluation pairing on cohomology. We then apply our observations from the previous paragraph.

[2] This operator is the unique "translation-invariant" BV Laplacian on the dg vector space $T^*[-1]M$.

Property (3) follows from the fact that $i : k \to \mathrm{Sym}^0 \hookrightarrow \mathcal{CQ}(M)$ is clearly a cochain map and thus induces a map of spectral sequences (using the filtration from above) which is an isomorphism on the first page. $\square$

Cotangent quantization is *not* a functor on $\mathrm{Perf}(k)$ because a morphism of complexes does not induce a morphism of "shifted cotangent bundles." If we are given a map $f : M \to N$, we automatically get a map $f^\vee : N^\vee \to M^\vee$, inducing a map on functions from $\mathscr{O}(N)$ to $\mathscr{O}(M)$. But we also need a map $g : N \to M$ to construct a map on the functions of the cotangent bundles. We could restrict to strict isomorphisms in $Perf(R)$, so that $g = f^{-1}$, but this requirement is unnatural from the derived viewpoint. Instead, we work with the weaker, homotopical notion of quasi-isomorphism. For M, N perfect complexes, let $\iota : Qis(M,N) \subset \mathrm{Hom}_k(M,N)$ denote the subset of quasi-isomorphisms.

DEFINITION 3.1.9. Let $\mathrm{Perf}(k)^{iso}$ denote the simplicial subcategory of $\mathrm{Perf}(k)$ with the same objects but the morphisms given by the pullback along ι:

$$\begin{tikzcd} \mathrm{Perf}(k)^{iso}(M,N) \arrow[r, equal] & \iota^{-1}(\mathrm{Perf}(k)(M,N)) \arrow[r] \arrow[d] & \mathrm{Perf}(k)(M,N) \arrow[d] \\ & Qis(M,N) \arrow[r, hook, "\iota"] & \mathrm{Hom}_k(M,N) \end{tikzcd} .$$

We will prove the following proposition in a more general setting in section 3.3.

PROPOSITION 3.1.10. *Cotangent quantization is a functor* $\mathcal{CQ} : \mathrm{Perf}(k)^{iso} \to \mathrm{Pic}(k)$ *such that*

(1) $\mathcal{CQ}(0) = k$ *and*
(2) $\mathcal{CQ}(M \oplus N) \cong \mathcal{CQ}(M) \otimes \mathcal{CQ}(N)$.

Thus, $\mathcal{CQ}$ *is a symmetric monoidal functor.*

This result in combination with proposition 3.1.8 says that $\mathcal{CQ}$ is a determinant functor.

3.2. Recollections

Before embarking on the study of $\mathcal{CQ}$ over more interesting rings, we introduce and recall various definitions. We discuss what it means for a dg R-module to be perfect, dualizable, or invertible.

3.2.1. Notation and context. For R a commutative dg algebra over k, let $\mathrm{Ch}(R)_{ab}$ denote the category whose objects are cochain complexes of R-modules and whose morphisms $\mathrm{Hom}_R(M,N)$ are

R-linear cochain maps.[3] In this section, we will work with a fixed but arbitrary dg algebra R, and so $\otimes$ will denote $\otimes_R$.

There is an enhancement of $\mathrm{Ch}(R)_{ab}$ to a simplicially-enriched category $\mathrm{Ch}(R)$ as follows. For M, N dg R-modules, the simplicial set of morphisms $\mathrm{Ch}(R)(M, N)$ has n-simplices given by $\mathrm{Hom}_R(M, N \otimes_k \Omega^*(\Delta^n))$, where $\Omega^*(\Delta^n)$ denotes the k-linear de Rham forms on Δ^n,[4] and the obvious face and degeneracy maps. All of our other simplicial categories, like $\mathrm{Perf}(R)$ and $\mathrm{Pic}(R)$, will be simplicial subcategories of this category $\mathrm{Ch}(R)$.

For M, N in $\mathrm{Ch}(R)$, let $Qis(M, N)$ denote the set of quasi-isomorphisms from M to N. As there is an inclusion

$$\iota : Qis(M, N) \hookrightarrow \mathrm{Hom}_R(M, N) = \mathrm{Ch}(R)(M, N)_0$$

into the zero-simplices, we can ask for the pullback

$$\begin{tikzcd}
\mathrm{Ch}(R)^{iso}(M, N) \arrow[r, equal] & \iota^{-1}(\mathrm{Ch}(R)(M, N)) \arrow[r] \arrow[d] & \mathrm{Ch}(R)(M, N) \arrow[d] \\
& Qis(M, N) \arrow[r, hook, "\iota"] & \mathrm{Hom}_R(M, N).
\end{tikzcd}$$

We then define $\mathrm{Ch}(R)^{iso}$ as the simplicial subcategory of $\mathrm{Ch}(R)$ with the same objects but taking $\mathrm{Ch}^{iso}(M, N)$ as the morphism space between complexes M and N.

3.2.2. Perfect modules. For ordinary commutative algebras (i.e., not dg), there are several equivalent definitions of *perfect R-modules* of which the most explicit is "quasi-isomorphic to a bounded complex of finitely-generated projective R-modules." We define this condition of *perfection* on modules by concrete properties rather than functorial properties because of our focus on an explicit construction. There is a substantial literature about perfect modules, with alternate characterizations. A succinct reference for our version is part III of [KM95].

DEFINITION 3.2.1. A module $M \in \mathrm{Ch}(R)$ is a *finite cell module* if there exists a finite sequence of R-modules

$$0 = M_0 \xrightarrow{i_1} M_1 \xrightarrow{i_2} \cdots \xrightarrow{i_n} M_n = M$$

such that for any $1 \leq k \leq n$, the cone of $i_k : M_{k-1} \to M_k$ is quasi-isomorphic to a shift of the free R-module R.

DEFINITION 3.2.2. A *retract* of a module M is a module N with cochain maps $i : N \to M$ and $q : M \to N$ such that $q \circ i$ is chain-homotopic to $\mathbb{1}_N$.

We now define the desired object.

[3] A cochain map has degree zero and commutes with the differentials.

[4] There's a bit of freedom to choose a model here. In general, polynomial de Rham forms work well. Over $\mathbb{R}$ or $\mathbb{C}$, one can use the more familiar smooth de Rham forms.

DEFINITION 3.2.3. A *strictly perfect complex* over R (equivalently, *strictly perfect R-module*) is a module $M \in \mathrm{Ch}(R)$ that is a retract of a finite cell module. A perfect complex is a module quasi-isomorphic to a strictly perfect complex.

For simplicity, we will always work directly with strictly perfect R-modules. From hereon, when we write perfect, we mean strictly perfect.

DEFINITION 3.2.4. Let $\mathrm{Perf}(R)$ denote the full simplicially-enriched subcategory of $\mathrm{Ch}(R)$ whose objects are strictly perfect complexes. Let $\mathrm{Perf}(R)^{iso}$ denote the full simplicially-enriched subcategory of $Ch(R)^{iso}$ whose objects are strictly perfect complexes.

LEMMA 3.2.5. *Given a morphism of commutative dg algebras $f : R \to S$, base-change preserves perfection, so the functor $S \otimes_R -$ sends $\mathrm{Perf}(R)$ to $\mathrm{Perf}(S)$.*

3.2.3. Dualizability. We recall the following definitions of the *dual* of a module over any commutative dg algebra R.

DEFINITION 3.2.6. A module $M \in \mathrm{Ch}(R)$ is *dualizable* if there exists a module $N \in \mathrm{Ch}(R)$ equipped with a *coevaluation* morphism

$$c : R \to M \otimes N$$

and an *evaluation* morphism

$$e : N \otimes M \to R$$

and chain homotopies $h_M : M \to M$ and $h_N : N \to N$ such that

$$\mathbb{1}_M - (\mathbb{1}_M \otimes e) \circ (c \otimes \mathbb{1}_M) = [d_M, h_M]$$

and

$$\mathbb{1}_N - (e \otimes \mathbb{1}_N) \circ (\mathbb{1}_N \otimes c) = [d_N, h_N].$$

We call N a *dual* to M.

This notion is well-behaved thanks to the following.

LEMMA 3.2.7. *There is a canonical homotopy equivalence between any two duals to M.*

PROOF. Let N and N' be duals to M, with (c, e, h_M, h_N) and $(c', e', h'_M, h_{N'})$ the respective duality data. Then we get morphisms

$$\phi := (e \otimes \mathbb{1}_{N'}) \circ (\mathbb{1}_N \otimes c') : N \to N'$$

and

$$\phi' := (e' \otimes \mathbb{1}_N) \circ (\mathbb{1}_{N'} \otimes c) : N' \to N.$$

We wish to show that

$$\phi' \circ \phi = (e' \otimes \mathbb{1}_N) \circ (\mathbb{1}_{N'} \otimes c) \circ (e \otimes \mathbb{1}_{N'}) \circ (\mathbb{1}_N \otimes c')$$

is chain homotopic to $\mathbb{1}_N$.

Observe that $\phi' \circ \phi$ can also be expressed as

$$(e \otimes \mathbb{1}_N) \circ (\mathbb{1}_M \otimes e') \circ (c' \otimes \mathbb{1}_M) \circ (\mathbb{1}_N \otimes c).$$

(This assertion is easiest to see when one draws the pictures *à la* topological field theory.) Thus, we know that

$$[d_N, (e \otimes \mathbb{1}_N) \circ h'_M \circ (\mathbb{1}_N \otimes c)] = (e \otimes \mathbb{1}_N) \circ \mathbb{1}_M \circ (\mathbb{1}_N \otimes c) - \phi' \circ \phi.$$

As $(e \otimes \mathbb{1}_N) \circ (\mathbb{1}_N \otimes c)$ is chain homotopic to $\mathbb{1}_N$, we see that $\phi' \circ \phi$ is chain homotopic to $\mathbb{1}_N$, since homotopy equivalence is an equivalence relation.

An identical argument shows that $\phi \circ \phi'$ is homotopic to $\mathbb{1}_{N'}$. $\square$

Perfection meshes nicely with duality. For M, N in $\mathrm{Ch}(R)$, let $\mathrm{Hom}^*_R(M, N)$ denote the cochain complex of graded morphisms from M to N equipped with the usual differential (so $d(\phi) = d_N \circ \phi \pm \phi \circ d_M$).

LEMMA 3.2.8. *For M a strictly perfect R-module, the R-module $M^\vee := \mathrm{Hom}^*_R(M, R)$ is a dual with the canonical evaluation map*

$$\begin{array}{rccc} e: & M^\vee \otimes M & \to & R \\ & \lambda \otimes x & \mapsto & \lambda(x) \end{array}.$$

Thus perfect modules are dualizable.

PROOF. This is a standard fact. For a proof, see e.g. lemma 5.7 of part III of [KM95]. $\square$

We now prove a property crucial to making $\mathcal{C}Q$ into a functor.

LEMMA 3.2.9. *Let M and N be strictly perfect R-modules. If $f : M \to N$ is a quasi-isomorphism and N' a dual to N with data c and e, then N' is a dual to M with evaluation map $\tilde{e} = e \circ (\mathbb{1}_{N'} \otimes f)$.*

PROOF. As $\mathbb{1}_{N'} \otimes f : N' \otimes M \to N' \otimes N$ is a quasi-isomorphism, there exists $\tilde{c}$ in $N' \otimes M$ such that $\mathbb{1}_{N'} \otimes f(\tilde{c})$ is cohomologous to c. Pick such an element and denote by da the degree 0 boundary in $N' \otimes N$ such that $\mathbb{1}_{N'} \otimes f(\tilde{c}) = c + da$.

We need to show that $\tilde{c}$ and $\tilde{e}$ provide duality data. Consider the composition

$$f \circ (\mathbb{1}_M \otimes \tilde{e}) \circ (\tilde{c} \otimes \mathbb{1}_M) : M \to N.$$

Then

$$\begin{aligned} f \circ (\mathbb{1}_M \otimes \tilde{e}) \circ (\tilde{c} \otimes \mathbb{1}_M) &= (\mathbb{1}_N \otimes \tilde{e}) \circ ((c + da) \otimes \mathbb{1}_M) \\ &= (\mathbb{1}_N \otimes (e \circ \mathbb{1}_{N'} \otimes f)) \circ ((c + da) \otimes \mathbb{1}_M) \\ &= (\mathbb{1}_N \otimes e) \circ (c \otimes \mathbb{1}_N) \circ f + \text{boundary term} \end{aligned}$$

so that the composition equals f up to a chain homotopy. As f is a quasi-isomorphism, we know that

$$f \circ - : \mathrm{Hom}^*_R(M, M) \to \mathrm{Hom}^*_R(M, N)$$

is also a quasi-isomorphism. Thus the element $(\mathbb{1}_M \otimes \tilde{e}) \circ (\tilde{c} \otimes \mathbb{1}_M)$ must be chain homotopic to the identity $\mathbb{1}_M$ as its composition with f is chain homotopic to f. $\square$

3.2.4. Invertible modules.

DEFINITION 3.2.10. A complex $M \in \mathrm{Ch}(R)$ is *invertible* if there exists a complex $N \in \mathrm{Ch}(R)$ and a quasi-isomorphism $M \otimes N \xrightarrow{\simeq} R$. A *strictly invertible* module is an R-module N such that the underlying graded module $N^\#$ of the underlying graded algebra $R^\#$ is projective and rank one.

One way to construct a strictly invertible R-module is to pick $n \in \mathbb{Z}$ and an element $A \in R^1$ (i.e., an element of cohomological degree 1) such that $d_R(A) + A^2 = 0$. Then $(R[n], d_R + A)$ is invertible with inverse $(R[-n], d_R - A)$.

DEFINITION 3.2.11. For any commutative dg k-algebra R, let $\mathrm{Pic}(R)$ denote the full simplicially-enriched subcategory of $\mathrm{Ch}(R)$ whose objects are complexes quasi-isomorphic to invertible R-modules.

3.3. Properties of cotangent quantization over any commutative dg algebra

Let R denote a commutative dg algebra. From section 3.2.2, we know what both $\mathrm{Perf}(R)$ and $\mathrm{Perf}(R)^{iso}$ are. We now observe that the description of cotangent quantization in construction 3.1.4 and 3.1.6 works for perfect R-modules once we choose a dual M' for each $M \in \mathrm{Perf}(R)$.

DEFINITION 3.3.1. Given a strictly perfect module M and a dual M' with evaluation morphism e, we define the *cotangent quantization* $\mathcal{CQ}(M, M', e)$ to be the cochain complex

$$(\mathrm{Sym}(M' \oplus M[1]), d' + d[1] + \Delta_e)$$

where $d' + d[1]$ denotes the extension of the differential $d' + d[1]$ on $M' \oplus M[1]$ as a derivation and Δ_e denotes the BV Laplacian that extends the degree 1 Poisson bracket $\{-,-\}_e$ on $M' \oplus M[1]$ arising from the pairing e (see construction 3.1.6).

We wish to show that $\mathcal{CQ}$ is a functor on $\mathrm{Perf}(R)^{iso}$. There are two tricky issues to grapple with. First, we must verify that we get the same answer (up to homotopy equivalence) for any choice of dual. Second, we need to show that quasi-isomorphisms between perfect complexes are sent to quasi-isomorphisms between their quantizations.

By lemma 3.2.9, we know that if we have a quasi-isomorphism $f : M \xrightarrow{\simeq} N$ and N' a dual to N with evaluation e, then we can make N' a dual to M with evaluation $\tilde{e} := e \circ \mathbb{1}_{N'} \otimes f$. It is easy to see that extending $F := \mathbb{1}_{N'} \oplus f[1]$ as a map of graded commutative algebras

$$F : \mathrm{Sym}(N' \oplus M[1]) \to \mathrm{Sym}(N' \oplus N[1]),$$

we obtain a map $F : \mathcal{CQ}(M, N', \tilde{e}) \to \mathcal{CQ}(N, N', e)$ because F preserves the Poisson brackets (and hence BV Laplacians) by construction. This map F is manifestly a quasi-isomorphism.

It remains to show the following.

LEMMA 3.3.2. *Let (M, M_1', e_1, c_1) and (M, M_2', e_2, c_2) be two duals to a perfect module M. There is a canonical quasi-isomorphism $\Phi : \mathcal{CQ}(M, M_1', e_1) \xrightarrow{\simeq} \mathcal{CQ}(M, M_2', e_2)$.*

PROOF. We proceed in two steps. Let $\phi : M_1' \to M_2'$ denote the canonical quasi-isomorphism between the duals. Observe that we can equip M_1' with a different evaluation pairing as follows. Set $\tilde{e}_2 := e_2 \circ \phi \otimes \mathbb{1}_M : M_1' \otimes M \to R$.

First, observe that $\mathcal{CQ}(M, M_1', \tilde{e}_2)$ is quasi-isomorphic to $\mathcal{CQ}(M, M_2', e_2)$ via the map $\phi \oplus \mathbb{1}_{M[1]}$ because the Poisson brackets agree on the nose by construction.

Second, we construct a homotopy equivalence between $\mathcal{CQ}(M, M_1', e_1)$ and $\mathcal{CQ}(M, M_1', \tilde{e}_2)$.

There is a natural degree 0 map

$$p : \left(M_1' \oplus M[1]\right)^{\otimes 2} \to R$$

yielding a chain homotopy equivalence between the degree 1 Poisson brackets (restricted to Sym^1)

$$\{-,-\}_{e_1} \text{ and } \{-,-\}_{\tilde{e}_2} : \left(M_1' \oplus M[1]\right)^{\otimes 2} \to R.$$

We can extend p to a degree 0 Poisson bracket $\{-,-\}_p$ on the commutative dg algebra $\mathrm{Sym}(M_1' \oplus M[1])$ with differential $\partial = d' + d[1]$.

Define an endomorphism Δ_p of $\mathrm{Sym}(M_1' \oplus M[1])$ as follows. On Sym^0 and Sym^1, we have Δ_p vanish. On Sym^2, we set $\Delta_p(ab) = p(a \otimes b)$. Then we recursively define Δ_p by imposing the relation

$$\Delta_p(a \cdot b) = \Delta_p(a) \cdot b + a \cdot \Delta_p(b) + \{a, b\}_p,$$

in analogy with the BV Laplacian (cf. equation (2)). By construction,

$$\left[\partial, \Delta_p\right] = \Delta_{e_1} - \Delta_{\tilde{e}_2}.$$

Note as well that

$$[\Delta_{e_1}, \Delta_p] = 0 = [\Delta_{\tilde{e}_2}, \Delta_p].$$

We use this endomorphism to obtain the desired map.

Consider the endomorphism

$$\exp(\Delta_p) : a \mapsto e^{\Delta_p} a = \sum_{n=0}^{\infty} \frac{1}{n!} \Delta_p^n(a).$$

Although this definition seems to involve an infinite sum, any element a is annihilated by some finite power of Δ_p, so we only ever take a finite sum. A straightforward computation then verifies that

$$[\partial, \exp(\Delta_p)] = \exp(\Delta_p) \circ \Delta_{e_1} - \Delta_{\tilde{e}_2} \circ \exp(\Delta_p),$$

and hence that $\exp(\Delta_p)$ is a cochain map from $\mathcal{C}Q(M, M_1', e_1)$ to $\mathcal{C}Q(M, M_1', \tilde{e}_2)$.

To see that this map is part of a homotopy equivalence, observe that $\exp(-\Delta_p)$ provides an inverse. $\square$

From hereon we fix a dual $M' := M^\vee = \mathrm{Hom}_R^*(M, R)$, evaluation e_M, etc, for every perfect module M. Set $\mathcal{C}Q(M) := \mathcal{C}Q(M, M^\vee, e_M)$. Let $\pi_0 \,\mathrm{Perf}(R)^{iso}$ and $\pi_0 \,\mathrm{Ch}(R)^{iso}$ denote the categories (*not* simplicially-enriched) with the same objects but the morphism sets given by the connected components of the morphism spaces in the simplicially-enriched category. We have shown the following proposition.

PROPOSITION 3.3.3. *Cotangent quantization $\mathcal{C}Q$ is a functor from $\pi_0 \,\mathrm{Perf}(R)^{iso}$ to $\pi_0 \,\mathrm{Ch}(R)^{iso}$, sending M to $\mathcal{C}Q(M)$ and a quasi-isomorphism $f : M \to N$ to the composition*

$$\mathcal{C}Q(M, M^\vee, e_M) \to \mathcal{C}Q(M, N^\vee, e_N \circ \mathbb{1}_{N'} \otimes f) \to \mathcal{C}Q(N, N^\vee, e_N).$$

Moreover, it is symmetric monoidal, sending $\oplus$ to $\otimes$.

We would like to lift this functor to a functor between the simplicially-enriched categories.

THEOREM 3.3.4. *Cotangent quantization $\mathcal{C}Q$ is a functor from $\mathrm{Perf}(R)^{iso}$ to $\mathrm{Ch}(R)^{iso}$, lifting the functor from the previous proposition.*

PROOF. We need to show that given a morphism $f : M \to N \otimes_k \Omega^*(\Delta^n)$ whose restriction to each 0-simplex of Δ^n is a quasi-isomorphism, there is a morphism $\mathcal{C}Q(f) : \mathcal{C}Q(M) \to \mathcal{C}Q(N) \otimes_k \Omega^*(\Delta^n)$ whose restriction to each 0-simplex of Δ^n is a quasi-isomorphism.

Recall that when $n = 0$, the essence of the argument was that $N^\vee$ pulls back to a dual of M along f and that any two duals are quasi-isomorphic. We model our approach to the $n > 0$ case on this argument.

Note that

$$\mathrm{Hom}_R(M, N \otimes_k \Omega^*(\Delta^n)) \cong \mathrm{Hom}_{R \otimes_k \Omega^*(\Delta^n)}(M \otimes_k \Omega^*(\Delta^n), N \otimes_k \Omega^*(\Delta^n)). \tag{3}$$

To compress the notation, we denote $R_n := R \otimes_k \Omega^*(\Delta^n)$ and $X_n := X \otimes_R R \otimes_k \Omega^*(\Delta^n)$ for any R-module X. We thus view f as a map of R_n-modules:

$$f : M_n \to N_n.$$

By hypothesis, f is a quasi-isomorphism when restricted to the vertices of Δ^n, and hence f is a quasi-isomorphism itself.

We now invoke the argument from the $n = 0$ case. We know $N_n^\vee$ is a dual to M_n over R_n and so we get a map

$$\mathcal{C}Q(f) : \mathcal{C}Q(M_n) \to \mathcal{C}Q(N_n),$$

using cotangent quantization over R_n. Because $\mathcal{C}Q(X_n) \cong \mathcal{C}Q(X) \otimes_k \Omega^*(\Delta)$ for any perfect R-module X, we can use the isomorphism (3) to make $\mathcal{C}Q(f)$ an n-simplex in $\mathrm{Ch}(R)^{iso}(M, N)$. $\square$

3.4. Invertibility survives over artinian dg algebras

Cotangent quantization over k had the appealing property that $\mathcal{C}Q(M)$ was always *invertible*. Thus cotangent quantization over k provided a determinant functor. We now extend this property to a larger class of categories close to $\mathrm{Perf}(k)$. The idea is simple: we work with the category of perfect modules for algebras that are "close to k," namely artinian k-algebras.

Before delving into the derived world, we review the underived version. Recall that a local artinian algebra A is a k-algebra possessing a unique maximal ideal $\mathfrak{m}$ such that

(1) the residue field $A/\mathfrak{m}$ is k and
(2) $\mathfrak{m}$ is nilpotent, so that there is some nonnegative integer N such that $\mathfrak{m}^N = 0$.

In other words, $A = k \oplus \mathfrak{m}$ where $\mathfrak{m}$ is a nilpotent "thickening" of k.

There are several ways to generalize the notion of "artinian" to the derived setting. We give proofs using two of the notions and discuss the relation with a more general notion in the final subsection. For whatever notion we use, we use the same notation for the associated category of artinian algebras.

DEFINITION 3.4.1. Let $dgArt_{/k}$ denote the simplicially-enriched category in which an object is an artinian commutative dg algebra over k and in which a morphism space $dgArt_{/k}(R, S)$ has n-simplices given by augmentation-preserving morphisms of commutative dg algebras from R to $S \otimes_k \Omega^*(\Delta^n)$.

3.4.1. The strictest notion of artinian dg algebra.

DEFINITION 3.4.2. A *strictly artinian* commutative dg algebra A is a commutative dg algebra such that the underlying graded algebra is artinian.

Let M be a perfect module over a strictly artinian commutative algebra R. We now prove that $\mathcal{C}Q(M)$ is invertible (i.e., $\mathcal{C}Q$ is a functor to $\mathrm{Pic}(R)$) by using the homological perturbation lemma.

LEMMA 3.4.3. *For a perfect module M, $\mathcal{C}Q(M)$ is quasi-isomorphic to an invertible R-module.*

PROOF. Pick a set of generators for M so that we have an explicit description

$$M^{\#} = \bigoplus_{i=1}^{N} R^{\#}[n_i]$$

for some list of integers n_i and the differential has the form

$$d_M = \sum_i d_R[n_i] + A_k + A_{\mathfrak{m}},$$

where A_k is a matrix whose entries live in k and $A_{\mathfrak{m}}$ is a matrix whose entries live in $\mathfrak{m}$.

Consider the perfect module $\widetilde{M}$ which has the same graded components but the differential is just $\sum d_R[n_i] + A_k$. Clearly

$$\widetilde{M} = R \otimes_k \bigoplus_{i=1}^{N} k[n_i] = R \otimes_k \widetilde{M}_k,$$

where $\widetilde{M}_k$ is a finite-dimensional graded k-vector space. Let $\widetilde{M}^{\vee}$ denote the dual

$$R \otimes_k \bigoplus_{i=1}^{N} k[-n_i] = R \otimes_k \widetilde{M}_k^{\vee}$$

with the obvious pairing. As $\widetilde{M}_k$ is a perfect k-module, we can construct a homotopy equivalence over k:

$$\begin{tikzcd} H^*\mathcal{C}Q(\widetilde{M}_k) \arrow[r, shift left, "\widetilde{\iota}_k"'] & \mathcal{C}Q(\widetilde{M}_k) \arrow[l, shift left, "\widetilde{\pi}_k"'] \arrow[loop right, "\widetilde{\eta}_k"] \end{tikzcd}$$

Let $H^*\mathcal{C}Q(\widetilde{M}_k)$ be isomorphic to $k[m]$ where m is some integer. Tensoring with R, we obtain a homotopy equivalence

$$\begin{tikzcd} R[m] \arrow[r, shift right, "\widetilde{\iota}"'] & \mathcal{C}Q(\widetilde{M}) \arrow[l, shift right, "\widetilde{\pi}"'] \arrow[loop right, "\widetilde{\eta}"] \end{tikzcd}$$

We will now try to perturb this homotopy equivalence to a homotopy equivalence for M. We obtain M by adding $A_{\mathfrak{m}}$ to the differential of $\widetilde{M}$.

We need $1 - A_{\mathfrak{m}}\widetilde{\eta}$ to have an inverse. Recall that $A_{\mathfrak{m}}$ is a matrix whose elements live in $\mathfrak{m}$. Hence $A_{\mathfrak{m}}\widetilde{\eta}$ is an element of $\mathfrak{m}_R \mathbb{R}\mathrm{Hom}_R(\widetilde{M}, \widetilde{M})$. There exists N such that $\mathfrak{m}^N = 0$, and so we know that $(A_{\mathfrak{m}}\widetilde{\eta})^N = 0$. Thus

$$1 + A_{\mathfrak{m}}\widetilde{\eta} + \cdots + (A_{\mathfrak{m}}\widetilde{\eta})^{N-1}$$

is the desired inverse. We now apply the homological perturbation lemma to obtain a homotopy equivalence

$$\begin{tikzcd} (R[m], d_R + \alpha) \arrow[r, shift right, "\iota"'] & \mathcal{C}Q(M) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\eta"] \end{tikzcd}$$

with α some degree 1 element of R such that $d_R\alpha = 0$. Thus $\mathcal{C}Q(M)$ is homotopy equivalent to some invertible R-module. $\square$

3.4.2. A looser definition via square-zero extensions. There is another perspective on (underived) artinian algebras thanks to the fact that we can construct its "nilpotent thickening" in a very controlled way.

DEFINITION 3.4.4. A *square-zero extension* is a surjective map of algebras $f : B \to A$ such that the ideal $I = \ker f$ satisfies $I^2 = 0$.

LEMMA 3.4.5. *An algebra A is artinian if and only if it is given by a finite sequence of square-zero extensions*

$$A \to A/I^{(1)} \to A/I^{(2)} \to \cdots \to A/I^{(n)} = k$$

where each square-zero ideal $I^{(m)}/I^{(m-1)}$ of $A/I^{(m-1)}$ is finite-dimensional over k.

PROOF. An algebra arising from such an extension is clearly artinian because the kernel of the map to k is maximal and nilpotent (since it's finite dimensional).

Let A be an artinian algebra with $\mathfrak{m}^N = 0$. We construct the sequence of ideals $I^{(m)}$ as follows. Let $SZ(R)$ denote the largest square-zero ideal in an algebra R. Set $I^{(1)} = SZ(A)$. This is nonempty since $\mathfrak{m}^{N-1}$ is square-zero. We obtain $I^{(2)}$ by taking the preimage of $SZ(A/I^{(1)})$ under the quotient map $A \to A/I^{(1)}$. Iterate this process. It terminates in finitely many steps because A is finite-dimensional over k. ☐

Our second notion of a dg artinian algebra is modeled on this definition. We impose some restrictions that make it easy to induct on square-zero extensions even in the derived setting.

DEFINITION 3.4.6. A *cdga* is a commutative dg algebra (A, d) such that $A^i = 0$ for $i > 0$.

There is natural model category structure on cdgas where the fibrations are level-wise surjections.

DEFINITION 3.4.7. A *dg square-zero extension* is a fibration of cdgas $f : B \to A$ such that the kernel $\ker f$ is square-zero (on the nose).

DEFINITION 3.4.8. An *artinian* cdga A is given by a finite sequence of dg square-zero extensions

$$A \xrightarrow{q_1} A_{(1)} \xrightarrow{q_2} \cdots \xrightarrow{q_n} A_{(n)} = k$$

where the cohomology of each ideal $H^n \ker q_m$ is finite-dimensional and vanishes for $n << 0$.

Our goal is to prove the following. Note that a strictly invertible module over an artinian cdga A is simply a shift $A[n]$ as $A^1 = 0$.

THEOREM 3.4.9. *Let $f : B \to A$ be a dg square-zero extension of artinian cdgas. Let M be a B-module such that $M \otimes_B A$ is invertible over A. Then M is invertible over B.*

COROLLARY 3.4.10. *For A an artinian cdga and M a perfect A-module, $\mathcal{CQ}(M)$ is invertible over A.*

PROOF OF COROLLARY. We use "artinian induction." Pick a description of A as a sequence of dg square-zero extensions

$$A \xrightarrow{q_1} A_{(1)} \xrightarrow{q_2} \cdots \xrightarrow{q_n} A_{(n)} = k.$$

We know $\mathcal{C}Q(M)$ is invertible if $\mathcal{C}Q(M) \otimes A_{(1)}$ is invertible, and so on until we have the condition $\mathcal{C}Q(M)$ is invertible if $\mathcal{C}Q(M) \otimes k$ is invertible. But we have already proved this base case. □

Because we will tensor over cdgas several times, we recall the definition.

DEFINITION 3.4.11. Let M and N be B-modules. The *tensor product* $M \otimes_B N$ is any B-module quasi-isomorphic to the bar complex $Bar(M, B, N)$. Recall that this is given by the total complex of the double complex

$$\bigoplus_{n \geq 0} M \otimes_k \underbrace{B \otimes_k \cdots \otimes_k B}_{n \text{ times}} \otimes_k N[n]$$

where the "new" differential (i.e., not the internal differential of the complex for each n) is given by taking the alternating sum of the obvious multiplications of pairs of adjacent elements.

The proof of the theorem breaks down into a few lemmas. Let $I = \ker(f : B \to A)$. Let M/I denote $M \otimes_B A$ and let $I \cdot M$ denote $M \otimes_B I$.

Suppose that we have a quasi-isomorphism $\phi : A \to M/I$. (By applying a shift, we can put the strictly invertible module in degree 0.)

LEMMA 3.4.12. *The obstruction to lifting ϕ to a map $\tilde{\phi} : B \to M$ lives in $H^1(I \cdot M)$.*

PROOF. We want to find $\tilde{\phi} : B \to M$ a quasi-isomorphism such that

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\tilde{\phi}"'] & A \arrow[d, "\phi"] \\ M \arrow[r, "f_M"] & M/I \end{tikzcd}$$

commutes.

Pick $m \in M$ such that $f_M(m) = \phi(1_A) \in M/I$. We would like to set $\tilde{\phi}(1_B) = m$ and thus obtain the desired $\tilde{\phi}$. We need to show that $d_M m = 0$ because we require $d_M \tilde{\phi}(1) = \tilde{\phi}(d_B 1) = 0$. Observe that we know

$$f_M(d_M m) = d_{M/I} f_M(m) = d_{M/I} \phi(1_A) = 0$$

so $d_M m \in I \cdot M$. Clearly, $[d_M m] = 0$ in $H^1(M)$ but it might be nonzero in $H^1(I \cdot M)$.

If $[d_M m] = 0$ in $H^1(I \cdot M)$, there exists $n \in I \cdot M$ such that $d_M n = d_M m$. Then we obtain the desired $\tilde{\phi}$ by setting $\tilde{\phi}(1_B) = m - n$. Otherwise, $\tilde{\phi}$ does not exist. □

LEMMA 3.4.13. *The obstruction vanishes as $H^1(I \cdot M) = 0$.*

PROOF. The ideal I has a canonical A-module structure. Pick a k-linear splitting

$$\begin{tikzcd} B \arrow[r, "f"'] & A \arrow[l, bend right, "s"'] \end{tikzcd}$$

and define $a \bullet i = s(a)i$ for $a \in A$ and $i \in I$. For any other splitting s', we see that $s(a) - s'(a) \in I$ so that $s(a)i - s'(a)i = 0$ since $I^2 = 0$.

As $I \cdot M := I \otimes_B M$, we see

$$I \cdot M \simeq (I \otimes_A A) \otimes_B M \simeq I \otimes_A (A \otimes_B M) = I \otimes_A M/I.$$

Using the bar complex, we obtain a spectral sequence

$$H^* I \otimes^{\mathbb{L}}_{H^* A} H^*(M/I) \Rightarrow H^*(I \cdot M)$$

via the filtration $F^N = \bigoplus_{n \geq N} I \otimes_k A^{\otimes n} \otimes_k (M/I)$. Observe that

$$H^m(I) = 0 = H^m(A) = H^m(M/I)$$

for all $m > 0$. Hence $H^1(I \cdot M) = 0$. $\square$

Thus we have a commuting diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\tilde{\phi}"'] & A \arrow[d, "\phi"] \\ M \arrow[r, "f_M"] & M/I \end{tikzcd}$$

where ϕ is a quasi-isomorphism. The theorem then follows once we prove that $\tilde{\phi}$ is a quasi-isomorphism.

PROPOSITION 3.4.14. *Let $\phi : M \to N$ be a map of B-modules such that the reduction*

$$\phi_A : M \otimes_B A \to N \otimes_B A$$

is a quasi-isomorphism. Then ϕ is a quasi-isomorphism.

PROOF. Consider the I-adic filtration of M and N. This induces a map of spectral sequences. On the first page, we have

$$\begin{array}{ccc} H^*(M/I) & \to & H^*(N/I) \\ \oplus & & \oplus \\ H^*(I \cdot M) & \to & H^*(I \cdot N) \end{array}$$

where M/I denotes $M \otimes_B A$ and so on. We know that the top layer is an isomorphism by hypothesis. Thus, once we show that bottom layer is an isomorphism, we know that ϕ is an isomorphism.

Because $I \cdot M \simeq I \otimes_A M/I$, we get another map of spectral sequences using the natural filtration on the bar complex (see the proceeding proof). On the first page, we get

$$H^* I \otimes^{\mathbb{L}}_{H^* A} H^*(M/I) \to H^* I \otimes^{\mathbb{L}}_{H^* A} H^*(N/I)$$

which is an isomorphism. $\square$

3.4.3. Comparing our notion of artinian dg algebra to others. There are several other definitions one might suggest for the dg generalization of artinian. Being *cohomologically artinian* is probably the most obvious from a derived perspective. Explicitly, it means the following.

DEFINITION 3.4.15. A *c-artinian algebra* is a cdga A such that

(1) $H^0(A)$ is artinian as a k-algebra,
(2) $H^i(A) = 0$ for $i << 0$,
(3) $\dim_k H^i(A) < \infty$ for all i, and
(4) the composition $k \to A \to H^0(A) \to k$ is the identity on k.

We would like every c-artinian algebra to be quasi-isomorphic to an artinian cdga, the notion we used in subsection 3.4.2. This is true for a large class of c-artinian algebras.[5]

The following proposition indicates that our definition of an artinian cdga is (hopefully) adequate. In particular, we can recover anything about maps of cdgas into this large class of c-artinian algebras using our notion. As nice moduli functors are studied locally via maps of cdgas into "artinian-type algebras," our notion works well for formal moduli problems.

PROPOSITION 3.4.16. *Every c-artinian algebra such that A^0 is finite-dimensional possesses a quasi-isomorphism to an artinian cdga.*

Let A be a c-artinian algebra. Observe that there is a canonical map $A \to H^0A$ of cdgas, as A is concentrated in non-positive degrees. We would like to extend this map to a sequence

$$\begin{tikzcd}
 & & & & & A \arrow[dllll, bend right] \arrow[dlll, bend right] \arrow[dl, bend right] \arrow[d] & \\
\cdots \arrow[r] & A_{[i]} \arrow[r] & \cdots \arrow[r] & A_{[-2]} \arrow[r] & A_{[-1]} \arrow[r] & A_{[0]} \arrow[r, equal] & H^0A
\end{tikzcd}$$

where each $A_{[i]}$ is an artinian cdga and is a dg square-zero extension from $A_{[i+1]}$.

Building this sequence depends on some simple observations. The unpalatable hypothesis that A^0 is finite-dimensional only appears at the very end (all the intermediate results hold for all c-artinian algebras), and we flag its appearance.

First, there is a natural filtration on A.

LEMMA 3.4.17. *The subcomplex*

$$A_{\leq i} := d(A^{i-1}) \oplus \bigoplus_{j<i} A^j,$$

[5] It probably holds in general, modulo some homotopical algebra that I do not yet know.

is a differential ideal of A.

PROOF. The subspace $\bigoplus_{j<i} A^j$ is clearly an ideal and its image under d lives in $A_{\leq i}$. Now consider an element $b = dc \in d(A^{i-1})$. For any $a \in A^0$, we see that

$$d(ac) = (da)c + a(dc) = 0 + ab,$$

so $ab \in A_{\leq i}$. $\square$

Let $A_{(i)}$ denote the cdga $A/A_{\leq i}$. Note that there exists some N such that $A \simeq A_{(N)}$ as there exists N such that $H^n A = 0$ for all $n < N$. These $A_{(i)}$ are almost, but not quite, the $A_{[i]}$ we desire; we will need a few more square-zero extensions as we get close to $H^0 A$.

The following observation nearly proves the proposition.

LEMMA 3.4.18. *For $i \leq -1$, the quotient map $q_{i-1} : A_{(i-1)} \to A_{(i)}$ is a dg square-zero extension and the kernel has finite-dimensional cohomology concentrated in degree $i-1$.*

PROOF. The kernel $I := \ker q_{i-1}$ is the two step complex

$$A^{i-1}/d(A^{i-2}) \to d(A^{i-1})$$

concentrated in degrees $i-1$ and i. The cohomology is $H^{i-1}(A)$ in degree $i-1$, and it is finite-dimensional as A is c-artinian. Moreover, when $i \leq -2$, it is clearly square-zero since the product of two elements of I has degree between $2i-2$ and $2i$, which are both less than $i-1$ by hypothesis.

The $i = -1$ case is different. The trouble is that two elements of cohomological degree -1 multiply to an element of degree -2, so that the kernel of the map $q_{-2} : A_{(-2)} \to A_{(-1)}$ might not be square-zero. Observe, however, that given da, db elements of degree -1 in A, we have

$$da \cdot db = d(a \cdot db) \Rightarrow da \cdot db = 0 \in A_{(-2)}$$

because this product is in $d(A^{-3})$. $\square$

We are now very close to having the desired sequence of square-zero extensions. We need to find a sequence of square-zero extensions between $A_{(-1)}$ and $H^0 A = A_{(0)}$. Observe that the map $A_{(-1)} \to A' := A_{(-1)}/H^{-1}A$ is a square-zero extension. It remains to show that $A' \to A_{(0)} = H^0 A$ is a square-zero extension. *Here the unpalatable hypothesis appears.*

LEMMA 3.4.19. *As A^0 is finite-dimensional, the map $A' \to H^0 A$ is given by a finite sequence of dg square-zero extensions.*

PROOF. Let J denote the differential ideal in A' generated by boundaries. There is some N such that $J^N = 0$ because J is finite-dimensional. We thus have a sequence of dg square-zero extensions

$$A' \to A'/J^{N-1} \to A'/J^{N-2} \to \cdots \to A'/J = H^0 A$$

just as in the ordinary artinian case. $\square$

CHAPTER 4

Factorization algebras

Just as a space is mirrored by its algebra of functions, the fields in a field theory are reflected by the *observables* of the theory. By an observation, we mean a measurement of a field, and an observable is a possible measurement. Naively, if $\mathscr{E}$ denotes the fields, then the observables of the classical field theory are $\mathscr{O}(\mathscr{E})$, the ring of functions on the fields. (We momentarily defer fixing a notion of "function.") Even better, if $\mathscr{E}$ denotes the *sheaf* of fields on the manifold M, then the classical field theory has a *cosheaf* of commutative algebras

$$\mathrm{Obs}^{\mathrm{cl}} : U \mapsto \mathscr{O}(\mathscr{E}(U)),$$

assigning to each open U the ring of functions on the fields $\mathscr{E}(U)$ on this open. We get a commutative algebra on each open due to the fact that, for a classical system, we can take two measurements with overlapping support.[1]

We also want to describe the observables of the quantum field theory. A characteristic feature of quantum systems is that it is incoherent to take two measurements with overlapping support. Hence we might expect that the quantum observables assign *just* a vector space $\mathrm{Obs}^{\mathrm{q}}(U)$ to each open set U but that we still have a way to combine measurements with *disjoint support*. In other words, we expect there is a "multiplication map"

$$m_{U,V;W} : \mathrm{Obs}^{\mathrm{q}}(U) \otimes \mathrm{Obs}^{\mathrm{q}}(V) \to \mathrm{Obs}^{\mathrm{q}}(W)$$

if $U \cap V = \emptyset$ and $U, V \subset W$. (As an example of such a map from an actual theory, consider scalar field theory and the two-point function $\langle \phi(x)\phi(y) \rangle$, which multiplies the measurements "value at x" and "value at y" to give a measurement on the whole manifold.)

Our aim in this section is pin down a precise definition for the structure possessed by quantum observables. We will define a local-to-global object called a *factorization algebra* that lives on a manifold, assigns a vector space to each open, possesses "multiplication maps" for the inclusion of disjoint opens into bigger opens, and has a gluing axiom. To justify this definition, we show in the next chapter how the BV formalism — applied to free fields — generates interesting examples of factorization algebras.

Factorization algebras are useful beyond the setting of QFT, however, and have a strong relationship with the topology of manifolds. We won't pursue that relationship here, although it is an active area of research (see [**Lurb**], [**Lur09**], [**Fra**]). Instead, we will exhibit the relationship

[1] Colloquially, we might say "we can make simultaneous measurements."

between factorization algebras on 1-manifolds and familiar homological constructions with associative and Lie algebras. (These results have natural extensions to n-manifolds and E_n-algebras.) With this language in hand, we will be able to make precise how holomorphic field theories recover vertex algebras in chapter 6.

4.1. Definitions

Like a sheaf, a factorization algebra lives on a space and takes values in a category, although a factorization algebra will require the target category to be symmetric monoidal. For this section M will denote a Hausdorff space, although our examples are always smooth manifolds, as these are the spaces we most care about here. The category $\mathcal{C}^{\otimes}$ will denote a symmetric monoidal category, closed under small colimits. Our two favorite examples are *dgVect*, the dg category of cochain complexes of vector spaces over $\mathbb{C}$ with the usual tensor product, and *dgNuc*, the dg category of cochain complexes of nuclear vector spaces with the completed projective tensor product.

DEFINITION 4.1.1. A *prefactorization algebra* $\mathcal{F}$ on M with values in $\mathcal{C}^{\otimes}$ consists of the following data:

- for each open $U \subset M$, an object $\mathcal{F}(U) \in \mathcal{C}$;
- for each inclusion $U \overset{\iota}{\subset} V$, a morphism $\mathcal{F}(\iota) : \mathcal{F}(U) \to \mathcal{F}(V)$;
- for any finite collection $U_1, \dots, U_k$ of pairwise disjoint opens inside an open V, a morphism
$$\mathcal{F}(U_1, \dots, U_k; V) : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(V),$$
that is equivariant under reordering of the opens;
- the natural coherences or associativities among these structure maps, e.g., if $U_1, U_2 \subset V \subset W$ with the U_i's disjoint, then we have a commuting diagram
$$\begin{tikzcd} \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[d] \\ & \mathcal{F}(W) \end{tikzcd}$$
encoding the transitivity of inclusion of opens;[2]
- $\mathcal{F}(\emptyset) = \mathbb{1}_{\mathcal{C}}$.

REMARK 4.1.2. There are other ways to phrase this concept — e.g., as an algebra over a certain colored operad of open sets in M, as a symmetric monoidal functor out of some symmetric monoidal category constructed out of open sets — depending on the reader's taste. See [CG] for a few options.

[2]We could weaken this requirement to being homotopy coherent — and this can be quite useful! — but our examples satisfy coherence on the nose and we want to minimize the machinery we use.

EXAMPLE 4.1.3. Every associative algebra A defines a prefactorization algebra $\mathcal{F}_A$ on $\mathbb{R}$, as follows. To each open interval (a,b), we set $\mathcal{F}_A((a,b)) := A$. To any open set $U = \coprod_j I_j$, where each I_j is an open interval, we set $\mathcal{F}((a,b)) := \bigotimes_j A$.[3] The structure maps simply arise from the multiplication map for A. Figure (1) displays the structure of $\mathcal{F}_A$. Notice the resemblance to the notion of an E_1 algebra.

value on an interval

structure map for two intervals in a larger interval

FIGURE 1. Structure of $\mathcal{F}_A$

REMARK 4.1.4. For any prefactorization algebra $\mathcal{F}$, the object $\mathcal{F}(U)$ is *pointed*. The empty set $\emptyset$ has empty intersection with itself, so $\mathcal{F}(\emptyset)$ is $\mathbb{1}_{\mathcal{C}}$ as a commutative algebra in $\mathcal{C}^{\otimes}$. The map $\mathcal{F}(\emptyset) \to \mathcal{F}(V)$ equips $\mathcal{F}(V)$ with a distinguished element $\mathbb{1}_V$. This pointedness allows us to define infinite tensor products, extending the trick from the example above. For $U = \cup_{i \in I} U_i$ a disjoint union of infinitely many components, we define

$$\mathcal{F}(U) := \mathrm{colim}_{\text{finite } J \subset I}\, \mathcal{F}(U_J), \text{ where } \mathcal{F}(U_J) := \otimes_{j \in J} \mathcal{F}(U_j)$$

where the inclusion of finite sets $J \subset J'$ sends x in $\mathcal{F}(U_J)$ to $x \otimes \mathbb{1}_{U_{J'-J}}$.

A factorization algebra satisfies a gluing axiom, just like a sheaf, but it has a slightly different flavor due to the different nature of the structure maps. For a sheaf F, we can recover its value on an open by knowing its behavior on a cover. The crucial property of a cover is that the primordial restriction maps $F(U) \to F_x$ (i.e., those mapping down to a stalk) factor through the cover since every point x lives in some element of the cover. For a factorization algebra, we need a refined notion of cover which likewise captures the most basic structure maps: the inclusion maps from a finite set of points $\mathcal{F}_{x_1} \otimes \cdots \otimes \mathcal{F}_{x_n} \to \mathcal{F}(U)$.

DEFINITION 4.1.5. An open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of U is *factorizing* if for any finite set of points $\{x_1, \ldots, x_n\}$ in U, there exists a collection of pairwise disjoint opens $\{U_{i_1}, \ldots, U_{i_k}\}$ from $\mathfrak{U}$ such that $\{x_1, \ldots, x_n\} \subset U_{i_1} \cup \cdots \cup U_{i_k}$.

[3] One can take infinite tensor products of unital algebras (see, for instance, exercise 23, chapter 2 [AM69]). The idea is simple. Given an infinite set I, consider the poset of finite subsets of I, ordered by inclusion. For each finite subset $J \subset I$, we can take the tensor product $A^J := \bigotimes_{j \in J} A$. For $J \hookrightarrow J'$, we define a map $A^J \to A^{J'}$ by tensoring with the identity $1 \in A$ for every $j \in J' \backslash J$. Then A^I is the colimit over this poset.

To phrase our gluing axiom concisely, we introduce some notation. If $\{U_i\}_{i\in I}$ is a cover, let PI denote the collection of all finite subsets $\alpha \subset I$ such that $U_i \cap U_j = \emptyset$ for any distinct $i, j \in \alpha$. We define

$$\mathcal{F}(\alpha) := \bigotimes_{i\in\alpha} \mathcal{F}(U_i)$$

and

$$\mathcal{F}(\alpha_0, \ldots, \alpha_m) := \bigotimes_{i_0\in\alpha_1,\ldots,i_m\in\alpha_m} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_m}).$$

Note that there is a natural map $\mathcal{F}(\alpha, \beta) \to \mathcal{F}(\alpha)$ (and likewise for β). More generally, there is a natural map

$$d_j : \mathcal{F}(\alpha_0, \ldots, \alpha_m) \to \mathcal{F}(\alpha_0, \ldots, \widehat{\alpha}_j, \ldots, \alpha_m),$$

where the hat indicates removal, just as in the usual Čech complex. Repeated indices lead to maps such as

$$f_0 : \mathcal{F}(\alpha_0, \alpha_1, \ldots, \alpha_m) \to \mathcal{F}(\alpha_0, \alpha_0, \alpha_1, \ldots, \alpha_m).$$

This simplicial structure makes the following construction is natural.

DEFINITION 4.1.6. The *Čech complex* $\check{C}(\mathfrak{U}, \mathcal{F})$ is the simplicial object in $\mathcal{C}$ with n-simplices

$$\coprod_{(\alpha_0,\ldots,\alpha_n)\in PI^{n+1}} \mathcal{F}(\alpha_0, \ldots, \alpha_n)$$

and the natural face maps f_j and degeneracy maps d_j. If $\mathcal{C}$ is, for instance, dg vector spaces, we obtain a double complex

$$\cdots \to \bigoplus_{\alpha_0,\alpha_1} \mathcal{F}(\alpha_0, \alpha_1) \xrightarrow{d_0-d_1} \bigoplus_{\alpha} \mathcal{F}(\alpha)$$

just as with the usual Čech complex for (co)sheaves. We abusively call the total complex the Čech complex, as well.

DEFINITION 4.1.7. A *factorization algebra* is a prefactorization algebra $\mathcal{F}$ satisfying the *locality axiom*: for every open set U and any factorizing cover $\mathfrak{U}$, the natural map

$$\operatorname{colim} \check{C}(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is a weak homotopy equivalence. As we are working in a homotopical context, colim denotes the *homotopy* colimit.

REMARK 4.1.8. We have given the homotopical version of factorization algebra. Of course, if $\mathcal{C}$ is more strict in nature (e.g., usual vector spaces, not dg vector spaces), then the natural version of the locality axiom is that

$$\operatorname{coeq}\left(\coprod_{\alpha_0,\alpha_1} \mathcal{F}(\alpha_0, \alpha_1) \rightrightarrows \coprod_{\alpha} \mathcal{F}(\alpha)\right) \to \mathcal{F}(U)$$

is an isomorphism. We always work in this thesis in the homotopical context.

4.2. Associative algebras as factorization algebras on $\mathbb{R}$

As we saw in example 4.1.3, every associative algebra A defines a prefactorization algebra $\mathcal{F}_A$ on the real line $\mathbb{R}$. On each open interval I, $\mathcal{F}_A(I) := A$, and the structure maps are given by multiplication in A. For instance, given two subintervals $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ inside $J = (a, b)$, with $b_1 \leq a_2$, the structure map is

$$\begin{array}{ccc} \mathcal{F}_A(I_1) \otimes \mathcal{F}_A(I_2) & \to & \mathcal{F}_A(J) \\ \shortparallel & & \shortparallel \\ A \otimes A & & A \\ \Psi & & \Psi \\ a \otimes b & \mapsto & a \cdot b \end{array}$$

Our goal is to show that $\mathcal{F}_A$ is, in fact, a factorization algebra and, moreover, that it is easy to characterize the class of factorization algebras on $\mathbb{R}$ that correspond to associative algebras.

NOTE 4.2.1. *This section invokes definitions only defined later in this chapter. Hopefully the meaning is clear from context. We are violating logical but not pedagogical order in arranging the material in this order.*

DEFINITION 4.2.2. A factorization algebra $\mathcal{F}$ on a manifold M is *locally constant* if the structure map

$$\mathcal{F}(\mathbb{D}) \to \mathcal{F}(\mathbb{D}')$$

is a weak equivalence (e.g., quasi-isomorphism for *dgVect*) for every inclusion of contractible opens $\mathbb{D} \hookrightarrow \mathbb{D}'$. We say $\mathcal{F}$ is a *strict* locally constant factorization algebra if every such structure map is an isomorphism (e.g., for *dgVect* a cochain map that is an isomorphism of graded vector spaces).

REMARK 4.2.3. A more general version of locally constant factorization algebra exists (not just on manifolds) but we state the condition in terms of disks because it is simple. A factorization algebra is determined by its behavior on a factorizing basis (see section 4.7) and the collection of all contractible opens in M forms such a factorizing basis.

THEOREM 4.2.4. *The functor $\mathcal{F}_{-} : A \mapsto \mathcal{F}_A$ gives an equivalence of categories between* strict *associative algebras and* strict *locally constant factorization algebras on* $\mathbb{R}$.

REMARK 4.2.5. Much stronger versions of this theorem hold — on $\mathbb{R}^n$ too — if one uses ideas from homotopical algebra (see, for instance, Lurie's work on E_n algebras [**Lurb**]). But I want to show what one can do bare-handed.

The proof is a sequence of (mostly) simple reductions. The first result is by far the hardest, so we defer its proof to the next section, namely corollary 4.3.6.

LEMMA 4.2.6. *For A an associative algebra, the prefactorization algebra $\mathcal{F}_A$ on $\mathbb{R}$ is in fact a factorization algebra.*

We now need to show that the image of $\mathcal{F}_-$ is essentially surjective on strict locally constant factorization algebras.

Let $\mathcal{V}$ be a strict locally constant factorization algebra on $\mathbb{R}$. Observe that for any two intervals I and J in $\mathbb{R}$, we obtain a canonical isomorphism $\tau_{I\to J} : \mathcal{V}(I) \to \mathcal{V}(J)$ as follows. Let $\phi_K : \mathcal{V}(K) \to \mathcal{V}(\mathbb{R})$ denote the structure map for the inclusion of an interval $K \subset \mathbb{R}$. Then $\tau_{I\to J} = \phi_J^{-1} \circ \phi_I$. (This is a "translation" map arising from the picture displayed in figure (2).) Thus we can "rigidify" $\mathcal{V}$ as follows.

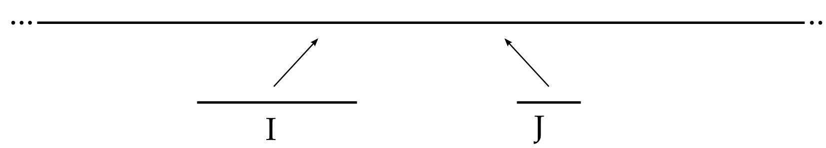


FIGURE 2. Inclusion of two intervals into whole line

LEMMA 4.2.7. *Any strict locally constant factorization algebra $\mathcal{V}$ on $\mathbb{R}$ is isomorphic to a factorization algebra $\mathcal{V}'$ on $\mathbb{R}$ that*

(1) assigns the same vector space V to every open interval, and
(2) assigns the identity $\mathcal{V}'(I) = V \xrightarrow{1_V} V = \mathcal{V}'(J)$ for the inclusion of any interval $I \hookrightarrow J$ into another interval.

In this section, we call factorization algebras *rigid* if it satisfies (1) and (2), like $\mathcal{V}'$. Thanks to this lemma, we can always view any strict locally constant factorization algebra as rigid, so we will be loose with our terminology elsewhere.

PROOF. We construct $\mathcal{V}'$ as follows. Set $V := \mathcal{V}(\mathbb{R})$. Given an interval $I \subset \mathbb{R}$, let $\phi_I^{-1} : V \to \mathcal{V}(I)$ denote the inverse to the structure map $\phi_I : \mathcal{V}(I) \to V = \mathcal{V}(\mathbb{R})$. We will use these inverses to construct the structure maps for $\mathcal{V}'$. For any finite tuple of disjoint intervals $I_1, \dots, I_n \subset J$, consider the structure map

$$m : \mathcal{V}(I_1) \otimes \cdots \otimes \mathcal{V}(I_n) \to \mathcal{V}(J).$$

We define the corresponding structure map for $\mathcal{V}'$ to be

$$m' := \phi_J^{-1} \circ m \circ (\phi_{I_1}^{-1} \otimes \cdots \otimes \phi_{I_n}^{-1}).$$

We define a map of prefactorization algebras $\Phi : \mathcal{V}' \to \mathcal{V}$ as follows. For any interval I, we define

$$\Phi(I) = \phi_I^{-1} : \mathcal{V}'(I) = V \to \mathcal{V}(I).$$

For unions of disjoint intervals, we simply take the appropriate tensor product of these maps. The associativity of the structure maps of $\mathcal{V}$ then implies the associativity of those for $\mathcal{V}'$. $\square$

We thus restrict our attention to rigid factorization algebras. For such $\mathcal{V}$, the translation maps τ are simply the identity. We now show that "all the multiplication maps $V^{\otimes n} \to V$ are the same."

LEMMA 4.2.8. *Let $\mathcal{V}$ be a rigid factorization algebra $\mathcal{V}$. Fix a coordinate on $\mathbb{R}$. Given any tuple of n disjoint intervals $I_1, \ldots, I_n$ contained in a larger interval J and ordered so that I_j is always to the left of I_{j+1}, the structure map*

$$m : \mathcal{V}(I_1) \otimes \cdots \otimes \mathcal{V}(I_n) \to \mathcal{V}(J)$$

is equal to the structure map

$$m_n : \mathcal{V}((0, 1/2)) \otimes \mathcal{V}((1, 3/2)) \otimes \cdots \otimes \mathcal{V}((n-1, n-1/2)) \to \mathcal{V}(0, n)$$

under the given identification of $\mathcal{V}(I)$ with V for each interval I.

We want to identify any tuple of intervals with our "preferred intervals," as in figure (3).

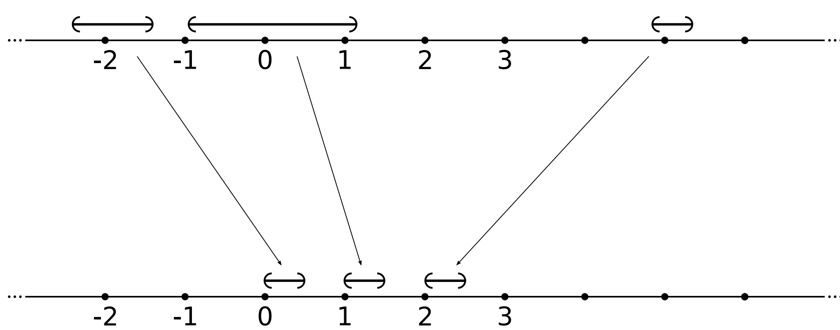


FIGURE 3. Identifying three intervals with the "preferred intervals"

This identification can be constructed by a sequence of "translations." For the example in figure (4), we show such a sequence.

PROOF. By a sequence of "translations," we intertwine m with m_n. As the translations are simply identity maps, however, we see the maps $m : V^{\otimes} \to V$ and $m_n : V^{\otimes n} \to V$ are identical. $\square$

COROLLARY 4.2.9. *For $\mathcal{V}$ a rigid factorization algebra, the vector space V with the map $m_2 : V^{\otimes 2} \to V$ is an associative algebra.*

PROOF. The associativity of the structure maps of $\mathcal{V}$ imply that m_2 is a strict associative product. $\square$

We thus see that the rigid factorization algebras are the image of $\mathcal{F}_-$ and form a skeletal subcategory of the strict locally constant factorization algebras on $\mathbb{R}$.

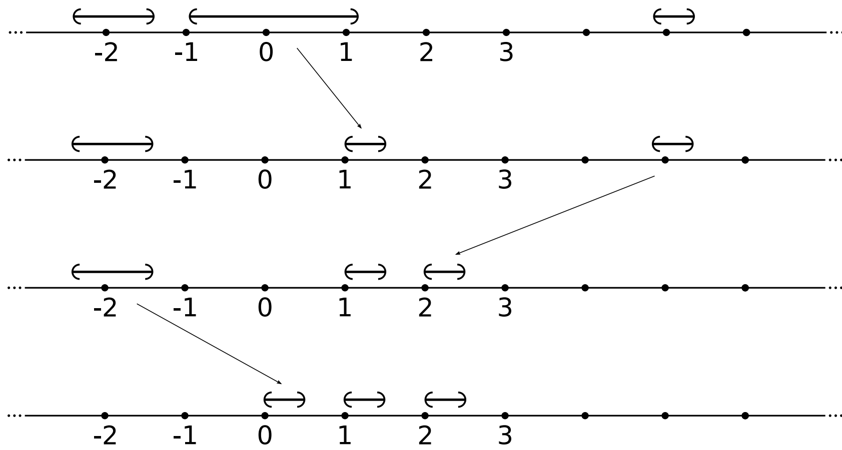


FIGURE 4. A sequence of translations constructing the identification

4.3. Associative algebras and the bar complex

We have two goals here:

(1) to prove that the prefactorization algebra $\mathcal{F}_A$ on $\mathbb{R}$ arising from an associative algebra A satisfies the locality axiom;
(2) to relate the locality axiom for the factorization algebras on $\mathbb{R}$ to familiar constructions from homological algebra, like derived tensor product and Hochschild homology.

The most useful result we'll prove is the following.

PROPOSITION 4.3.1. *For A a dg algebra, M a right A-module, and N a left A-module, let $\mathcal{F}_{(M,A,N)}$ the constructible prefactorization algebra on $[0,1]$ assigning M to every interval $[0,x)$, N to every interval $(x,1]$, and A to every interval (x,y). It is a factorization algebra and its global sections are*

$$\mathcal{F}_{(M,A,N)}([0,1]) \simeq M \otimes_A^{\mathbb{L}} N.$$

Thus we can compute global sections using the bar complex.

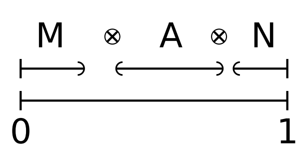


FIGURE 5. The structure of $\mathcal{F}_{M,A,N}$

REMARK 4.3.2. For us, a *constructible* factorization algebra $\mathcal{F}$ on a space X means there is a decomposition $X = \bigsqcup_i X_i$ into finitely many disjoint, locally closed subsets X_i, each of which is a manifold, such that $\mathcal{F}\big|_{X_i}$ is locally constant. Again, this is not the most general definition but suffices for our purposes.

We build up to this result somewhat indirectly, by proving a result about tensor algebras (i.e., free algebras) and then using resolutions.

PROPOSITION 4.3.3. *For I an interval (with or without endpoints) and F be a cosheaf on I with values in dgVect, there is a factorization algebra $\mathcal{T}_F$ on I such that for each open interval U,*

$$\mathcal{T}_F(U) := T(F(U)) = \bigoplus_{n=0}^{\infty} F(U)^{\otimes n}.$$

In short, for an interval U, $\mathcal{T}_F(U)$ is the tensor algebra of $F(U)$.

As the proof of the proposition is unpleasantly long, we defer it to the end of the section and immediately describe its consequences.

COROLLARY 4.3.4. *Let $V_0 \overset{f_0}{\leftarrow} V \overset{f_1}{\rightarrow} V_1$ be a diagram in dgVect. There is an associated constructible cosheaf $\mathcal{V}$ on $I = [0,1]$ assigning V_0 to every interval $[0,x)$, V_1 to every interval $(x,1]$, and V to every interval (x,y). The associated factorization algebra $\mathcal{T} = T(\mathcal{V})$ has global sections*

$$\mathcal{T}([0,1]) = T(V_0) \otimes_{T(V)} T(V_1).$$

In particular, given a vector space V, the locally constant prefactorization algebra $\mathcal{F}_{T(V)}$ is, in fact, a factorization algebra.

We now use homological algebra to piggyback on proposition 4.3.3 to get a useful, general result. Recall that a *semi-free algebra* (R,d) is a dg algebra such that the underlying graded algebra $R^\#$ is a tensor algebra.

LEMMA 4.3.5. *For (R,d) a semi-free algebra, the locally constant prefactorization algebra $\mathcal{F}_R$ is a factorization algebra.*

PROOF. For any open U and any factorizing cover $\mathfrak{U}$, we need to show that the map

$$\iota : \check{C}(\mathfrak{U}, \mathcal{F}_R) \to \mathcal{F}_R(U)$$

is a quasi-isomorphism. The Čech complex is naturally viewed as a double complex with horizontal differential given by the structure maps — for example, we have

$$\mathcal{F}(\alpha_0,\alpha_1) \overset{d_0-d_1}{\rightarrow} \mathcal{F}(\alpha_0) \oplus \mathcal{F}(\alpha_1)$$

and so on — and the vertical differential given by the internal differential for each $\vec{\alpha}$. We can likewise view the cone of ι as a double complex where we adjoin $\mathcal{F}(U)$ as the rightmost column.

Consider now the spectral sequence on this double complex $Cone(\iota)$ whose first page is given by using the horizontal (i.e., Čech) differential. We know the horizontal complex is acyclic by proposition 4.3.3, so the sequence vanishes on the first page. □

COROLLARY 4.3.6. *For (A, d_A) a dg algebra, the locally constant prefactorization algebra $\mathcal{F}_A$ is a factorization algebra.*

PROOF. Pick a semi-free resolution (R, d_R) of A. This means that (R, d_R) is semi-free and there is a surjective quasi-isomorphism $q : R \to A$.

We already know that $\mathcal{F}_R$ is a factorization algebra. We also know that q induces a map of prefactorization algebras $q : \mathcal{F}_R \to \mathcal{F}_A$ that is a quasi-isomorphism on every open. For any open U and any factorizing cover $\mathfrak{U}$, we thus have a commuting diagram

$$\begin{tikzcd} \check{C}(\mathfrak{U}, \mathcal{F}_R) \arrow[r] \arrow[d] & \mathcal{F}_R(U) \arrow[d] \\ \check{C}(\mathfrak{U}, \mathcal{F}_A) \arrow[r] & \mathcal{F}_A(U) \end{tikzcd}$$

where the top row and right column are quasi-isomorphisms. We now show that the left column is a quasi-isomorphism via a spectral sequence.

Observe that there is a natural filtration of the Čech complex by "index." Namely, let

$$F^k \check{C}(\mathfrak{U}, \mathcal{F}) := \bigoplus_{n \leq k} \bigoplus_{\vec{\alpha} \in PI^{n+1}} \mathcal{F}(\vec{\alpha}).$$

The associated graded components are then

$$F^k / F^{k-1} = \bigoplus_{\vec{\alpha} \in PI^{k+1}} \mathcal{F}(\vec{\alpha}).$$

Using this filtration, we see that the left column map induces a quasi-isomorphism on the first page of the spectral sequence because $q : \mathcal{F}_R(\vec{\alpha}) \to \mathcal{F}_A(\vec{\alpha})$ is a resolution for every $\vec{\alpha}$. Hence the left column map is a quasi-isomorphism. $\square$

A parallel argument implies the following.

COROLLARY 4.3.7. *For A a dg algebra, M a right A-module, and N a left A-module, let $\mathcal{F}_{(M,A,N)}$ the constructible prefactorization algebra on $[0,1]$ assigning M to every interval $[0, x)$, N to every interval $(x, 1]$, and A to every interval (x, y). It is a factorization algebra and its global sections are*

$$\mathcal{F}_{(M,A,N)}([0,1]) \simeq M \otimes^{\mathbb{L}}_A N.$$

Thus we can compute global sections using the bar complex.

Pushforward then implies a simple relationship with Hochschild homology.

COROLLARY 4.3.8. *Parametrizing S^1 by $\theta \in [0, 2\pi)$, we have a natural map $p : S^1 \to [-1, 1]$ sending θ to $\cos\theta$. Then*

$$\mathcal{F}_A(S^1) = p_* \mathcal{F}_A([-1, 1]) \simeq A \otimes^{\mathbb{L}}_{A \otimes A^{op}} A$$

as $P_* \mathcal{F}_A$ is isomorphic to the factorization algebra $\mathcal{F}_{(A, A\otimes A^{op}, A)}$ on $[-1, 1]$.

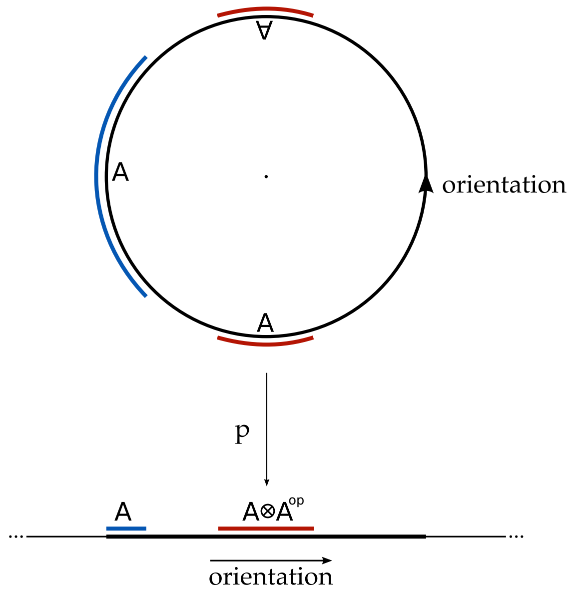


FIGURE 6. The projection p onto the x-axis and the pushforward $p_*\mathcal{F}_A$

PROOF OF PROPOSITION 4.3.3. We describe the prefactorization algebra structure before verifying the locality axiom.

Note that we still need to define $\mathcal{T}_F$ on opens other than an interval.We use heavily the fact the $\mathbb{R}$ is orderable. Fix an orientation on I (e.g., by picking a coordinate). This orientation induces a partial ordering on opens:

$$U < V \iff \forall x \in U, y \in V,\ x < y.$$

Given any finite tuple $\{U_1, \ldots, U_n\}$ of disjoint open intervals, there is a unique permutation σ of indices that puts them in order,

$$U_{\sigma(1)} < \cdots < U_{\sigma(n)}.$$

We define

$$\mathcal{T}_F(U_1 \cup \cdots \cup U_n) := T(F(U_{\sigma(1)})) \otimes \cdots \otimes T(F(U_{\sigma(n)})).$$

Given an infinite collection of disjoint open intervals, we take the colimit over all finite subsets, as discussed in remark 4.1.4.

We now describe the structure maps of the prefactorization algebra. Let V be an open containing a finite tuple $\{U_1, \ldots, U_n\}$ of disjoint open intervals. Let σ denote the permutation that "puts the opens in order." The structure map is given by the composition

$$\begin{array}{ccccc} T(F(U_1)) \otimes \cdots \otimes T(F(U_n)) & \xrightarrow{\sigma} & T(F(U_{\sigma(1)})) \otimes \cdots \otimes T(F(U_{\sigma(n)})) & & \\ & & \downarrow & & \\ & & T(F(U_1 \cup \cdots \cup U_n)) & \to & T(F(V)) \end{array}$$

where the first map uses the Koszul rule of signs to permute vectors and the third map is simply the functor T (the free algebra functor) applied to the cosheaf structure map $F(U_1 \cup \cdots \cup U_n) \to F(V)$.

We now verify locality. Let $U \subset I$ be an open and $\mathfrak{U} = \{V_j\}_{j \in J}$ be a factorizing cover of U. We need to prove that we have a quasi-isomorphism

$$\check{C}(\mathfrak{U}, \mathcal{T}_F) \simeq \mathcal{T}_F(U),$$

where $\check{C}$ denotes the Čech complex from definition 4.1.6. (Our strategy mimics the proof of theorem 4.5.1 that $\mathrm{Sym}\, F$ is a factorization algebra.)

We introduce some notation to organize the combinatorics. Let $U \sqcup V$ indicate that the union is of disjoint opens U and V. Given $\alpha \in PI := PI(J)$, set $U_\alpha := \sqcup_{j \in \alpha} V_j$. For $\vec{\alpha} = (\alpha_0, \ldots, \alpha_N) \in PI^{N+1}$, set

$$U(\vec{\alpha}) := \cap_{i=0}^{N} U_{\alpha_i}.$$

Observe that we can swap intersections past unions to obtain

$$U(\vec{\alpha}) = \sqcup_{j_0 \in \alpha_0, \ldots, j_N \in \alpha_N} (\cap_{i=0}^{N} V_{j_i}),$$

because the opens $\cap_{i=0}^{N} V_{j_i}$ are pairwise disjoint. Thus, by the definition of $\mathcal{T}_F$, we have a natural isomorphism

$$\mathcal{T}_F(\vec{\alpha}) = \bigotimes_{j_0 \in \alpha_0, \ldots, j_N \in \alpha_N} \mathcal{T}_F(\cap_{i=0}^{N} V_{j_i}) \cong \mathcal{T}_F(U(\vec{\alpha})),$$

where the isomorphism "puts the components in order."

Under this isomorphism, the structure maps preserve tensor power and hence so does the differential in the Čech complex, as we now explain. By "tensor power," we mean that an element $w \in V^{\otimes n} \subset T(V)$ is of tensor power n. Observe that for a disjoint union $U \sqcup U'$ with $U < U'$, we can view $\mathcal{T}_F(U \sqcup U') = T(F(U)) \otimes T(F(U'))$ as a vector subspace of $T(F(U \sqcup U'))$. For $w \otimes w' \in \mathcal{T}_F(U \sqcup U')$, with w of tensor power n and w' of tensor power n', the image in $T(F(U \sqcup U'))$ has tensor power $n + n'$. On every open U, we equip $\mathcal{T}_F(U)$ with this natural "total grading" by tensor power and write $\mathcal{T}_F(U) = \bigoplus_{n \geq 0} \mathcal{T}_F(U)^{\langle n \rangle}$, where the superscript $\langle n \rangle$ indicates the subspace with grading $\langle n \rangle$.

Decompose the Čech complex into a sum of complexes over tensor power. It now remains to show that

$$(\dagger) \qquad \operatorname{colim}\left(\cdots \to \bigoplus_{\vec{\alpha} \in PI^2} \mathcal{T}_F(U(\vec{\alpha}))^{\langle n \rangle} \to \bigoplus_{\alpha} \mathcal{T}_F(U_\alpha)^{\langle n \rangle} \right) \to \mathcal{T}_F(U)^{\langle n \rangle}$$

is a quasi-isomorphism for every n.

Our method is to relate $\mathcal{T}_F^{\langle n \rangle}$ to $F^{\boxtimes n}$, the cosheaf on the product space $I^n = \underbrace{I \times \cdots \times I}_{n \text{ times}}$ given by the n-fold external tensor product of F. In particular, we define a subset $U^{\langle n \rangle} \subset I^n$ for each open $U \subset I$ such that $F^{\boxtimes}(U^{\langle n \rangle}) = \mathcal{T}_F^{\langle n \rangle}(U)$.

DEFINITION 4.3.9. For $U \subset I$, there is a unique decomposition $U = \sqcup_{j \in J} I_j$, where the I_j are disjoint open intervals. The index set J is totally ordered by the relation that $j < j'$ if $I_j < I_{j'}$. Let $[n] := \{0 < 1 < \cdots < n-1\}$ denote the model totally ordered set with n elements. We define

$$J^{\langle n \rangle} := \{ f : [n] \to J \ : \ f(0) \leq f(1) \leq \cdots \leq f(n-1) \} .$$

In other words, $J^{\langle n \rangle}$ consists of the monotonically increasing functions from $[n]$ into J. For $f \in J^{\langle n \rangle}$, we define

$$U^f := \prod_{i=0}^{n-1} I_{f(i)} \subset I^n.$$

Finally, we define

$$U^{\langle n \rangle} := \bigsqcup_{f \in J^{\langle n \rangle}} U^f.$$

Note that this is a disjoint union of connected opens.

This definition is the bridge between $\mathcal{T}_F^{\langle n \rangle}$ and $F^{\boxtimes n}$.

LEMMA 4.3.10. *For every open $U \subset I$, we have $\mathcal{T}_F(U)^{\langle n \rangle} = F^{\boxtimes n}(U^{\langle n \rangle})$.*

PROOF OF LEMMA. We compute

$$\begin{aligned} F^{\boxtimes n}(U^{\langle n \rangle}) &= F^{\boxtimes n}\Big(\bigsqcup_{f \in J^{\langle n \rangle}} U^f \Big) \\ &= \bigoplus_{f \in J^{\langle n \rangle}} F^{\boxtimes n}(U^f) \\ &= \bigoplus_{f \in J^{\langle n \rangle}} \bigotimes_{i=0}^{n-1} F(I_{f(i)}) \\ &= \mathcal{T}_F(U)^{\langle n \rangle}. \end{aligned}$$

For the last step, recall that $\mathcal{T}_F(U) = \bigotimes_{j \in J} T(F(I_j))$. ☐

We claim that the cosheaf gluing axiom for $F^{\boxtimes n}$ implies the desired quasi-isomorphism (†). Suppose the following facts, which we will prove below:

(1) the opens $\{U_\alpha^{\langle n \rangle}\}_{\alpha \in PI}$ form a cover for $U^{\langle n \rangle}$, and
(2) $U_\alpha^{\langle n \rangle} \cap U_\beta^{\langle n \rangle} = (U_\alpha \cap U_\beta)^{\langle n \rangle}$ for all α, β.

By assertion (2), we see that for any $\vec{\alpha} \in PI^{N+1}$, we have $U(\vec{\alpha})^{\langle n \rangle} = \bigcap_{i=0}^{N} U_{\alpha_i}^{\langle n \rangle}$. In the diagram (†), we replace every $\mathcal{T}_F^{\langle n \rangle}(U(\vec{\alpha}))$ with $F^{\boxtimes}(\bigcap_{i=0}^{N} U_{\alpha_i}^{\langle n \rangle})$. We then obtain the diagram

$$\operatorname{colim}\left(\cdots \to \bigoplus_{(\alpha_0,\alpha_1)\in PI^2} F^{\boxtimes n}(U_{\alpha_0}^{\langle n \rangle} \cap U_{\alpha_1}^{\langle n \rangle}) \to \bigoplus_{\alpha} F^{\boxtimes n}(U_{\alpha}^{\langle n \rangle}) \right) \to F^{\boxtimes n}(U^{\langle n \rangle})$$

This is precisely the cosheaf Čech diagram for $F^{\boxtimes n}$ on $U^{\langle n \rangle}$ using the cover $\{U_{\alpha}^{\langle n \rangle}\}_{\alpha \in PI}$. As this diagram holds, we obtain the locality axiom.

It remains to verify assertions (1) and (2).

We know (1) by the factorizing property of the basis $\mathfrak{U}$. Given any point $x = (x_1, \ldots, x_n) \in U^{\langle n \rangle}$, there exists $\alpha \in PI$ such that $\{x_1, \ldots, x_n\} \in U_{\alpha}$, and so $x \in U_{\alpha}^{\langle n \rangle}$.

We obtain (2) as a consequence of a more general fact.

LEMMA 4.3.11. *Let U and V be open subsets of I. Then $U^{\langle n \rangle} \cap V^{\langle n \rangle} = (U \cap V)^{\langle n \rangle}$ for any n.*

PROOF OF LEMMA. Let $U = \sqcup_{j \in J} U_j$ and $V = \sqcup_{k \in K} V_k$ be the unique decompositions into open intervals. Both J and K are totally ordered by the orientation of I. Then

$$U \cap V = \bigsqcup_{(j,k)\in L} U_j \cap V_k,$$

where

$$L := \{(j,k) \in J \times K \ : \ U_j \cap V_k \neq \emptyset\}.$$

Note that L also obtains a total ordering from the orientation of the interval I.

Note that $L^{\langle n \rangle} \hookrightarrow J^{\langle n \rangle} \times K^{\langle n \rangle}$ because $h(i) \in J \times K$ for every $i \in [n]$.

Given $(f, g) \in J^{\langle n \rangle} \times K^{\langle n \rangle}$, we compute

$$U^f \cap V^g = \prod_{i=0}^{n} (U_{f(i)} \cap V_{g(i)}),$$

because the cartesian product commutes with intersections. Hence $U^f \cap V^g \neq \emptyset$ if and only if $(f, g) \in L^{\langle n \rangle}$.

Thus we see

$$
\begin{aligned}
U^{\langle n\rangle} \cap V^{\langle n\rangle} &= \left(\bigsqcup_{f\in J^{\langle n\rangle}} U^f\right) \cap \left(\bigsqcup_{g\in K^{\langle n\rangle}} V^g\right) \\
&= \bigsqcup_{(f,g)\in J^{\langle n\rangle}\times K^{\langle n\rangle}} U^f \cap V^g \\
&= \bigsqcup_{(f,g)\in L^{\langle n\rangle}} U^f \cap V^g \\
&= (U\cap V)^{\langle n\rangle},
\end{aligned}
$$

which is what needed to be proved. □

Thus (2) follows directly from the lemma. □

4.4. The category of factorization algebras

In this section, we explain how prefactorization algebras and factorization algebras form categories. In fact, they naturally form multicategories (or colored operads). We also explain how these multicategories are enriched in simplicial sets when the (pre)factorization algebras take values in cochain complexes.

4.4.1. Morphisms and the category structure.

DEFINITION 4.4.1. A morphism of prefactorization algebras $\phi : F \to G$ consists of a map $\phi_U : F(U) \to G(U)$ for each open $U \subset M$, compatible with the structure maps. That is, for any open V and any finite collection $U_1, \ldots, U_k$ of pairwise disjoint open sets, each contained in V, the following diagram commutes:

$$
\begin{array}{ccc}
F(U_1)\otimes\cdots\otimes F(U_k) & \xrightarrow{\phi_{U_1}\otimes\cdots\otimes\phi_{U_k}} & G(U_1)\otimes\cdots\otimes G(U_k) \\
\downarrow & & \downarrow \\
F(V) & \xrightarrow{\phi_V} & G(V)
\end{array}
$$

Likewise, all the obvious associativity relations are respected.

REMARK 4.4.2. When our prefactorization algebras take values in cochain complexes, we require the ϕ_U to be cochain maps, i.e., they each have degree 0 and commute with the differentials.

DEFINITION 4.4.3. On a space X, we denote the category of prefactorization algebras on X taking values in the symmetric monoidal category $\mathcal{C}$ by $PreFA(X,\mathcal{C})$. The category of factorization algebras, $FA(X,\mathcal{C})$, is the full subcategory whose objects are the factorization algebras.

Throughout the thesis, we will want to say when two (pre)factorization algebras are equivalent. Here are two notions we use repeatedly.

DEFINITION 4.4.4. A morphism $\phi : F \to G$ of prefactorization algebras (with values in $dgVect$) is a *quasi-isomorphism* if it is a quasi-isomorphism on every open. It is an *opens-wise homotopy equivalence* if on each open U, the morphism $\phi(U) : F(U) \to G(U)$ extends to a chain homotopy equivalence, though we do not require compatibility between the structure maps and the equivalences.

REMARK 4.4.5. A full theory of factorization algebras would encompass good notions of equivalence, a characterization of FA as a localization of $PreFA$ (hopefully), and much more. We make no pretensions to providing such a theory in this text. The notion of opens-wise homotopy equivalence, for instance, is not even an equivalence relation. It just happens to be a stronger property than quasi-isomorphism that we can verify explicitly in several cases.

N.B. 4.4.6. Even if these notions were well-behaved, we are often working with cochain complexes of *topological* vector spaces, a notoriously awkward setting. Homological algebra and topological vector space mix uneasily, and in [**CG**] we are pursuing an alternative with diffeological spaces. In this thesis, we will usually either explicitly construct *continuous* homotopy equivalences (e.g., with the Atiyah-Bott lemma, see lemma 5.2.13) or work with complexes whose cohomology is well-behaved topologically.

4.4.2. The multicategory structure. There is a natural tensor product on $PreFA(X, \mathcal{C})$, as follows. Let F, G be prefactorization algebras. We define $F \otimes G$ by

$$F \otimes G(U) := F(U) \otimes G(U),$$

and we simply define the structure maps as the tensor product of the structure maps. For instance, if $U \subset V$, then the structure map is

$$F(U \subset V) \otimes G(U \subset V) : F \otimes G(U) = F(U) \otimes G(U) \to F(V) \otimes G(V) = F \otimes G(V).$$

DEFINITION 4.4.7. Let $PreFA_{mc}(X, \mathcal{C})$ denote the multicategory arising from the symmetric monoidal product on $PreFA(X, \mathcal{C})$. That is,

$$PreFA_{mc}(F_1, \cdots, F_n; G) := PreFA(F_1 \otimes \cdots \otimes F_n, G).$$

Factorization algebras inherit this multicategory structure.

4.4.3. Enrichment over simplicial sets. Recall that cochain complexes are enriched over simplicial sets as follows. For K a simplicial set,

$$sSets(K, \mathrm{Maps}(A, B)) = dgVect(A, \Omega^*(K) \otimes B),$$

where $dgVect(A, B)$ denotes the cochain maps, i.e., degree zero maps that commute with the differentials, and $\Omega^*(K)$ denotes the de Rham complex on the geometric realization of K. In particular, the n-simplices of $\mathrm{Maps}(A, B)$ are precisely $dgVect(A, \Omega^*(\Delta^n) \otimes B)$. Let $\underline{dgVect}$ denote the category whose objects are cochain complexes and whose morphisms from A to B is the mapping space $\mathrm{Maps}(A, B)$. This category is enriched over simplicial sets.

We use this same method to enrich prefactorization algebras over simplicial sets. Given prefactorization algebras F, G taking values in cochain complexes, define $\mathrm{Maps}(F, G)$ as follows. An n-simplex ϕ in $\mathrm{Maps}(F, G)$ consists of a map $\phi_U \in dgVect(F(U), \Omega^*(\Delta^n) \otimes G(U))$ for each open $U \subset X$, compatible with the structure maps.

4.4.4. Factorization algebras with structures.

DEFINITION 4.4.8. A *commutative* factorization algebra $\mathcal{F}$ with values in $\mathcal{C}^\otimes$ is

(1) a prefactorization algebra taking values in $CAlg(\mathcal{C})$, the category of commutative algebras in $\mathcal{C}$ with the usual tensor product $\otimes$ as the symmetric monoidal structure;
(2) a factorization algebra in $\mathcal{C}$ after applying the forgetful functor to $\mathcal{C}$ on every open.

By theorem 4.5.1 below, the factorization algebra $\mathrm{Sym}\,\mathcal{F}$ of a cosheaf $\mathcal{F}$ is a commutative factorization algebra.

4.5. General construction methods for factorization algebras

Generating prefactorization algebras is fairly easy but verifying the locality axiom is nontrivial, so it is convenient to have procedures that provide examples. Sometimes these examples are boring in themselves, but we obtain nontrivial factorization algebras by deforming them. BV quantization, for instance, is a systematic procedure for finding meaningful deformations. In this section, we introduce two sources of factorization algebras, one using commutative algebras and the other Lie algebras.

THEOREM 4.5.1. *For every cosheaf of dg vector spaces $\mathcal{F}$, the precosheaf* $\mathrm{Sym}\,\mathcal{F}$ *is a factorization algebra. Moreover,* Sym *defines a functor from cosheaves to factorization algebras.*

PROOF. It is straightforward to show that $\mathrm{Sym}\,\mathcal{F}$ is a prefactorization algebra. We simply need to obtain the structure maps. Note that for a finite collection of disjoint opens $U_1, \dots, U_k$, there is a canonical isomorphism

$$\mathcal{F}(U_1 \cup \cdots \cup U_k) \cong \mathcal{F}(U_1) \oplus \cdots \oplus \mathcal{F}(U_k)$$

and hence a canonical isomorphism

$$\mathrm{Sym}\,\mathcal{F}(U_1 \cup \cdots \cup U_k) \cong \mathrm{Sym}\,\mathcal{F}(U_1) \otimes \cdots \otimes \mathrm{Sym}\,\mathcal{F}(U_k).$$

Thus, if the opens are all contained in the open V, we get the structure map

$$\mathrm{Sym}\,\mathcal{F}(U_1) \otimes \cdots \otimes \mathrm{Sym}\,\mathcal{F}(U_k) \cong \mathrm{Sym}\,\mathcal{F}(U_1 \cup \cdots \cup U_k) \to \mathrm{Sym}\,\mathcal{F}(V).$$

The coherences of the cosheaf induce those for the prefactorization algebra.

We now verify the locality axiom.

Our first reduction is to observe that all the structure maps preserve the "algebraic degree": that is, the structure map above is simply the direct sum over m of the maps

$$\mathrm{Sym}^m \mathcal{F}(U_1 \cup \cdots \cup U_k) \to \mathrm{Sym}^m \mathcal{F}(V).$$

Thus, it suffices to verify the locality axiom independently for each m.

Explicitly, that means the following. For each $\alpha \in PI$, let $U_\alpha = \coprod_{i \in \alpha} U_i$. Note that

$$\mathrm{Sym}\, \mathcal{F}(\alpha, \beta) = \bigotimes_{i \in \alpha, j \in \beta} \mathrm{Sym}\, \mathcal{F}(U_i \cap U_j) \cong \mathrm{Sym}\, \mathcal{F}(U_\alpha \cap U_\beta),$$

and likewise for the case with $\alpha_0, \ldots, \alpha_n$. The Sym^m level of the locality axiom becomes the requirement that

$$\mathrm{colim}\left(\cdots \to \oplus_{\alpha,\beta \in PI} \mathrm{Sym}^m \mathcal{F}\left(U_\alpha \cap U_\beta\right) \to \oplus_{\gamma \in PI} \mathrm{Sym}^m \mathcal{F}\left(U_\gamma\right)\right) \to \mathrm{Sym}^m \mathcal{F}(U)$$

is a quasi-isomorphism.

Second, observe that

$$\mathcal{F}(U)^{\otimes m} = \mathcal{F}^{\boxtimes m}(U^m)$$

where $\mathcal{F}^{\boxtimes m}$ is the cosheaf on $U^m = \underbrace{U \times \cdots \times U}_{m \text{ times}}$ obtained as the external product of $\mathcal{F}$ with itself m times.

Thus it is enough to show that

$$\mathcal{F}^{\boxtimes m}(U^m) = \mathrm{colim}\left(\cdots \to \oplus_{\alpha,\beta \in PI} \mathcal{F}^{\boxtimes m}\left((U_\alpha \cap U_\beta)^m\right) \to \oplus_{\gamma \in PI} \mathcal{F}^{\boxtimes m}\left(U_\gamma^m\right)\right).$$

Our cover $\mathfrak{U}$ is a factorizing cover. This means that, for every finite set of points $x_1, \ldots, x_k \in M$ we can find disjoint open subsets $U_{i_1}, \ldots, U_{i_k}$ in the cover $\mathfrak{U}$ with $x_i \in U_{i_i}$. This implies that the subsets of U^m of the form $(U_\alpha)^m$, where $\alpha \in PI$, cover U^m. Further,

$$(U_\alpha)^m \cap (U_\beta)^m = (U_\alpha \cap U_\beta)^m.$$

The desired isomorphism now follows from the fact that $F^{\boxtimes m}$ is a cosheaf on M^m.

Every morphism of cosheaves clearly induces a morphism of prefactorization algebras, and hence factorization algebras. $\square$

Now we consider the Lie-theoretic method.

DEFINITION 4.5.2. A *Lie-structured cosheaf* of vector spaces $\mathfrak{g}$ is a precosheaf of dg Lie algebras that is a cosheaf of dg vector spaces (after applying the forgetful functor $Forget : dgLie \to dgVect$).

We have two favorite examples of Lie-structured cosheaves: let $\mathfrak{g}$ be a dg Lie algebra, then

(1) define $\mathfrak{g}^M := \Omega^*_{M,c} \otimes \mathfrak{g}$ to be the cosheaf of compactly-supported, $\mathfrak{g}$-valued de Rham forms on a smooth manifold M;
(2) define $\mathfrak{g}^{M_{\bar\partial}} := \Omega^{0,*}_{M,c} \otimes \mathfrak{g}$ be the cosheaf of compactly-supported, $\mathfrak{g}$-valued Dolbeault forms on a complex manifold M.

Both these examples will reappear throughout the text.

THEOREM 4.5.3. *For every Lie-structured cosheaf $\mathfrak{g}$, applying the functor of Chevalley-Eilenberg chains $C_*\mathfrak{g}$ to each open*

$$U \mapsto (\mathrm{Sym}(\mathfrak{g}(U)[1]), d_{CE})$$

is a factorization algebra in dg vector spaces. We denote this factorization algebra by $C_*\mathfrak{g}$. Moreover C_* defines a functor from Lie-structured cosheaves to factorization algebras.

We call $C_*\mathfrak{g}$ the *enveloping factorization algebra* of $\mathfrak{g}$.

REMARK 4.5.4. Every cosheaf of dg vector spaces is a Lie-structured cosheaf where we assign an *abelian* dg Lie algebra to each open. Hence this theorem is a very direct generalization of theorem 4.5.1.

PROOF. Consider the filtration on the prefactorization algebra

$$F^i C_*\mathfrak{g} := \mathrm{Sym}^{\leq i}(\mathfrak{g}[1]).$$

We will use the spectral sequence induced by this filtration to show $C_*(\mathfrak{g})$ is a factorization algebra. For any factorizing cover $\mathfrak{U}$ of an open U, the structure maps induce a map

$$\check{C}(\mathfrak{U}, C_*\mathfrak{g}) \to C_*\mathfrak{g}(U)$$

and hence a map of spectral sequences. The first page of these spectral sequences is given by forgetting the Lie algebra structure on $\mathfrak{g}$ and simply viewing $\mathfrak{g}[1]$ as a cosheaf in dg vector spaces and then applying the functor Sym. Hence by theorem 4.5.1 the map on the first page is an isomorphism. The original map is thus a quasi-isomorphism. □

REMARK 4.5.5. For the manifold $\mathbb{R}^n$, the factorization algebra $C_*\mathfrak{g}^{\mathbb{R}^n}$ provides the E_n *enveloping algebra* of $\mathfrak{g}$. In the next section, we prove a shadow of this assertion by showing that for $\mathfrak{g}$ a graded Lie algebra, $C_*\mathfrak{g}^{\mathbb{R}}$ recovers the universal enveloping algebra (i.e., the E_1 enveloping algebra) $U\mathfrak{g}$ of $\mathfrak{g}$. To prove the full assertion, one needs to use the full power of ∞-categories and homotopical algebra, which is far beyond our scope. A proof should follow quite directly from results in [**Lurb**] and [**Fra**]. We now sketch the idea with no pretense of rigor.

Recall that the formality of the E_n operad in characteristic zero states that the E_n operad is equivalent to the P_n operad (i.e., the operad describing commutative dg algebras with a Poisson bracket of degree $1-n$). There is a forgetful functor from P_n algebras to dg Lie algebras by forgetting the commutative product and shifting the complex down by $n-1$. There is an adjoint to this forgetful functor, giving the "enveloping" P_n algebra of a dg Lie algebra. Explicitly, the enveloping

P_n algebra of $(\mathfrak{g}, d)$ is $(\mathrm{Sym}(\mathfrak{g}[1-n]), d)$ where we extend d as a derivation and extend the shifted bracket of $\mathfrak{g}$ to obtain the Poisson bracket. Formality then tells us that we can lift the enveloping P_n algebra to an E_n algebra.

A simple computation shows that $C_*\mathfrak{g}^{\mathbb{R}^n}$, when viewed as an E_n algebra, has the correct commutative product and shifted Lie bracket. In particular, consider the structure map given by the inclusion of a small n-disk nested inside a thickened $n-1$-sphere into a large n-disk. This structure map encodes the shifted Lie bracket.

4.6. A novel construction of the universal enveloping algebra

Let $\mathfrak{g}$ be a graded Lie algebra (i.e., a dg Lie algebra with zero differential). Recall that $\mathfrak{g}^{\mathbb{R}}$ denotes the Lie-structured cosheaf on $\mathbb{R}$ that assigns $(\Omega^*_c(U) \otimes \mathfrak{g}, d)$ to each open U, with d the exterior derivative. Our main result shows how to construct the universal enveloping algebra $U\mathfrak{g}$ using theorem 4.5.3.

PROPOSITION 4.6.1. *Let $\mathcal{H}$ denote the cohomology prefactorization algebra of $C_*\mathfrak{g}^{\mathbb{R}}$, the enveloping factorization algebra of $\mathfrak{g}^{\mathbb{R}}$. That is, we take the cohomology of every open and every structure map, so*

$$\mathcal{H}(U) = H^*(C_*\mathfrak{g}^{\mathbb{R}}(U))$$

for any open U. Then $\mathcal{H}$ is isomorphic to $\mathcal{F}_{U\mathfrak{g}}$, the factorization algebra for the universal enveloping algebra of $\mathfrak{g}$.

We break the proof into a sequence of lemmas. First, we obtain a kind of PBW result (showing the proposition is plausible).

LEMMA 4.6.2. *On an open interval U, the vector space $\mathcal{H}(U)$ has a natural filtration F such that* $\mathrm{Gr}_F\, \mathcal{H}(U) \cong \mathrm{Sym}\, \mathfrak{g}$.

PROOF. Let $\tilde{F}$ denote the filtration on $C_*\mathfrak{g}^{\mathbb{R}}(U)$ where $\tilde{F}^k = \mathrm{Sym}^{\leq k}(\mathfrak{g}^{\mathbb{R}}(U)[1])$. Then F is the induced filtration on its cohomology $\mathcal{H}(U)$. Now consider the spectral sequence induced by $\tilde{F}$. Its first page is the cohomology of the complex $(\mathrm{Sym}(\mathfrak{g}^{\mathbb{R}}(U)[1]), d_{dR})$, with d_{dR} the exterior derivative. This cohomology is precisely $\mathrm{Sym}\, \mathfrak{g}$, by the compactly-supported Poincaré lemma.

For $\mathfrak{g}$ an ordinary Lie algebra (i.e., concentrated in degree 0), we see the spectral sequence collapses because the first page is concentrated in degree 0.

For $\mathfrak{g}$ a graded Lie algebra, we see that the differential vanishes on every higher page as follows. For any nontrivial element on the first page, a lift to $C_*\mathfrak{g}^{\mathbb{R}}(U)$ lives in $\mathrm{Sym}(\Omega^1(U) \otimes \mathfrak{g})$. The full differential vanishes on any such element because the wedge of two 1-forms is always zero. ∎

Now we show $\mathcal{H}$ is locally constant and hence corresponds to some associative algebra A.

LEMMA 4.6.3. *For any inclusion $i : U \subset V$ of an interval into an interval, the associated structure map $\mathcal{H}(i)$ of $\mathcal{H}$ is an isomorphism. Hence $\mathcal{H}$ is a locally constant factorization algebra.*

PROOF. Apply the filtration from the previous lemma to obtain a morphism of spectral sequences. We know that $i_!$ (extension by zero) induces a quasi-isomorphism from $\Omega_c^*(U)$ to $\Omega_c^*(V)$, so this morphism of spectral sequences is an isomorphism on the first page. Thus the structure map

$$C_*\mathfrak{g}^{\mathbb{R}}(i) : C_*\mathfrak{g}^{\mathbb{R}}(U) \to C_*\mathfrak{g}^{\mathbb{R}}(V)$$

is a quasi-isomorphism, implying the lemma. □

Together, these lemmas imply that A is isomorphic to $\operatorname{Sym}\mathfrak{g}$ as a vector space. We wish to show that $\mathfrak{g}$ generates A as an algebra. To make this precise, we introduce some notation that allows us to define an inclusion map $\iota_U : \mathfrak{g} \to \mathcal{H}(U)$ for each interval U.

Pick a bump function ϕ on $\mathbb{R}$ such that

- $\phi \geq 0$;
- $\operatorname{supp}(\phi) \subset (0, 1)$;
- $\int_{\mathbb{R}} \phi(t)\, dt = 1$.

To each interval $U = (a, b)$, we then associate the 1-form

$$\alpha_U := \frac{U}{b-a}\phi\left(\frac{t-a}{b-a}\right) dt,$$

which has support in U and integrates to 1, by construction. Observe that for any $X \in \mathfrak{g}$, the element $\alpha_U \otimes X$ is a cocycle in $C_*\mathfrak{g}^{\mathbb{R}}(U)$ whose cohomology class $[\alpha_U \otimes X]$ goes to X in $\operatorname{Gr}_F \mathcal{H}(U)$. Define ι_U by $X \mapsto [\alpha_U \otimes X]$. Note that for an inclusion $i : U \subset V$ of intervals, we have $\iota_V = \mathcal{H}(i) \circ \iota_U$ by construction. Thus there is a well-defined map $\iota : \mathfrak{g} \to A$.

LEMMA 4.6.4. *Viewing $\mathfrak{g}$ as the image of ι, it generates the algebra A.*

PROOF. Let $X^\nu := X_1 X_2 \cdots X_n$ be an arbitrary element of $\operatorname{Sym}^n \mathfrak{g}$. We will use ι to obtain an element Ξ in A whose image in $\operatorname{Gr}_F A$ is precisely this element X^ν.

Pick an interval U and a collection of subintervals

$$U_1 < U_2 < \cdots < U_n.$$

(Recall from section 4.3 that $I < J$ if every element of I is less than every element of J.) Consider the element

$$\iota_{U_1}(X_1) \otimes \cdots \otimes \iota_{U_n}(X_n) \in \mathcal{H}(U_1) \otimes \cdots \otimes \mathcal{H}(U_n)$$

and apply the structure map

$$m : \mathcal{H}(U_1) \otimes \cdots \otimes \mathcal{H}(U_n) \to \mathcal{H}(U)$$

to obtain a representative for Ξ (under the identification between $\mathcal{H}(U)$ and A). At the cochain level, we have

$$(\alpha_{U_1} \otimes X_1) \otimes \cdots \otimes (\alpha_{U_n} \otimes X_n) \in C_*\mathfrak{g}^{\mathbb{R}}(U_1) \otimes \cdots \otimes C_*\mathfrak{g}^{\mathbb{R}}(U_n).$$

Under the analogous structure map for $C_*\mathfrak{g}^{\mathbb{R}}$, we view $(\alpha_{U_1} \otimes X_1) \otimes \cdots \otimes (\alpha_{U_n} \otimes X_n)$ as an element of $\mathrm{Sym}^n(\Omega^1_c(U) \otimes \mathfrak{g})$ by extending the forms α_{U_j} by zero. This element is closed and hence descends to some cohomology class. Using the filtration/spectral sequence from earlier, we see that it corresponds precisely to X^ν. □

We now show that these generators satisfy the same relations as $U\mathfrak{g}$. Let $\bullet$ denote multiplication in A.

LEMMA 4.6.5. *For all $X, Y \in \mathfrak{g}$, we have the following relation in A:*

$$\iota(X) \bullet \iota(Y) - \iota(Y) \bullet \iota(X) = \iota([X,Y]).$$

Thus A is isomorphic to $U\mathfrak{g}$.

PROOF. We will obtain this relation by showing that the elements on either side of the equality represent the same cohomology classes. Thus, we work at the cochain level (i.e., in $C_*\mathfrak{g}^{\mathbb{R}}$).

Pick intervals $U_1 < U_2 < U_3$ inside a bigger interval V. Let m denote the structure map

$$C_*\mathfrak{g}^{\mathbb{R}}(U_1) \otimes C_*\mathfrak{g}^{\mathbb{R}}(U_2) \otimes C_*\mathfrak{g}^{\mathbb{R}}(U_3) \to C_*\mathfrak{g}^{\mathbb{R}}(V).$$

We want to compute

$$m((\alpha_{U_1} \otimes X) \otimes (\alpha_{U_2} \otimes Y) \otimes 1) - m(1 \otimes (\alpha_{U_2} \otimes Y) \otimes (\alpha_{U_3} \otimes X))$$

and see that it is cohomologous to $\alpha_{U_2} \otimes [X,Y]$. This implies the relation for $\mathcal{H}$, at the level of cohomology.

Let

$$\Phi(t) = \int_{-\infty}^t \alpha_{U_1} - \int_{-\infty}^t \alpha_{U_3}$$

denote a compactly supported function on V. Consider the element $\Phi \otimes X \cdot \alpha_{U_2} \otimes Y$ in $\mathrm{Sym}^2(\Omega^*_c(V) \otimes \mathfrak{g}[1])$. We compute

$$\begin{aligned} d_{C_*}(\Phi \otimes X \cdot \alpha_{U_2} \otimes Y) &= (d_{dR}\Phi) \otimes X \cdot \alpha_{U_2} \otimes Y - \Phi\alpha_{U_2} \otimes [X,Y] \\ &= \alpha_{U_1} \otimes X \cdot \alpha_{U_2} \otimes Y - \alpha_{U_3} \otimes X \cdot \alpha_{U_2} \otimes Y - \alpha_{U_2} \otimes [X,Y] \\ &= \alpha_{U_1} \otimes X \cdot \alpha_{U_2} \otimes Y - \alpha_{U_2} \otimes Y \cdot \alpha_{U_3} \otimes X - \alpha_{U_2} \otimes [X,Y] \end{aligned}$$

since $\Phi\big|_{U_2} \equiv 1$ and all the elements are cohomologically degree 0. Hence, in $\mathcal{H}(V)$, we see that

$$X \bullet Y - Y \bullet X - [X,Y] = 0,$$

as it is a boundary. □

REMARK 4.6.6. In chapter 6 on vertex algebras, we will use a variant of this construction for Σ a Riemann surface to construct the Kac-Moody vertex algebras.

4.7. Extension from a factorizing basis

4.7.1. Factorization algebras defined on a factorizing basis. Let X be a topological space, and let $\mathfrak{U}$ be a basis for X, which is closed under taking finite intersections. It is well-known that there is an equivalence of categories between sheaves on X and sheaves which are only defined for open sets in the basis $\mathfrak{U}$. In this section we will prove a similar statement for factorization algebras. This will allow us to perform several useful formal constructions with factorization algebras, such as gluing.

DEFINITION 4.7.1. A *factorizing basis* for X is a basis $\mathfrak{U}$ of open sets of X which is closed under finite intersections and is also a factorizing cover for X.

Let $\mathfrak{U}$ be a factorizing basis.

DEFINITION 4.7.2. A $\mathfrak{U}$-prefactorization algebra $\mathcal{F}$ is like a factorization algebra, except that $\mathcal{F}(U)$ is only defined for $U \in \mathfrak{U}$. A $\mathfrak{U}$-factorization algebra is a $\mathfrak{U}$-prefactorization algebra with the property that, for all $U \in \mathfrak{U}$ and all factorizing covers $\mathfrak{V}$ of U consisting of open sets in $\mathfrak{U}$,

$$\check{C}(\mathfrak{V}, \mathcal{F}) \simeq \mathcal{F}(U),$$

where $\check{C}(\mathfrak{V}, \mathcal{F})$ denotes the Čech complex described earlier in section 4.1.

In this section we will show that any $\mathfrak{U}$-factorization algebra on X extends to a factorization algebra on X. This extension is unique up to quasi-isomorphism.

Let $\mathcal{F}$ be a $\mathfrak{U}$-factorization algebra. Let us define a prefactorization algebra $i_*^{\mathfrak{U}}\mathcal{F}$ on X by

$$i_*^{\mathfrak{U}}(\mathcal{F})(V) = \check{C}(\mathfrak{U}_V, \mathcal{F}),$$

for each $V \subset X$ open. Here $\mathfrak{U}_V$ is the cover of V consisting of those open subsets in the cover $\mathfrak{U}$ which are contained in V.

LEMMA 4.7.3. *With this definition, $i_*^{\mathfrak{U}}(\mathcal{F})$ is a factorization algebra whose restriction to open sets in the cover $\mathfrak{U}$ is quasi-isomorphic to $\mathcal{F}$.*

PROOF. We need to check that if $\mathfrak{W}$ is a factorizing cover of $V \subset X$, then

$$i_*^{\mathfrak{U}}(\mathcal{F})(V) \simeq \check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})).$$

Before we prove this, we need a lemma. Let $\mathfrak{U}_{\mathfrak{W}}$ be the cover of V consisting of open sets in $\mathfrak{U}$ which are subordinate to $\mathfrak{W}$.

LEMMA 4.7.4. *For any $\mathfrak{U}$-prefactorization algebra $\mathcal{F}$, the natural map*

$$\check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})) \to \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F})$$

is a quasi-isomorphism.

PROOF. Before we check this, let us recall the notation we used when discussing Čech complexes. Let $P\mathfrak{U}$ denote the set of subsets $\alpha \subset \mathfrak{U}$, where for each distinct $i, j \in \alpha$, U_i and U_j are disjoint. If $\alpha \in P\mathfrak{U}$ we will let

$$U_\alpha = \amalg_{i \in \alpha} U_i.$$

If $\alpha_1, \ldots, \alpha_k \in P\mathfrak{U}$, we will let

$$\mathcal{F}(\alpha_1, \ldots, \alpha_k) = \oplus_{i_1 \in \alpha_1, \ldots, i_k \in \alpha_k} \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_k}).$$

With this notation, if $W \subset M$, then

$$i_*^{\mathfrak{U}}(\mathcal{F})(W) = \oplus_{\alpha_1, \ldots, \alpha_r \in \mathfrak{U}_W} \mathcal{F}(\alpha_1, \ldots, \alpha_r)[r-1]$$

where $\mathfrak{U}_W$ refers to the cover of W consisting of open sets in $\mathfrak{U}$ which lie in W.

Let us define a filtration on $i_*^{\mathfrak{U}}(\mathcal{F})$ by saying that

$$F^i i_*^{\mathfrak{U}}(\mathcal{F}) = \oplus_{r \leq i} \oplus_{\alpha_1, \ldots, \alpha_r \in \mathfrak{U}_W} \mathcal{F}(\alpha_1, \ldots, \alpha_r)[r-1].$$

This filters $i_*^{\mathfrak{U}}(\mathcal{F})$ as a prefactorization algebra.

There is a natural map

$$\check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F})) \to \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}).$$

Let us filter $\check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F}))$ by the filtration coming from $i_*^{\mathfrak{U}}(\mathcal{F})$. Let us filter $\check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F})$ in the same way that we filtered $i_*^{\mathfrak{U}}(\mathcal{F})$. The map preserves the filtration.

Thus, to prove that this map is a quasi-isomorphism, it suffices to show that it is on the associated graded.

The complex $\mathrm{Gr}^n \check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F}))$ breaks up as a direct sum of pieces corresponding to tuples $\alpha_1, \ldots, \alpha_n \in P\mathfrak{U}_{\mathfrak{W}}$, as follows. If $\beta \in P\mathfrak{W}$ and $\alpha \in P\mathfrak{U}_{\mathfrak{W}}$, say $\alpha \subset \beta$ if $U_\alpha \subset U'_\beta$. Then,

$$\mathrm{Gr}^n \check{C}(\mathfrak{W}, i_*^{\mathfrak{U}}(\mathcal{F}))$$
$$= \bigoplus_{\alpha_1, \ldots, \alpha_n \in P\mathfrak{U}} \mathcal{F}(\alpha_1, \ldots, \alpha_n)[n-1] \otimes \left(\bigoplus_{\substack{\beta_1, \ldots, \beta_m \in P\mathfrak{W} \\ \alpha_i \subset \beta_j \text{ all } i,j}} \mathbb{C} \cdot (\beta_1, \ldots, \beta_k) \right).$$

Here $(\beta_1, \ldots, \beta_k)$ denotes a vector in degree $-k$. This is a direct sum decomposition of cochain complexes.

On the other hand,

$$\mathrm{Gr}^n \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}) = \oplus_{\alpha_1, \ldots, \alpha_n \in P\mathfrak{U}_{\mathfrak{W}}} \mathcal{F}(\alpha_1, \ldots, \alpha_n)$$

Thus, to prove the lemma, we need to verify that the complex

$$\oplus_{\substack{\beta_1,\dots,\beta_m \in P\mathfrak{W} \\ \alpha_i \subset \beta_j \text{ all } i,j}} \mathbb{C} \cdot (\beta_1, \dots, \beta_k)$$

has homology $\mathbb{C}$ if all $\alpha_i \in P\mathfrak{U}_{\mathfrak{W}}$, and zero otherwise.

It is clear that the complex is zero if all α_i are not in $P\mathfrak{U}_{\mathfrak{W}}$. So let us assume that all α_i are in $P\mathfrak{U}_{\mathfrak{W}}$. Then, the complex is simply the simplicial chain complex on the infinite simplex with vertices $\beta \in P\mathfrak{U}$ such that $\cup U_{\alpha_i} \subset U_\beta$. This is of course contractible. $\square$

It remains to shows that the natural map

$$\check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}) \to \check{C}(\mathfrak{U}_V, \mathcal{F})$$

is a quasi-isomorphism. (Here, as before, $\mathfrak{U}_V$ refers to the cover of V consisting of sets in $\mathfrak{U}$ which lie in V).

To see that this map is a quasi-isomorphism, observe that by another application of the sub-lemma there is a quasi-isomorphism

$$\check{C}(\mathfrak{U}, i_*^{\mathfrak{U}_{\mathfrak{W}}}(\mathcal{F})) \simeq \check{C}(\mathfrak{U}_{\mathfrak{W}}, \mathcal{F}).$$

Here $i_*^{\mathfrak{U}_{\mathfrak{W}}}$ refers to the prefactorization algebra on V obtained by extending $\mathcal{F}$, as before, but now considered as a $\mathfrak{U}_{\mathfrak{W}}$-factorization algebra.

Now the fact that $\mathcal{F}$ is a $\mathfrak{U}$-factorization algebra implies that, for all $U \in \mathfrak{U}$, the natural map

$$\check{C}(\mathfrak{U}_{\mathfrak{W}} \cap \mathfrak{U}_U, \mathcal{F}) \to \mathcal{F}(U)$$

is a quasi-isomorphism.

It follows that the natural map

$$\check{C}(\mathfrak{U}, i_*^{\mathfrak{U}_{\mathfrak{W}}}(\mathcal{F})) \to \check{C}(\mathfrak{U}, \mathcal{F})$$

is a quasi-isomorphism, as desired. $\square$

4.8. Pushforward and Pullback

So far, we have only discussed factorization algebras on a fixed manifold M. It is useful to understand how factorization algebras can be moved between spaces along a smooth map $f : M \to N$. Just as with sheaves, pushforwards are straightforward. Pullbacks, however, are more subtle and are only easily constructed when f is an embedding.

4.8.1. Pushing forward factorization algebras. A crucial feature of factorization algebras is that they push forward nicely. Let M and N be topological spaces admitting factorizing covers and let $f : M \to N$ be a continuous map. Given a factorizing cover $\mathfrak{U} = \{U_\alpha\}$ of an open $U \subset N$, let $f^{-1}\mathfrak{U} = \{f^{-1}U_\alpha\}$ denote the preimage cover of $f^{-1}U \subset M$. Observe that $f^{-1}\mathfrak{U}$ is factorizing: given a finite collection of points $\{x_1, \ldots, x_n\}$ in $f^{-1}U$, the image points $\{f(x_1), \ldots, f(x_n)\}$ can be covered by a disjoint collection of opens $U_{\alpha_1}, \ldots, U_{\alpha_k}$ in $\mathfrak{U}$ and hence $f^{-1}U_{\alpha_1}, \ldots, f^{-1}U_{\alpha_k}$ is a disjoint collection of opens in $f^{-1}\mathfrak{U}$ covering the x_j.

DEFINITION 4.8.1. Given a factorization algebra $\mathcal{F}$ on a space M and a continuous map $f : M \to N$, the *pushforward factorization algebra $f_*\mathcal{F}$* on N is defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Note that for the map to a point $f : M \to pt$, the pushforward factorization algebra $f_*\mathcal{F}$ is simply the global sections of $\mathcal{F}$. We also call this the *factorization homology* of $\mathcal{F}$ on M. We sometimes denote this $FH(M, \mathcal{F})$.

REMARK 4.8.2. It can be convenient to compute sheaf cohomology by iterating pushforwards. For instance, the Serre spectral sequence computes the sheaf cohomology on the total space of a fiber bundle by first pushing forward the sheaf to the base space and then pushing forward to a point. Likewise, it can be helpful to compute factorization homology by iterating pushforwards and thus obtaining spectral sequences.

4.8.2. A case where pushforwards commute. The notion of pushforward intertwines nicely with our general construction methods.

PROPOSITION 4.8.3. *Let $\mathfrak{g}$ be a Lie-structured cosheaf on a topological space M. Let $f : M \to N$ be a continuous map. Then the factorization algebra $C_*(f_*\mathfrak{g})$ is naturally isomorphic to $f_*(C_*\mathfrak{g})$.*

PROOF. On an open $U \subset N$,

$$C_*(f_*\mathfrak{g})(U) = C_*(\mathfrak{g}(f^{-1}(U))) = f_*(C_*\mathfrak{g})(U).$$

Likewise, the structure maps are identical. $\square$

4.8.3. Pulling back factorization algebras. Let $\mathcal{F}$ be a factorization algebra on M. Let $U \subset M$ be an open subset. Then we can restrict $\mathcal{F}$ to a factorization algebra $\mathcal{F}\big|_U$ on U, whose value on an open subset $V \subset U$ is simply $\mathcal{F}(V)$.

In this section we will discuss a generalization of this construction. We will not try to define pull-backs for arbitrary maps, but only for open immersions.

Let $f : N \to M$ be an open immersion. Let $\mathfrak{U}_f$ be the cover of N consisting of those open subsets $U \subset N$ with the property that

$$f\mid_U : U \to f(U)$$

is a homeomorphism. (To say that f is an open immersion means that sets of this form cover N).

Now, $\mathfrak{U}_f$ is a factorizing basis for N. Let us define a $\mathfrak{U}_f$-prefactorization algebra $f^*\mathcal{F}$ by

$$f^*\mathcal{F}(U) = \mathcal{F}(f(U))$$

if $U \in \mathfrak{U}_f$.

LEMMA 4.8.4. *$f^*\mathcal{F}$ is a $\mathfrak{U}_f$-factorization algebra.*

PROOF. We need to verify that if $U \in \mathfrak{U}_f$, and $\mathfrak{V}$ is a factorizing cover of U by elements of $\mathfrak{U}_f$, that

$$\check{C}(\mathfrak{V}, f^*\mathcal{F}) \simeq f^*\mathcal{F}(U) = \mathcal{F}(f(U)).$$

Now, $f(\mathfrak{V})$ is a factorizing cover of $f(U)$, and

$$\check{C}(\mathfrak{V}, f^*\mathcal{F}) = \check{C}(f(\mathfrak{V}, \mathcal{F}).$$

The result follows from the fact that $\mathcal{F}$ is a factorization algebra on M. $\square$

So far we have defined $f^*\mathcal{F}$ as a $\mathfrak{U}_f$-factorization algebra. We can extend (see section 4.7) $f^*\mathcal{F}$ to an actual factorization algebra, which we will continue to call $f^*\mathcal{F}$.

CHAPTER 5

Free fields and their observables

5.1. Introduction

In this chapter we will show how the viewpoint of BV quantization articulated in chapter 2 carries over precisely to one class of field theories. These are the simplest kind of field theories — what we'll call the *free theories* — and most interesting field theories (such as Chern-Simons, Yang-Mills, or even ϕ^4) do not fall into this class. Nonetheless, they play a special role in perturbative QFT because every interacting theory is studied as a deformation of a free theory. It thus behooves us to obtain an understanding of free theories that is as clear and thorough as possible. Moreover, many mathematical structures that appear complicated or obscure in the interacting cases have clean, elegant interpretations in the free setting. For instance, we'll see familiar constructions like determinants of complexes and Heisenberg Lie algebras appear quite naturally in this context. One reason behind this relative simplicity is that we use only basic homological algebra and analysis for free fields; Feynman diagrams and other constructions from physics do not appear, which makes free fields particularly accessible to a mathematician. Indeed, one goal of this thesis is to see exactly how much one can accomplish without requiring the full machinery of perturbative field theory developed in, for instance, [**Cos11**].

Before embarking on the myriad definitions and constructions that constitute this chapter, we quickly discuss what field theory is and then overview what's accomplished in this chapter.

5.1.1. Classical field theory and deformations. Field theory has two aspects — the classical and the quantum — and classical field theory is essentially (a subset of) the study of partial differential equations. In practice, the equations studied in field theory usually arise from variational problems and hence are the Euler-Lagrange equations of some action functional. Understanding the spaces of solutions of a PDE is often quite difficult. *Perturbative* classical field theory, from our viewpoint (see [**CG**]), consists of studying the formal neighborhood of a fixed solution to a PDE in the moduli space of all solutions, and hence we describe the *formal* moduli space by a dg Lie algebra (or L_∞ algebra), following the correspondence

$$\{\text{formal moduli problems}\} \leftrightarrow \{\text{dg Lie algebras}\}$$

that is well-established in deformation theory. (We elaborate on this assertion in the next paragraph below.) We thus obtain the following dictionary between terminology in field theory (physics), PDE, and deformation theory (using the correspondence above):

physics	PDE	Lie theory
free theory	linear system	abelian dg Lie algebra
interacting theory	nonlinear system	nonabelian dg Lie algebra

It should be clear that free theories ought to be substantially simpler to understand.

5.1.2. An example. An example is in order, and we use the original motivating example: moduli of flat connections. Let M be a smooth manifold with a vector bundle $E \to M$ and a flat connection ∇. We want to study infinitesimal deformations of the connection ∇, where we only want deformations that are also flat. The space of all connections on E is simply $\Omega^1(\mathrm{End}\, E)$, where we identify the origin of this vector space with ∇. The space of flat connections is the subspace

$$\left\{ A \in \Omega^1(\mathrm{End}\, E) \;:\; (\nabla + A)^2 = 0 \right\},$$

although we should take into account the gauge-equivalence of different connections (i.e., how automorphisms of the bundle relate the connections). In summary, we want to describe the formal neighborhood of the origin up to gauge equivalence.

The flat connection ∇ on E induces a flat connection on End E, which we abusively denote ∇ as well. Consider the *de Rham complex* of End E

$$\mathfrak{g} := (\Omega^*(\mathrm{End}\, E), \nabla).$$

This sheaf naturally takes values in dg Lie algebras via the usual bracket on endomorphisms. The Maurer-Cartan equation in this situation is simply

$$\nabla A + \frac{1}{2}[A, A] = 0,$$

with $A \in \Omega^1(\mathrm{End}\, E)$, which is precisely the zero-curvature equation given earlier. Because we are interested in the *formal* moduli problem, however, we will focus on A that are "infinitesimal." Thus, we consider the functor

$$\mathrm{Def}_\nabla : dgArt \to sSets$$

sending a local dg Artinian ring $(R, \mathfrak{m})$ to the simplicial set $\mathrm{Def}_\nabla(R)_\bullet$ whose n-simplices are

$$MC(\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)),$$

which denotes families over the n-simplex Δ^n of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes \mathfrak{m}$. Explicitly, if A is a degree 1 element of $\Omega^*(\mathrm{End}\, E) \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$, then the Maurer-Cartan equation is

$$\nabla A + d_R A + d_{dR} A + \frac{1}{2}[A, A] = 0,$$

where d_R is the derivation on R and d_{dR} denotes the exterior derivative on the de Rham forms on the n-simplex. Although it looks complicated at first, this functor Def_∇ is a natural construction. It encodes, for instance, all the infinitesimal deformations of ∇ (these are the 0-simplices) and how these deformations are related by gauge transformations (or, more accurately, homotopies — these are the 1-simplices). As an example, consider $R = \mathbb{C}[\epsilon]/(\epsilon^2)$, the usual dual numbers (i.e., $|\epsilon| = 0$), and the 1-simplex Δ^1. A solution to the Maurer-Cartan equation is a 1-dimensional family

$$\epsilon A(t) + \epsilon B(t)\, dt, \text{ with } t \in [0,1],$$

where $A(t) \in \Omega^1(\mathrm{End}\, E)$ and $B(t) \in \Omega^0(\mathrm{End}\, E)$, satisfying the pair of differential equations

$$\frac{d}{dt} A + \nabla B = 0 \text{ and } \nabla A = 0.$$

Hence, the path of 1-forms $A(t)$ stays in the space of first-order deformations of ∇, and this path is simply the flow determined by the vector field $B(t)$ with values in the Lie algebra $\Omega^0(\mathrm{End}\, E)$.

5.1.3. Free fields in this perspective. Any sheaf of cochain complexes is a sheaf of *abelian* dg Lie algebras, so the formal moduli problem for a free theory is particularly simple to understand. The bracket is always zero, so the Maurer-Cartan equation asks for families of closed elements of degree 1. Notice that by evaluating our functor $\mathrm{Def}_{\mathfrak{g}}$ on the different versions of dual numbers

$$D_n := \mathbb{C}[\epsilon]/(\epsilon^2) \text{ where } |\epsilon| = n \in \mathbb{Z},$$

we recover basic cohomological information about the cochain complex $\mathfrak{g}$:

- the 0-simplices of $\mathrm{Def}_{\mathfrak{g}}(D_n)$ are the closed elements of cohomological degree $1 - n$ (e.g., when $n = 0$, we get the 1-cocycles of $\mathfrak{g}$; when $n = 1$, we get the 0-cocycles);
- the 1-simplices encode the coboundaries, so that π_0 of the simplicial set $\mathrm{Def}_{\mathfrak{g}}(D_n)$ is the cohomology group $H^{1-n}(\mathfrak{g})$.

In general, this functorial interpretation of a cochain complex — inside the broader context of dg Lie algebras as formal moduli problems — provides a geometric way of understanding much of homological algebra. Our aim in this paper is to apply this point of view to field theory when $\mathfrak{g}$ is abelian. Much of the effort in [CG] is aimed at extending this perspective to the nonabelian case — and hence to interacting field theories.

5.1.4. Quantization using the Batalin-Vilkovisky formalism. In section 2.4 of chapter 2 and in chapter 3, we studied the Batalin-Vilkovisky (BV) approach to quantization in the case of so-to-speak zero-dimensional free field theories: we defined BV quantization for any cochain complex, and a cochain complex can be viewed as a free theory over a point. Recall that so long as the complex (E, d) is dualizable (i.e., bounded and finite-dimensional in every degree) the cotangent quantization functor $\mathcal{CQ}(E)$ provides a determinant of E. As the determinant of a vector space is the natural home of its volume forms, we see that BV quantization realizes — in a precise, categorical form — the intuition of the path integral.

Here we want to apply this viewpoint to free theories over manifolds of higher dimensions. We raise the question: what happens if you apply the functor $\mathcal{CQ}$ to a sheaf of complexes, open set by open set? Another version of this question is: what is the local-to-global nature of the observables of a quantized free field theory?

5.2. Elliptic complexes and free BV theories

We take for granted the standard notion of an elliptic complex and the basic machinery used in working with elliptic complexes (such as parametrices). With that language in hand, we can provide the definition of a Batalin-Vilkovisky theory *à la* Costello, which is simply a special kind of elliptic complex.

DEFINITION 5.2.1. A *free BV theory* on a manifold M consists of the following data:

- a finite rank, $\mathbb{Z}$-graded vector bundle (or super vector bundle) E on M;
- a vector bundle map $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}_M$ that is fiberwise nondegenerate, antisymmetric, and of cohomological degree -1; this local pairing induces a pairing on compactly-supported sections
$$\langle -, - \rangle : \mathscr{E}_c \otimes \mathscr{E}_c \to \mathbb{C},$$
$$\langle s_0, s_1 \rangle = \int_{x \in M} \langle s_0(x), s_1(x) \rangle_{loc};$$
- a differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 such that
  (1) $(\mathscr{E}, Q)$ is an elliptic complex;
  (2) Q is skew self adjoint with respect to the pairing, i.e., $\langle s_0, Q s_1 \rangle = -(-1)^{|s_0|} \langle Q s_0, s_1 \rangle$.

The action functional associated to this theory is
$$S(\phi) := \int_M \langle \phi, Q\phi \rangle$$
where ϕ is a compactly-supported section of E. To be more accurate, we should say that the classical field theory given by S looks for fields ϕ (not necessarily compactly supported) that are critical points of S with respect to perturbations by compactly supported sections. In other words, there is a 1-form dS with respect to the foliation of $\mathscr{E}$ by $\mathscr{E}_c$ and we are looking for ϕ such that $dS_\phi = 0$. Note that S is quadratic so that dS is linear. This relationship is why quadratic actions correspond to free field theories.

REMARK 5.2.2. In [Cos11], another condition is included: there exists a "gauge-fixing operator" $Q^* : \mathscr{E} \to \mathscr{E}$ of cohomological degree -1 such that

(1) $(Q^*)^2 = 0$;
(2) Q^* is self adjoint for the pairing;
(3) $D = [Q, Q^*]$ is a generalized Laplacian on M for the vector bundle E.

This third condition allows one to use heat kernel asymptotics, and these are crucial for Costello's approach to renormalization. The constructions in this chapter (and much of the thesis), however, do not rely on a gauge-fixing operator, so we drop this condition until chapter 7, where it becomes relevant.

5.2.1. Examples of free theories. Elliptic complexes are ubiquitous in geometry, and so we have a wealth of examples to consider. Typically the complexes themselves do not have an obvious meaning but their cohomology groups (or at least H^0) often do. For instance, the de Rham complex can be viewed as a resolution of the constant functions, and the Dolbeault complex can be viewed as a resolution of holomorphic functions. When these complexes arise from an action functional (as with most field theories), these cohomology groups encode the solutions to the Euler-Lagrange equations. We might say that these equations tells us what the theory *means*. Often these complexes live in families — e.g., by varying through spaces of connections or complex structures — and it is an interesting question to ask how the theories vary in these families.

EXAMPLE 5.2.3 (The scalar field). Let M be a smooth, compact manifold with Riemannian metric g. Denote by Dens the density line bundle over M, by dvol the canonical Riemannian volume form on M, and by Δ_g the Laplace-Beltrami operator. The complex

$$\begin{array}{ccc} C^\infty_M & \xrightarrow{\mathrm{dvol}\cdot\Delta_g} & \mathrm{Dens}_M \\ \phi & \mapsto & D\phi := \left(\Delta_g \phi\right)\mathrm{dvol} \end{array}$$

concentrated in degrees 0 and 1 is $(\mathscr{E}, D)$. The action functional is

$$S(\phi) := \int_M \phi D\phi.$$

The classical solutions are harmonic functions on M in degree 0 and densities modulo Laplacians of densities in degree 1.

EXAMPLE 5.2.4 (The $\beta\gamma$ system). Let M be a Riemann surface. Let $\mathscr{E} = \Omega^{0,*} \oplus \Omega^{1,*}$ with the total complex concentrated in degrees 0 and 1. Denote an element of $\Omega^{0,*}$ by γ and an element of $\Omega^{1,*}$ by β. The pairing is just "wedge and integrate":

$$\langle \gamma_0 + \beta_0, \gamma_1 + \beta_1 \rangle = \int_M \gamma_0 \wedge \beta_1 + \beta_0 \wedge \gamma_1,$$

where the integral denotes integration of the $\mathrm{d}z\,\mathrm{d}\bar{z}$ term. The integral simply vanishes, by definition, on a differential form that is not in $\Omega^{1,1}$. The action is then

$$S(\gamma, \beta) = \int_M \beta \wedge \bar{\partial}\, \gamma = \frac{1}{2}\langle \gamma + \beta, \bar{\partial}(\gamma + \beta) \rangle.$$

The classical solutions are the holomorphic functions and holomorphic 1-forms on M in degree 0. If M is closed, we obtain the higher cohomology of $\mathscr{O}$ and Ω^1_{hol} in degree 1. We obviously obtain a moduli of theories by running over variations of complex structure.

EXAMPLE 5.2.5 (The *bc* system). Let M be a Riemann surface. Let $\mathscr{E} = \Pi\Omega^{0,*} \oplus \Pi\Omega^{1,*}$, the pure odd complex, and denote an element of $\Omega^{0,*}$ by c and an element of $\Omega^{1,*}$ by b. The pairing is just "wedge and integrate":

$$\langle c_0 + b_0, c_1 + b_1 \rangle = \int_M c_0 \wedge b_1 + b_0 \wedge c_1.$$

Note that we use the Koszul rule of signs here in conjunction with the usual sign conventions for differential forms. Thus, following the convention in [DEF+99], if $c \in \Pi\Omega^{0,0}$ and $b \in \Pi\Omega^{1,0}$,

$$c \wedge b = (-1)^{0 \cdot 1}(-1)^{1 \cdot 1} b \wedge c = -b \wedge c,$$

where the first sign is the differential form sign and the second sign is the super-sign. The action is then

$$S(c, b) = \int_M b \wedge \bar{\partial} c = \langle b + c, \bar{\partial}(b + c) \rangle.$$

The classical solutions are again the holomorphic functions and holomorphic 1-forms on M. We obviously obtain a moduli of theories by running over variations of complex structure.

EXAMPLE 5.2.6 (The chiral free fermion *aka* an instance of abelian holomorphic Chern-Simons). Let M be a Riemann surface for which there exists a nowhere-vanishing holomorphic volume form dvol, which we fix. For example, consider M an elliptic curve $\mathbb{C}/\Lambda$ and let $\mathrm{dvol} = \mathrm{d}z$. Then $\mathscr{E} = \Pi\Omega^{0,*}$, the pure odd complex, is a free BV theory where the symplectic pairing is given by

$$\langle -, - \rangle : \Omega_c^{0,*} \otimes \Omega_c^{0,*} \to \mathbb{C},$$

$$\langle \phi, \psi \rangle = \int_M \phi \wedge \psi \, \mathrm{dvol}\,.$$

The action is

$$S(\psi) = \langle \psi, \bar{\partial} \psi \rangle = \int_M \psi \wedge \bar{\partial} \psi \, \mathrm{dvol}\,.$$

We obviously obtain a moduli of theories by running over variations of complex structure and holomorphic volume form.

REMARK 5.2.7. One way to spice up these examples is to tensor with a vector space with a pairing or, even better, twist with a vector bundle. For instance, if V denotes a holomorphic vector bundle and $V^\vee$ its dual holomorphic bundle on M, then we can twist the $\beta\gamma$ system by letting $\mathscr{E} = \Omega^{0,*}(V) \oplus \Omega^{1,*}(V^\vee)$ and we include the evaluation pairing between V and $V^\vee$ as part of the pairing $\langle -, - \rangle$.

EXAMPLE 5.2.8 (Abelian Chern-Simons). Let M be an oriented smooth 3-manifold. Let $\mathscr{E} = \Omega^*[1]$ be a shifted copy of the usual de Rham complex, with the pairing "wedge and integrate."

5.2.2. A useful lemma. In working with elliptic complexes, one often switches between smooth and distributional sections. We will prove here a useful lemma — a variant of a result of Atiyah and Bott [**AB67**] — that pins down the relationship between the smooth and distributional complexes. It will play a crucial role in our construction of the observables of a free field theory. To prove this lemma, we will need to introduce some basic machinery from the theory of elliptic complexes for which [**AB67**] is a good reference.

In [**AB67**], Atiyah and Bott show that for an elliptic complex $(\mathscr{E}, Q)$ on a compact, closed manifold M, with $\mathscr{E}$ the smooth sections of a $\mathbb{Z}$-graded vector bundle, the inclusion

$$(\mathscr{E}, Q) \hookrightarrow (\overline{\mathscr{E}}, Q)$$

into the elliptic complex of distributional sections is a homotopy equivalence. The argument follows from the existence of *parametrices* for elliptic operators. We need a simple variant of this result on open sets of a manifold.

5.2.2.1. *Parametrices.* As we are working with smooth or distributional sections, we have the Schwartz kernel theorem, which allows us to pass freely between continuous linear operators $F \in \mathrm{Hom}(\mathscr{E}, \mathscr{F})$ and their (integral) kernels $K_F \in \overline{\mathscr{E}^!} \otimes \mathscr{F}$. Among all continuous linear operators, elliptic operators have a special property: they admit "inverses up to smoothing operators," and we call such an "inverse" a *parametrix* for the elliptic operator. We now make these statements precise.

DEFINITION 5.2.9. An operator $S : \mathscr{E} \to \mathscr{F}$ is *smoothing* if its kernel K_S is a smooth, i.e., a smooth section of the vector bundle $E^! \boxtimes F$ on $M \times M$, where $E^! = E^\vee \otimes \mathrm{Dens}$ is the fiberwise-dual vector bundle to E twisted with the density (or orientation) line bundle.

We want to focus on kernels whose support is controlled and small in the appropriate sense, so we introduce the following technical definition.

DEFINITION 5.2.10. A subset $X \subset M^n$ is *proper* if the projection maps $\pi_j : X \subset M^n \to M$ are proper for all $j = 1, \ldots, n$. A function (or section, etc) over M^n has *proper support* if its support is proper.

DEFINITION 5.2.11. A *parametrix* for the elliptic complex $(\mathscr{E}, Q)$ on M is a continuous linear operator $P : \mathscr{E} \to \mathscr{E}$ of cohomological degree -1 such that $[Q, P] = 1_{\mathscr{E}} + S$, where S is a smoothing operator whose kernel has proper support.

PROPOSITION 5.2.12. *For M compact, every elliptic complex $(\mathscr{E}, Q)$ has a parametrix P. Moreover, there exists a* pseudodifferential *parametrix.*

This is proposition (6.1) in [**AB67**]. The adjective "pseudodifferential" implies that the kernel of P is smooth away from the diagonal in $M \times M$.

5.2.2.2. *The Atiyah-Bott result generalized.*

LEMMA 5.2.13. *Let $E \to M$ be a $\mathbb{Z}$-graded vector bundle on a closed smooth manifold M. Let $\mathscr{E}$ denote the sheaf of smooth sections and let $\overline{\mathscr{E}}$ denote the sheaf of noncompactly supported distributional sections. Let Q be a differential operator on $\mathscr{E}$ of cohomological degree 1 such that $(\mathscr{E}, Q)$ is an elliptic complex on M. Then on any open set $U \subset M$, there is a homotopy equivalence*

$$(\mathscr{E}|_U, Q) \hookrightarrow (\overline{\mathscr{E}}|_U, Q).$$

PROOF. Pick a parametrix P for $(\mathscr{E}(M), Q)$ and let S denote the associated smoothing operator. Let ϕ be a cut-off function on $M \times M$ such that ϕ is 1 in a small neighborhood of the diagonal and has proper support. In particular, we require ϕ to have proper support on $U \times U$. Consider the kernel $K_\Phi = \phi K_P$ and let Φ denote the associated operator.

Then Φ defines a parametrix for Q by the following computation:

$$\begin{aligned}
[Q, \Phi] &= Q \circ \Phi + \Phi \circ Q \\
&= R + \phi[Q, P]
\end{aligned}$$

where the term R arises because Q is a differential operator and, by the Leibniz rule, there will be a contribution involving its (possibly subtle) action on ϕ. The support of the kernel K_R is away from the diagonal but proper in $U \times U$ since $\phi \equiv 1$ in some neighborhood of the diagonal. Hence any derivative of ϕ will vanish in a neighborhood of the diagonal. We see that R is smoothing because P is a parametrix, and so K_P and its derivative will be smooth away from the diagonal. Continuing, we compute

$$\begin{aligned}
[Q, \Phi] &= R + \phi K_{1_{\mathscr{E}}} + \phi S \\
&= 1_{\mathscr{E}} + T
\end{aligned}$$

where $T = R + \phi S$ is a smoothing operator with proper support away from the diagonal.

Note that $\phi K_{1_{\mathscr{E}}} = K_{1_{\mathscr{E}}}$, since it is the delta function along the diagonal and ϕ is 1 in a neighborhood of the diagonal. Moreover, we have shown that the commutator has proper support in $U \times U$ since all the terms do.

The existence of this parametrix with proper support in $U \times U$ gives us a chain homotopy equivalence on the distributional complex $(\overline{\mathscr{E}}(U), Q)$ between the identity and T. As T is smoothing, however, the image of T is contained in the image of the inclusion $i : \mathscr{E}(U) \hookrightarrow \overline{\mathscr{E}}(U)$. Hence T defines an inverse to i, up to homotopy. $\square$

5.3. Observables as a factorization algebra

The language of factorization algebras clarifies the meaning of BV quantization: it is a version of deformation quantization for *field* theories. Our goal in this section is to formulate and prove a precise and simple incarnation of this idea in the context of free fields. To state this theorem, Theorem 5.3.5, we need to introduce some terminology.

5.3.1. The algebra of BV quantization. As discussed at the beginning of chapter 4, the observables of a quantum field theory should form a factorization algebra that assigns merely a vector space to each open set. We expect the quantum observables to be a deformation of the classical observables as a factorization algebra. Thus, on each open set, we are deforming the commutative dg algebra $\mathrm{Obs}^{cl}(U)$ to a dg vector space $\mathrm{Obs}^{q}(U)$. The BV formalism provides a very controlled way to construct such a deformation, and we now introduce the relevant mathematical structures. All the relevant structures are described in greater detail in [**CG**].

The basic picture is as follows. Recall from the setting of ordinary deformation quantization that if a family of associative algebras A^q over the formal disk $\mathrm{Spec}\,\mathbb{R}[[\hbar]]$ is commutative at the origin, then this commutative algebra actually has a Poisson bracket arising from the commutator of the associative algebra. This Poisson bracket "remembers" the direction of deformation from its commutative structure toward the associative structure. We might expect that a similar situation holds for a family of cochain complexes over $\mathbb{R}[[\hbar]]$ that restricts to a commutative dg algebra at the origin. Indeed, that commutative dg algebra will have a homotopy Poisson bracket of degree 1 that remembers how to deform. In the BV formalism, one typically works with a strict version of this Poisson structure.

DEFINITION 5.3.1. A *Pois*$_0$ *algebra* $(A, d, \{-,-\})$ is a commutative dg algebra (A, d) over $\mathbb{R}$ equipped with a Poisson bracket $\{-,-\}$ of cohomological degree 1. Explicitly, the bracket is a skew symmetric map $\{-,-\} : A \otimes A \to A$ of degree 1, closed with respect to the differential, and a biderivation.

We now want to characterize what it means to be in a one-dimensional family of cochain complexes that restricts to a commutative algebra. In the BV formalism, we work with the following strict structure.

DEFINITION 5.3.2. A *Beilinson-Drinfeld (BD) algebra* $(A, d, \{-,-\})$ is a commutative graded algebra A, flat as a module over $\mathbb{R}[[\hbar]]$, equipped with a degree 1 Poisson bracket such that

$$d(ab) = (da)b + (-1)^{|a|}a(db) + \hbar\{a, b\}. \tag{4}$$

Observe that given a BD algebra A^q, we can restrict to "$\hbar = 0$" by setting

$$A_{\hbar=0} := A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/(\hbar).$$

Note that the induced differential on A_0 is a derivation, so that A_0 is a Pois_0 algebra! Likewise, when we restrict to "$\hbar \neq 0$" by setting

$$A_{\hbar\neq 0} := A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)),$$

we obtain just a cochain complex.

DEFINITION 5.3.3. A *BV quantization* of a Pois_0 algebra A is a BD algebra A^q such that $A_{\hbar=0} = A$.

Our goal is to realize this idea in the setting of factorization algebras, so we need to say what it means to be a Pois_0 or BD factorization algebra. We take the following simple approach. Note that Pois_0 algebras (respectively, BD algebras) form a symmetric monoidal category in a straightforward way.

DEFINITION 5.3.4. A Pois_0 *factorization algebra* (respectively, BD factorization algebra) $\mathcal{F}$ is a prefactorization algebra taking values in the symmetric monoidal category of Pois_0 algebras (respectively, BD algebras) such that $\mathcal{F}$ is a factorization algebra when we forget down to the category of cochain complexes.

We can now state our main theorem.

THEOREM 5.3.5 (Central theorem of free field quantization). *A free BV theory* $(M, \mathscr{E}, Q, \langle -, - \rangle)$ *has a canonical* Pois_0 *factorization algebra of classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ *and a canonical BD factorization algebra of quantum observables* $\mathrm{Obs}^{\mathrm{q}}$.

We prove this theorem by constructing these factorization algebras in the subsections below.

5.3.2. The classical observables. The classical observables ought to assign $\mathscr{O}(\mathscr{E}(U))$ to each open set U. As $\mathscr{E}$ is a sheaf of dg nuclear spaces, it is natural to take an algebraist's approach and set

$$\mathscr{O}(\mathscr{E}(U)) := \mathrm{Sym}(\mathscr{E}(U)^\vee),$$

where $\mathscr{E}(U)^\vee$ denotes the compactly supported distributions dual to the smooth sections $\mathscr{E}(U)$ and Sym means the symmetric algebra constructed in $dgNuc$. The differential Q naturally induces a differential on $\mathscr{E}(U)^\vee$, which we abusively denote Q as well. Continuing the abuse, we extend this differential to a derivation on $\mathscr{O}(\mathscr{E}(U))$ that we denote Q. Hence $\mathscr{O}(\mathscr{E})$, as given, defines a cosheaf of dg commutative algebras.

We would like the classical observables to have a Pois_0 structure, though, and here we run into a problem. Because $\mathscr{E}$ is infinite-dimensional, the pairing $\langle -, - \rangle$ on $\mathscr{E}$ does not induce the desired Poisson bracket on $\mathscr{O}(\mathscr{E})$. There is thankfully a simple, natural fix known as "smearing observables."[1]

Let $E^!$ denote the vector bundle $E^\vee \otimes \mathrm{Dens}_M$ on M. Then distributional sections $\overline{\mathscr{E}^!}$ of this bundle are precisely the distributions dual to smooth sections of E. The differential Q on $\mathscr{E}$ naturally induces a differential Q on $\mathscr{E}^!$ and makes it into an elliptic complex.

[1]For interacting theories, this fix does not work, which leads to many of the complications in [CG].

DEFINITION 5.3.6. The *classical observables* of the free theory $(M, \mathscr{E}, Q, \langle -,-\rangle)$ are the commutative factorization algebra

$$\mathrm{Obs}^{\mathrm{cl}} : U \mapsto (\mathrm{Sym}\, \mathscr{E}_c^!(U), Q).$$

Note that this is a factorization algebra by theorem 4.5.1.

LEMMA 5.3.7. *The inclusion* $\mathrm{Obs}^{\mathrm{cl}} \hookrightarrow \mathscr{O}(\mathscr{E})$ *is an opens-wise continuous homotopy equivalence of commutative factorization algebras. In particular, it is a quasi-isomorphism.*

PROOF. By lemma 5.2.13, we know that on each open U, we have a continuous homotopy equivalence

$$\mathscr{E}_c^!(U) \hookrightarrow \overline{\mathscr{E}^!}_c(U) = \mathscr{E}(U)^\vee.$$

Applying the functor Sym, we obtain a continuous homotopy equivalence

$$\mathrm{Obs}^{\mathrm{cl}}(U) = \mathrm{Sym}\, \mathscr{E}_c^!(U) \hookrightarrow \mathrm{Sym}\, \overline{\mathscr{E}^!}_c(U) = \mathscr{O}(\mathscr{E}(U)).$$

Thus the map of factorization algebras is a quasi-isomorphism but also an opens-wise continuous homotopy equivalence. $\square$

There is a natural degree 1 Poisson bracket defined on each open set U as follows. Recall that our free theory has a pairing $\langle -,-\rangle_{loc} : E \otimes E \to \mathrm{Dens}_M$ on vector bundles. As it is fiberwise nondegenerate, we use it to define a pairing

$$\langle -,-\rangle^!_{loc} : E^! \otimes E^! \to \mathrm{Dens}_M$$

that is also fiberwise nondegenerate and skew-symmetric. It has cohomological degree -1. We thus obtain a skew-symmetric pairing

$$\{-,-\} : \mathscr{E}_c^! \otimes \mathscr{E}_c^! \to \mathbb{C} \quad \text{where} \quad \lambda \otimes \mu \mapsto \int_{x \in M} \langle \lambda(x), \mu(x)\rangle^!_{loc}.$$

Extend this pairing as a biderivation to $\mathrm{Obs}^{\mathrm{cl}}$ by using the Leibniz rule. We have shown the following.

PROPOSITION 5.3.8. *The classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ *have a canonical* Pois_0 *algebra structure.*

5.3.3. The quantum observables. Our model, as we construct the quantum observables, is the BV quantization functor from section 2.4. The Pois_0 structure on $\mathrm{Obs}^{\mathrm{cl}}$ induces a canonical BV Laplacian Δ and we use it to deform the differential on $\mathrm{Obs}^{\mathrm{cl}}$. Explicitly, on each open U, we define a second-order differential operator Δ of cohomological degree 1 by setting $\Delta \equiv 0$ on $\mathrm{Sym}^{\leq 1} \mathscr{E}_c^!(U)$ and $\Delta(xy) = \{x, y\}$ for $x, y \in \mathscr{E}_c^!(U)$. We extend to higher symmetric powers via the relation

$$\Delta(xy) = (\Delta x)y + (-1)^{|x|} x(\Delta y) + \{x, y\}.$$

The compatibility of the Poisson brackets with the structure maps implies that Δ commutes with the structure maps.

DEFINITION 5.3.9. The *quantum observables* of the free theory $(M, \mathscr{E}, Q, \langle -,- \rangle)$ are the prefactorization algebra

$$\mathrm{Obs}^{\mathrm{q}} : U \mapsto (\mathrm{Sym}(\mathscr{E}_c^!(U))[\hbar], Q + \hbar\Delta).$$

THEOREM 5.3.10. *The prefactorization algebra* $\mathrm{Obs}^{\mathrm{q}}$ *satisfies the locality axiom and is hence a factorization algebra.*

PROOF. Consider the filtration of the prefactorization algebra

$$F^k \mathrm{Obs}^{\mathrm{q}} = \mathrm{Sym}^{\leq k}(\mathscr{E}_c^!)[\hbar].$$

The differential $Q + \hbar\Delta$ preserves this filtration since Q preserves each symmetric power and Δ sends F^k to F^{k-2} by construction. For any open U and factorizing cover $\mathfrak{U}$, we have a map

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^{\mathrm{q}}) \to \mathrm{Obs}^{\mathrm{q}}(U),$$

thanks to the structure maps of the prefactorization algebra. Applying the filtration, we get a map of spectral sequences. The map on the first page is

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^{\mathrm{cl}} \otimes \mathbb{C}[\hbar]) \to \mathrm{Obs}^{\mathrm{cl}}(U) \otimes \mathbb{C}[\hbar],$$

which is a quasi-isomorphism, as $\mathrm{Obs}^{\mathrm{cl}}$ is a factorization algebra. Thus the map on the original complexes is a quasi-isomorphism. □

Note that $\mathrm{Obs}^{\mathrm{q}}$ is almost a factorization algebra in BD algebras: we work with polynomials in $\hbar$ rather than power series, so that the natural completion of $\mathrm{Obs}^{\mathrm{q}}$ is a BD algebra. Working with polynomials has the appealing aspect that we can "evaluate" $\hbar$ at finite, nonzero values. When we "set $\hbar = 0$," we still recover $\mathrm{Obs}^{\mathrm{cl}}$ and hence have a quantization of $\mathrm{Obs}^{\mathrm{cl}}$.

5.4. BV quantization as a Heisenberg Lie algebra construction

In the study of quantum mechanics on $\mathbb{R}^n$, particularly with quadratic Hamiltonians, the Heisenberg Lie algebra and (quotients of) its universal enveloping algebra play a central role. There is a partial generalization of this story to *all* free theories, as we now explain.

Observe that the classical observables are defined as the symmetric algebra of the 1-symplectic vector space given by the *linear* observables $\mathscr{E}_c^!$. We would like to realize the quantum observables as the Chevalley-Eilenberg complex for the Lie algebra *homology* of a Heisenberg Lie algebra constructed from the linear observables.

5.4.1. The algebraic construction. Recall that for any symplectic vector space $(V, \langle -,- \rangle)$, the *Heisenberg Lie algebra* $\mathcal{H}_V$ is the central extension

$$0 \to \mathbb{C} \cdot \hbar \to \mathcal{H}_V \to V \to 0$$

with Lie bracket

$$[v, v'] := \langle v, v' \rangle \hbar,$$

where $\hbar$ denotes the central element we have adjoined. This definition clearly works for dg vector spaces with symplectic pairing, as well.

DEFINITION 5.4.1. A 1-*Poisson vector space* is a dg vector space (V, d) equipped with a pairing

$$\{-,-\} : V \otimes V \to \mathbb{C}$$

of cohomological degree 1 such that

- (skew-symmetry) $\{x, y\} = -(-1)^{|x||y|}\{y, x\}$,
- (compatibility with d) $\{dx, y\} = (-1)^{|x|}\{x, dy\}$.

We say it is 1-*symplectic* if $\{x, -\}$ is a nonzero linear functional for every nonzero $x \in V$.

DEFINITION 5.4.2. A *Poisson map* $\phi : V \to W$ of 1-Poisson vector spaces is a cochain map such $\{\phi(x), \phi(y)\}_W = \{x, y\}_V$ for all $x, y \in V$. We denote the category of 1-Poisson vector spaces by 1-$PoisVect$.

We can apply the Heisenberg construction to such vector spaces as well.

DEFINITION 5.4.3. The *Heisenberg Lie algebra* $\mathcal{H}_V$ of a 1-Poisson vector space (V, d) is the central extension

$$0 \to \mathbb{C} \cdot \hbar \to \mathcal{H}_V \to V[-1] \to 0$$

where $\hbar$ denotes the central element we have adjoined and it has cohomological degree 1. The Lie bracket is

$$[v, v'] := \{v, v'\}\hbar,$$

with $v, v' \in V$.

REMARK 5.4.4. The shift $V[-1]$ is justified by the proposition below, where we show how to obtain a BD algebra from the Heisenberg Lie algebra. Note that in the definition of the Lie bracket, we view v, v' as living in $V[-1]$ on the left hand side but in V on the right hand side.

PROPOSITION 5.4.5. *The Lie algebra homology complex* $(\mathrm{Sym}(\mathcal{H}_V[1]), d_{CE})$*, where* d_{CE} *includes both* d *on* V *and the action of* $\mathcal{H}_V$ *on the trivial module, is a BD algebra.*

PROOF. Forgetting the differential, the underlying graded module is simply a symmetric algebra. Notice that $\hbar$ now has degree 0. The differential satisfies, by construction,

$$\begin{aligned} d_{CE}(ab) &= (da)b + (-1)^{|a|}a(db) + [a, b] \\ &= (da)b + (-1)^{|a|}a(db) + \hbar\{a, b\} \end{aligned}$$

for any $a, b \in V$. We extend the pairing $\{-,-\}$ on V to a biderivation on Sym to obtain the requisite Poisson bracket. $\square$

REMARK 5.4.6. This proof points out an intriguing aspect of the Lie algebra homology complex. For any dg Lie algebra $(\mathfrak{g}, d)$, the commutative dg algebra $(\mathrm{Sym}(\mathfrak{g}[1]), d)$ is a Pois_0 algebra whose Poisson bracket is given by shifting down the Lie bracket of $\mathfrak{g}$ and extending it as a biderivation. Thus the Chevalley-Eilenberg complex $(\mathrm{Sym}(\mathfrak{g}[1]), d + d_{CE})$ can be viewed as a BD quantization! We never explicitly take advantage of this observation in the thesis, but it suggests possible directions of research.

5.4.2. The factorization algebra version. As usual, we now obtain a factorization algebra by applying this construction to a cosheaf, via the general construction in theorem 4.5.3. That is, on each open set, we apply the algebraic construction above. Let $(\mathcal{V}, d)$ be a 1-Poisson-structured cosheaf (i.e., a precosheaf of 1-Poisson vector spaces where the underlying precosheaf of dg vector spaces *is* a cosheaf). By theorem 4.5.3, we will obtain a factorization algebra $C_*\mathcal{H}_{\mathcal{V}}$ by composing the Lie algebra homology functor with the Heisenberg Lie algebra construction.

PROPOSITION 5.4.7. *There is a functor*

$$Heis : 1\text{-}Poisson\ structured\ cosheaves \to factorization\ algebras$$

given by $(\mathcal{V}, d) \mapsto C_*\mathcal{H}_{\mathcal{V}}$.

PROOF. The construction on objects is straightforward. We see that if $\mathcal{V}$ is a 1-Poisson-structured cosheaf, then $\mathcal{H}_{\mathcal{V}}$ is a Lie-structured cosheaf. Thus theorem 4.5.3 tells us $C_*\mathcal{H}_{\mathcal{V}}$ is a factorization algebra.

Likewise, given a morphism of cosheaves, we clearly obtain a morphism of the Chevalley-Eilenberg complexes, compatible with all the structure maps. $\square$

5.4.3. The "pushforwards commute" theorem. The following result is useful for studying the observables of a free field theory.

COROLLARY 5.4.8. *Given a continuous map $f : M \to N$, we have $f_*(Heis(V)) \cong Heis(f_*(V))$ for every 1-Poisson structured cosheaf V on M.*

PROOF. Simply apply proposition 5.4.8. $\square$

5.5. BV quantization as a determinant functor

Thanks to section 2.4 and chapter 3, we know that BV quantization of finite-dimensional linear systems provides a Pfaffian functor. This suggests an appealing interpretation of Obs^q for a free field $\mathscr{E}$: the factorization algebra Obs^q is a *local* Pfaffian of the elliptic complex $\mathscr{E}$. Of course, $\mathrm{Obs}^q(U)$ is typically an enormous cochain complex, just as $\mathscr{E}$ often has an infinite-dimensional space of solutions locally. Globally, however, we have an appealing result.

PROPOSITION 5.5.1. *For $\mathscr{E}$ a free field on a closed manifold M, there is a quasi-isomorphism*

$$\mathrm{Obs}^{\mathrm{q}}(M) \cong \mathcal{BVQ}(H^*\mathscr{E}(M)).$$

In particular, for $\mathscr{F}$ an arbitrary elliptic complex, we take the free cotangent quantization and obtain a quasi-isomorphism

$$\mathrm{Obs}^{\mathrm{q}}(M) \cong \mathcal{C}Q(H^*\mathscr{F}(M)) \cong \det(H^*\mathscr{F}(M))[d(\mathscr{F}(M)) + \chi(\mathscr{F}(M))],$$

where $\chi(\mathscr{F}(M))$ is the Euler characteristic of $\mathscr{F}(M)$ and

$$d(\mathscr{F}(M)) = -\sum_n (2n+1)(\dim_k H^{2n}(\mathscr{F}(M)) + \dim_k H^{2n+1}(\mathscr{F}(M))).$$

(This function of the Betti numbers of $\mathscr{F}(M)$ arose in chapter 3.)

This result is a straightforward corollary of our earlier work in chapters 2 and 3. We develop a relationship with index theorems and torsion of elliptic complexes in chapter 7.

5.6. Implications for interacting theories

Although our focus is on free fields, our results have some strong consequences for interacting theories. As this section is a detour from the trajectory of this chapter, we will take for granted the definitions and terminology of [Cos11] to keep the digression brief. (For an overview in this thesis, see section 7.4 in chapter 7.)

THEOREM 5.6.1. *Let $(E, \langle -,- \rangle, Q)$ be a free BV theory on a closed manifold M. Let $\{I[\Phi]\}$ be an effective interaction satisfying the quantum master equation, i.e., an* interacting *BV theory. For any choice of parametrix Φ, the global quantum observables*

$$\mathrm{Obs}^{\mathrm{q}}[\Phi] := (\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]], Q + \frac{1}{2}\{I[\Phi], -\}_\Phi + \hbar\Delta_\Phi)$$

have cohomology isomorphic to $\mathbb{C}[[\hbar]]$ concentrated in degree

$$d_Q := -\sum_n (2n+1)\dim_{\mathbb{C}} H^{2n+1}(\mathscr{E}, Q).$$

PROOF. As usual, we use a spectral sequence argument. Define the D-degree of an element $\alpha \in \hbar^m \mathrm{Sym}^n(\mathscr{E}^\vee)$ to be $2m+n$. Equip $\mathrm{Obs}^{\mathrm{q}}[\Phi]$ with the filtration

$$F^k_D := \{\alpha \;:\; D(\alpha) \geq k\} = \sum_{n=0}^{\lfloor k/2 \rfloor} \hbar^n \mathrm{Sym}^{\geq k-2n}(\mathscr{E}^\vee).$$

For the induced spectral sequence, the first page is

$$(\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]], Q + \hbar\Delta_\Phi),$$

which we know has cohomology $\mathbb{C}[[\hbar]]$ concentrated in degree d_Q by using the proof of proposition 2.4.11.[2] Thus the spectral sequence collapses. $\square$

This result, while simple to prove, is rather remarkable: it says that *on a closed manifold, global observables "take values in* $\mathbb{C}[[\hbar]]$." No matter how complicated the observable that we cook up — perhaps we pick a 45-point function — when we look at its image in global observables, we get an element of a $\mathbb{C}[[\hbar]]$-line, which we can justifiably interpret as the "expectation value."

5.7. Theories with a Poincaré lemma

Just as an elliptic complex may be easy to understand locally but record interesting global information (e.g., the de Rham complex), some theories yield factorization algebras that are simple locally but interesting globally. We introduce here some definitions and theorems useful for understanding such theories.

DEFINITION 5.7.1. A free field has a *Poincaré lemma* if there is a decomposition of $\mathscr{E}$ into a direct sum of elliptic complexes $\oplus_j(\mathscr{E}_{(j)}, Q_{(j)})$ and a factorizing basis such that for any open U in the basis, each complex $(\mathscr{E}_{(j)}(U), Q_{(j)})$ is acyclic above its lowest degree.

We call an open *permissible* if it is in this factorizing basis. We say the Poincaré lemma *holds* on a permissible open.

EXAMPLE 5.7.2. Let $\mathscr{E} = T^*[-1](\Omega^*_M)$ be the shifted cotangent bundle of the de Rham complex on M. Then the usual Poincaré lemma implies that $\mathscr{E}$ has a Poincaré lemma. Here $\mathscr{E}$ decomposes into a direct sum $\Omega^* \oplus \Omega^*_{or}[\dim M - 1]$, where Ω^*_{or} denotes the de Rham complex twisted by the orientation local system. The permissible opens are contractible opens.

REMARK 5.7.3. We make the caveat about *some* class of contractible opens based on the example of the Dolbeault complex. The theory of several complex variables tells us the U needs to be pseudoconvex for $\mathscr{O}(U)$ to be the cohomology of $(\Omega^{0,*}(U), \bar{\partial})$. A good class of permissible opens is given by open Stein submanifolds.

Having a Poincaré lemma leads to an appealing simplification of the observables. To understand this simplification, however, requires some basic functional analysis.

Recall the following statement, which is a consequence of the Hahn-Banach theorem for locally convex topological vector spaces (see theorem 3.6 of [**Rud91**]).

LEMMA 5.7.4. *If λ is a continuous linear functional on a subspace W of a locally convex space V, then there exists a continuous linear functional Λ on V such that $\Lambda\big|_W = \lambda$.*

[2] That proof relies merely on the fact that the cohomology of the linear observables, in this case $(\mathscr{E}^\vee, Q)$, is finite-dimensional.

For us, the crucial consequence of this lemma is the following fact (see [**Ser53**]).

PROPOSITION 5.7.5. *Let $E \xrightarrow{u} F \xrightarrow{v} G$ be an exact sequence (i.e.,* $\ker v = \operatorname{im} u$*) of continuous linear maps between Fréchet spaces. Then the sequence $G^\vee \xrightarrow{v^*} F^\vee \xrightarrow{u^*} E^\vee$ is also exact, where $(-)^\vee$ denotes the functor "take the continuous linear dual."*

PROOF. Given $\lambda \in F^\vee$ such that $u^*\lambda = 0$, we wish to find $\mu \in G^\vee$ such that $v^*\mu = \lambda$. Note that since λ vanishes on $u(E) \subset F$, it induces a linear functional $\tilde{\lambda}$ on $F/u(E)$. We want to obtain a linear functional $\tilde{\mu}$ on $v(F) \subset G$ such that $\tilde{\lambda} = \tilde{\mu} \circ v$. Hahn-Banach would then imply that we can extend $\tilde{\mu}$ to a linear functional μ on G and we would know $\lambda = v^*\mu$.

Observe that there is a natural continuous linear map $\tilde{v} : F/u(E) \to v(F)$. Although it is bijection by definition, we need to verify it is a *homeomorphism* so that we can construct a continuous linear inverse. As F is Fréchet and $u(E)$ is a closed subspace, the quotient $F/u(E)$ is also Fréchet. The open mapping theorem (see, e.g., theorem 2.11 of [**Rud91**]) implies that $\tilde{v}$ is open and hence a homeomorphism. Thus, $\tilde{v}$ has a continuous linear inverse, which allows us to construct $\tilde{\mu}$. $\square$

We now return to field theories. Let $(\mathscr{E}, Q)$ be a free field theory with a Poincaré lemma. On an open U for which the Poincaré lemma holds, there is a subspace $K_{(j)}(U) \hookrightarrow \mathscr{E}_{(j)}(U)$ such that

$$0 \to K_{(j)}(U) \overset{\iota}{\hookrightarrow} \mathscr{E}_{(j)}^{m_{(j)}}(U) \xrightarrow{Q} \cdots$$

is acyclic. Here $m_{(j)}$ denotes the lowest degree in which $\mathscr{E}_{(j)}(U)$ is nonzero. Note that all the spaces are Fréchet as the spaces in $\mathscr{E}_{(j)}(U)$ are smooth sections of a vector bundle and $K_{(j)}$ is a closed subspace since it is the kernel of Q. Hence proposition 5.7.5 implies that the dual sequence is acyclic as well:

$$\cdots \to \left(\mathscr{E}_{(j)}^{m_{(j)}(U)}\right)^\vee \xrightarrow{\iota^*} K_{(j)}(U)^\vee \to 0.$$

We thus obtain the following result.

LEMMA 5.7.6. *There is a map of dg commutative algebras*

$$i : \mathrm{Obs}^{\mathrm{cl}}(U) \to \mathrm{Sym}(\bigoplus_j K_{(j)}(U)^\vee)$$

for each permissible open U. The differential on the right hand term is zero.

PROOF. The only subtlety here is that $\mathrm{Obs}^{\mathrm{cl}}$ uses the compactly-supported smooth sections of $E^!$, not the compactly-supported distributional sections. But the Atiyah-Bott lemma 5.2.13 implies that the inclusion

$$((\mathscr{E}^!_{(j)})_c(U), Q) \hookrightarrow \left((\mathscr{E}(U)_{(j)})^\vee, Q\right) = (\overline{(\mathscr{E}^!_{(j)})_c}(U), Q)$$

is a homotopy equivalence. The composition of ι^* with this inclusion induces the desired map i of dg commutative algebras. $\square$

Observe that $\mathrm{Sym}(\bigoplus_j K_{(j)}^\vee))$ defines a factorization algebras on the factorizing basis of permissible opens. We denote the extension (see section 4.7) to a factorization algebra on M by $\mathcal{K}^{cl}$. We thus have the following.

LEMMA 5.7.7. *There is a quasi-isomorphism of factorization algebras $i : \mathrm{Obs}^{\mathrm{cl}} \to \mathcal{K}^{cl}$.*

We would like to have a similar result for the quantum observables.

DEFINITION 5.7.8. Let $\Delta_{\mathcal{K}}$ denote the BV Laplacian induced on $\mathcal{K}^{cl}(U)$ where U is a permissible open. We define

$$\mathcal{K}^q(U) := (\mathrm{Sym}(\bigoplus_j K_{(j)}(U)^\vee)[\hbar], \hbar\Delta_{\mathcal{K}}).$$

Then $\mathcal{K}^q$ is a factorization algebra on the factorizing basis of permissible opens. We denote the extension to a factorization algebra on M by $\mathcal{K}^q$ as well.

To see it is a factorization algebra, use the spectral sequence given by the filtration by powers of $\hbar$. The map i extends to a map of complexes $i^q : \mathrm{Obs}^{\mathrm{q}}(U) \to \mathcal{K}^q(U)$ on any permissible open U. This filtration then gives a map of spectral sequences between $\mathrm{Obs}^{\mathrm{q}}(U)$ and $\mathcal{K}^q(U)$. Because this map is an isomorphism on the first page, we see that $H^* \mathrm{Obs}^{\mathrm{q}}(U) = H^* \mathcal{K}^q(U)$.

LEMMA 5.7.9. *There is a quasi-isomorphism of factorization algebras $i^q : \mathrm{Obs}^{\mathrm{q}} \to \mathcal{K}^q$.*

CHAPTER 6

Free holomorphic field theories and vertex algebras

In this chapter, we study some examples of factorization algebras that live on Riemann surfaces. The central goal is to find ways to understand, as a human, the huge amount of data encoded by these objects. We will find that if we focus on the simplest structure maps — such as inclusions of disjoint disks into a big disk — we recover the data of a vertex algebra. More precisely, the vertex algebra appears at the level of cohomology, i.e., working with the cohomology prefactorization algebra $H^*\mathcal{F}$ of the factorization algebra $\mathcal{F}$. Because a factorization algebra is a manifestly geometric object — after all, it lives on a manifold — this construction of vertex algebras provides helpful, motivating pictures for the axioms. Conversely, the explicit, algebraic nature of vertex algebras makes computations of some structure maps much simpler.

We now outline the contents of this chapter. Sections 6.1 to 6.3 examine the free $\beta\gamma$ system, its factorization algebra of quantum observables, and the associated vertex algebra. Although this example is quite simple, it shows how one can start with an action functional and rigorously recover a vertex algebra. Moreover, we indicate how this construction relates the work of Costello ([**Cosa**]) to constructions of chiral differential operators ([**GMS00**], [**MSV99**], [**KV04**], [**Che**], among others). In section 6.4, we construct factorization algebras whose associated vertex algebras are known as the affine Kac-Moody vertex algebras. This construction does not involve the BV formalism but has instead a beautiful deformation-theoretic interpretation. Finally, our examples live on every Riemann surface and hence we discuss what it means to have a factorization algebra on the site of Riemann surfaces.

REMARK 6.0.10. We make some polemical remarks. The juxtaposition with vertex algebras vivifies two compelling aspects of the formalism of factorization algebras. First, it provides a rigorous relationship between vertex algebras and the pictures and language used by physicists when discussing conformal field theory. We are not merely extracting axioms so that the physicists' formal power series manipulations become mathematical; we are recovering all the formulas from manifestly geometric constructions, so that aspects like associativity come for free. More aggressively, I might say that factorization algebras provide a direct embodiment of the physical thinking. Second, the factorization algebras live on Riemann surfaces from the beginning. By contrast, it is a fair amount of work to connect vertex algebras with Riemann surfaces (with [**FBZ04**] as a key example) and the interpretations can be quite subtle. Our construction of a factorization algebra from a Kac-Moody Lie algebra in section 6.4 has a simple interpretation via moduli of bundles and recovers in a very clean way the affine Kac-Moody vertex algebra as the local behavior.

6.1. The $\beta\gamma$ system

This chapter focuses on one of the simplest holomorphic field theories and a few variants. We describe these BV theories in order of increasing complexity.

6.1.1. The massless $\beta\gamma$ system. Let $M = \mathbb{C}$, the complex line, and let $\mathscr{E} = \left(\Omega^{0,*}_M \oplus \Omega^{1,*}_M, \bar\partial\right)$ be the Dolbeault complex of functions and 1-forms as a sheaf on M.[1] Following the convention of physicists, we denote by γ an element of $\Omega^{0,*}$ and by β an element of $\Omega^{1,*}$. The pairing $\langle -, - \rangle$ is

$$\begin{array}{ccc} \mathscr{E}_c \otimes \mathscr{E}_c & \to & \mathbb{C}, \\ (\gamma_0 + \beta_0) \otimes (\gamma_1 + \beta_1) & \mapsto & \int_{\mathbb{C}} \gamma_0 \wedge \beta_1 + \beta_0 \wedge \gamma_1. \end{array}$$

Thus we have the data of a free BV theory. The action functional for the theory is

$$S(\gamma, \beta) = \langle \gamma + \beta, \bar\partial(\gamma + \beta) \rangle = 2 \int_M \beta \wedge \bar\partial \gamma$$

The Euler-Lagrange equation is simply $\bar\partial\gamma = 0 = \bar\partial\beta$. One should think of $\mathscr{E}$ as the "derived space of holomorphic functions and 1-forms on M." Note that this theory is well-defined on any Riemann surface, and one can study how it varies over the moduli space of curves.

REMARK 6.1.1. One can add d copies of $\mathscr{E}$ (equivalently, tensor $\mathscr{E}$ with $\mathbb{C}^d$) and let S_d be the d-fold sum of the action S on each copy. The Euler-Lagrange equations for S_d picks out "holomorphic maps γ from M to $\mathbb{C}^d$ and holomorphic sections β of $\Omega^1_M(\gamma^* T_{\mathbb{C}^d})$."

6.1.2. The massive $\beta\gamma$ system. We have the same basic input data except that the differential Q changes slightly. Fix a nonzero complex number m that we'll call "mass." The elliptic complex is $\mathscr{E} = \left(\Omega^{0,*}_M \oplus \Omega^{1,*}_M, \bar\partial - m\, d\bar z\right)$.[2] The Euler-Lagrange equation is then $\partial\gamma/\partial\bar z = m\gamma$ and $\partial\beta/\partial\bar z = m\beta$. Notice that on $M = \mathbb{C}$ there is a homotopy equivalence of complexes between the massless and massive $\beta\gamma$ systems:

$$\begin{tikzcd} \left(\Omega^{0,*}_M \oplus \Omega^{1,*}_M, \bar\partial\right) \arrow[r, bend left, "e^{m\bar z}\cdot"] & \left(\Omega^{0,*}_M \oplus \Omega^{1,*}_M, \bar\partial - m\, d\bar z\right) \arrow[l, bend left, "e^{-m\bar z}\cdot"] \end{tikzcd}.$$

This equivalence disappears on an elliptic curve because the constant functions provide global holomorphic functions (the massless case) and, by contrast, there are no global functions f such that $\partial f/\partial\bar z = mf$, at least for generic m. In particular, we see that although the factorization algebras for the massless and massive systems are isomorphic on small open sets, they differ globally.

[1] We simply recall example 5.2.4.

[2] We could instead modify the connection $\bar\partial$ by some holomorphic function f, but only the constant functions will extend to define theories on a compact Riemann surface, and so we restrict ourselves to simply adding a mass term.

6.1.3. **Abelian holomorphic Chern-Simons.** We now make a minor modification of this theory that drastically enlarges its scope. Fix $M = \mathbb{C}$. Now fix a dg manifold $X = (X_0, \mathscr{O}_X)$ and let $\mathfrak{g}$ be a sheaf of abelian dg Lie algebras on X.[3] Let $\mathfrak{g}^\vee$ denote the cochain complex that is the $\mathscr{O}_X$-linear dual to $\mathfrak{g}$.

There is a sheaf on X of free BV theories on M constructed as follows: to an open $U \subset X_0$ and an open $V \subset M$, we get the elliptic complex

$$\mathscr{E}(U, V) = \left(\Omega_M^{0,*}(V) \otimes \mathfrak{g}[1](U) \oplus \Omega_M^{1,*}(V) \otimes \mathfrak{g}^\vee[-1](U), \bar{\partial} + d_{\mathfrak{g}} \right),$$

where $d_{\mathfrak{g}}$ denotes the differential on the complex $\mathfrak{g}(U)$. This complex $\mathscr{E}_c(U, V)$ has a symplectic pairing

$$\langle \alpha, \beta \rangle := \int_V \langle \alpha(z), \beta(z) \rangle,$$

where we also include the evaluation pairing between $\mathfrak{g}$ and $\mathfrak{g}^\vee$. For a more thorough discussion of holomorphic Chern-Simons (including the nonabelian case), see [**Cosa**].

6.1.3.1. *Sigma models.* There is a particular choice of X and $\mathfrak{g}$ that is both nontrivial and accessible; it encodes a nonlinear sigma model as a case of holomorphic Chern–Simons. Our discussion here is a quick gloss of the formalism developed in [**Cosa**].

Let U be an open set in $\mathbb{C}^n$, equipped with the holomorphic coordinates $z_1, \ldots, z_n$. Let $X = (U, \Omega_U^*)$, the so-called "de Rham space of U." There is a sheaf on X that recovers the holomorphic structure of U, constructed as follows. Let $\mathscr{J}$ denote the sheaf of ∞-jets of holomorphic functions on U; this sheaf has a natural flat connection and hence a canonical D_U-module structure (in fact, a D_U algebra structure). This sheaf consists of the smooth sections of an infinite-rank vector bundle whose fiber at a point $p \in U$ is naturally identified with $\mathbb{C}[[z_1, \ldots, z_n]]$, as the jet of a function at a point p is simply the Taylor series at p once we've chosen local coordinates. In more formal language, there is a natural filtration on $\mathscr{J}$ by "order of vanishing;" at every point $p \in U$, the stalk $\mathscr{J}_p$ is a module over $C^\infty_{U,p}$, and we consider the filtration by powers of the maximal ideal $\mathfrak{m}_p$ of functions vanishing at p, namely $F^k \mathscr{J}_p := \mathfrak{m}_p^k \mathscr{J}_p$. Consider the quotient map

$$q : F^1 \mathscr{J} \to F^1 \mathscr{J} / F^2 \mathscr{J} \cong \left(T_U^{1,0} \right)^\vee,$$

which just records the "first derivative" component of a holomorphic function. Once we pick a splitting σ of the quotient map, we obtain a map of commutative algebras

$$\sigma : \widehat{\mathrm{Sym}}_{C^\infty_U} \left(T_U^{1,0} \right)^\vee \to \mathscr{J}$$

which is actually an isomorphism, as one can check locally. This isomorphism then equips the symmetric algebra with a flat connection.

[3] In other words, $\mathfrak{g}$ is just a sheaf of cochain complexes on X_0 that is $\mathscr{O}_X$-linear.

In our case, there is a natural splitting to use because we have global coordinates. Note that at each point p we can express a germ $f \in \mathscr{J}_p$ as a "formal power series"

$$\sum_{\alpha \in \mathbb{N}^n} f_\alpha \otimes z^\alpha \in C^\infty_{U,p} \otimes \mathbb{C}[[z_1, \ldots, z_n]],$$

where, for instance, the jet of a holomorphic function ϕ is given by its Taylor series at p:

$$\sum_\alpha \frac{1}{\alpha!} \left(\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \right) \cdots \left(\frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} \right) \phi(p) \otimes z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

What we have done is define σ by sending our frame of global holomorphic 1-forms $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ to the elements $1 \otimes z_1, \ldots, 1 \otimes z_n$ in $\mathscr{J}$. From hereon, we will always use this splitting whenever we work with this example.

We now construct the L_∞ algebra over X. Take the de Rham complex of $\mathscr{J}$, $dR(\mathscr{J}) := \Omega^*_U \otimes_D \mathscr{J}$. Note that it is a commutative dg Ω^*_U algebra. There is an isomorphic de Rham complex $dR\left(\widehat{\mathrm{Sym}}_{C^\infty_U}\left(T^{1,0}_U\right)^\vee\right)$ via the splitting σ, and we can interpret it as the Chevally-Eilenberg cochain complex of an L_∞ algebra over X, namely

$$\mathfrak{g} := \Omega^*_U \otimes T^{1,0}_U[-1]$$

with the L_∞ brackets induced by the flat connection borrowed from $\mathscr{J}$. In particular, this L_∞ algebra is abelian (albeit curved) because our use of global coordinates insures that the differential sends Sym^1 to $\mathrm{Sym}^{\leq 1}$.

As explained in [Cosa], the *classical* abelian holomorphic Chern-Simons theory on the Riemann surface M arising from $\mathfrak{g}$ encodes the holomorphic maps from M to U — to be accurate, the formal neighborhood of the constant maps inside the full mapping space. One goal of this chapter is to understand the quantized theory.

REMARK 6.1.2. Although we discussed the case of an open $U \subset \mathbb{C}^n$, we can apply this technique to more interesting targets. Suppose we have a target space Y that admits a cover by opens U_i such that each U_i is biholomorphic to an open in $\mathbb{C}^n$ and the patching maps are *affine* (i.e., on each overlap, we glue by a locally constant translation-and-linear transformation). As an example, any complex torus $\mathbb{C}^n/\Lambda$ provides such a Y. Then we can do descent from the abelian holomorphic Chern-Simons theory on the cover $\coprod_i U_i$ down to Y.

6.2. The quantum observables of the $\beta\gamma$ system

The factorization algebra Obs^q of quantum observables assigns to each open $U \subset \mathbb{C}$, the cochain complex

$$\mathrm{Obs}^q(U) := \left(\mathrm{Sym}\left(\Omega^{1,*}_c(U)[1] \oplus \Omega^{0,*}_c(U)[1]\right)[\hbar], Q + \hbar\Delta\right),$$

where $Q = \bar{\partial}$ for the massless system and $Q = \bar{\partial} - m\,\mathrm{d}\bar{z}$ for the massive system. By the isomorphism 6.1.2, it's enough to focus on the massless case for opens in $\mathbb{C}$. We now unpack what information Obs^q encodes by examining some simple open sets and the cohomology $H^* \mathrm{Obs}^q$ on those open sets. As usual, the meaning of a complex is easiest to garner through its cohomology.

REMARK 6.2.1. We want to recover the explicit formulas that appear in the literature on the $\beta\gamma$ system, particularly the vertex operation. *We thus fix a coordinate z on $\mathbb{C}$ but all the constructions are defined invariantly.* In particular, the structure maps of the factorization algebra are given in a coordinate-independent way (they arise from the structure maps of the Dolbeault complex as a sheaf). The coordinate simply gives us insight into how the structure maps work.

6.2.1. Analytic preliminaries.

We remind the reader of some facts from the theory of several complex variables (references for this material are [**GR65**], [**For91**], and [**Ser53**]). Note that we have already discussed the some of this material in section 5.7, notably the Hahn-Banach theorem (see lemma 5.7.4) and duality for short exact sequences (see proposition 5.7.5). We then use these facts to describe the cohomology of the observables.

PROPOSITION 6.2.2. *Every open set $U \subset \mathbb{C}$ is Stein* [**For91**]. *As the product of Stein manifolds is Stein, every product $U^n \subset \mathbb{C}^n$ is Stein.*

REMARK 6.2.3. Behnke and Stein [**BS49**] proved that every noncompact Riemann surface is Stein, so the arguments we develop here extend farther than we exploit them.

We need a particular instance of Cartan's theorem B about coherent analytic sheaves [**GR65**].

THEOREM 6.2.4 (Cartan's Theorem B). *For X a Stein manifold,*

$$H^k(\Omega^{p,*}(X), \bar{\partial}) = \begin{cases} 0, & k \neq 0 \\ \Omega^p_{hol}(X), & k = 0, \end{cases}$$

where $\Omega^p_{hol}(X)$ denotes the holomorphic p-forms on X.

We use a corollary first noted by Serre [**Ser53**]. Note that we use the Fréchet topology on Ω^p_{hol}, obtained as a closed subspace of $\Omega^{p,0}(X)$.

COROLLARY 6.2.5. *For X a Stein manifold of complex dimension n, the* compactly-supported *Dolbeault cohomology is*

$$H^k(\Omega^{p,*}_c(X), \bar{\partial}) = \begin{cases} 0, & k \neq n \\ (\Omega^{n-p}_{hol}(X))^\vee, & k = n, \end{cases}$$

where $(\Omega^{n-p}_{hol}(X))^\vee$ denotes the continuous linear dual to holomorphic $(n-p)$-forms on X.

PROOF. The Atiyah-Bott lemma (see lemma 5.2.13) shows that the inclusion

$$(\Omega^{p,*}_c(X), \bar{\partial}) \hookrightarrow (\overline{\Omega^{p,*}_c(X)}, \bar{\partial})$$

is a chain homotopy equivalence. (Recall that the bar denotes "distributional sections.") As $\overline{\Omega_c^{p,k}(X)}$ is the continuous linear dual of $\Omega^{n-p,k}(X)$, it suffices to prove the desired result for the continuous linear dual complex.

Consider the acyclic complex

$$0 \to \Omega_{hol}^{n-p}(X) \xhookrightarrow{i} \Omega^{n-p,0}(X) \xrightarrow{\bar{\partial}} \Omega^{n-p,1}(X) \to \cdots \to \Omega^{n-p,n}(X) \to 0.$$

For all $k \geq 0$ we see that

$$\bar{\partial}(\Omega^{n-p,k}) = \ker(\bar{\partial} : \Omega^{n-p,k+1} \to \Omega^{n-p,k+2})$$

so that their duals are also equal. This proves the vanishing claim. Now consider the sequence of maps

$$\Omega^{n-p,1}(X)^\vee \xrightarrow{\bar{\partial}} \Omega^{n-p,0}(X)^\vee \xrightarrow{i^*} \Omega_{hol}^{n-p}(X)^\vee.$$

By the Hahn-Banach theorem (see lemma 5.7.4), every continuous linear functional on a closed subspace $V \subset W$ can be extended to a continuous linear functional on W. Hence i^* is surjective. Now suppose $i^*(\omega) = 0$. Then ω descends to a continuous linear functional on $\Omega^{n-p,0}(X)/\Omega_{hol}^{n-p}(X)$ as $\Omega_{hol}^{n-p}(X)$ is a closed subspace. This quotient space is isomorphic to $\bar{\partial}(\Omega^{n-p,0}(C)) \subset \Omega^{n-p,1}(X)$, so by Hahn-Banach, we can extend ω to a continuous linear functional ω' on $\Omega^{n-p,1}(X)$. Thus $\bar{\partial}(\omega') = \omega$ by construction. Thus the sequence is *exact*. (This is an instantiation of proposition 5.7.5.) $\square$

LEMMA 6.2.6. *Let $U \subset \mathbb{C}$ be an open. The cohomology of the linear observables $H^*(\Omega_c^{1,*}(U)[1] \oplus \Omega_c^{0,*}(U)[1])$ is concentrated in degree 0 and is the continuous linear dual of the holomorphic functions and 1-forms $\mathscr{O}(U) \oplus \Omega_{hol}^1(U)$. Hence, the cohomology of the classical observables is likewise concentrated in degree 0 and consists of*

$$H^0\,\mathrm{Obs}^{\mathrm{cl}}(U) \cong \mathrm{Sym}\left(\mathscr{O}(U)^\vee \oplus \Omega_{hol}^1(U)^\vee\right).$$

PROOF. The result about linear observables follows from our results above. Now observe that the differential Q on the observables sends Sym^k to itself for all k. Thus, for instance, we need to show that

$$H^*(\mathrm{Sym}^k(\Omega_c^{1,*}(U)[1]), \bar{\partial}) = \mathrm{Sym}^k(\mathscr{O}(U)^\vee),$$

concentrated in degree 0. We see that $(\Omega_c^{1,*}(U)[1])^{\otimes k} \cong \Omega_c^{k,*}(U^k)[k]$, and by Serre's result, we see that

$$H^*((\Omega_c^{1,*}(U)[1])^{\otimes k}) \cong \mathscr{O}(U^k)^\vee \cong (\mathscr{O}(U)^\vee)^{\otimes k},$$

concentrated in degree 0. Taking into account the account of the symmetric group S_k, we obtain the desired statement about Sym^k. $\square$

In consequence, we find that we understand the cohomology of the *quantum* observables, as a graded vector space.

COROLLARY 6.2.7. *We have*

$$H^*(\mathrm{Obs}^q(U)) \cong \mathrm{Sym}\left(\mathscr{O}(U)^\vee \oplus \Omega^1_{hol}(U)^\vee\right) \otimes \mathbb{C}[\hbar],$$

as vector spaces, by the spectral sequence on Obs^q *arising from the filtration by powers of* $\hbar$.

6.2.2. Disks. Let $\mathbb{D}_R(x)$ denote the open disk $\{z \in \mathbb{C} : |z - x| < R\}$ (see figure (1)). For brevity's sake, we will simply denote this disk by $\mathbb{D}$ in this subsection, but will use all the decorations when multiple disks are in use. *Notice that the choice of coordinate has entered.*

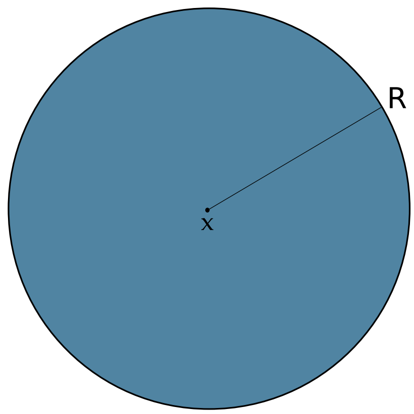


FIGURE 1. Disk centered at x

Consider first the *classical* observables $\mathrm{Obs}^{cl}(\mathbb{D})$, or rather its cohomology $H^* \mathrm{Obs}^{cl}(\mathbb{D})$. These are supposed to describe measurements one can make of holomorphic functions and 1-forms inside the disk $\mathbb{D}$. Since a holomorphic function ϕ is such a rigid object, any measurement can be expressed in terms of the power series

$$\phi(z) = \phi(x) + \partial_z\phi(x)(z - x) + \frac{1}{2}\partial_z^2\phi(x)(z - x)^2 + \cdots,$$

i.e., in terms of the value of ϕ at x and the value of all its holomorphic derivatives $\partial_z^n\phi$ at x. More precisely, any linear observable $\lambda \in H^* \mathrm{Obs}^{cl}(\mathbb{D})$ can be expressed as an infinite sum

$$\lambda(\phi) = \sum_n \lambda_n \left(\frac{1}{n!}\partial_z^n\phi\right)(x)$$

where the series must be convergent for every holomorphic function ϕ on the disk $\mathbb{D}$. Hence any observable is some formal power series in such linear functionals. In particular, we see that all observables are generated by the delta function δ_x and its holomorphic derivatives $\partial_z^n\delta_x$.

By invoking lemma 6.2.6 and corollary 6.2.7, we have proofs of our intuitive assertions above.

LEMMA 6.2.8. *The cohomology of the linear observables* $H^*(\Omega_c^{1,*}(\mathbb{D})[1] \oplus \Omega_c^{0,*}(\mathbb{D})[1])$ *is concentrated in degree 0 and is the continuous linear dual of the holomorphic functions and 1-forms* $\mathscr{O}(\mathbb{D}) \oplus$

$\Omega^1_{hol}(\mathbb{D})$. *Hence, the cohomology of the classical observables is likewise concentrated in degree 0 and consists of*

$$H^* \operatorname{Obs}^{cl}(\mathbb{D}) \cong \operatorname{Sym}\left(\mathscr{O}(\mathbb{D})^\vee \oplus \Omega^1_{hol}(\mathbb{D})^\vee\right).$$

COROLLARY 6.2.9. *We have*

$$H^*(\operatorname{Obs}^q(\mathbb{D})) \cong \operatorname{Sym}\left(\mathscr{O}(\mathbb{D})^\vee \oplus \Omega^1_{hol}(\mathbb{D})^\vee\right) \otimes \mathbb{C}[\hbar],$$

as vector spaces, by the spectral sequence on Obs^q *arising from the filtration by powers of* $\hbar$.

6.2.3. Annuli. Let $\mathbb{A}_{r<R}(x)$ denote the open annulus $\{z \in \mathbb{C} : r < |z-x| < R\}$ (see figure (2) below). For brevity's sake, we will simply denote this disk by $\mathbb{A}$ in this subsection, but will use all the decorations when multiple disks are in use.

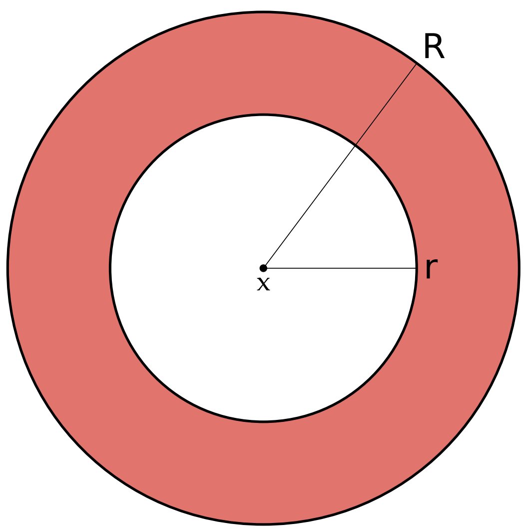


FIGURE 2. Annulus centered at x

Consider first the *classical* observables $\operatorname{Obs}^{cl}(\mathbb{A})$, or rather its cohomology $H^* \operatorname{Obs}^{cl}(\mathbb{A})$. These are supposed to describe measurements one can make of holomorphic functions and 1-forms inside the disk $\mathbb{A}$. Since a holomorphic function ϕ is such a rigid object, any measurement can be expressed in terms of its Laurent series

$$\phi(z) = \cdots + a_{-1}(z-x)^{-1} + a_0 + a_1(z-x) + a_2(z-x)^2 + \cdots.$$

More precisely, any linear observable $\lambda \in H^* \operatorname{Obs}^{cl}(\mathbb{A})$ can be expressed as an infinite sum

$$\lambda(\phi) = \sum_{n \geq 0} \lambda_n \left(\frac{1}{n!}\partial_z^n \phi\right)(x) + \sum_{n>0} \lambda_{-n}\left((z-x)^n \phi\right)(x)$$

where the series must be convergent for every holomorphic function ϕ on the annulus $\mathbb{A}$. Hence any observable is some formal Laurent series in such linear functionals. In particular, we see that all observables are generated by the delta function δ_x and its derivatives $\partial_z^n \delta_x$ and $\delta_x \circ ((z-x)^n \cdot -)$. (Multiplication by $(z-x)$ is essentially an inverse to the derivation ∂_z.)

Again, lemma 6.2.6 and corollary 6.2.7 allow us to understand the cohomology of the observables on an annulus.

6.2.4. The structure maps as a kind of multiplication. We want to interpret the structure maps of the factorization algebra as defining "multiplications of observables parametrized by opens." It is easiest if we consider the simplest structure maps. The basic idea is that the observables on nested annuli provide a "holomorphic associative algebra" and the inclusion of an annuli wrapping around a disk leads to a "module" for this algebra.

6.2.4.1. *The annuli as an associative algebra.* First, consider two nested annuli inside a larger annulus: $\mathbb{A}_1 := \mathbb{A}_{r_1<R_1}(0)$, $\mathbb{A}_2 := \mathbb{A}_{r_2<R_2}(0)$ and $\mathbb{A} := \mathbb{A}_{r<R}(0)$ where

$$0 < r \leq r_1 < R_1 \leq r_2 < R_2 \leq R.$$

Pictorially we have figure (3).

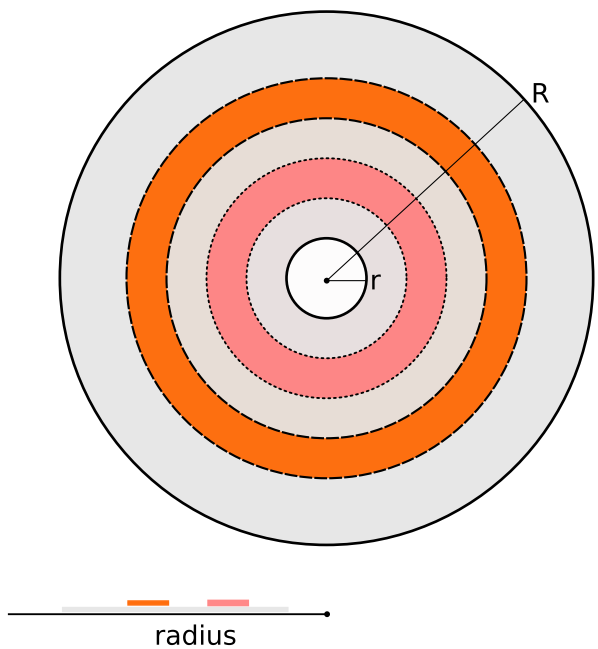


FIGURE 3. Nested annuli centered at the origin

We then have the structure map

$$m : \mathrm{Obs}(\mathbb{A}_2) \otimes \mathrm{Obs}(\mathbb{A}_1) \to \mathrm{Obs}(\mathbb{A})$$

and we denote the image of $\mathcal{O}_2 \otimes \mathcal{O}_1$ by $\mathcal{O}_2 \bullet \mathcal{O}_1$. We use Obs here to denote either $\mathrm{Obs}^{\mathrm{cl}}$ or $\mathrm{Obs}^{\mathrm{q}}$, since they both possess such a structure map. Moving outward radially corresponds to multiplying from right to left in this notation. We call this *radial ordering*.[4]

To understand the nature of this "multiplication," it's easiest to study the structure map at the level of cohomology and see what it does to the simplest observables.

DEFINITION 6.2.10. On any annulus $\mathbb{A}(x)$ centered at the point x, let $c_n(x)$ denote the linear functional

$$c_n(x) : \gamma \in \mathscr{O}(\mathbb{A}) \mapsto \begin{cases} (\partial_z^n \gamma)(x), & n \geq 0, \\ ((z-x)^{-n}\gamma)(x) & n < 0. \end{cases}$$

Likewise, let $b_n(x)$ denote the linear functional

$$b_n(x) : \beta \, dz \in \Omega^1_{hol}(\mathbb{A}) \mapsto \begin{cases} (\partial_z^n \beta)(x), & n \geq 0, \\ ((z-x)^{n}\beta)(x) & n < 0. \end{cases}$$

These observables simply read off the Laurent coefficients of holomorphic fields $\gamma \in \mathscr{O}(\mathbb{A}(x))$ or $\beta \in \Omega^1_{hol}(\mathbb{A}(x))$. We call them the *distinguished annular observables*.

We now describe how the structure map m from above behaves on these simple observables. The cohomology of the classical observables $H^*(\mathrm{Obs}^{\mathrm{cl}}(\mathbb{A}))$ is simply functions on the holomorphic fields, so we simply recover the product in a symmetric algebra. For example, $b_m(0) \bullet c_n(0) = b_m(0)c_n(0)$ where

$$b_m(0)c_n(0) : (\gamma, \beta dz) \mapsto (b_m(0)(\beta dz))(c_n(0)(\gamma)),$$

for $\gamma \in \mathscr{O}(\mathbb{A})$ and $\beta dz \in \Omega^1_{hol}(\mathbb{A})$. In other words, we apply $b_m(0)$ and $c_n(0)$ separately and then multiply their outputs.

On the quantum observables, however, we discover something more complicated. *For simplicity of notation, we make the origin* 0 *the center of every annulus in the arguments below. Hence we replace* $c_k(0)$ *by* c_k *and* $b_k(0)$ *by* b_k *as well.*

LEMMA 6.2.11. *Using the radial ordering to order "multiplication of observables," we find that*

$$c_m \bullet c_n - c_n \bullet c_m = 0 = b_m \bullet b_n - b_n \bullet b_m$$

and

$$c_m \bullet b_n - b_n \bullet c_m = \frac{\hbar}{8\pi} \delta_{m,-n-1},$$

where $\delta_{a,b}$ *denotes the Kronecker delta.*

[4] It helps to picture the ray parametrizing radius as pointing the left, laying along the negative reals so that the moving from left to right corresponds to decreasing radius. This possibly perverse-looking choice is motivated by the desire to be consistent with the usual vertex algebra literature.

Before embarking on the proof, we remark on a simple but important corollary. Radial ordering allows us to send any finite sequence, such as $c_{i_1}, c_{i_2}, \ldots, c_{i_n}$, to an element in $H^* \operatorname{Obs}^q(\mathbb{A})$. Explicitly we pick a sequence of nested, nonoverlapping annuli $\mathbb{A}_{i_1}, \ldots, \mathbb{A}_{i_n} \subset \mathbb{A}$ and we take the image of the element

$$\begin{array}{ccc} H^* \operatorname{Obs}^q(\mathbb{A}_{i_1}) \otimes \cdots \otimes H^* \operatorname{Obs}^q(\mathbb{A}_{i_n}) & \to & H^* \operatorname{Obs}^q(\mathbb{A}), \\ \cup & & \cup \\ c_{i_1} \otimes \cdots \otimes c_{i_n} & \mapsto & c_{i_1} \bullet \cdots \bullet c_{i_n}. \end{array}$$

Let T denote the tensor algebra over $\mathbb{C}[\hbar]$ generated by the elements $\{c_m, b_n\}_{m,n\in\mathbb{Z}}$ (simply as a vector space, with no topology). Then, by the procedure above, we get a linear map $T \to H^* \operatorname{Obs}^q(\mathbb{A})$. Let $\mathcal{A}$ denote the image of this map. Notice that, as a subspace of the annular observables, it is preserved by any inclusion of annuli.

COROLLARY 6.2.12. *These relations make the vector space $\mathcal{A}$ into a Weyl algebra over $\mathbb{C}[\hbar]$ with generators $\{c_m, b_n\}_{m,n\in\mathbb{Z}}$ and c_m conjugate to b_{-m-1}.*

PROOF. The proof of this lemma is along the lines of the proof that $H^* C_* \mathfrak{g}^{\mathbb{R}}$ recovers the universal enveloping algebra $U\mathfrak{g}$ (see proposition 4.6.1).

First, observe that there is a systematic way to lift the observables c_k and b_k to cocycles in $\operatorname{Obs}^q(\mathbb{A})$ for any annulus $\mathbb{A} = \mathbb{A}_{r<R}(0)$. Pick a bump function $\phi(\rho)$ of the radius $\rho = \sqrt{z\bar{z}}$ such that

(1) $\int_0^\infty \phi(\rho)\rho \, d\rho = 1$, and
(2) $\phi(\rho) \geq 0$ with the support of ϕ contained in (r, R).

Then define

$$\widetilde{c_k} := \phi(\rho) z^{-k} \frac{\mathrm{d}z \, \mathrm{d}\bar{z}}{4\pi i} = \phi(\rho)\rho^{-k} e^{-ik\theta} \cdot \rho \frac{d\theta \, d\rho}{2\pi}.$$

Observe that

$$\begin{aligned} \int_{\mathbb{A}} z^n \widetilde{c_k} &= \int_r^R \phi(\rho)\rho^{1+n-k} d\rho \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-k)\theta} d\theta \\ &= \delta_n^k. \end{aligned}$$

Hence, viewing $\widetilde{c_k}$ as a distribution (by the rule "multiply and integrate"), we see that its cohomology class $[\widetilde{c_k}]$ is precisely c_k in cohomology. We similarly use

$$\widetilde{b_k} := -\phi(\rho) z^{-k} \frac{\mathrm{d}\bar{z}}{4\pi i}$$

to obtain a linear functional on holomorphic 1-forms $\beta \, \mathrm{d}z$. (To be annoyingly pedantic, we chose this sign so that the integration $\int_{\mathbb{A}} \widetilde{b_k} \wedge \beta \, \mathrm{d}z$ has the correct sign.)

Suppose we have three annuli $\mathbb{A}_1 = \mathbb{A}_{r_1>R_1}(0)$, $\mathbb{A}_2 = \mathbb{A}_{r_2>R_2}(0)$, and $\mathbb{A}_3 = \mathbb{A}_{r_3>R_3}(0)$ contained in an annulus $\mathbb{A} := \mathbb{A}_{r>R}(0)$ where

$$r \geq r_1 > R_1 \geq r_2 > R_2 \geq r_3 > R_3 \geq R > 0.$$

We have two structure maps

$$m_{12} : \mathrm{Obs}^{\mathrm{q}}(\mathbb{A}_1) \otimes \mathrm{Obs}^{\mathrm{q}}(\mathbb{A}_2) \to \mathrm{Obs}^{\mathrm{q}}(\mathbb{A})$$

and

$$m_{23} : \mathrm{Obs}^{\mathrm{q}}(\mathbb{A}_2) \otimes \mathrm{Obs}^{\mathrm{q}}(\mathbb{A}_3) \to \mathrm{Obs}^{\mathrm{q}}(\mathbb{A}).$$

If we pick lifts of c_m for $\mathbb{A}_1$ and $\mathbb{A}_3$, denoted c_m^1 and c_m^3 respectively, and a lift b_n^2 for $\mathbb{A}_2$, we want to show that

$$[m_{12}(c_m^1 \otimes b_n^2) - m_{23}(b_n^2 \otimes c_n^3)] = \hbar\delta_{m,-n-1}$$

in the cohomology $H^* \mathrm{Obs}^{\mathrm{q}}(\mathbb{A})$. (Here $\delta_{a,b}$ is the Kronecker delta.)

To make the computations a bit easier, we use a minor modification of the lifts defined above.

Pick a bump function ϕ_j such that

(1) $\int_0^\infty \phi_j(\rho^2)\rho\, d\rho = 1$, and
(2) $\phi_j(\rho^2) \geq 0$ with the support of $\phi_j(\rho^2)$ contained in (r_j, R_j).

Define

$$c_m^j := \phi_j(\rho^2) z^{-m} \frac{\mathrm{d}z\, \mathrm{d}\bar{z}}{4\pi i}$$

for $j = 1$ or 3 and

$$b_n^2 := \phi_2(\rho^2) z^{-n} \frac{\mathrm{d}\bar{z}}{-4\pi i}.$$

These are lifts as well.

We want some $\alpha \in \mathrm{Obs}^{\mathrm{q}}(\mathbb{A})$ whose image under the differential $\bar{\partial} + \hbar\Delta$ is the difference

$$m_{12}(c_m^1 \otimes b_n^2) - m_{23}(b_n^2 \otimes c_n^3) - \hbar\delta_{m,-n-1},$$

so that we get the desired relation in cohomology.

Set

$$\Phi(\rho^2) := \int_0^{\rho^2} \phi_1(s) - \phi_3(s)\, ds.$$

Observe that

$$\bar{\partial}\left(-\Phi(\rho^2) z^{-1-m} \frac{\mathrm{d}z}{4\pi i}\right) = \frac{\partial\Phi}{\partial(\rho^2)}\, z^{-m} \frac{\mathrm{d}z\, \mathrm{d}\bar{z}}{4\pi i} = c_m^1 - c_m^3$$

(or, rather, their image in $\mathrm{Obs}^{\mathrm{q}}(\mathbb{A})$) because

$$\partial_{\bar{z}} = \frac{\partial(\rho^2)}{\partial\bar{z}} \frac{\partial}{\partial(\rho^2)} + \frac{\partial(\theta)}{\partial\bar{z}} \frac{\partial}{\partial\theta}$$

and

$$\frac{\partial\Phi}{\partial(\rho^2)} = \phi_1(\rho^2) - \phi_3(\rho^2).$$

Hence define

$$\alpha := -\Phi(z\bar{z}) z^{-1-m} \frac{\mathrm{d}z}{4\pi i} \cdot b_n^2.$$

We apply the differential $\bar{\partial} + \hbar\Delta$ for $\mathrm{Obs}^{\mathrm{q}}$:

$$\begin{aligned}
\bar{\partial}\,\alpha + \hbar\Delta\alpha &= (\partial_{\bar{z}}\Phi(z\bar{z}))\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{4\pi i}\cdot b_n^2 - \frac{\hbar}{4\pi i}\{\Phi(z\bar{z})z^{-1-m}\,\mathrm{d}z, b_n^2\} \\
&= z\frac{\partial\Phi}{\partial(\rho^2)}\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{4\pi i}\cdot b_n^2 - \frac{\hbar}{4\pi i}\int_{\mathbb{A}}\Phi(\rho^2)\phi_2(\rho^2)z^{-n-1-m}\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{-4\pi i} \\
&= m_{12}(c_m^1 \otimes b_n^2) - m_{23}(b_n^2 \otimes c_m^3) - \frac{\hbar}{8\pi}\delta_{m,-n-1}.
\end{aligned}$$

In the first line, we used the relation between the BV Laplacian Δ and the BV bracket. In the second line, we used the computation about Φ above and the definition of the BV bracket. In the third line we used the fact that $\Phi\big|_{\mathbb{A}_2} \equiv 1$.

In short,

$$c_m \bullet b_n - b_n \bullet c_m = \frac{\hbar}{8\pi}\delta_{m,-n-1}$$

in cohomology, where we use $\bullet$ to denote multiplication via radial ordering.

A similar argument shows that the brackets between c_m and c_n and between b_m and b_n vanish, because the BV Laplacian acts by zero on the analogous term α for those cases. □

Let $\rho : \mathbb{C} - \{0\} \to \mathbb{R}_{>0}$ denote the radial projection map $z \mapsto |z|$.

DEFINITION 6.2.13. Let $\rho_*\widetilde{H^*\,\mathrm{Obs}^{\mathrm{q}}}$ denote the prefactorization algebra in *vector spaces* that assigns

$$H^*\,\mathrm{Obs}^{\mathrm{q}}(\rho^{-1}(I))$$

to every open $I \subset \mathbb{R}_{>0}$. The structure maps are borrowed from the factorization algebra in *nuclear spaces* $\rho_* H^*\,\mathrm{Obs}^{\mathrm{q}}$ via the inclusions of the algebraic tensor product $\otimes_{alg}$ into the completed projective tensor product $\otimes$ of nuclear spaces. For example, for U, V disjoint opens inside W in $\mathbb{C}$, we have

$$\begin{tikzcd}
H^*\,\mathrm{Obs}^{\mathrm{q}}(U) \otimes_{alg} H^*\,\mathrm{Obs}^{\mathrm{q}}(V) \arrow[d, hook] \arrow[dr] & \\
H^*\,\mathrm{Obs}^{\mathrm{q}}(U) \otimes H^*\,\mathrm{Obs}^{\mathrm{q}}(V) \arrow[r] & H^*\,\mathrm{Obs}^{\mathrm{q}}(W).
\end{tikzcd}$$

This yields a structure map

$$\rho_*\widetilde{H^*\,\mathrm{Obs}^{\mathrm{q}}}(I) \otimes_{alg} \rho_*\widetilde{H^*\,\mathrm{Obs}^{\mathrm{q}}}(J) \to \rho_*\widetilde{H^*\,\mathrm{Obs}^{\mathrm{q}}}(K)$$

for every pair of disjoint opens I, J inside an open K in $\mathbb{R}_{>0}$.

To summarize our work above, we have shown the following. Let $\mathcal{F}_{\mathcal{A}}$ denote the locally constant prefactorization algebra on $\mathbb{R}> 0$ arising from the Weyl algebra $\mathcal{A}$.

THEOREM 6.2.14. *There is a map of prefactorization algebras on the positive reals* $\mathbb{R}_{>0}$

$$\iota : \mathcal{F}_{\mathcal{A}} \to \rho_*\widetilde{H^*\,\mathrm{Obs}^{\mathrm{q}}}.$$

As the inclusion ι on each open is a dense subspace (with respect to the topology on $\rho_* H^* \operatorname{Obs}^{\mathrm{q}}$), the structure maps of $\mathcal{F}_{\mathcal{A}}$ determine the structure maps of the factorization algebra $\rho_* H^* \operatorname{Obs}^{\mathrm{q}}$.

The locally constant factorization algebra $\mathcal{F}_{\mathcal{A}}$ corresponds to an honest associative algebra, but $\rho_* H^* \operatorname{Obs}^{\mathrm{q}}$ cares about the radial width of an annulus, as it provides observables on the holomorphic functions and 1-forms on that annulus. It thus provides a more sensitive tool for measuring such fields. It provides a kind of holomorphic "envelope" of $\mathcal{F}_{\mathcal{A}}$.

6.2.4.2. *The disk as a module.* We again begin by describing the simplest observables.

DEFINITION 6.2.15. On any disk $\mathbb{D}(x)$ centered at the point x, let $c_n(x)$ denote the linear functional

$$c_n(x) : \gamma \in \mathscr{O}(\mathbb{D}) \mapsto (\partial_z^n \gamma)(x).$$

Likewise, let $b_n(x)$ denote the linear functional

$$b_n(x) : \beta\, dz \in \Omega^1_{hol}(\mathbb{D}) \mapsto (\partial_z^n \beta)(x).$$

These observables simply read off the Taylor coefficients of holomorphic fields $\gamma \in \mathscr{O}(\mathbb{D}(x))$ or $\beta \in \Omega^1_{hol}(\mathbb{D}(x))$. We call them the *distinguished disk observables*.

Throughout this section, all disks and annuli will be centered at the origin, so we simplify notation and denote $c_m(0)$ by c_m and $b_m(0)$ by b_m.

Let T now denote the tensor algebra over $\mathbb{C}[\hbar]$ generated by $\{c_m, b_n\}_{m,n\in\mathbb{N}}$ (notice the change in index set from $\mathbb{Z}$ to $\mathbb{N}$). There is a linear map from T to $H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{D})$ given by radially ordering these generators in nested, nonoverlapping annuli that sit inside $\mathbb{D}$. Let $\mathcal{V}$ denote its image.

Our main result here is the following.

LEMMA 6.2.16. *The vector space $\mathcal{V}$ is the left module of the Weyl algebra $\mathcal{A}$,*

$$\mathcal{V} = \operatorname{Ind}_{\mathcal{A}^-}^{\mathcal{A}} \mathbb{C} = \mathcal{A} \otimes_{\mathcal{A}^-} \mathbb{C},$$

given by the induction of the trivial module $\mathbb{C}$ for the subalgebra

$$\mathcal{A}^- := \mathbb{C}[\ldots, c_{-2}, c_{-1}, \ldots, b_{-2}, b_{-1}][\hbar] \hookrightarrow \mathcal{A},$$

namely the subalgebra generated by the observables that measure the strictly polar parts of Laurent functions and 1-forms.

This module structure is a consequence of the structure map for a disk nested inside an annulus, both sitting inside a bigger disk. Let $\mathbb{D} = \mathbb{D}_r(0)$, $\mathbb{A} = \mathbb{A}_{r'<R'}$ and $\mathbb{D}_{big} = \mathbb{D}_R(0)$ where $0 < r \leq r' < R' \leq R$. (See figure (4).) We want to understand the structure map

$$m : \operatorname{Obs}(\mathbb{A}) \otimes \operatorname{Obs}(\mathbb{D}) \to \operatorname{Obs}(\mathbb{D}_{big})$$

for both the classical and quantum observables. As in the annulus case above, we use the fact that $\mathcal{V}$ is dense in $H^* \operatorname{Obs}(\mathbb{D})$ to get a feel for this structure map.

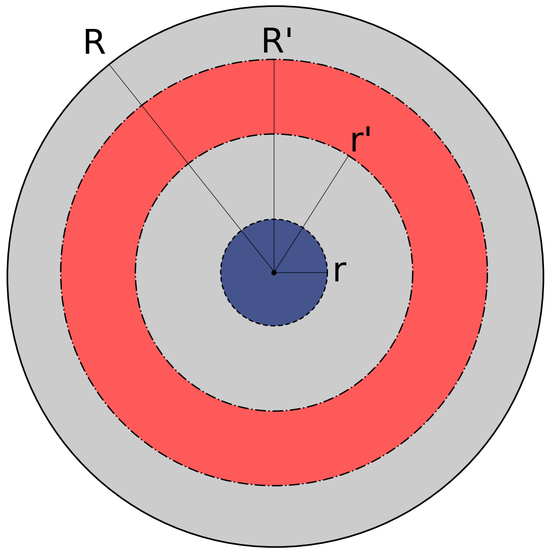


FIGURE 4. Disk and annulus nested inside larger disk

More concretely, we are saying that the observables c_k and b_k act by zero if $k < 0$. This should be plausible since when we apply an observable like c_{-1} to a holomorphic function γ, it returns zero because γ has no negative powers of z in its Laurent expansion: it's a power series, after all!

PROOF. As usual, we obtain the multiplication maps for $\mathcal{A}$, and the action of $\mathcal{A}$ on $\mathcal{V}$, by borrowing the structure maps from $H^* \operatorname{Obs}^q$. We work out those structure maps by picking lifts to Obs^q, applying its structure maps, and then taking cohomology.

First, consider the inclusion $\iota : \mathbb{A} \hookrightarrow \mathbb{D}_{big}$. The associated structure map

$$H^* \operatorname{Obs}^q(\mathbb{A}) \to H^* \operatorname{Obs}^q(\mathbb{D}_{big})$$

satisfies

$$c_k \mapsto \begin{cases} c_k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

and

$$b_k \mapsto \begin{cases} b_k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

by arguments modeled on the proof of lemma 6.2.11.

Pick a bump function ϕ such that

(1) $\int_0^\infty \phi(\rho^2)\rho \, d\rho = 1$, and
(2) $\phi(\rho^2) \geq 0$ with the support of $\phi(\rho^2)$ contained in (r', R').

Define

$$\widetilde{c_k} := \phi(\rho^2) z^{-k} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{4\pi i}.$$

Then the cohomology class of $\widetilde{c_k}$ in $H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{A})$ is c_k. Now define

$$\Phi(\rho^2) = \int_{\rho^2}^{\infty} \phi(s)\, ds.$$

Set $\alpha = \Phi(\rho^2) z^{-1-k}\, \mathrm{d}z/4\pi$ so that

$$\bar{\partial}\,\alpha = z\frac{\partial \Phi}{\partial(\rho^2)} \cdot z^{-1-k}\, \mathrm{d}z/4\pi = \widetilde{c_k}.$$

Note that this 1-form is well-defined only when $-1-k \geq 0$, so we see that $\widetilde{c_k}$ is a boundary in $\operatorname{Obs}^{\mathrm{q}}$ precisely when $k \leq -1$. The argument for the b_k's is identical.

Second, we can use this construction to compute the action of $\mathcal{A}$ on $\mathcal{V}$. We have a structure map

$$m : \operatorname{Obs}^{\mathrm{q}}(\mathbb{A}) \otimes \operatorname{Obs}^{\mathrm{q}}(\mathbb{D}) \to \operatorname{Obs}^{\mathrm{q}}(\mathbb{D}_{big})$$

and we ask for the cohomology class of $m(\widetilde{c_n} \otimes 1)$ for 1 the "constant term" in $\operatorname{Obs}^{\mathrm{q}}(\mathbb{D})$ (i.e., in Sym^0) and $\widetilde{c_n}$ the lift of c_n to $\operatorname{Obs}^{\mathrm{q}}(\mathbb{A})$ used above. We see that when $n < 0$, we have $(\bar{\partial} + \hbar\Delta)(\alpha \otimes 1) = \widetilde{c_n} \otimes 1$, using α as above. Hence $m(\widetilde{c_n} \otimes 1)$ vanishes in cohomology when $n < 0$. Again, an identical argument works for b_n in place of c_n with $n < 0$.

Just for clarity's sake, note that $m(\widetilde{c_m} \otimes \widetilde{b_n})$ yields $\hbar\delta_{m,-n-1}/8\pi$ for $m \geq 0$. The element b_n in $H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{D})$ can be obtained as the image of b_n in $H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{A}')$ for some annulus $\mathbb{A}' \subset \mathbb{D}$. Hence we use the commutation relation from lemma 6.2.11 to swap the product $c_m \cdot b_n$ with $b_n \cdot c_m + \hbar\delta_{m,-n-1}/8\pi$ as "products of annuli" $\mathbb{A}'$ and $\mathbb{A}$. The image of c_m from $H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{A}')$ into $H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{D})$ then vanishes, so only the term $b_n \cdot c_m$ vanishes. $\square$

6.2.4.3. *State-field correspondence: disks include into annuli.* We now study the structure map associated to the inclusion of a disk into an annulus. Let $\mathbb{A} = \mathbb{A}_{r<R}(0)$ and $\mathbb{D} = \mathbb{D}_s(x)$ where $r < |x| - s$ and $|x| + s < R$ so that $\mathbb{D} \subset \mathbb{A}$. See figure (5). The structure map

$$\operatorname{Obs}^{\mathrm{q}}(\mathbb{D}) \to \operatorname{Obs}^{\mathrm{q}}(\mathbb{A})$$

encodes the "state-field correspondence" of vertex algebras, a theme we revive in the next section.

LEMMA 6.2.17. *The structure map*

$$H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{D}) \to H^* \operatorname{Obs}^{\mathrm{q}}(\mathbb{A})$$

satisfies

$$c_k(x) \mapsto \frac{1}{k!} \sum_{n\in\mathbb{Z}} \left(\prod_{j=1}^{k} (n-j+1) \right) x^{n-k} c_n(0) = \sum_{n\in\mathbb{Z}} \binom{n}{k} x^{n-k} c_n(0)$$

and

$$b_k(x) \mapsto \frac{1}{k!} \sum_{n\in\mathbb{Z}} \left(\prod_{j=1}^{k} (n-j+1) \right) x^{n-k} b_n(0) = \sum_{n\in\mathbb{Z}} \binom{n}{k} x^{n-k} b_n(0),$$

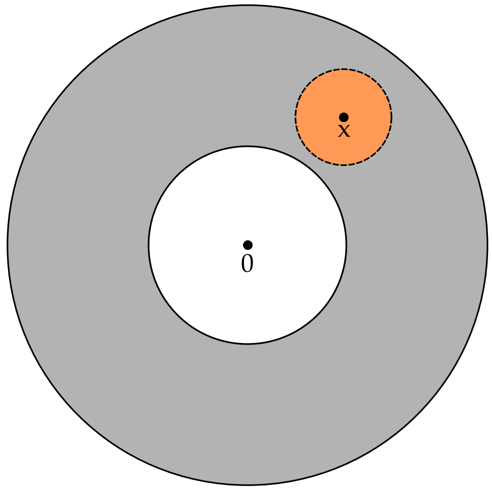


FIGURE 5. A disk inside an annulus

where $k \geq 0$.

REMARK 6.2.18. Note that for $k > n \geq 0$, we have $\binom{n}{k} = 0$.

PROOF. We do the case for the $c_k(x)$'s. Let γ be a holomorphic function on $\mathbb{A}$ and let

$$\gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^n$$

be its Laurent series. Then

$$c_k(x)(\gamma) = \frac{1}{k!}(\partial_z^k \gamma)(x) = \frac{1}{k!} \sum_{n \in \mathbb{Z}} \left(\prod_{j=1}^{k} (n - j + 1) \right) x^{n-k} \gamma_n,$$

and since $\gamma_n = c_n(0)(\gamma)$, we obtain the lemma. □

Combining this result with lemmas 6.2.11 and 6.2.16, we obtain a description of the structure maps for two disjoint disks including into a larger disk. Let $\mathbb{D}_1 = \mathbb{D}_r(0)$, $\mathbb{D}_2 = \mathbb{D}_s(x)$, and $\mathbb{D}_{big} = \mathbb{D}_R(0)$ where $r < |x| - s$ and $|x| + s < R$. See figure (6).

COROLLARY 6.2.19. *Under the structure map*

$$m : H^* \mathrm{Obs}^{\mathrm{q}}(\mathbb{D}_1) \otimes H^* \mathrm{Obs}^{\mathrm{q}}(\mathbb{D}_2) \to H^* \mathrm{Obs}^{\mathrm{q}}(\mathbb{D}_{big}),$$

we have

$$m(c_k(0) \otimes c_l(x)) = \sum_{n=l}^{\infty} \binom{n}{l} x^{n-l} c_k(0) c_n(0),$$

$$m(b_k(0) \otimes b_l(x)) = \sum_{n=l}^{\infty} \binom{n}{l} x^{n-l} b_k(0) b_n(0),$$

and

$$m(c_k(0) \otimes b_l(x)) = \binom{-k-1}{l} \frac{\hbar}{8\pi} \frac{1}{x^{k+1}} + \sum_{n=l}^{\infty} \binom{n}{l} x^{n-l} c_k(0) b_n(0),$$

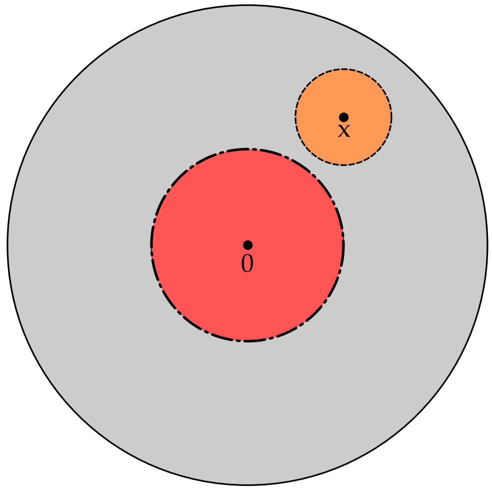


FIGURE 6. Two small disks inside larger disk

$$m(b_k(0) \otimes c_l(x)) = \binom{-k-1}{l} \frac{\hbar}{8\pi} \frac{1}{x^{k+1}} + \sum_{n=l}^{\infty} \binom{n}{l} x^{n-l} b_k(0) c_n(0).$$

REMARK 6.2.20. The products are examples of *operator product expansions*, where we express the product of two observables with support at distinct points in terms of a sum of observables supported at one of the points. Notice that each product has two types of terms: those with nonnegative powers of x — which agree with the product for the classical observables — and those with a negative power of x — the "quantum correction." These corrections are divergent as $x \to 0$ and constitute the interesting part of the "short distance behavior" of the observables. The convergent piece is usually called the "normally ordered product."

6.3. Recovering a vertex algebra

Our goal in this section is to demonstrate by example how the data of a vertex algebra is encoded by the factorization algebra of the $\beta\gamma$ system (massive or massless). We begin by reviewing the basics of vertex algebras in the first subsection, with the $\beta\gamma$ vertex algebra as a running example. In the second subsection, we explicate how the factorization algebra, at the level of cohomology, encodes the $\beta\gamma$ vertex algebra. As a result, it's natural to view the factorization algebra as a derived enrichment of the vertex algebra structure. One appealing feature about factorization algebras is that they are manifestly geometric — they live on Riemann surfaces from the very beginning — whereas it takes some work to recover geometric objects from vertex algebras. On the other hand, vertex algebras are a bit simpler to construct and to understand because they are algebraic in nature.

6.3.1. Review of vertex algebras. We recall the definition of a vertex algebra and various properties as given in [FBZ04].

DEFINITION 6.3.1. An element $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n}$ in $\operatorname{End} V[[z, z^{-z}]]$ a *field* if, for each $v \in V$, there is some N such that $a_j v = 0$ for all $j > N$.

DEFINITION 6.3.2 (Definition 1.3.1, [FBZ04]). A *vertex algebra* is the following data:

- a vector space V over $\mathbb{C}$ (the *state space*);
- a nonzero vector $|0\rangle \in V$ (the *vacuum vector*);
- a shift operator $T : V \to V$ (the *shift operator*);
- a linear map $Y(-, z) : V \to \operatorname{End} V[[z, z^{-1}]]$ sending every a to a field (the *vertex operation*);

subject to the following axioms:

- (*vacuum axiom*) $Y(|0\rangle, z) = \mathbb{1}_V$ and $Y(v, z)|0\rangle \in v + zV[[z]]$ for all $v \in V$;
- (*translation axiom*) $[T, Y(v, z)] = \partial_z Y(v, z)$ for every $v \in V$ and $T|0\rangle = 0$;
- (*locality axiom*) for any pair of vectors $v, v' \in V$, there exists a nonnegative integer N such that $(z - w)^N [Y(v, z), Y(v', w)] = 0$ as an element of $\operatorname{End} V[[z^{\pm 1}, w^{\pm 1}]]$.

Notice that the translation operator can be recovered from the vertex operation. Moreover, there is a powerful "reconstruction" theorem that provides simple criteria to uniquely construct a vertex algebra given "generators and relations."

THEOREM 6.3.3 (Reconstruction, Theorem 4.4.1, [FBZ04]). *Let V be a complex vector space equipped with a nonzero vector $|0\rangle$, an endomorphism T, a countable ordered set $\{a^\alpha\}_{\alpha \in S}$ of vectors, and fields*

$$a^\alpha(z) = \sum_{n \in \mathbb{Z}} a^\alpha_{(n)} z^{-n-1}$$

such that

(1) for all α, $a^\alpha(z)|0\rangle = a^\alpha + O(z)$;
(2) $T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all α;
(3) all fields $a^\alpha(z)$ are mutually local;
(4) V is spanned by the vectors

$$a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)} |0\rangle$$

with the $j_i < 0$.

Then, using the formula

$$Y(a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)} |0\rangle, z) := \frac{1}{(-j_1 - 1)! \cdots (-j_m - 1)!} : \partial_z^{-j_1 - 1} a^{\alpha_1}(z) \cdots \partial_z^{-j_m - 1} a^{\alpha_m}(z) :$$

to define a vertex operation, we obtain a well-defined and unique vertex algebra $(V, |0\rangle, T, Y)$ satisfying conditions (1)-(4) and $Y(a^\alpha, z) = a^\alpha(z)$.

Here $: a(z)b(w) :$ denotes the *normally ordered product* of fields, defined as

$$: a(z)b(w) := a(z)_+ b(w) + b(w)a(z)_-$$

where

$$a(z)_+ := \sum_{n\geq 0} a_n z^n \text{ and } a(z)_- := \sum_{n<0} a_n z^n.$$

Normal ordering eliminates various "divergences" that appear in naively taking products of fields.

Although vertex algebras are not (typically) associative algebras, they possess an important "associativity" property, known as the "operator product expansion."

PROPOSITION 6.3.4. *Let V be a vertex algebra. For any $v_1, v_2, v_3 \in V$, we have the following equality in $V((w))((z-w))$:*

$$Y(v_1, z)Y(v_2, w)v_3 = Y(Y(v_1, z-w)v_2, w)v_3.$$

6.3.2. How to relate the factorization algebra to the $\beta\gamma$ vertex algebra. We now describe the vertex algebra associated to the free $\beta\gamma$ system and explain its relationship to the factorization algebra already constructed. We follow [FBZ04], notably chapters 11 and 12, to make the dictionary clear.

DEFINITION 6.3.5. Let A denote the Weyl algebra generated by elements a_n, a_n^*, for $n \in \mathbb{Z}$ satisfying the commutation relations

$$[a_m, a_n] = 0 = [a_m^*, a_n^*], \quad \text{for all } m, n \in \mathbb{Z},$$

and

$$[a_m, a_n^*] = \delta_{m,-n}, \quad \text{for all } m, n \in \mathbb{Z}.$$

Let V denote the left A-module $\mathrm{Ind}_{A_+}^{A} \mathbb{C}$, where A_+ is the commutative subalgebra of A generated by the a_n for $n \geq 0$ and the a_n^* with $n > 0$.

DEFINITION 6.3.6. The $\beta\gamma$ *vertex algebra* has state space V, vacuum vector 1, and the vertex operator satisfies

$$Y(a_{-1}, z) = \sum_{n\in\mathbb{Z}} a_n z^{-1-n}$$

and

$$Y(a_0^*, z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n}.$$

By theorem 6.3.3 above, these determine the vertex algebra.

REMARK 6.3.7. If we want the $\beta\gamma$ vertex algebra in d variables (this corresponds to the $\beta\gamma$ system of maps into $\mathbb{C}^d$), then we add an extra index $a_{i,n}$ and $a_{i,n}^*$ with $i \in \{1, \ldots, d\}$ and $n \in \mathbb{Z}$. We modify the commutation relations above so that, for instance,

$$[a_{i,m}, a_{j,n}^*] = \delta_{i,j}\delta_{m,-n}.$$

In other words, just take the d-fold tensor product of A and A_+ above and then mimic the construction.

These formulas should remind the reader of the formulas from section 6.2. Explicitly, we have a dictionary

Vertex algebra	Factorization algebra
A	$\mathcal{A}$
V	$\mathcal{V}$
a_n	c_{-1-n}
a_n^*	b_{-n}

that is precise if we require $\hbar = 8\pi$. In other words, both the vertex and factorization algebras for the $\beta\gamma$ system are controlled by a Weyl algebra and an induced module. As a synopsis, we have a putative correspondence:

Vertex algebra	Factorization algebra
state space V	distinguished observables in a disk $\mathbb{D}$
vacuum vector $\|0\rangle$	unit observable 1 = "no observation"
vertex operator $Y(-,z)$	structure map for two disks, $\mathbb{D}(0)$ and $\mathbb{D}(z)$, including into bigger disk

and under this correspondence, many basic properties of vertex algebras (such as associativity) can be guessed because they are inherent properties of a factorization algebra. This correspondence will guide our constructions for other examples of factorization algebras.

REMARK 6.3.8. The only significant difference between the vertex algebra and the basic observables of the factorization algebra is the indexing scheme. In the factorization algebra, we chose the index to reflect which Laurent coefficient is measured: for $n \geq 0$, for instance,

$$c_n(\gamma) = \frac{1}{n!} \left(\partial_z^n \gamma\right)\big|_0,$$

so that c_0 is the delta function at the origin. In the vertex algebra setting, the index reflects the construction of the observable using residues:

$$a_n(\gamma) = \int_{|z|^2=1} \gamma(z) z^n \, dz,$$

so that a_{-1} is exactly the delta function at the origin. Thus c_0 and a_{-1} denote the same observable.

Summarizing our results from the previous section, we obtain the following.

PROPOSITION 6.3.9. *The distinguished observables $\mathcal{V}$ (for any disk $\mathbb{D}$ centered at the origin) are equipped, by the "state-field correspondence" of subsection 6.2.4.3, with the data of a vertex algebra that is isomorphic to the $\beta\gamma$ vertex algebra.*

6.3.3. The case of abelian holomorphic Chern-Simons. Recall from section 6.1.3.1 that we can describe a sigma model of holomorphic maps from a Riemann surface Σ into an open set U of $\mathbb{C}^n$

as a variant of the $\beta\gamma$ system. In fact, the same technique works for any target complex manifold X that is a "holomorphically affine manifold" (i.e., possessing a flat holomorphic connection on its tangent bundle), such as a complex torus. We then obtain a sheaf on U of factorization algebras on Σ, and this sheaf naturally has a flat connection (inherited from the flat connection on jets of holomorphic functions on U). Take the global horizontal sections of this sheaf of factorization algebras to obtain a factorization algebra on Σ. By copying our arguments for the free $\beta\gamma$ system, one can compute the vertex operation and prove the following.

PROPOSITION 6.3.10. *The distinguished observables $\mathcal{V}$ (for any disk $\mathbb{D}$ centered at the origin) are equipped, by the "state-field correspondence," with the data of a vertex algebra that is isomorphic to the vertex algebra of chiral differential operators on U.*

This fact should be utterly unsurprising: in essence, chiral differential operators of a complex manifold X are constructed on a local patch by picking holomorphic coordinates $U \hookrightarrow \mathbb{C}^n$ and then working with the free $\beta\gamma$ system in n variables (see, e.g., [Che]). The challenge is to glue together these coordinate-dependent descriptions. It is hard to believe that chiral differential operators for X are not recovered from the factorization algebra of quantum observables for Costello's holomorphic Chern-Simons theory [Cosa] as they possess precisely the same obstructions to such gluing and the same gerbe of gluings if the obstruction is cohomologically trivial. To give a careful proof of this relationship is a problem in Gelfand-Kazhdan formal geometry, and so it falls outside the purview of this thesis.

6.3.4. Other properties of a vertex algebra. Our dictionary between the factorization and vertex algebra of the $\beta\gamma$ system arose by picking a global coordinate z on $\mathbb{C}$ and then showing that the structure maps of the factorization algebra recovered the formulas for the vertex operation. We have not discussed several other important aspects of the theory of vertex algebras, such as conformal weight/dimension, actions of the Virasoro algebra, modules, or conformal blocks. All these features have their analogues in the factorization setting, and we intend to give a careful articulation of them in future work.

We give a quick example. The $\beta\gamma$ system is defined on any Riemann surface because it is constructed using the Dolbeault complex. We thus get a sheaf of factorization algebras on the moduli of curves whose fiber at a point Σ is the factorization algebra for the $\beta\gamma$ system on Σ. One can ask how infinitesimally varying the complex structure affects the structure of this factorization algebra, and the central charge measures this variation. In particular, recall that by proposition 5.5.1, $\mathrm{Obs}^q(\Sigma)$ is (a cohomological shift of) the determinant of $H^*(\Sigma, \mathscr{O})$. Thus the $\beta\gamma$ system naturally constructs a line bundle on the moduli of curves. The central charge should be the number c such that our $\beta\gamma$ line bundle is the c-fold tensor power of the Hodge bundle, which generates line bundles on the moduli of curves.

6.4. Vertex algebras from Lie algebras

We provide another class of examples in this section, using the enveloping algebra construction (see section 4.5) to build factorization algebras from Lie algebras without any use of field theory. As above, these factorization algebras recover vertex algebras by looking at the simplest structure maps. Notably, we recover the Heisenberg vertex algebra, the free fermion vertex algebra, and the affine Kac-Moody vertex algebras. These methods can be applied, however, to any dg Lie algebra, so there is a plethora of new, unexplored factorization algebras provided by this construction.

The input data is the following:

- Σ a Riemann surface;
- $\mathfrak{g}$ a Lie algebra (for simplicity, we stick to ordinary Lie algebras like $\mathfrak{sl}_2$);
- a $\mathfrak{g}$-invariant symmetric pairing $\kappa : \mathfrak{g}^{\otimes 2} \to \mathbb{C}$.

From this data, we obtain a Lie-structured cosheaf on Σ,

$$\mathfrak{g}^{\Sigma} : U \mapsto (\Omega_c^{0,*}(U) \otimes \mathfrak{g}, \bar{\partial}),$$

that is, a cosheaf of dg vector spaces that is a precosheaf of dg Lie algebras. When κ is nontrivial (though not necessarily nondegenerate), we obtain an interesting central extension on each open:

$$\mathfrak{g}_{\kappa}^{\Sigma} : U \mapsto (\Omega_c^{0,*}(U) \otimes \mathfrak{g}, \bar{\partial}) \oplus \underline{\mathbb{C}} \cdot \mathbf{K},$$

where $\underline{\mathbb{C}}$ denotes the locally constant cosheaf on Σ and $\mathbf{K}$ is a central element of cohomological degree 1 such that

$$[\alpha \otimes X, \beta \otimes Y]_{\kappa} := \alpha \wedge \beta \otimes [X, Y] - \frac{1}{2\pi i} \left(\int_U \partial\alpha \wedge \beta \right) \kappa(X, Y)\mathbf{K},$$

with $\alpha, \beta \in \Omega^{0,*}(U)$ and $X, Y \in \mathfrak{g}$. (These constants are chosen to match with the use of κ for the affine Kac-Moody algebra below.)

REMARK 6.4.1. The dg Lie algebra $\mathfrak{g}^{\Sigma}(U)$ has a natural interpretation in deformation theory: it describes “deformations *with compact support in* U of the trivial G-bundle on Σ.” For U an annulus, it is closely related to the affine Grassmannian: we are modifying how we glue the trivial bundle “outside the annulus” to the trivial bundle “inside the annulus.” A choice of κ has an interpretation in terms of a $\mathbb{C}^{\times}$-gerbe so that $\mathfrak{g}_{\kappa}^{\Sigma}(U)$ describes κ-twisted deformations with compact support in U.

By theorem 4.5.3, we obtain factorization algebras on Σ by applying the Chevalley-Eilenberg functor C_*. We define

$$\mathcal{F}_{\kappa} := C_* \mathfrak{g}_{\kappa}^{\Sigma} : U \mapsto C_*(\mathfrak{g}_{\kappa}^{\Sigma}(U)) = (\mathrm{Sym}(\Omega_c^{0,*}(U) \otimes \mathfrak{g}[1])[\mathbf{K}], \bar{\partial} + d_{CE}),$$

where $\mathbf{K}$ now has cohomological degree 0 in the Lie algebra homology complex.

REMARK 6.4.2. Given a dg Lie algebra $(\mathfrak{g}, d)$, we interpret $C_*\mathfrak{g}$ as the "distributions with support on the closed point of the formal space $B\mathfrak{g}$." Hence, our factorization algebras $\mathcal{F}_\kappa(U)$ describes the κ-twisted distributions supported at the point in $Bun_G(\Sigma)$ given by the trivial bundle on Σ.

The main result of this section is that the vertex algebra associated to $\mathcal{F}_\kappa$ is precisely the same as the vertex algebra for the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$. Recall that this Lie algebra is the central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$,

$$0 \to \mathbb{C} \cdot \mathbf{K} \to \widehat{\mathfrak{g}}_\kappa \to L\mathfrak{g} \to 0$$

where

$$[f(t) \otimes X, g(t) \otimes Y]_\kappa := f(t)g(t) \otimes [X, Y] + (\mathrm{Res}_{t=0} f\, dg)\kappa(X, Y)\mathbf{K}$$

for $X, Y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$. Here $\mathbf{K}$ has cohomological degree 0.

We now make a precise statement of the main result. From hereon, Σ will denote the Riemann surface $\mathbb{C}$, which we equip with a distinguished coordinate z. Although the factorization algebra is manifestly coordinate-independent, we use the coordinate to establish a relationship with vertex algebras.

We introduce some notation.

- Let $U = U\widehat{\mathfrak{g}}_\kappa$ denote the universal enveloping algebra of the affine Kac-Moody algebra.
- Let $\mathcal{U}$ denote the locally constant prefactorization algebra on $\mathbb{R}_{>0}$ given by this associative algebra.
- Let $V = V_\kappa = \mathrm{Ind}_{U_+}^{U} \mathbb{C}$ be the left U-module induced up from the subalgebra $U_+ \subset U$ generated by elements of the form $X \otimes t^n$, with $n \geq 0$.
- Let $\mathcal{V}$ denote the locally constant prefactorization algebra on $\mathbb{R}_{\geq 0}$ that agrees with $\mathcal{U}$ on $\mathbb{R}_{>0}$ but assigns V to any open interval of the form $[0, a)$.
- Let $\rho : \mathbb{C} \to \mathbb{R}_{\geq 0}$ sends z to $|z|$.
- Let $\widetilde{\rho_* H^* \mathcal{F}_\kappa}$ denote the prefactorization algebra in *vector spaces* on $\mathbb{R}_{\geq 0}$ induced from the pushforward factorization algebra $\rho_* H^* \mathcal{F}_\kappa$ in *nuclear spaces*. (See definition 6.2.13 for the description.)

THEOREM 6.4.3. *There is a map of prefactorization algebras*

$$\iota : \mathcal{V} \to \widetilde{\rho_* H^* \mathcal{F}_\kappa}.$$

On every open $I \subset \mathbb{R}_{\geq 0}$, this map ι is a dense inclusion with respect to the topology on $\rho_* H^* \mathcal{F}_\kappa(I)$ so that the structure maps of $\mathcal{V}$ determine those of $\rho_* H^* \mathcal{F}_\kappa$.

Using the correspondence described in section 6.3, we obtain the following.

COROLLARY 6.4.4. *Using this map ι, the vector space $V = V_\kappa$ is equipped with a vertex algebra structure, and this vertex algebra is isomorphic to the vertex algebra associated to the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$.*

One can prove this theorem in a way completely analogous to our work on the $\beta\gamma$ system. We will use a different approach as a chance to exhibit other methods.

6.4.1. The case of level zero. When the level κ is zero — i.e., when we are studying $\mathcal{F} := C_*\mathfrak{g}^\Sigma$ — there is a more efficient way to relate $\mathcal{F}$ to the vertex algebra. In brief, we construct a factorization algebra (in dg nuclear spaces) $\mathcal{L}$ on $\mathbb{R}_{\geq 0}$ such that

- there is a natural map of factorization algebras $F : \mathcal{L} \to \rho_*\mathcal{F}$,
- the map F is an inclusion and induces a dense inclusion on the level of cohomology, and
- the cohomology prefactorization algebra $H^*\mathcal{L}$ is precisely $\mathcal{F}_V$.

Hopefully this approach will help in constructing the factorization algebras underlying a broad class of vertex algebras.

The idea motivating the definition of $\mathcal{L}$ is quite simple. Recall that for an annulus $\mathbb{A} := \mathbb{A}_{r<R}(0)$, the linear functional

$$b_n : f\,\mathrm{d}z = \sum_{n\in\mathbb{Z}} f_n z^n\,\mathrm{d}z \mapsto f_n$$

that reads off the n^{th} Laurent coefficient has nice representatives in $\Omega_c^{0,1}(\mathbb{A})$ of the form

$$\widetilde{b_n} = \phi(|z|)z^{-n}\,\mathrm{d}\bar{z} \text{ with } \phi \in C_c^\infty((r,R))$$

because

$$\int_{\mathbb{A}} \widetilde{b_n} \wedge z^k\,\mathrm{d}z = 2i \int_r^R \phi(r) r^{k-n+1}\,dr \cdot \int_0^{2\pi} e^{i(k-n)\theta} d\theta$$

and the integral over θ yields $\delta_{k,n}$ (so we just have to scale everything by a constant depending on ϕ). The point is that we can describe representatives of b_n that have the form

$$\text{(something depending on radius)} \times \text{(some power of } z\text{)}.$$

It's easy to characterize, on each annulus $\mathbb{A}$, the Lie subalgebra of $\mathfrak{g}^\Sigma(U)$ with this property.

DEFINITION 6.4.5. Let $L\mathfrak{g}^{\mathbb{R}_{>0}}$ be the Lie-structured cosheaf on $\mathbb{R}_{>0}$ that assigns

$$(\Omega_c^*(U) \otimes L\mathfrak{g}, d)$$

to each open $U \subset \mathbb{R}_{>0}$.

Observe that given an element α of $\Omega_c^*((r,R))$, the pullback $\rho^*\alpha$ is a compactly-supported form on $\mathbb{A}_{r<R}$. Let $[\rho^*\alpha]^{0,*}$ denote the component in $\Omega_c^{0,*}(\mathbb{A})$. For example, given a function f, the pullback $(\rho^* f)(z) = f(|z|)$ is already in $\Omega^{0,0}(\mathbb{A})$. By contrast, if s denotes the coordinate on $\mathbb{R}_{>0}$, then

$$\rho^* d(s^2) = \rho^*(2s\,ds) = d(\rho^* s^2) = d(z\bar{z}) = z\,\mathrm{d}\bar{z} + \bar{z}\,\mathrm{d}z,$$

so

$$[\rho^*(f(s^2)2s\,ds)]^{0,*} = f(|z|^2) z\,\mathrm{d}\bar{z}.$$

We now use this map to define a map of Lie-structured cosheaves.

LEMMA 6.4.6. *For each open $U \subset \mathbb{R}_{>0}$, define the map*

$$\begin{array}{rccc} F(U): & (\Omega_c^*(U) \otimes L\mathfrak{g}, d) & \to & \rho_* \mathfrak{g}^\Sigma(U) \\ & \alpha \otimes Xt^m & \mapsto & \left(z^m [\rho^* \alpha]^{0,*}\right) \otimes X \end{array}$$

for $\alpha \in \Omega_c^*(U)$ and $Xt^m \in L\mathfrak{g}$. Then we have a map of Lie-structured cosheaves

$$F : L\mathfrak{g}^{\mathbb{R}_{>0}} \to \rho_* \mathfrak{g}^\Sigma.$$

This map of Lie-structured cosheaves automatically induces a map of the associated enveloping factorization algebras

$$F : C_* L\mathfrak{g}^{\mathbb{R}_{>0}} \to \rho_*(C_* \mathfrak{g}^\Sigma) = \mathcal{F}\big|_{\mathbb{R}_{>0}},$$

and it is a dense inclusion at the level of cohomology by construction (we get all the $b_n \otimes X$, for instance).

We now examine disks. Although the cosheaf $L\mathfrak{g}^{\mathbb{R}_{\geq 0}}$ is well-defined, the map F does not extend to intervals of the form $[0, a)$. The problem is simple: we would have, with $m > 0$,

$$F(\alpha \otimes Xt^{-m}) = z^{-m} [\rho^* \alpha]^{0,*} \otimes X,$$

but z^{-m} is meromorphic at the origin, so the image is ill-defined if α is nonzero at the origin. There is a straightforward modification by simply excluding this possibility.

DEFINITION 6.4.7. Let $L\mathfrak{g}_+^{\mathbb{R}_{\geq 0}}$ denote the Lie-structured cosheaf on $\mathbb{R}_{\geq 0}$ that agrees with $L\mathfrak{g}^{\mathbb{R}_{>0}}$ on $\mathbb{R}_{>0}$ and assigns to an interval $[0, a)$, the Lie algebra

$$\left(\Omega_c^*([0, a)) \otimes \mathfrak{g}[t] \right) \oplus \left(j_! \Omega_c^*((0, a)) \otimes \mathfrak{g}[t^{-1}] \right),$$

where $j_! \Omega_c^*((0, a))$ denotes forms on $[0, a)$ with support in $(0, a)$.[5]

Extending our previous work, we obtain the following.

LEMMA 6.4.8. *There is a map of Lie-structured cosheaves on $\mathbb{R}_{\geq 0}$*

$$F : L\mathfrak{g}_+^{\mathbb{R}_{\geq 0}} \to \rho_* \mathfrak{g}^\Sigma$$

and, abusing notation, of factorization algebras

$$F : C_* L\mathfrak{g}_+^{\mathbb{R}_{\geq 0}} \to \mathcal{F} = \rho_* C_* \mathfrak{g}^\Sigma.$$

By construction, it is an inclusion and a dense inclusion at the level of cohomology prefactorization algebras.

We want to show that theorem 6.4.3 at level zero is a corollary of this lemma. By construction, $C_* L\mathfrak{g}^{\mathbb{R}_{>0}}$ is precisely the factorization algebra associated to the universal enveloping algebra $U(L\mathfrak{g})$. We need to understand the behavior of the cohomology prefactorization algebra on intervals containing the boundary.

[5]We use $j_!$ to denote the "extension by zero" functor for the inclusion $j : (0, a) \to [0, a)$.

LEMMA 6.4.9. *The cohomology prefactorization algebra $H^*C_*L\mathfrak{g}_+^{\mathbb{R}_{\geq 0}}$ assigns the right $U(L\mathfrak{g})$-module $\mathrm{Ind}_{\mathfrak{g}[t]}^{L\mathfrak{g}}\mathbb{C}$ to each interval $[0,a)$.*

PROOF. Notice that $H^*(\Omega_c^*([0,a)),d)=0$, so that nonnegative powers of t vanish cohomologically, and $H^*(j_!\Omega_c^*((0,a)))$ is zero except for $H^1=\mathbb{R}$, so that negative powers of t survive. This implies that for an inclusion $(a',b')\subset[0,a)$, the structure map for the prefactorization algebra has the property

$$Xt^m \mapsto \begin{cases} 0, & m\geq 0 \\ Xt^m, & m<0 \end{cases}.$$

This implies the claim. □

6.4.2. The case of nontrivial level. It would be nice to extend the method for zero level to nontrivial level. Unfortunately, I cannot find a way to relate the κ-central extension of $L\mathfrak{g}^{\mathbb{R}_{>0}}$ and the cosheaf $(\widehat{\mathfrak{g}}_\kappa)^{\mathbb{R}_{>0}}$. Instead, we use a deformation-theoretic approach. In outline, we construct a central extension of $L\mathfrak{g}^{\mathbb{R}_{\geq 0}}$ that maps into $\rho_*\mathfrak{g}_\kappa^\Sigma$, and then we observe that the cohomology prefactorization algebra $H^*C_*\widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$ is locally constant and hence corresponds to an associative algebra. This algebra is a deformation of $U(L\mathfrak{g})$ and we pick out the deformation by looking at generators.

DEFINITION 6.4.10. Let $\widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$ denote the Lie-structured cosheaf on $\mathbb{R}_{>0}$ that assigns to an open interval $U\subset\mathbb{R}_{>0}$, the central extension

$$0\to\mathbb{C}\cdot\mathbf{K}\to\widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa(U)\to L\mathfrak{g}^{\mathbb{R}_{>0}}(U)\to 0$$

where

$$[\alpha\otimes Xt^m,\beta\otimes Yt^n]_\kappa := \alpha\wedge\beta\otimes[X,Y]t^{m+n}-\frac{1}{2\pi i}\left(\int_{\rho^{-1}U}\partial(z^m[\rho^*\alpha]^{0,*})\wedge z^n[\rho^*\beta]^{0,*}\right)\kappa(X,Y)\mathbf{K}$$

and $\mathbf{K}$ has cohomological degree one.

We deliberately chose this extension to obtain the following lemma.

LEMMA 6.4.11. *There is a map $F_\kappa:\widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa\to\rho_*\mathfrak{g}_\kappa^\Sigma$ of Lie-structured cosheaves, where on an interval U, we have*

$$F(U):\alpha\otimes Xt^m+a\mathbf{K}\mapsto\left(z^m[\rho^*\alpha]^{0,*}\right)\otimes X+a\mathbf{K}.$$

PROOF. We need to verify that the central extensions are compatible. Let $U=(r,R)$. Given $\alpha,\beta\in\Omega_c^*(U)$ and $X,Y\in\mathfrak{g}$, we need to show that

$$[F(\alpha\otimes Xt^m),F(\beta\otimes Yt^n)]_\kappa = F([\alpha\otimes Xt^m,\beta\otimes Yt^n]_\kappa).$$

We know

$$[\rho^*(\alpha\wedge\beta)]^{0,*}=[\rho^*\alpha]^{0,*}\wedge[\rho^*\beta]^{0,*},$$

so it remains to check the $\mathbf{K}$ component. But this holds by definition. □

Restricting to $\rho : \mathbb{C}^\times \to \mathbb{R}_{>0}$, we obtain a natural map of factorization algebras

$$F : C_* \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa \to \rho_* \mathcal{F}_\kappa$$

that is dense at the level of cohomology. Thus the following result helps us to understand what we've constructed.

LEMMA 6.4.12. *The cohomology prefactorization algebra $H^* C_* \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$ is isomorphic to the factorization algebra associated to the universal enveloping algebra $U\widehat{\mathfrak{g}}_\kappa$.*

PROOF. Let $\mathcal{H}_\kappa$ denote the locally constant factorization algebra $H^* C_* \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$. Let H_κ denote the corresponding associative algebra. We want to recognize H_κ as an enveloping algebra.

We make some remarks that simplify the problem.

(1) The map of Lie-structured cosheaves

$$\iota : \underline{\mathbb{C}}\mathbf{K} \hookrightarrow \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$$

induces a map of factorization algebras

$$\iota : C_* \underline{\mathbb{C}}\mathbf{K} \hookrightarrow C_* \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$$

and hence of cohomology factorization algebras and also associative algebras

$$H^* \iota : \mathbb{C}[\mathbf{K}] \to H_\kappa,$$

where everything here is now in degree 0.

(2) In fact, H_κ is a free module over $\mathbb{C}[\mathbf{K}]$. Consider the spectral sequence on $C_* \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$ given by filtering in powers of $L\mathfrak{g}^{\mathbb{R}_{>0}}$:

$$F^k := \mathbb{C}[\mathbf{K}] \otimes \mathrm{Sym}^{\leq k} L\mathfrak{g}^{\mathbb{R}_{>0}}.$$

This spectral sequence collapses on the first page and shows that the cohomology associated graded is a free module over $\mathbb{C}[\mathbf{K}]$.

(3) The map of Lie-structured cosheaves

$$q : \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa \to L\mathfrak{g}^{\mathbb{R}_{>0}}$$

induces a map of factorization algebras, so that we see $H_\kappa \otimes_{\mathbb{C}[\mathbf{K}]} \mathbb{C} \cong U(L\mathfrak{g})$.

Putting these together, we see that H_κ is a well-behaved deformation of $U(L\mathfrak{g})$ over $\mathbb{C}[\mathbf{K}]$ and it suffices to describe the deformed multiplication on generators like Xt^m.

We now compute the commutator of elements Xt^m and Yt^n, with $X, Y \in \mathfrak{g}$. As this is the umpteenth time we've used this argument, we write it tersely. Pick intervals $I = (a, b)$, $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$, $I_1 = (a_3, b_3)$, where

$$0 < a \leq a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \leq b.$$

Pick bump functions ϕ_j such that $\phi_j(s^2)$ has support in I_j and $\int_{I_j} \phi_j(s^2) 2s\, ds = 1$. We define

$$\tilde{X}_1 = \phi_1(s^2)2s\, ds \otimes Xt^m, \tilde{X}_3 = \phi_3(s^2)2s\, ds \otimes Xt^m, \text{ and } \tilde{Y} = \phi_2(s^2)2s\, ds \otimes Yt^n,$$

which provide representatives of Xt^m on I_1 and I_3 and of Yt^n on I_2 in $C_* \widehat{L\mathfrak{g}^{\mathbb{R}_{>0}}}_\kappa$. Define

$$\Phi(s^2) = \int_0^{s^2} \phi_1(t) - \phi_3(t)\, dt.$$

Then, on I, we find

$$\begin{aligned} d(\Phi(s^2) \otimes Xt^m \cdot \phi_2(s^2)2s\, ds \otimes Yt^n) \quad = \quad & \phi_1(s^2)2s\, ds \otimes Xt^m \bullet \phi_2(s^2)2s\, ds \otimes Yt^n \\ & -\phi_2(s^2)2s\, ds \otimes Yt^n \bullet \phi_3(s^2)2s\, ds \otimes Xt^m. \end{aligned}$$

We also compute

$$\begin{aligned} & \tfrac{1}{2\pi i} \int_{\rho^{-1}I} \partial(z^m \rho^* \Phi(s^2)) \wedge z^n [\rho^* \phi_2(s^2) 2s\, ds]^{0,*} \\ = \quad & \tfrac{1}{2\pi i} \int_{\rho^{-1}I} (m z^{m-1} \Phi(|z|^2) + z^m \bar{z} \Phi'(|z|^2))\, \mathrm{d}z \wedge z^n (\phi_2(|z|^2) z\, \mathrm{d}\bar{z}) \\ = \quad & \tfrac{1}{2\pi i} \int_{\rho^{-1}I} m z^{m+n} \Phi(|z|^2) \phi_2(|z|^2)\, \mathrm{d}z\, \mathrm{d}\bar{z} \\ = \quad & -\tfrac{1}{2\pi i} im \int_I r^{m+n} \phi_2(r^2) 2r\, dr \int_0^{2\pi} e^{i(m+n)\theta} d\theta \\ = \quad & -m\delta_{m,-n} \end{aligned}$$

where we used the fact that $\Phi(s^2)$ is the constant 1 on I_2 to replace $\Phi\phi_2$ by ϕ_2 and to drop the term $\Phi'\phi_2$. Putting these computations together, we see that at the level of cohomology, we have

$$Xt^m \bullet Yt^n - Yt^n \bullet Xt^m - [Xt^m, Yt^n]_\kappa = 0,$$

which is precisely the relation for $U\widehat{\mathfrak{g}}_\kappa$. Thus H_κ is the desired deformation of $U(L\mathfrak{g})$. □

6.5. Definitions and a conjecture

We now have several examples of factorization algebras that locally reproduce well-known vertex algebras. Our goal in this section is to provide a systematic vocabulary for such examples and to make a precise conjecture about the relationship between factorization algebras and vertex algebras.

So far in this thesis, we have only discussed a factorization algebra living on a fixed manifold. But we would like to talk about factorization algebras that are more general in nature, living on some large class of manifolds, much as the de Rham sheaf lives on any smooth manifold. Such an extension of our notions will be useful in physics: the most important action functionals are well-defined on large classes of manifolds (e.g., the Yang-Mills action functional lives on many 4-manifolds) and it is fruitful to study the ensemble of theories given by one of these functionals all at once. For instance, we introduce below a definition of "holomorphic field theory" and hope it captures what a physicist might call the "chiral sector of a conformal field theory." Although we only consider theories associated to Riemann surfaces, it is straightforward to generalize this approach to define supersymmetric or Euclidean or Riemannian or topological field theories.

DEFINITION 6.5.1. Let $\mathcal{R}S$ denote the category whose objects are complex manifolds of complex dimension 1 (without boundary) and whose morphisms are holomorphic embeddings.

We now define a factorization algebra on this category.[6]

DEFINITION 6.5.2. A *holomorphic factorization algebra* (on Riemann surfaces) $\mathcal{F}$ consists of the following data,

- for each Riemann surface $S \in \mathrm{Ob}\mathcal{R}S$, a factorization algebra $\mathcal{F}_S$ on S, and
- for each holomorphic embedding $\phi : S \hookrightarrow S'$, an isomorphism of factorization algebras $\mathcal{F}_\phi : \mathcal{F}_S \to \phi^*\mathcal{F}_{S'}$,

and satisfying the natural compatibility requirements,

- the identity $1_S : S \to S$ goes to the identity $\mathcal{F}_1 = 1_{\mathcal{F}_S}$, and
- a composable sequence of embeddings $S \overset{\phi}{\hookrightarrow} S' \overset{\psi}{\hookrightarrow} S''$ leads to a commuting diagram

$$\begin{tikzcd} \mathcal{F}_S \arrow[r, "\mathcal{F}_\phi"] \arrow[rr, bend right, "\mathcal{F}_{\psi\circ\phi}"'] & \phi^*\mathcal{F}_{S'} \arrow[r, "\phi^*\mathcal{F}_\psi"] & \phi^*\psi^*\mathcal{F}_{S''}. \end{tikzcd}$$

REMARK 6.5.3. It is manifest that a holomorphic factorization algebra is determined by its behavior on an open disk. This feature is an avatar of the "cobordism hypothesis."

PROPOSITION 6.5.4. *The factorization algebras constructed in this chapter —* $\mathrm{Obs}^{\mathrm{cl}}$ *and* $\mathrm{Obs}^{\mathrm{q}}$ *for the $\beta\gamma$ system and the $\mathcal{F}_\mathfrak{g}$ and $\mathcal{F}_\kappa$ for a Lie algebra $\mathfrak{g}$ and pairing κ — are holomorphic.*

PROOF. All of these examples are constructed atop the cosheaf $\Omega_c^{0,*}$ of compactly-supported Dolbeault forms, so that we obtain the requisite maps $\mathcal{F}_\phi$ via the natural pullback isomorphisms on differential forms. □

Connecting this definition to our notion of field theories is simple: a *holomorphic* field theory is a field theory whose factorization algebra of (classical and) quantum observables is holomorphic. Here is a precise version of this idea.

DEFINITION 6.5.5. A *holomorphic field theory* (on Riemann surfaces) consists of the following data:

- a free BV theory on every Riemann surface S (i.e., a $\mathbb{Z}$-graded vector bundle $\pi_S : E_S \to S$, and so on);
- for every holomorphic embedding $\phi : S \to T$, an isomorphism of vector bundles $E_\phi : E_S \to \phi^* E_T$ inducing a compatibility between the remaining data of the free theory;

[6]For simplicity, we avoid giving the obvious definition using fibered categories. Developing a nice theory of such factorization algebras, however appealing, is a detour from the main goals of this thesis.

- a choice of interaction term $I_S \in \mathscr{O}_{loc}(\mathscr{E}_S)$ for every Riemann surface S and compatibilities under pullbacks (i.e., $I_S = \phi^* I_T$).

CONJECTURE 6.5.6. *Given a holomorphic field theory whose associated free theory satisfies a Poincaré lemma concentrated in degree 0, there is a vertex algebra recovered from the quantum observables on a disk.*

REMARK 6.5.7. The $\beta\gamma$ system should lead to a large class of examples where this conjecture holds: take the interacting field theories whose free theory is the $\beta\gamma$ system and whose interaction terms are purely holomorphic (i.e., only depend on holomorphic derivatives of the fields). Results of Li [**Li**] imply that such interaction terms do not require counterterms, so that the interacting field theories are relatively easy to construct. On $\mathbb{C}$, the Poincaré lemma still holds so the factorization algebra of quantum observables is still concentrated in degree 0. Moreover, the structure maps are only modified by interaction terms that are holomorphic in nature.

CHAPTER 7

An index theorem

Given a free *classical* cotangent field theory with an action of an L_∞ algebra $\mathfrak{g}$ on the underlying elliptic complex, there is an obstruction (*aka* anomaly) to $\mathfrak{g}$-invariant quantization which encodes the renormalized trace of the $\mathfrak{g}$-action on the underlying elliptic complex. If the obstruction vanishes, the partition function is a function of $\mathfrak{g}$. Both the obstruction and the partition functions have explicit descriptions that we state in the form of an "index theorem." What is most exciting about this index theorem is that all these structures exist *locally* on the manifold where the theory lives, thanks to the sheaf-theoretic nature of Costello's construction of perturbative QFT [**Cos11**] and the language of factorization algebras.

7.0.1. Phrasing the question. Before stating the theorem, we want to introduce the kind of question it addresses. We describe the question in three forms, in the languages of derived geometry, Lie theory, and physics. Essentially, we simply explore what factorization algebras contribute to the long-standing relationship between determinants of elliptic complexes and the path integral for free theories.

7.0.2. Phrasing the question in the style of derived geometry. We sketch the idea heuristically, discussing objects that should exist but have not yet been constructed mathematically. With a suitably restricted class of objects, our very local index theorem will make the idea precise.

On a manifold M, there is a moduli space of free classical field theories, where the components are labelled by a choice of fiber bundle on M — the fields are sections of the bundle — and where each component consists of the quadratic *local* functionals on that space of fields. In fact, we get a presheaf of moduli spaces on M because bundles restrict from larger opens to smaller opens and so do local functionals (since they're local!). One might hope that there is a likewise a presheaf of moduli spaces given by quantum field theories, and that there is a map of presheaves called *dequantization* or "take the classical limit."

As usual, we study a formal geometry, or deformation-theoretic, version of this problem. Let us fix a free BV theory on M, so that the bundle is actually a vector bundle and our action functional arises from a quadratic pairing and an elliptic complex. Denote the theory by $\mathscr{E}$. Consider the formal neighborhood $\widehat{\mathscr{E}}$ of this theory inside the moduli space of free classical BV theories.

(This formal neighborhood is very similar the formal neighborhood of $\mathscr{E}$ inside the space of elliptic complexes on M.) We know how to BV quantize $\mathscr{E}$ and so we ask if we can extend the quantization to this neighborhood and if we can describe the structure of such a family of quantizations.

A more precise version of the problem goes as follows. Each deformation of $\mathscr{E}$ on an open U has a natural quantization and so we get the quantum observables on U for the quantum theory of the deformed classical field theory. Thus, we get a presheaf $Quant_{\mathscr{E}}$ of formal moduli spaces on M with

$$Quant_{\mathscr{E}}(U) := \left\{ \begin{array}{c} \text{the space of factorization algebras on } U \text{ that} \\ \text{arise as quantizations of deformations of } \mathscr{E}\big|_U \end{array} \right\}.$$

Although this presheaf is quite complicated, we can extract from it something more familiar. Suppose M is a closed manifold. Then we know that the global quantum observables $\mathrm{Obs}^q{}_{\mathscr{E}}(M)$ provide a kind of Pfaffian line for the elliptic complex $\mathscr{E}(M)$. If we consider the global quantum observables for each deformation, we say, speaking casually, that we get a Pfaffian line bundle over the formal neighborhood $\widehat{\mathscr{E}}$. We can ask whether this line bundle is trivial (i.e., compute its Chern class) and, if so, whether there are natural secondary characteristic classes.

These global questions first appear in Quillen's study of the determinant line of the $\bar{\partial}$ operator over the moduli of holomorphic line bundles on a fixed Riemann surface [Kvi85].[1] Using the language of factorization algebras, we can ask similar questions locally on M. Namely, can we trivialize the presheaf of spaces $Quant_{\mathscr{E}}$ over the presheaf $\widehat{\mathscr{E}}$? If we can, what structure do we obtain by a choice of trivialization?

7.0.3. Phrasing the question in the style of Lie theory. The Koszul duality between formal moduli spaces and dg Lie (equivalently, L_∞) algebras allows us to phrase the problem in a different way. Let $\mathfrak{g}$ be a sheaf of dg Lie algebras on M with an action

$$\rho : \mathfrak{g} \otimes \mathscr{E} \to \mathscr{E}$$

preserving the -1-symplectic pairing on $\mathscr{E}$. Then the quantum observables of the free theory $\mathrm{Obs}^q{}_{\mathscr{E}}(U)$ becomes a $\mathfrak{g}(U)$-module for every open U in M. It is natural to ask for invariants of the $\mathfrak{g}$-module structure of $\mathrm{Obs}^q{}_{\mathscr{E}}$. Since the global observables are cohomologically one-dimensional, a natural global invariant is to ask for the character of $\mathrm{Obs}^q{}_{\mathscr{E}}(M)$ as a representation of $\mathfrak{g}(M)$ (i.e., the trace on this one-dimensional space). This character is a kind of index (see section 7.1 below). There is also a section $\mathcal{O}(\rho)$ (of cohomological degree 1) of the sheaf $C^*\mathfrak{g}$ on M that provides a local description of this character. When the character is zero, we know that $\mathrm{Obs}^q{}_{\mathscr{E}}(M)$ is a trivializable $\mathfrak{g}$-module and we can ask to construct an explicit trivialization.

[1] Because Quillen published this paper in a Russian journal, the translator gave his name as "Kvillen" and, oddly, this is the author listed in MathSciNet.

7.0.4. Phrasing the question in the style of physics. There is a third description of the problem that has a more physical flavor. Given a formal family of theories around $\mathscr{E}$, we can describe the family by the inclusion of a background field. In other words, we couple $\mathscr{E}$ to another set of fields $\mathfrak{g}$ that we treat as classical fields. For each solution to the Euler-Lagrange equations for the coupled system, we can ask to quantize just the $\mathscr{E}$ fields, with the component $X \in \mathfrak{g}$ fixed. When X is zero, we have a canonical quantization but it may not be possible to quantize for all choices of X. This is the obstruction element that appears in the index theorem. If we can quantize in families over the values of X, we can ask how the partition function varies.

REMARK 7.0.8. The broadest form of these questions is far beyond what we accomplish in the theorem, and a fuller answer would require a substantial enhancement of the machinery used in this thesis. An answer that included interacting theories and factorization algebras would provide a broad extension of the notion of an index theorem.

7.1. A motivating example

Before approaching the general theorem, we begin with a simple example which directly recovers the index of an elliptic complex. Let $(\mathscr{E}, Q)$ be an elliptic complex on a closed manifold M and let $T^*[-1]\mathscr{E}$ denote the elliptic complex $(\mathscr{E} \oplus \mathscr{E}^![-1], Q + Q^!)$. Then $\mathbb{G}_m$ acts, as usual, by multiplication:

$$\begin{array}{ccc} \mathbb{G}_m \times T^*[-1]\mathscr{E} & \to & T^*[-1]\mathscr{E}, \\ (\lambda, f \oplus g) & \mapsto & \lambda f \oplus \lambda g. \end{array}$$

This complex $T^*[-1]\mathscr{E}$ naturally forms a free BV theory and hence we can BV quantize it. Our first question is how $\mathbb{G}_m$ acts on the global observables $\mathrm{Obs}^q(M)$ of this theory.

7.1.1. Determinants and the index. Recall that the *determinant* of a finite-dimensional vector space V is defined to be the one-dimensional vector space $\det V := \Lambda^{\dim V} V$ and of a finite-dimensional, $\mathbb{Z}$-graded vector space V^* is the one-dimensional vector space

$$\det V^* := \bigotimes_{n \in \mathbb{Z}} (\det V^n)^{(-1)^n},$$

where $L^{-1} := L^\vee$ for L a one-dimensional vector space.

For any vector space, there is a natural action of $\mathbb{G}_m$ given by scalar multiplication. If we take the most naive version of the action on $\det V^*$, we see that $\mathbb{G}_m$ acts by

$$\mathbb{G}_m \ni \lambda \mapsto \prod_n (\lambda^{\dim V^n})^{(-1)^n} = \lambda^{\chi(V)},$$

where $\chi(V)$ is the Euler characteristic of V.[2] Since the Euler characteristic is precisely the natural generalization of the index of an operator (viewed as a two-term complex), we see how determinants recover the index.

7.1.2. Back to elliptic complexes. It does not make sense *a priori* to take the determinant of an infinite-dimensional vector space but an elliptic complex on a closed manifold has finite-dimensional cohomology. Hence the *determinant of an elliptic complex* on a closed manifold is defined as

$$\det(\mathscr{E}, Q) := \det H^*(\mathscr{E}(M), Q).$$

Thus, the action of $\mathbb{G}_m$ on the determinant of an elliptic complex recovers the usual index of an elliptic complex.

In section 2.4, we saw that for a finite-dimensional $\mathbb{Z}$-graded vector space V,

$$H^* \mathcal{C}Q(V) \cong \det V[d(V)]$$

where $d(V)$ is a shift depending on the Betti numbers of V.[3] Applying this result in the context of elliptic complexes, we see that

$$H^* \operatorname{Obs}^{\mathrm{q}}(M) \cong \det(\mathscr{E})[d(\mathscr{E})].$$

Thus, the action of $\mathbb{G}_m$ on the global quantum observables of a free theory recovers the index of the elliptic complex.

With the language of factorization algebras, though, we obtain an enhancement of this observation. The quantum observables $\operatorname{Obs}^{\mathrm{q}}$ provide a local-to-global object that recovers the determinant, and $\mathbb{G}_m$ acts on this factorization algebra. Thus we have a "geometric" way to recover the index: we can compute the global observables using the locality axiom. By construction, the two methods of computing global observables — using analysis or using a gluing procedure — agree. This fact is akin to the Hirzebruch-Riemann-Roch theorem, which identifies a sheaf-theoretic computation (the Euler characteristic of sheaf cohomology for an $\mathscr{O}_X$-module) with a topological computation (the integral of a characteristic class for the module).

7.2. A precise statement of the theorem

Let $(\mathscr{E}, Q, \langle -, - \rangle)$ be a free BV theory on a smooth manifold M.

The basic new ingredient, *vis à vis* the rest of this thesis, is a sheaf of dg Lie algebras (or L_∞ algebras) $\mathscr{L}$ on M such that $\mathscr{E}$ is a $\mathscr{L}$-module. We want this sheaf to fit naturally with the structure of a BV theory, so we impose some reasonable conditions on $\mathscr{L}$.

[2]There are other natural actions. For instance, if one views a graded vector space as already a representation of $\mathbb{G}_m$ with the n^{th} component acted on by λ^n, we would get a different action on $\det V^*$.

[3]Explicitly, $d(V) = -\sum_n (2n+1)(\dim H^{2n}(V) + \dim H^{2n+1}(V))$. If one views the determinant as concentrated in degree $\chi(V)$, then one simply subtracts $\chi(V)$ from $d(V)$ to get the appropriate shift.

DEFINITION 7.2.1. A *local L_∞ algebra* on M consists of the following data:

(1) A $\mathbb{Z}$-graded vector bundle L on M whose space of smooth sections will be denoted $\mathscr{L}$.
(2) A differential operator $d : \mathscr{L} \to \mathscr{L}$ of cohomological degree 1 such that $d^2 = 0$.
(3) A collection of polydifferential operators (of cohomological degree 0) for each $n \geq 1$

$$\ell_n : \mathscr{L}^{\otimes n} \to \mathscr{L}[2-n]$$

which are alternating and make $\mathscr{L}$ into an L_∞ algebra. Note that $\ell_1 = d$.

We say $\mathscr{L}$ is *elliptic* if $(\mathscr{L}, d)$ is an elliptic complex.

EXAMPLE 7.2.2. Our elliptic complex $(\mathscr{E}, Q)$ has a natural elliptic dg Lie algebra associated to it. Let $\mathrm{End}(E)$ denote the vector bundle with fiber $\mathrm{End}(E)_x$ given by $\mathrm{End}(E_x)$. The sheaf of smooth sections $\mathscr{E}nd(E)$ is a sheaf of associative algebras by fiberwise multiplication, and so, using the commutator for the bracket, $\mathscr{E}nd(E)$ is a sheaf of graded Lie algebras. The operator $[Q, -]$ makes it into a sheaf of dg Lie algebras.

EXAMPLE 7.2.3. For E a vector bundle with flat connection ∇, take $\mathscr{L} = (\Omega^*(\mathrm{End}\, E), \nabla^{\mathrm{End}})$.

Let $D_{\mathscr{E}}$ denote the sheaf of differential operators on E (i.e., mapping $\mathscr{E}$ to itself).

DEFINITION 7.2.4. A *local representation* (or module) of the elliptic L_∞ algebra $\mathscr{L}$ on an elliptic complex $(\mathscr{E}, Q)$ is an L_∞ module such that the brackets

$$\rho_n : \mathscr{L}^{\otimes n-1} \otimes \mathscr{E} \to \mathscr{E}[2-n]$$

are all polydifferential operators. Equivalently, we have a collection of maps

$$\rho_n : \mathscr{L}^{\otimes n} \to D_{\mathscr{E}}[2-n]$$

forming a map of L_∞ algebras.

Consider the natural formal moduli space $B\mathscr{L}$ associated to $\mathscr{L}$. A Maurer-Cartan element for $A \in dgArt$ is a degree 1 element of $\mathscr{L} \otimes \mathfrak{m}_A$ satisfying

$$\sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(X^{\otimes n}) = 0.$$

Under the representation ρ, we obtain a differential operator

$$\rho(X) = \sum_{n=1}^{\infty} \rho_n(X^{\otimes n})$$

on $\mathscr{E}$. As X is a Maurer-Cartan element, we obtain the following equation:

$$\begin{aligned}(Q + \rho(X))^2 &= Q^2 + [Q, \rho(X)] + \frac{1}{2}[\rho(X), \rho(X)] \\ &= 0.\end{aligned}$$

In other words, $B\mathscr{L}(A)$ is a simplicial set parametrizing certain deformations of $\mathscr{E}$ as an elliptic complex.[4]

EXAMPLE 7.2.5 (Example 7.2.3 continued). For E a vector bundle with flat connection ∇ and $\mathscr{L} = (\Omega^*(\operatorname{End} E), \nabla^{\mathrm{End}})$, the space $B\mathscr{L}$ describes deformations of ∇ as a flat connection on E.

The very local index theorem addresses the question of whether we can $\mathscr{L}$-equivariantly quantize $\mathscr{E}$. Equivalently, it asks whether we can trivialize the family of quantizations over the formal moduli space $B\mathscr{L}$ (which is really a presheaf of formal moduli spaces over our manifold M). The obstruction to such quantization — the "index" — is a local object on M. It lives in a sheaf that we now construct.

DEFINITION 7.2.6. A *local functional* Φ for smooth sections $\mathscr{F}$ of a vector bundle F on the manifold M is an element of $\mathscr{O}(\mathscr{F}) := \prod_{n\geq 0}(\mathscr{F}^{\otimes n})^{\vee}_{S_n}$ such that each Taylor component $\Phi_n \in \operatorname{Sym}^n \mathscr{F}^{\vee}$ has the form

$$\Phi_n(s) = \sum_{j=1}^{k} \int_M D_{j,1}(s) \cdots D_{j,n}(s)\mu_j,$$

with the $D_{j,m}$ arbitrary differential operators from $\mathscr{F}$ to $C^\infty(M)$ and μ_j a smooth measure on M. Equivalently, each Φ_n is a distribution supported on the small diagonal $M \subset M^n$ whose wavefront set is the conormal bundle to the small diagonal.

Because a local functional only depends on local data (notably the jets of a section s of $\mathscr{F}$), it restricts from larger to smaller opens. Hence, local functionals on $\mathscr{F}$ form a sheaf denoted $\mathscr{O}_{loc}(\mathscr{F})$ on M.

DEFINITION 7.2.7. For $\mathscr{L}$ a local L_∞ algebra on M, let $C^*_{loc}\mathscr{L}$ denote the *local* Chevalley-Eilenberg complex of $\mathscr{L}$. This sheaf on M assigns $\mathscr{O}_{loc}(\mathscr{L}[1])$ equipped with the differential arising from the L_∞ bracket on $\mathscr{L}$.

REMARK 7.2.8. There is another natural description using D_M-modules. Let $J(L)$ denote the left D_M-module of jets of sections of L (with $J(d)$ as its differential) and set

$$J(L)^* := \operatorname{Hom}_{C^\infty_M}(J(L), C^\infty_M),$$

the sheaf of continuous linear maps of C^∞_M-modules. There is a natural D_M-algebra

$$\mathscr{O}(BJ(L)) := \prod_{n\geq 0} \operatorname{Sym}^n(J(L)^*[-1]),$$

where the symmetric powers are taken over C^∞_M. The L_∞ bracket on $\mathscr{L}$ induces a natural derivation on this algebra. We then define

$$C^*_{loc}\mathscr{L} := \operatorname{Dens}_M \otimes_{D_M} \mathscr{O}(BJ(L)),$$

[4]Costello's work on the Witten genus shows how we can encode even manifolds as dg Lie algebras, so we can see quite sophisticated objects with these things.

with the *un*derived tensor product. By lemma 6.6.2 in chapter 5 of [**Cos11**], we also know

$$(C^*_{loc}\mathscr{L})_{red} = \mathrm{Dens}_M \otimes^{\mathbb{L}}_{D_M} \mathscr{O}(BJ(L))_{red},$$

where the subscript *red* denotes the reduced complex (i.e., drop the Sym^0 component).

The very local index theorem consists of several interlocking statements. In essence, we describe the obstruction to constructing an $\mathscr{L}$-equivariant BV quantization of $T^*[-1]\mathscr{E}$. This obstruction (the "index") is local in nature, but its global section on a closed manifold can be computed analytically, organized as Feynman diagram computations. (The global statement is closest to the usual index theorems.) If the obstruction is cohomologically trivial, there exists an $\mathscr{L}$-equivariant BV quantization of $T^*[-1]\mathscr{E}$, and so we obtain a family of factorization algebras over $B\mathscr{L}$. Globally on a closed manifold, this family is equipped with a distinguished section over $B\mathscr{L}(M)$ which provides a generalization of the torsion of $\mathscr{E}$. Again, one can compute this distinguished section using analysis or with the local-to-global structure of the factorization algebras. By comparison with the usual families index theorem, we are building something local on the *total* space and not just the *base*.

THEOREM 7.2.9 (Local statements). *Let $(\mathscr{E}, Q)$ be an elliptic complex equipped with a local representation of the local L_∞ algebra $\mathscr{L}$.*

- *The obstruction $\mathcal{O}$ to $\mathscr{L}$-equivariant quantization of the cotangent theory is a section of the sheaf $C^*_{loc}(\mathscr{L})_{red}$ with cohomological degree 1. It is given by the trace of an action of $\mathscr{L}$ on $T^*[-1]\mathscr{E}$.*
- *If the obstruction is cohomologically trivial, we obtain a family of factorization algebras for M living over $B\mathscr{L}$ that we denote $\mathrm{Obs}^q \to B\mathscr{L}$. There is a distinguished section of cohomological degree 0 of this family.*

We obtain the strongest statements about global observables under a reasonable hypothesis on $(\mathscr{E}, Q)$ described in section 7.6. (Essentially, this hypothesis says that there exists an "adjoint" Q^* to Q such that $[Q, Q^*]$ is a self-adjoint, generalized Laplacian with nonnegative eigenvalues.)

PROPOSITION 7.2.10 (Global observables). *For M a closed manifold and $(\mathscr{E}, Q)$ satisfying the hypothesis in section 7.6, we have the following results for global observables.*

- *The cohomology class $[\mathcal{O}]$ in $H^*(C^*(\mathscr{L}(M))_{red})$ is given by the trace of the action of $H^*\mathscr{L}(M)$ on $H^*\,\mathrm{Obs}^q{}_{\mathscr{E}}(M)$, which is a shift of* $\det H^*(\mathscr{E}(M))$.
- *There is a distinguished section of cohomological degree 0 for the bundle of BD algebras* $\mathrm{Obs}^q(M) \to B\mathscr{L}(M)$. *This section provides a definition of the ratio* $\det(Q - X) / \det(Q)$.

For these global objects, there are two ways to describe them. We can use sheaf theory to obtain $[\mathcal{O}]$ or we can give an explicit Feynman diagram description. Likewise, the distinguished section can be constructed using the factorization algebra or with Feynman diagrams. In other words, we have a local-to-global construction or an analytic construction. The identification between the two methods is the clearest sense in which the theorem resembles the usual index theorems.

NOTE 7.2.11. *Some of the objects in this theorem are a priori ill-defined, such as the trace of the action of $\mathscr{L}$ or the determinant of Q, because we are working with infinite-dimensional vector spaces. As usual, we remedy this problem by introducing* renormalized *versions of these concepts. In Costello's formalism, however, the choices involved in constructing these renormalizations (typically a choice of "gauge-fix," such as an operator d^*) form a simplicial set $\mathcal{GF}$ — usually contractible — and the simplicial set of quantizations $\mathcal{Q}uant$ is a fibration over $\mathcal{GF}$. Thus we have complete control of all choices up to homotopy.*[5]

7.3. Setting up the problem

The input to our construction is the following. We have a smooth manifold M and $(\mathscr{E}, Q)$ an elliptic complex on M. Let $(\mathscr{L}, d_{\mathscr{L}})$ be an elliptic L_∞ algebra with ρ a representation of $\mathscr{L}$ on $\mathscr{E}$. We now phrase our family of elliptic complexes in the language of QFT.

7.3.1. Rephrasing the problem as an elliptic moduli problem. Our preferred way of describing a classical BV theory, as discussed in the introduction to chapter 5, is with the language of elliptic moduli problems. (Below, we'll restate everything in the language of action functionals.) Because $\mathscr{E}$ is a $\mathscr{L}$-module, we naturally obtain an $\mathscr{L}$-action on $\mathscr{E}^!$. Set

$$\mathscr{F} = \mathscr{L} \oplus \mathscr{E}[-1] \oplus \mathscr{E}^![-2],$$

viewed as a split-square zero extension of $\mathscr{L}$. This sheaf provides an elliptic moduli problem that is straightforward to describe.

Recall that $\mathscr{L}$ describes "solutions to the Maurer-Cartan equation for $\mathscr{L}$." More precisely, for each dg Artinian ring A, we ask for X in $\mathscr{L} \otimes \mathfrak{m}_A$ satisfying the Maurer-Cartan equation

$$\sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(X^{\otimes n}) = 0.$$

We write $X \in MC(\mathscr{L}, A) = B\mathscr{L}(A)$.

Because $\mathscr{F}$ is a split-square zero extension of $\mathscr{L}$, every solution to its Maurer-Cartan equation forgets down to a solution for the Maurer-Cartan equation of $\mathscr{L}$. In other words, we have a natural map $B\mathscr{F} \to B\mathscr{L}$. The fiber of this map at a solution $X \in MC(\mathscr{L}, A)$ is given by solutions to the equation

$$(Q + \rho(X))s = 0$$

with $s \in T^*[-1]\mathscr{E}$. That is, the fiber is given by the kernel of $Q + \rho(X)$, which is we view as a deformation of the free BV theory for Q. Summing up, we view $B\mathscr{F}$ as a family of free BV theories parametrized by the space $B\mathscr{L}$.

[5] This is, in essence, what Ray and Singer do [RS71]. They choose a Riemannian metric and then show that it doesn't matter at the end.

The commutative algebra of functions on $\mathscr{F}$ provides the observables of this classical field theory. Because we have a nonabelian elliptic L_∞ algebra, we use the Chevalley-Eilenberg cochain complex (completed, as Koszul duality requires). Equivalently, one can say that we have an interacting field theory so that we are studying a formal moduli problem and hence use "formal power series on fields."

DEFINITION 7.3.1. The observables of this classical field theory are the cosheaf of commutative algebras assigning to each open U, the commutative algebra

$$\mathrm{Obs}^{\mathrm{cl}}(U) = C^*(\mathscr{L}(U), \widehat{\mathrm{Sym}}(\mathscr{E}(U)^\vee \oplus (\mathscr{E}^!(U))^\vee[1])),$$

where C^* denotes the Chevalley-Eilenberg cochain complex for the L_∞ algebra $\mathscr{L}(U)$ acting on its module $\widehat{\mathrm{Sym}}(\mathscr{E}^\vee \oplus (\mathscr{E}^!)^\vee[1])(U)$. We write the Chevalley-Eilenberg differential as

$$d_{\mathscr{L}} + d_{T^*[-1]\mathscr{E}} + Q,$$

where the first term encodes the action of $\mathscr{L}$ on itself, the second term encodes the action of $\mathscr{L}$ on the module, and the last denotes the internal differential Q of the $\mathscr{L}$-module.

7.3.2. Rephrasing the problem using action functionals. Let $\mathscr{F}[1] = \mathscr{L}[1] \oplus \mathscr{E} \oplus \mathscr{E}^![-1]$ denote a space of fields, where $T^*[-1]\mathscr{E}$ are the fields we will quantize and $\mathscr{L}[1]$ provide "background classical fields" that we will not quantize.[6] Note the shift, which is simply a consequence of the fact that the fields are the tangent space to $B\mathscr{F}$.

The action functional is

$$\begin{aligned} S(X, \phi, \psi) &= \langle \psi, Q\phi \rangle + \langle \psi, \rho_1(X \otimes \phi) \rangle + \frac{1}{2}\langle \psi, \rho_2(X \otimes X \otimes \phi) \rangle + \cdots \\ &= \langle \psi, Q\phi \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \langle \psi, \rho_n(X^{\otimes(n-1)} \otimes \phi) \rangle \\ &= \langle \psi, (Q + \rho(X))\phi \rangle \\ &= S_0(\phi, \psi) + I(X, \phi, \psi), \end{aligned}$$

with $X \in \mathscr{L}[1]$, $\phi \in \mathscr{E}$, and $\psi \in \mathscr{E}^![-1]$. This action functional describes an *interacting* field theory where the interaction term I is linear in both $\mathscr{E}$ and $\mathscr{E}^!$.

We want to include a constraint on the background fields, namely that they are points of $B\mathscr{L}$ (equivalently, solutions to the Maurer-Cartan equation for $\mathscr{L}$). For such X, we obtain a classical

[6]In [Cos11], chapter 2, section 13, there is a discussion of such background fields, called there "non-propagating fields." The goal is to include local parameters (e.g., a dependence on a Riemannian metric, viewed as a background field) for a quantum field theory. We discuss these issues in more detail in section 7.4.

free BV theory where the "quadratic term" arises from the operator $Q + \rho(X)$. Our action functional thus defines a family of free field theories over the formal moduli space $B\mathscr{L}$. With this idea in mind, we provide a definition that captures this constraint.[7]

DEFINITION 7.3.2. The observables of this classical field theory are the cosheaf of commutative algebras assigning to each open U, the commutative algebra

$$\mathrm{Obs}^{\mathrm{cl}}(U) = C^*(\mathscr{L}(U), \widehat{\mathrm{Sym}}(\mathscr{E}(U)^\vee \oplus (\mathscr{E}^!(U))^\vee[1])),$$

where C^* denotes the Chevalley-Eilenberg cochain complex for the L_∞ algebra $\mathscr{L}(U)$ acting on its module $\widehat{\mathrm{Sym}}(\mathscr{E}^\vee \oplus (\mathscr{E}^!)^\vee[1])(U)$. We write the Chevalley-Eilenberg differential as

$$d_{\mathscr{L}} + \{S, -\} = d_{\mathscr{L}} + \{I, -\} + Q,$$

where the first term encodes the action of $\mathscr{L}$ on itself, the second term encodes the action of $\mathscr{L}$ on the module, and the last denotes the derivation Q of the module.

REMARK 7.3.3. In contrast with our earlier work on free theories, we've extended our observables by including the dependence on $\mathscr{L}$ and switched to using distributional duals because we have an interacting theory. Of course, we can also view this as describing the $\mathscr{L}$-module structure of the classical observables on $T^*[-1]\mathscr{E}$.

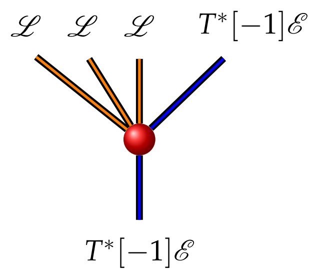
Notice that we have a natural map from $C^*\mathscr{L}$ to $\mathrm{Obs}^{\mathrm{cl}}$, so that we can view our classical observables as a family of commutative algebras over the formal moduli space $B\mathscr{L}$. We can thus view $\mathrm{Obs}^{\mathrm{cl}}$ as parametrizing a family of deformation problems over this moduli space: for each X solving the Maurer-Cartan equation for $\mathscr{L}$, we are asking for solutions to the equation $Q + \rho(X)$.

7.3.3. Feynman diagrams and obstructions. We now describe this theory using Feynman diagram, as these pictures make clear the structures we want to focus upon. (From the elliptic L_∞ perspective, we're simply drawing the usual pictures for the L_∞ brackets of $\mathscr{F}$.) For each nontrivial bracket

$$\rho_{n+1} : \mathscr{L}^{\otimes n} \otimes \mathscr{E} \to \mathscr{E},$$

there is an the interaction term I_{n+2} which corresponds to a $(n+2)$-valent vertex (in red) with an n-fold input from $\mathscr{L}$ (the orange edges) and two inputs from $T^*[-1]\mathscr{E}$. The case with $n = 3$ is pictured below.

[7]There is another way to impose this constraint, perhaps more natural from the QFT perspective. We add antighosts $\mathscr{L}^![-2]$ in addition to the ghosts $\mathscr{L}[1]$ but treat the ghosts and antighosts as background fields. This approach simplifies some aspects of the construction but complicates others. In the end, both achieve the same end.



7.3.4. BV quantization. We now review how to BV quantize an *interacting* field theory. Because one cannot multiply distributions, the naive Poisson bracket and BV Laplacian no longer exist, so there are analytic issues to resolve. We defer those issues momentarily. Suppose the bracket and BV Laplacian existed in the naive sense. Then BV quantization means finding an action functional $S_q = S + I_q$ depending on a formal parameter $\hbar$ such that

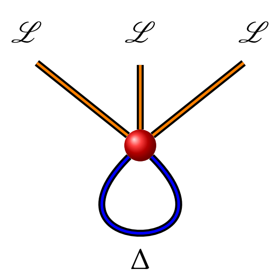
(1) $S_q \mod \hbar = S$ and
(2) S_q satisfies the quantum master equation (QME)

$$\{S_q, S_q\} + \hbar \Delta S_q = 0.$$

How does our initial action functional S fail to satisfy the QME? For $X \in \mathscr{L}$ satisfying the Maurer-Cartan equation, if we plug our action functional into the QME, we find

$$\{S, S\} + \hbar \Delta S = (Q + \rho(X))^2 + \hbar \Delta I = \hbar \Delta I.$$

Thus ΔI is the only term that might not vanish. As the BV Laplacian uses the pairing between $\mathscr{E}$ and $\mathscr{E}^!$, we see that ΔI_5, for instance, has the following form.



The sum $\sum_n \Delta I_n$ thus computes the trace of X acting on $\mathscr{E}$. Hence, the first obstruction to quantizing our classical action is a natural invariant of the action of $\mathscr{L}$ on $\mathscr{E}$!

This naive approach is essentially correct. The only problem is that the kernel for Δ is distributional and hence cannot be applied directly to the I_n, which are also distributional. We circumvent this problem by using renormalization techniques, which provide a systematic way to mollify Δ. Equivalently, we replace our observables by a homotopy equivalent $\mathscr{L}$-module for which it does make sense to take the trace.

The proof will follow quickly once we have the machinery from [**Cos11**] (with small modifications developed in [**CG**]) in place, so we review the relevant results in the next section and then give the proof in the subsequent section.

7.4. Background about BV theories and renormalization

We provide a synopsis of the main definitions and results from [**Cos11**] relevant to the proof.

7.4.1. A condition on free BV theories. To use the results of [**Cos11**], it is necessary that Q possesses an "adjoint" Q^* so that $D = [Q, Q*]$ is a generalized Laplacian. This condition insures the existence of nice asymptotic expansions for the propagator (a parametrix for D) that underlies many of the results in the book. Here is the precise definition.

DEFINITION 7.4.1. A *gauge fixing operator* on a free BV theory $\mathscr{E}$ is an operator $Q^* : \mathscr{E} \to \mathscr{E}$ such that

(1) Q^* has cohomological degree -1, with $(Q^*)^2 = 0$, and self-adjoint for the pairing $\langle -, - \rangle$ on $\mathscr{E}$,
(2) the commutator
$$D := [Q, Q^*]$$
is a generalized Laplacian.

This condition restricts our attention to a subset of all elliptic complexes but is fairly weak in practice. Many examples are given in [**Cos11**]. Typically, the space of such gauge-fixes is contractible if not it is not empty.

NOTE 7.4.2. *From hereon, a free BV theory will always possess a gauge-fixing operator. We discuss the dependence on this choice in subsection 7.4.6 below.*

7.4.2. Recollections on parametrices. We recall some definitions from [**Cos11**] and [**CG**]. In this subsection, let $\mathscr{E}$ denote an arbitrary free BV theory possessing gauge-fixes. Fix a gauge-fixing operator Q^* for Q and consider the degree 0 operator $D = [Q, Q^*]$.

DEFINITION 7.4.3. A *parametrix* Φ is an $F \boxtimes F$-valued distribution on $M \times M$ such that

(1) Φ is symmetric across the diagonal,
(2) Φ has cohomological degree 1,
(3) Φ is closed under the differential $Q \otimes 1 + 1 \otimes Q$,
(4) Φ has proper support (so under the two projections onto M, the support is proper),
(5) $(D \otimes 1)\Phi - \delta_M$ is a smooth section of $F \boxtimes F$ on $M \times M$ where δ_M refers to the delta function on the diagonal in $M \times M$ tensored with the identity section.

Thus $(D \otimes 1)\Phi - \delta_M$ is an element of $\mathcal{F}(M) \otimes \mathcal{F}(M)$.

We use a parametrix in several ways. First, we will use it to give a definition of a *pre-theory*, which is an effective field theory that may not satisfy the quantum master equation. Then we construct the desired Poisson bracket and BV Laplacian, which allows us to discuss the QME and provide a definition of a *theory*.

7.4.3. RG flow and pre-theories. The collection of possible *effective* interaction terms for a theory is the following.

DEFINITION 7.4.4. Let $\mathscr{O}(\mathscr{E})[[\hbar]]^+$ denote the subset of $\widehat{\mathrm{Sym}}(\mathcal{F}^\vee)[[\hbar]]$ given by elements that are cubic modulo $\hbar$.

The following operation relates different "effective actions," as we'll see below.

DEFINITION 7.4.5. For Φ a parametrix, let $P(\Phi) := (Q^* \otimes 1)\Phi$ be the associated *propagator*. For any two parametrices Φ and Φ', the *renormalization group (RG) operator* is the map

$$W(\Phi - \Phi', F) := \hbar \log(\exp(\hbar \partial_{P(\Phi)-P(\Phi')}) \exp(F/\hbar)),$$

for any $F \in \mathscr{O}(\mathscr{E})[[\hbar]]^+$.

REMARK 7.4.6. As explained in chapter 2, section 3 of [Cos11], the RG operator is another way of writing a Feynman diagram expansion (since we take a logarithm, this is the graph expansion with connected graphs). As explained there, we work with stable graphs: each vertex is labelled by a nonnegative integer called its "internal genus" and any vertex of genus 0 must be at least trivalent. The genus of a graph is the sum of its first Betti number and the sum of the internal genera of the vertices. In the Feynman graph expansion, a stable graph γ gives a term weighted by $h^{genus(\gamma)}$. Thus the h^0 term of the RG operator corresponds to the tree-level Feynman expansion.

We view the different parametrices as providing different "levels of resolution" for measurements. A pre-theory consists of a coherent system of action functionals where we recover the same behavior at different levels of resolution.

DEFINITION 7.4.7. Given a free BV theory $\mathscr{E}$, a *pre-theory* is a collection of functionals

$$\{I[\Phi]\} \in \mathscr{O}(\mathscr{E})[[\hbar]]^+,$$

where Φ ranges over parametrices for the free theory, such that the following hold:

(1) For two parametrices Φ and Φ', we have

$$W(\Phi - \Phi', I[\Phi']) = I[\Phi],$$

which is well-defined because $P(\Phi) - P(\Phi')$ is smooth on $M \times M$.

(2) The functionals $I[\Phi]$ satisfy the following locality axiom. For any pair of nonnegative integers (g,k) and open neighborhood U of the small diagonal $M \subset M^k$, there exists a parametrix Φ such that the support

$$\operatorname{Supp} I_{g,k}[\Phi'] \subset U$$

for all parametrices Φ' with $\operatorname{Supp}\Phi' \subset \operatorname{Supp}\Phi$, where $I_{g,k}$ denotes the component in $\operatorname{Sym}^k(\mathscr{E}^\vee)$ weighted by $\hbar^g$.

(3) The functionals $I[\Phi]$ all have smooth first derivative (i.e., the directional derivative with respect to smooth sections $\phi \in \mathscr{E}$ extends to a map on distributional sections).

One of the central theorems of [Cos11] is that the set of pre-theories is in bijection with a space of (more accurately, built from) local action functionals.

THEOREM 7.4.8. *Let $\widetilde{\mathscr{T}}^{(n)}$ denote the set of pre-theories modulo $\hbar^{n+1}$. Then, in a canonical way, $\widetilde{\mathscr{T}}^{(n+1)}$ is a principal bundle over $\widetilde{\mathscr{T}}^{(n)}$ for the abelian group $\mathscr{O}_{loc}(\mathscr{E})$ of local action functionals on $\mathscr{E}$. Furthermore, $\widetilde{\mathscr{T}}^{(0)}$ is canonically isomorphic to the space $\mathscr{O}_{loc}(\mathscr{E})^+$ of local action functionals that are at least cubic.*

This theorem follows from a less-canonical theorem that constructs an effective field theory for each local action functional by the addition of counterterms, depending upon a choice of *renormalization scheme* (i.e., a vector space decomposition of $C^\infty((0,\infty))$ into a direct sum $C^\infty([0,\infty)) \oplus Sing$, where $Sing$ is a space of functions that are singular at 0).

REMARK 7.4.9 (Background fields and pre-theories in families). In chapter 2, section 13 of [Cos11], a useful extension of theorem 7.4.8 is provided (see theorem 13.4.3). One considers $E = E_1 \oplus E_2$ a direct sum of $\mathbb{Z}$-graded bundles on M — the propagating and non-propagating fields, respectively — so that $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$. Given a graded manifold X, one defines a free BV theory over X with fields $\mathscr{E}$ to be the usual structure for $\mathscr{E}_1$ where all the data is linear over the graded commutative algebra $\mathscr{O}_X$. One then simply views the parametrices and so on as living on $\mathscr{E}$ by tensoring with identity on $\mathscr{E}_2$ in all the appropriate places. There is then a version of theorem 7.4.8 characterizing pre-theories over X where the abelian group is now $\mathscr{O}_{loc}(\mathscr{E}) \otimes \mathscr{O}_X$. Our proof of the index theorem relies on this enhanced version of theorem 7.4.8 but we suppress the concomitant notational overhead as it obfuscates the structure of the proof.

7.4.4. BV theories and the quantum master equation. A BV theory is a pre-theory satisfying a further condition on its action functional, known as the *quantum master equation* (QME). There is extensive discussion of this condition in [Cos11], [CG], and, of course, the rest of this thesis. We thus simply explain how to phrase this condition in this setting without justification for our interest in the QME.

Before we can discuss the QME, we need to have a BV Laplacian. Again, a parametrix is crucial.

DEFINITION 7.4.10. Let Φ be a parametrix. Let

$$K_\Phi := \delta_M - (D \otimes 1)\Phi$$

denote the associated Φ-*pairing*. The *BV Laplacian* associated to Φ is the differential operator

$$\Delta_\Phi := -\partial_{K_\Phi},$$

which acts on $\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)$ by contraction with $-K_\Phi$. The associated *Poisson bracket* is

$$\{a, b\}_\Phi := \Delta_\Phi(ab) - (\Delta_\Phi a)b - (-1)^{|a|}a(\Delta_\Phi b).$$

These objects provide the basic ingredients of a BD algebra. We want to equip the graded commutative algebra $\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]]$ with the structure of a BD algebra with the differential

$$d_\Phi := Q + \{I[\Phi], -\}_\Phi + \hbar\Delta_\Phi.$$

But it may be the case that this putative differential does not square to zero.

DEFINITION 7.4.11. A pre-theory satisfies the QME for parametrix Φ if $d_\Phi^2 = 0$. Equivalently, the condition is

$$QI[\Phi] + \frac{1}{2}\{I[\Phi], I[\Phi]\}_\Phi + \hbar\Delta_\Phi I[\Phi] = 0,$$

which is the more familiar form.

By the lemma below, we know that if a pre-theory satisfies the QME for Φ, it satisfies the QME for every parametrix. Hence, we introduce the following definitions.

DEFINITION 7.4.12. A *theory* is a pre-theory satisfying the QME for some parametrix. The *quantum observables* for parametrix Φ are the BD algebra

$$\mathrm{Obs}^q[\Phi] := (\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]], d_\Phi).$$

We denote the space of theories $\mathscr{T}$ and the space of theories modulo $\hbar^{n+1}$ by $\mathscr{T}^{(n)}$.

The crucial insight[8] is that the RG operator intertwines with the BD algebra structure.

LEMMA 7.4.13. *Given two parametrices Φ and Φ', we have the following equality:*

$$(Q + \hbar\Delta_\Phi)(e^{\hbar\partial_P}e^{F/\hbar}) = e^{\hbar\partial_P}(Q + \hbar\Delta_{\Phi'})e^{F/\hbar}$$

where $F \in \widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]]$ and $P = P_\Phi - P_{\Phi'}$. In consequence, we have

$$(Q + \hbar\Delta_\Phi)\exp(\hbar^{-1}W(P, F)) = \exp(\hbar^{-1}W(P, (Q + \hbar\Delta_{\Phi'})F)).$$

In particular, if a pre-theory satisfies the QME for parametrix Φ', it also satisfies the QME for parametrix Φ.

[8] More accurately, we should say "the crucial discovery by Costello."

PROOF. Observe that

$$\begin{aligned}(Q \otimes 1 + 1 \otimes Q)P(\Phi) &= (D \otimes 1)\Phi - (Q^* \otimes 1)(Q \otimes 1 + 1 \otimes Q)\Phi \\ &= (D \otimes 1)\Phi.\end{aligned}$$

Hence, we obtain a relationship between the operation on $\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]]$:

$$[\partial_P, Q] = \partial_{K_{\Phi'}} - \partial_{K_\Phi} = \Delta_\Phi - \Delta_{\Phi'}.$$

(This fact is easiest to see by drawing the graphical description of the statement.) As pure derivatives commute, we have

$$[\partial_P, \Delta_\Phi] = 0$$

and likewise for $\Delta_{\Phi'}$. Putting these facts together, we see that

$$\begin{aligned}(Q + \hbar\Delta_\Phi)e^{\hbar\partial_P} &= \sum_{n \geq 0} \frac{\hbar^n}{n!}(Q + \hbar\Delta_\Phi)\partial_P^n \\ &= Q + \sum_{n \geq 1} \frac{\hbar^n}{n!}(Q\partial_P + n\Delta_\Phi)\partial_P^{n-1} \\ &= Q + \sum_{n \geq 1} \frac{\hbar^n}{n!}(\partial_P Q + (n-1)\Delta_\Phi + \Delta_{\Phi'})\partial_P^{n-1} \\ &\quad\vdots \\ &= Q + \sum_{n \geq 1} \frac{\hbar^n}{n!}\partial_P^n(Q + n\Delta_{\Phi'}) \\ &= e^{\hbar\partial_P}(Q + \hbar\Delta_{\Phi'}).\end{aligned}$$

This computation gives the first two assertions of the lemma.

As the QME

$$QI[\Phi] + \frac{1}{2}\{I[\Phi], I[\Phi]\}_\Phi + \hbar\Delta_\Phi I[\Phi] = 0$$

is equivalent to the equation

$$(Q + \hbar\Delta_\Phi)e^{I[\Phi]/\hbar} = 0,$$

the computation also implies the final assertion of the lemma. □

This lemma implies a relationship between the observables with respect to different parametrices. It is a commonplace idea in physics that the observables of a theory are given by the tangent space to its action functional in the space of functionals. We thus need to use the derivative of the RG operator.

DEFINITION 7.4.14. Given a theory $\{I[\Phi]\}$, let $\partial W(\Phi - \Phi', I)$ denote the operator

$$F \mapsto \frac{d}{d\delta} W(\Phi - \Phi', I[\Phi'] + \delta F)$$

for $F \in \widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]]$ of cohomological degree n and δ of degree $-n$ with $\delta^2 = 0$.

COROLLARY 7.4.15. *The operator* $\partial W(\Phi - \Phi', I)$ *provides a homotopy equivalence*

$$\mathrm{Obs}^q[\Phi'] \xrightarrow{\simeq} \mathrm{Obs}^q[\Phi]$$

between the two BD algebra structures on $\widehat{\mathrm{Sym}}(\mathscr{E}^\vee)[[\hbar]]$.

PROOF. Apply the lemma to $I[\Phi'] + \delta F$ with $\delta^2 = 0$ and $|\delta| = -|F|$. $\square$

7.4.5. Obstructions to constructing a BV theory. Just as we construct the pre-theories inductively in powers of $\hbar$, it is natural to construct theories inductively. Thus, if we have a theory modulo $\hbar^n$, we might ask whether we can add a term $\hbar^n I^{(n)}$ that satisfies the QME modulo $\hbar^{n+1}$. In [Cos11], the following lemma gives a cohomological answer to this question (see chapter 5, section 11).

For a free BV theory $(\mathscr{E}, Q)$ and $\{I[\Phi]\}$ in $\mathscr{T}^{(n)}$ a theory modulo $\hbar^{n+1}$. Let $\{I_0[\Phi]\}$ denote the interaction term modulo $\hbar$, which defines the associated "effective" classical field theory. Pick an arbitrary lift $\{\widetilde{I}[\Phi]\}$ of this theory to a pretheory modulo $\hbar^{n+2}$. As this theory satisfies the QME for parametrix Φ modulo $\hbar^{n+1}$, the failure to satisfy the QME modulo $\hbar^{n+2}$ is

$$\mathcal{O}_{n+1}[\Phi] := \frac{1}{\hbar^{n+1}}\left(Q\widetilde{I}[\Phi] + \frac{1}{2}\{\widetilde{I}[\Phi], \widetilde{I}[\Phi]\}_\Phi + \hbar\Delta_\Phi \widetilde{I}[\Phi]\right).$$

We call this the *obstruction cocycle*. We now explain what cochain complex it lives in.

PROPOSITION 7.4.16. *For* δ *a parameter of cohomological degree* -1 *such that* $\delta^2 = 0$*, the functional*

$$I_0[\Phi] + \delta\mathcal{O}_{n+1}[\Phi]$$

satisfies the classical master equation for Φ *and the classical RG equation. Hence this functional is a classical BV theory and there thus exists a local functional* $I_0 + \delta\mathcal{O}_{n+1}$ *satisfying the CME for the local Poisson bracket. Thus* $\mathcal{O}_{n+1}$ *is a closed, degree 1 element of the cochain complex*

$$(\mathscr{O}_{loc}(\mathscr{E})_{red}, Q + \{I_0, -\}),$$

called the obstruction-deformation complex *for the classical theory* I_0.

7.4.6. The space of gauge-fixes. We now turn to a structural feature of this approach to QFT. To construct the quantization of a classical BV theory, we made a choice of a gauge-fixing operator Q^*. It is crucial to understand the dependence of our constructions on this choice. Ideally, any two choices will lead to equivalent spaces of BV quantizations, so that the choice is essentially unimportant. For comparison, note that theorem 7.4.8 provides a canonical space of pre-theories, but a choice of renormalization scheme yields a more concrete description of this space. Since the space of choices is a space of vector space splittings, we have an explicit understanding of how our choice impacts our constructions.

In chapter 5, section 10 of [Cos11], there is a theorem that addresses this issue, using the formalism of simplicial sets. One can construct a simplicial set $\mathscr{GF}(\mathscr{E}, Q)$ of gauge-fixes for a free BV theory $(\mathscr{E}, Q)$ and then construct a fibration of simplicial sets

$$\mathscr{T}(\mathscr{E}, Q) \to \mathscr{G}\mathscr{F}(\mathscr{E}, Q)$$

whose fiber over a point Q^* is the simplicial set of BV theories using that gauge-fixing. Moreover, in all examples currently studied, the space $\mathscr{G}\mathscr{F}(\mathscr{E}, Q)$ is contractible, so that the we see that the space of theories is essentially unique.

DEFINITION 7.4.17. A *family of gauge-fixing operators* over Δ^n for the free theory $(\mathscr{E}, Q)$ is a $\Omega^*(\Delta^n)$-linear differential operator

$$Q^* : \mathscr{E} \otimes \Omega^*(\Delta^n) \to \mathscr{E} \otimes \Omega^*(\Delta^n)$$

of cohomological degree -1 such that

(1) Q^* is self-adjoint for the $\Omega^*(\Delta^n)$-linear pairing

$$\langle -, - \rangle : \mathscr{E} \otimes \mathscr{E} \otimes \Omega^*(\Delta^n) \to \Omega^*(\Delta^n);$$

(2) $(Q^*)^2 = 0$;
(3) the operator $D = [Q + d_{dR}, Q^*]$ is a generalized Laplacian (i.e., a family of generalized Laplacians on M smoothly parametrized by Δ^n).

DEFINITION 7.4.18. The *space of gauge-fixes* $\mathscr{G}\mathscr{F}(\mathscr{E}, Q)$ is the simplicial set whose n-simplices are given by the set of Δ^n-families of gauge-fixes, with the obvious face and degeneracy maps.

Given a family of gauge-fixes, one can ask for parametrices, which are simply $\Omega^*(\Delta^n)$-linear versions of the above definition of parametrices. We obtain propagators and BV Laplacians in consequence.

We now define families of pre-theories analogously.

DEFINITION 7.4.19. Given a free BV theory $\mathscr{E}$, a *family of pre-theories* over Δ^n consists of a Δ^n-family of gauge-fixes Q^* and a collection of functionals

$$\{I[\Phi]\} \in \mathscr{O}(\mathscr{E})[[\hbar]]^+ \otimes \Omega^*(\Delta^n),$$

where Φ ranges over parametrices for the free theory, satisfying the conditions of pre-theory $\Omega^*(\Delta^n)$-linearly.

We denote the simplicial set of pre-theories by $\widetilde{\mathscr{T}}(\mathscr{E}, Q)$.

There is a natural map of simplicial sets $\widetilde{\mathscr{T}}(\mathscr{E}, Q) \to \mathscr{G}\mathscr{F}(\mathscr{E}, Q)$. By the families-version of theorem 7.4.8, we see that $\widetilde{\mathscr{T}}^{(n)}(\mathscr{E}, Q) \to \widetilde{\mathscr{T}}^{(n-1)}(\mathscr{E}, Q)$ is a principal bundle for the simplicial abelian group $\mathscr{O}_{loc}(\mathscr{E})$, whose n-simplices are given by Δ^n-families of local action functionals. As the classical pre-theories are independent of gauge-fix, we see that the base space has the form $\mathscr{G}\mathscr{F}(\mathscr{E}, Q) \times \mathscr{O}_{loc}(\mathscr{E})$.

We now describe families of solutions to the QME.

DEFINITION 7.4.20. A Δ^n-family of pre-theories *satisfies the QME for parametrix* Φ if

$$(Q + d_{dR} + \hbar\Delta_\Phi)\exp(I[\Phi]/\hbar) = 0.$$

Under RG flow, solutions to the QME are preserved. Thus, a *family of theories* is a family of pre-theories satisfying the QME for some parametrix. We denote the simplicial set of theories by $\mathscr{T}(\mathscr{E}, Q)$.

It is a corollary of the obstruction lemma from the previous subsection that the map $\mathscr{T}(\mathscr{E}, Q) \to \mathscr{G}\mathscr{F}(\mathscr{E}, Q)$ is a fibration.

7.5. The proof

The tools introduced in the preceding section 7.4 allow us to realize *mutatis mutandis* the naive idea sketched at the end of section 7.3.[9] Following the format in the preceding section, we construct

- a pre-theory for our classical BV theory (more accurately, a family of pre-theories for the family of classical theories over $B\mathscr{L}$),
- the obstruction to solving the QME, which lives in $C^*_{loc}(\mathscr{L})_{red}$, and
- the distinguished section of $B\mathscr{L}$ when the obstruction vanishes.

The work consists in examining the structure of the Feynman diagrams and invoking the appropriate theorems from [**Cos11**].

We discuss the meaning of the obstruction and the distinguished section for (global sections of) a closed manifold in the next section.

7.5.1. Counterterms and Feynman diagrams.

Recall that our classical interaction term is

$$I(X,\phi,\psi) = \sum_{n\geq 1} I_n(X,\phi,\psi) = \sum_{n\geq 1} \frac{1}{n!}\langle \psi, \rho_n(X^{\otimes(n-1)}\otimes\phi)\rangle,$$

with X in $\mathscr{L}[1]$, ϕ in $\mathscr{E}$, and ψ in $\mathscr{E}^![-1]$. By theorem 7.4.8, we know there exists a pre-theory corresponding to this interaction. We now describe this pre-theory using counterterms, so that we assume the choice of a renormalization scheme.

7.5.1.1. *Tree-level Feynman diagrams.* We begin by examining the $\hbar^0$ component of the RG flow operator on I. As discussed in remark 7.4.6, this corresponds to the tree-level Feynman diagram expansion. One can verify directly that tree-level Feynman diagrams never exhibit divergences.[10]

[9] Any renormalization procedure, in fact, should provide an approach to these issues.

[10] Here is a quick way to see this fact. Pick a root for the tree. If the tree has $n+1$ external edges, then it defines an operator $\mathscr{E}^{\otimes n} \to \mathscr{E}$ given by a sequence of multilinear operations (arising from the vertices) followed by applying the propagator sitting on each edge. At no point is a distribution paired with a distribution.

Thus, the RG operator

$$W(\Phi, F) \mod \hbar$$

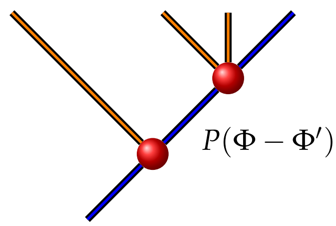
can be defined directly for a parametrix Φ even though it is distributional.

DEFINITION 7.5.1. For a parametrix Φ, set

$$I_t[\Phi] := W(\Phi, I) \mod \hbar.$$

The subscript t stands for "tree-level."

For our theory, the trees always have two external edges labeled by $T^*[-1]\mathscr{E}$ and the remaining edges labeled by $\mathscr{L}$. For example, shown below is a tree with two vertices given by the interaction terms I_4 and I_3.



The tree-level interactions define a "classical effective pre-theory." The following lemma gives a precise meaning to this notion.

LEMMA 7.5.2. *For a parametrix Φ, we have a complex*

$$\mathrm{Obs^{cl}}_\Phi := (\widehat{\mathrm{Sym}}(\mathscr{L}^\vee[-1] \oplus \mathscr{E}^\vee \oplus (\mathscr{E}^!)^\vee[1]), d_{\mathscr{L}} + Q + \{I[\Phi], -\}_\Phi),$$

describing the classical observables for parametrix Φ. *The tree-level RG operator $\partial W(\Phi, I) \mod \hbar$ is a quasi-isomorphism*

$$\mathrm{Obs^{cl}} \xrightarrow{\simeq} \mathrm{Obs^{cl}}_\Phi$$

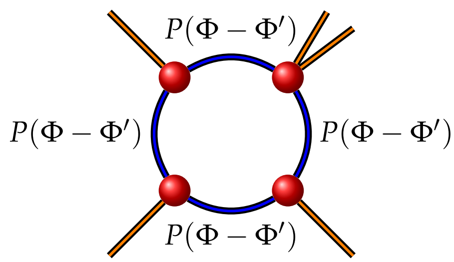
with strict inverse $\partial W(-\Phi, I) \mod \hbar$.

PROOF. The proof is analogous to the proof of lemma 7.4.13, where we showed that the operator $W(\Phi - \Phi', -)$ intertwines the BV Laplacians Δ_Φ and $\Delta_{\Phi'}$. Indeed, if we take this operator modulo $\hbar$, we obtain the desired quasi-isomorphism between $\mathrm{Obs^{cl}}_\Phi$ and $\mathrm{Obs^{cl}}_{\Phi'}$. But at the tree-level, we can take $\Phi' = 0$. $\square$

This lemma has a natural Lie-theoretic interpretation. The mod $\hbar$ RG operator provides a homotopy equivalence between two different $\mathscr{L}$-modules, namely the $\mathrm{Obs^{cl}}$ and $\mathrm{Obs^{cl}}_\Phi$.

7.5.1.2. *Wheels and divergences.* We now consider the full Feynman expansion. Thanks to the simple nature of our theory, only simple diagrams appear.

In contrast to the tree-level case, we cannot stick a parametrix into the RG operator. Fix two parametrices Φ and Φ'. Set $\tilde{I} = W(\Phi - \Phi', I)$. The terms of this (nonlocal) functional will indicate what kind of divergences can appear. For us, the Feynman diagrams come in two flavors. There are trees obtained by attaching copies of our vertices with the propagator, as discussed above. There are also wheels whose external legs all take inputs from $\mathscr{L}$. Pictured below is a wheel with four vertices (three copies I_3 and one copy of I_4).



Our interaction term thus has the form $\tilde{I} = \tilde{I}_t + \hbar \tilde{I}_w$, where $\tilde{I}_t$ is the term given by the trees and $\tilde{I}_w$ is the term given by wheels.

Trees cannot yield divergences as the support of Φ' gets arbitrarily close to the origin, but the wheels can have divergences. Thus the counterterm J we use to eliminate these divergences will, like the wheels, only be a function on $B\mathscr{L}$. We have thus shown the following.

LEMMA 7.5.3. *There exists a pre-theory $\{I[\Phi]\}$ corresponding to the local action functional I such that, for every parametrix Φ,*

$$I[\Phi] = I_t[\Phi] + \hbar I_w[\Phi]$$

with

$$I_t = W(\Phi, I) \mod \hbar$$

and $I_w[\Phi] \in \mathscr{O}(B\mathscr{L})$.

7.5.2. The obstruction to BV quantization. We now address the question of whether this pre-theory $\{I[\Phi]\}$ satisfies the QME. We define the *obstruction* $\mathcal{O}[\Phi]$ to be the failure to satisfy the QME for parametrix Φ. In our case, this involves the $\mathscr{L}$-action — which is independent of the parametrix Φ — in addition to the terms involving just $\mathscr{E}$ and $\mathscr{E}^![-1]$. We want to compute

$$(d_{\mathscr{L}} + \{I[\Phi], -\}_{\Phi} + Q + \hbar\Delta_{\Phi})^2,$$

as the desired differential on the quantum observables is

$$d_{\mathscr{L}} + \{I[\Phi], -\}_{\Phi} + Q + \hbar\Delta_{\Phi}.$$

The failure of this operator to square to zero is precisely the obstruction to satisfying the QME.

LEMMA 7.5.4. *The failure to satisfy the QME is given by*

$$\mathcal{O}[\Phi] = \Delta_\Phi I_t[\Phi] + d_{\mathscr{L}} I_w[\Phi].$$

This obstruction yields a closed, degree 1 element $\mathcal{O}$ of $C^*_{loc}(\mathscr{L}, \mathscr{O}_{loc}(T^*[-1]\mathscr{E}))_{red}$ under tree-level RG flow. As this local obstruction $\mathcal{O}$ only depends on $\mathscr{L}$, it also provides a closed, degree 1 element of $C^*_{loc}(\mathscr{L})_{red}$.

PROOF. Modulo $\hbar$ first, the square is zero, as shown by lemma 7.5.2. Thus, it remains to consider terms that depend on $\hbar$. As $I_w[\Phi]$ only depends on $\mathscr{L}$, it is annihilated by Q and Δ_Φ and cannot pair nontrivially through the bracket $\{-,-\}_\Phi$. Thus, squaring our putative differential, we are left with

$$d_{\mathscr{L}} I_w[\Phi] + \Delta_\Phi I_t[\Phi].$$

By lemma 7.4.16, we can apply the classical RG operator to obtain the local obstruction. This element lives in the subcomplex $C^*\mathscr{L}$. □

REMARK 7.5.5. This obstruction cocycle has a Lie-theoretic interpretation as the character of the $\mathscr{L}$-module $T^*[-1]\mathscr{E}$. We've used the tree-level RG flow to obtain an L_∞-module structure on $T^*[-1]\mathscr{E}$ for which it is possible to take a trace: the BV Laplacian Δ_Φ is smooth and hence we avoid the divergences that might appear from naively taking the trace with Δ_0. In the next section, we consider the situation on a closed manifold, where the interpretation with traces is manifest.

7.5.3. If the obstruction vanishes. We can BV quantize precisely when the obstruction is an exact element. Let $\{J[\Phi]\}$ be a degree 0 element of $C^*(\mathscr{L}, \mathscr{O}(T^*[-1]\mathscr{E}))$ such that

$$d_{\mathscr{L}} J[\Phi] + \{I[\Phi], J[\Phi]\}_\Phi + QJ = \mathcal{O}[\Phi],$$

namely an element making $\mathcal{O}[\Phi]$ exact and hence cohomologically trivial. Then we obtain a theory $\{I_q[\Phi]\} = \{I[\Phi] - \hbar J[\Phi]\}$ which satisfies the QME. (The most canonical thing to do is to work over the vector space of all choices of J.) We write

$$I_q[\Phi] := I^{(0)}[\Phi] + \hbar I^{(1)}[\Phi]$$

with $I^{(0)}[\Phi] = I_t[\Phi]$ the $\hbar^0$ term and $I^{(1)}[\Phi] = I_w[\Phi] - J[\Phi]$ the $\hbar^1$ term (i.e., the one-loop correction that insures the theory satisfies the QME).

For X a point in $B\mathscr{L}$ (i.e., a solution to the Maurer-Cartan equation for some dg Artinian ring), we obtain an element

$$I_q[\Phi](X) \in \widehat{\mathrm{Sym}}(\mathscr{E}^\vee \oplus (\mathscr{E}^!)^\vee[1])[\hbar]$$

which satisfies the QME for parametrix Φ by evaluating the interactiona t X. This provides a distinguished element of the quantum observables over the point X: let $\mathrm{Obs^q}_X[\Phi]$ denote

$$(\widehat{\mathrm{Sym}}(\mathscr{E}^\vee \oplus (\mathscr{E}^!)^\vee[1])[\hbar], Q + \hbar\Delta_\Phi + \{I_q[\Phi](X), -\}_\Phi)$$

denote the *quantum observables for the deformation of the free theory by X* at parametrix Φ.[11]

To summarize, a choice of trivialization J leads to a family of factorization algebras over the (pre)sheaf of formal moduli spaces $B\mathscr{L}$, both of which live on the manifold M. There is a natural section of this family given by the interaction term I_q.

7.6. Global statements

Suppose M is closed. We restrict our attention to global sections of the observables, as the obstruction and distinguished section are easier to interpret in this case.

The best statements follow from a stronger hypothesis on our operator $D = [Q, Q^*]$.

DEFINITION 7.6.1. A gauge-fix Q^* is *non-negative* if

(1) the operator D is symmetric for some hermitian metric on the vector bundle $E \oplus E^![-1]$,
(2) the eigenvalues of D are nonnegative, and
(3) we have a direct sum decomposition
$$T^*[-1]\mathscr{E} := \ker D \oplus \operatorname{im} Q \oplus \operatorname{im} Q^*.$$
This decomposition is as topological vector spaces.

As should be clear, these conditions are just an imposition of a well-behaved Hodge theorem. In particularly, $\ker D$ is canonically isomorphic to the Q-cohomology of $T^*[-1]\mathscr{E}$. *We will assume in this section that we have a non-negative gauge-fix.*

In this setting, we can use the heat kernel to provide parametrices whose interpretation is probably more familiar. Let K_t denote the kernel of the operator e^{-tD}, with $t \geq 0$. Then the kernel
$$\Phi_L := \int_0^L K_t \, dt$$
provides a parametrix for $L > 0$. Working with these parametrices is the default practice in [Cos11]. As M is closed, the limit
$$T = \lim_{L \to \infty} \int_0^L (Q^* \otimes 1) K_t \, dt$$
exists and provides a partial inverse to Q as follows. Consider the decomposition of $\mathscr{E}(M) \oplus \mathscr{E}^!(M)[-1]$ into eigenspaces V_λ of D. Then
$$QT = \int_0^\infty D e^{-tD} \, dt = \int_0^\infty e^{-tD} D \, dt,$$

[11] Note that we have polynomial powers in $\hbar$. For most interacting theories, one uses formal power series in $\hbar$ because the RG flow usually produces interaction terms $I[\Phi]$ with infinitely many powers of $\hbar$. For our theory here, we have seen that only $\hbar$ appears and so our theory is well-defined for finite $\hbar$ (not just infinitesimal).

so that for an eigenvector $v \in V_\lambda$ with $\lambda > 0$, we see

$$QTv = \int_0^\infty e^{-tD} Dv\, dt = \int_0^\infty \lambda e^{-t\lambda} v\, dt = v,$$

and for $v \in V_0$, we see $QTv = 0$. Hence QT is the operator projecting onto the positive eigenspaces and so T provides an inverse to Q away from the "harmonic fields" V_0. We denote by $\mathcal{H}$ the null space of D on $\mathscr{E}$. Thus, $V_0 = T^*[-1]\mathcal{H}$.

Thanks to the continuous homotopy equivalence between $T^*[-1]\mathscr{E}$ and $T^*[-1]\mathcal{H}$, we can transfer questions about the full space of fields (e.g., what is the obstruction?) to questions about this finite-dimensional space. Often, constructions become more intelligible on this smaller space.

7.6.1. Tree-level RG flow as L_∞ transfer. We begin with the tree-level term $I_t[\Phi]$, which is purely classical. Working with the heat flow parametrices Φ_L, we have a homotopy equivalence of complexes between the classical observables at scale 0 and scale L. (We drop the appearance of the manifold M in $\mathrm{Obs}^{\mathrm{cl}}(M)$ from hereon and denote global observables by $\mathrm{Obs}^{\mathrm{cl}}$.) In this case, the interaction term $I_t[L] := I_t[\Phi_L]$ encodes simply the homotopy transfer lemma for L_∞ modules: we start with

$$\mathrm{Obs^{cl}}_0 := (C^*(\mathscr{L}, \widehat{\mathrm{Sym}}(\mathscr{E}^\vee \oplus \mathscr{E}^{!\vee}[1])), d_{\mathscr{L}} + Q + \{I, -\})$$

and transfer to obtain a homotopy equivalent L_∞-module structure

$$\mathrm{Obs^{cl}}_L := (C^*(\mathscr{L}, \widehat{\mathrm{Sym}}(\mathscr{E}^\vee \oplus \mathscr{E}^{!\vee}[1])), d_{\mathscr{L}} + Q + \{I_t[L], -\}).$$

When $L = \infty$, we have a particularly succinct description.

Explicitly, the transfer works as follows. For I_t (as with any trees), we can pick one outer leg and view it as the root. The rooted tree then defines a map from the "inputs" (the legs excluding the distinguished leg) to the "output" (the root). In our case, let's fix the root as a leg associated to $T^*[-1]\mathscr{E}$ (hence the map goes from $T^*[-1]\mathscr{E}$ to $T^*[-1]\mathscr{E}$). A tree thus corresponds to a map that takes ϕ to $\rho_{i_n}(X) \circ P \circ \rho_{i_{n-1}}(X) \circ P \circ \cdots \circ \rho_{i_1}(X)(\phi)$, where P denotes the operator corresponding to the propagator $P(\Phi - \Phi')$ and ρ_{i_k} denotes the operation labeled by the k^{th} vertex, ordered so that the last vertex is attached to the root. Note that for $L = \infty$, $P = T$, which is Q^{-1} on the nonzero eigenspaces V_λ and $P = 0$ on V_0. When we sum over all the trees, viewed as operators on $T^*[-1]\mathscr{E}$ we obtain

$$I_t[\infty](X) = \rho(X) \circ \left(\sum_{n=0}^\infty (T \circ \rho(X))^n \right) = \rho(X) \circ (1 - T\rho(X))^{-1}.$$

At scale ∞, we have equipped the very small algebra generated just by $T^*[-1]\mathcal{H}$, the cohomology of $T^*[1-]\mathscr{E}(M)$ with a $\mathscr{L}(M)$-module structure that we denote $\widetilde{\rho}$.

7.6.2. The obstruction. Consider $\mathcal{O}[\infty]$. By definition,

$$\mathcal{O}[\infty] = \Delta_\infty I_t[\infty] + d_{\mathscr{L}} I_w[\infty],$$

but we are free to ignore the second term as it is a boundary. Recall that the kernel defining the BV Laplacian Δ_∞ is $K_\infty =$ "$\exp(-\infty D)$", whose corresponding operator is the identity on V_0 and annihilates the positive eigenspaces. Hence $\mathcal{O}[\infty]$ amounts to tracing $I_t[\infty]$ over $V_0 = T^*[-1]\mathcal{H}$. We see that

$$\mathcal{O}[\infty] = \mathrm{Tr}_{V_0}(\{I_t[\infty], -\}_\infty) = \mathrm{Tr}_{V_0}\left(\sum_{n\geq 0} \rho(X)(T\rho(X))^n\right)$$

by using the operator interpretation of the trees. Thus, the obstruction is the "character" of the $\mathscr{L}$-representation $\widetilde{\rho}$ on $H^*T^*[-1]\mathscr{E}(M)$.

The element $X \in \mathscr{L}$ may not be trace-class on the complex $\mathscr{E}$, so the naive character $\mathrm{Tr}\,\rho(X)$ may not exist. We have mollified everything by the renormalization process, however. This element $\mathcal{O}[\infty]$ thus provides a well-defined version of $\mathrm{Tr}\,\rho(X)$ on all of $\mathscr{E}$, as the homological perturbation lemma replaces $\mathscr{E}$ by $\mathcal{H}$, leading to these powers of $T(= Q^{-1})$.

Formally applying the familiar identity for geometric series, we can write

$$\mathcal{O}[\infty]\text{"}=\text{"}\,\mathrm{Tr}_{V_0}\left(\rho(X)\cdot(1 - T\rho(X))^{-1}\right).$$

This expression is suggestive. Continuing in this vein, we have

$$\frac{X}{1 - Q^{-1}X} = \frac{X}{Q^{-1}(Q - X)} = \frac{QX}{Q - X}$$

for instance.

7.6.3. If the obstruction vanishes. If the obstruction is cohomologically trivial, we obtain a theory satisfying the QME,

$$I_q[\Phi] = I^{(0)}[\Phi] + \hbar I^{(1)}[\Phi],$$

after picking a trivialization $J[\Phi]$ for $\mathcal{O}[\Phi]$. We want to describe the 1-loop correction $I^{(1)}[\infty]$ at scale ∞ and, in particular, its image in $\widehat{\mathrm{Sym}}(\mathcal{H}^\vee \oplus \mathcal{H}[1])[\hbar]$, the graded space of quantum observables on harmonic fields.

We have a quasi-isomorphism

$$\mathrm{Obs}^{\mathrm{q}}_\infty \xrightarrow{\simeq} \left(\widehat{\mathrm{Sym}}(\mathcal{H}^\vee \oplus \mathcal{H}[1])[\hbar], \widetilde{\rho}(X) + \hbar\{I^{(1)}[\infty], -\}_{T^*[-1]\mathcal{H}} + \hbar\Delta_{T^*[-1]\mathcal{H}}\right),$$

where, on the right hand side, we use the Poisson bracket and BV Laplacian for the harmonic fields.

Working formally, there is an appealing interpretation of $I^{(1)}[\infty]$. Suppose that there were no need to add to add a counterterm J. Then we see that

$$I^{(1)}[\infty] = I_w[\infty] = \sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{Tr}_{T^*[-1]\mathscr{E}}(T\rho(X))^n.$$

But we can interpret this last term formally as

$$\mathrm{Tr}\log(1 - T\rho(X))$$

and by the equality $\mathrm{Tr}\log = \log\det$ for finite matrices, we obtain formally

$$I^{(1)}[\infty] \text{“} = \text{”} \log\det(1 - Q^{-1}X) = \log\det(Q^{-1}(Q - X)) = \log\det(Q - X) - \log\det(Q).$$

Thus, the wheels provide a renormalized definition of the logarithm of the ratio of determinants between the free theory defined by Q and the theory defined by $Q - X$. Compare this formal description to Ray-Singer torsion, where the determinant of an elliptic complex is given by using the zeta-function definition of torsion.

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