Homework 6—Midterm Exam

Math 652
Spring 2020

Due Friday, April 10, 2020

1 Spectral Methods

First, we review a few facts about the (semi-discrete) Fourier transform defined in class on February 26. For any 1-periodic function $u : \mathbb{R} \to \mathbb{C}$, we define $\hat{u} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{u}(k) = \langle \psi_k, u \rangle = \int_{x=0}^{1} u(x) \exp(-2\pi i k x)\, dx.$$

Here, the functions $\{\psi_k, k \in \mathbb{Z}\}$ defined by

$$\psi_k(x) = \exp(2\pi i k x)$$

are called the *characters*. When we used the Fourier transform to solve the advection-diffusion equation, we proved that for any 1-periodic $u \in C^n$,

$$\widehat{\frac{d^n u}{dx^n}}(k) = (2\pi i k)^n \hat{u}(k).$$

We also recall that

$$\|u\|_{L^2} = \|\hat{u}\|_{\ell^2} = \left\{ \sum_{k \in \mathbb{Z}} \hat{u}(k)^2 \right\}^{\frac{1}{2}}.$$

You have recently verified something similar for the finite Fourier transform. The general result is sometimes called *Plancherel's theorem.* It is pretty easy to prove once you remember that the characters are orthonormal and that by the Fourier inversion formula

$$u = \sum_{k \in \mathbb{Z}} \hat{u}(k) \psi_k,$$

where the sum converges in L^2.

The Fourier transform provides a means of characterizing Sobolev spaces. We say that a 1-periodic function u is weakly differentiable if there exists a 1-periodic Du so that for any 1-periodic $v \in C^\infty$,

$$\int_{x=0}^{1} Du(x) v(x)\, dx = -\int_{x=0}^{1} u(x) v'(x)\, dx.$$

Let $H^n(\mathbb{R}/\mathbb{Z})$ be the set of all 1-periodic functions with n weak derivatives such that each weak derivative is in L^2. One can show that $u \in H^n(\mathbb{R}/\mathbb{Z})$ if and only if

$$\sum_{k\in\mathbb{Z}}(1+k^2)^n|\hat{u}(k)|^2 < \infty.$$

Problem 1 (2+1 points)**.**

1. *Show that if $u \in H^1(\mathbb{R}/\mathbb{Z})$, then*

$$\widehat{\frac{du}{dx}}(k) = 2\pi i k \hat{u}(k).$$

   Hint*: You can use the argument from class. You just have to check that all the steps work when u is only* weakly *differentiable.*

2. *Show that if $u \in H^1(\mathbb{R}/\mathbb{Z})$, then*

$$\sum_{k\in\mathbb{Z}}(1+k^2)|\hat{u}(k)|^2 < \infty.$$

I won't make you prove that $\sum_{k\in\mathbb{Z}}(1+k^2)^n|\hat{u}(k)|^2 < \infty$ implies $u \in H^n(\mathbb{R}/\mathbb{Z})$, but I will say that you now know enough analysis to do it, at least mostly.

We now consider the accuracy of a Galerkin method based on the Fourier characters. This is an example of a *spectral method*. Let

$$V_m = \operatorname{span}\{\psi_{-m}, \ldots, \psi_m\}.$$

Our goal is to estimate the best approximation error

$$\inf_{v\in V_m}\|u-v\|_2,$$

where u is the solution of some boundary value problem (or other similar variational problem).

Problem 2 (5+3 points)**.** *Let $u \in H^n(\mathbb{R}/\mathbb{Z})$. Define*

$$S_m u = \sum_{k=-m}^{m}\hat{u}(k)\psi_k$$

1. *Show that*

$$\|u - S_m u\|_2 = O\left(\frac{1}{m^{(n-\frac{1}{2})}}\right)$$

   in the limit as $m \to \infty$.

   Hint: *Observe that if $\sum_{k\in\mathbb{Z}}\hat{u}(k)^2k^{2n} < \infty$, then we can only have $|\hat{u}(k)|^2 > 1/k^{2n}$ for at most finitely many k, hence $|\hat{u}(k)|^2 \le C/k^{2n}$ for some constant C.*

2. *How would the above result change if you wanted to measure the error in the $H^1(\mathbb{R}/\mathbb{Z})$-norm instead of in L^2?*

Observe that if $u \in C^\infty$, then

$$\|u - S_m u\|_2 = O\left(\frac{1}{m^\ell}\right)$$

for all $\ell > 0$! In this case, one says that the approximations converge with *infinite order* or at *spectral speed.* By comparison, convergence of finite elements is quite slow. For example, if we take V to be the set of all periodic, piecewise affine functions with respect to the mesh $\{0, 1/N, 2/N, \ldots, 1\}$, then by the standard analysis based on the Bramble–Hilbert lemma, we have only the estimate

$$\|u - u_V\|_2 = O\left(\frac{1}{N^2}\right).$$

This is the best general estimate for a finite element method based on piecewise affine functions, no matter how smooth the solution u might be. Thus, we conclude that finite element methods do not converge at spectral speed. However, this does not mean that finite element methods are useless. Spectral methods lead to system matrices that are dense, so it can be very costly to solve the minimization problem. Also, for BVPs on domains of complicated shapes it can be difficult or impossible to choose an appropriate basis for a spectral method.

There is a more refined (but also much more complicated) explanation of the convergence properties of spectral methods in the book by Arieh Iserles; see the course website for a link to the online version. There are many things that can go wrong with spectral approximations, so the book is a worthwhile read.

2 Functional Analysis

Functional analysis is the study of infinite-dimensional vector spaces. It concerns generalizations of finite-dimensional results such as the rank-nullity theorem. For example, suppose that for some domain $\Omega \subset \mathbb{R}^d$ it can be shown that the zero function is the only solution of the differential equation

$$\Delta u = 0$$

within the space $C_0^\infty(\Omega)$. If $C_0^\infty(\Omega)$ were finite-dimensional, we could conclude that since $\ker \Delta = 0$, $\operatorname{Rg} \Delta = C_0^\infty(\Omega)$ by rank-nullity, and therefore there must be a unique solution of the equation $\Delta u = f$ for any $f \in C_0^\infty(\Omega)$. Unfortunately, however, $C_0^\infty(\Omega)$ is infinite-dimensional, so rank-nullity does not apply. Instead, to show existence and uniqueness of solutions, one can apply a generalization of rank-nullity called the Fredholm alternative to a weak formulation of the problem.

Functional analysis is the foundation for a modern understanding of numerical methods for PDEs. A complete exposition is beyond the scope of this class, but the following problems should explain why it is necessary.

Definition 1. Let V and W be normed vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Let $L : V \to W$ be a linear operator. We say that L is *bounded* if and only if there exists some $c > 0$ so that

$$\|Lx\|_W \leq c\|x\|_V.$$

Remark 1. One can show that a linear operator is bounded if and only if it is continuous. It's easy. You can do it as an exercise, if you like.

If V is finite-dimensional, then every linear operator $L : V \to W$ is bounded. However, when V is infinite-dimensional, a linear operator may be unbounded.

Problem 3 (3 points)**.** *Let V be the set of all continuously differentiable functions $u : [0, 1] \to \mathbb{R}$. Define*

$$\|v\|_V = \|v\|_\infty$$

for all $v \in V$. Let W be the set of all continuous functions, and define

$$\|w\|_W = \|w\|_\infty.$$

Let $L : V \to W$ by

$$Lv = \frac{dv}{dx}.$$

Show that L is unbounded.

Hint: *It will suffice to exhibit a sequence u_n of elements of V with $\|u_n\|_V = 1$ for all n so that $\|Lu_n\|_W \to \infty$ as $n \to \infty$.*

You have already seen another example of an unbounded operator. In the first semester of this course, you showed that the L^1 and L^∞ norms on $C([0, 1])$ were not equivalent. In particular, there exists a sequence u_n of functions in $C([0, 1])$ so that $\|u_n\|_{L^1} = 1$ but $\|u_n\|_{L^\infty} \to \infty$ as $n \to \infty$. Let V be the space $C([0, 1])$ equipped with the L^1-norm, and let W be the same space equipped with the L^∞-norm. According to your homework, the linear operator $i : V \to W$ by $iu = u$ is unbounded.

The rather trivial operator i above is an example of an *embedding*, i.e. an inclusion of one vector space into another that contains the original space. We now consider another nicer embedding. Recall the Sobolev embedding theorem from class:

Theorem 1 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary. If $r > d/2$, then there exists a $c > 0$ so that for any $u \in H^r(\Omega)$,*

$$\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{H^r(\Omega)}.$$

One consequence of the Sobolev embedding the following vitally important lemma which was used in class to develop error estimates for finite elements:

Problem 4 (3 points)**.** *Let $T \subset \mathbb{R}^2$ be a triangle. Show that the affine interpolant $I_T : H^2(T) \to H^1(T)$ is bounded, i.e. there exists a constant $C > 0$ so that*

$$\|I_T u\|_{H^1(T)} \leq C\|u\|_{H^2(T)}.$$

Hint: *Use the Sobolev embedding, and remember that you have a nice explicit formula for $I_T u$ in your notes from before break.*

Problem 5 (3 points, extra credit)**.** *Show that if $r > d/2$, then any $u \in H^r(\Omega)$ is continuous.* Hint*: Use the Sobolev embedding, remember that for any function $u \in H^r(\Omega)$ there exists a sequence u_n of C^∞ functions so that $\|u - u_n\|_{H^r(\Omega)} \to 0$ as $n \to \infty$, and remember that if a sequence of continuous functions converges uniformly, then the limit is continuous.*