Homework 5

Math 652
Spring 2020

Due Friday, March 27, 2020

Consider the initital value problem

$$x' = f(x) \text{ with } x(0) = y,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a scalar function. Assume that f is Lipschitz with constant $L > 0$, i.e.

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}$. Recall that Euler's method for the solution of the IVP reads

$$x_{n+1} = x_n + \Delta t f(x_n) \text{ with } x_0 = y.$$

We have shown that Euler's method is *stable*. In particular, for any $T > 0$, if w_n solves the perturbed recurrence relation

$$w_{n+1} = w_n + \Delta t f(w_n) + G_n \text{ with } z_0 = y,$$

where $|G_n| \leq \varepsilon$ for all $n = 0, \dots, \lfloor T/\Delta t \rfloor$, then

$$\max_{n=0,\dots,\lfloor T/\Delta t \rfloor} |w_n - x_n| \leq \frac{C\varepsilon}{L\Delta t} \exp(TL).$$

Moreover, Euler's method is *consistent* of order one. That is, the exact solution x of the IVP solves

$$x((n+1)\Delta t) = x(n\Delta t) + \Delta t f(x(n\Delta t)) + H_n,$$

where $|H_n| \leq C\Delta t^2$ for some constant $C > 0$ independent of Δt. The error estimate

$$\max_{n=0,\dots,\lfloor T/\Delta t \rfloor} |x(n\Delta t) - x_n| \leq \frac{C}{L} \exp(TL)\Delta t$$

follows. You will now apply the technique of modified equations to the study of Euler's method.

Problem 1 (5 + 3 points)**.**

1. *Define*

$$g(x) = \frac{f(x)}{1 + \frac{\Delta t}{2} f'(x)},$$

and consider the modified equation

$$z'(t) = g(z(t)) \text{ with } z(0) = y.$$

(Note that g is defined whenever Δt is sufficiently small, if we assume that f' is bounded.) Show that the Euler recurrence

$$x_{n+1} = x_n + \Delta t f(x_n) \text{ with } x_0 = y$$

for the original equation has second order *consistency error as a numerical method for the modified equation.*

2. *Prove that if z solves the modified equation, then*

$$\max_{n=0,\dots,\lfloor T/\Delta t \rfloor} |z(n\Delta t) - x_n| \leq \frac{C}{L} \exp(TL)\Delta t^2.$$

You may use any results proved in class or stated above.

Recall that the modified equation for the upwind method is the advection-diffusion equation

$$\frac{\partial v}{\partial t} = -a\frac{\partial v}{\partial x} + \frac{1}{2}a\Delta t \left(1 - \frac{a\Delta t}{\Delta x}\right) \frac{\partial^2 u}{\partial x^2}.$$

As a consequence, one expects that if upwind is used to solve the advection equation, waves spread out as they propagate. You will now verify that this is the case.

Problem 2 (5 points)**.** *Use the upwind method to solve the advection equation on the domain $[0, 1]$ with $a = 1$, with periodic boundary conditions, and with initial condition*

$$u_0(x) = \exp\left(-128\left|x - \frac{1}{2}\right|^2\right).$$

For $i \in \{4, 5, 6, 7\}$, run the upwind method using the parameters $\Delta x = 2^{-i}$, $\Delta t = \frac{1}{2}\Delta x$, and $T = 2$. For each i, plot the solution at an appropriate sequence of times, say $t = 0.25, 0.5, 0.75, \dots, 2$. Verify that the wave spreads as it propagates; notice that it spreads more slowly when Δt is small.

Finally, I would like you to resolve a few points from the first lecture on Ritz methods.

Problem 3 (3 points)**.** *Let $\{\phi_1, \dots, \phi_N\}$ be a basis for a subspace $V \leq C_0^1$. Let $a : \Omega \to \mathbb{R}$ be a continuously differentatiable function so that* $\min a > 0$*. Show that the matrix $M \in \mathbb{R}^{N\times N}$ with entries*

$$M_{k\ell} = \int_\Omega a\nabla\phi_k \cdot \nabla\phi_\ell \, dx$$

is positive definite. Hint: *You're going to have to use the Poincaré inequality.*