Homework 3

Math 652
Spring 2020

Due Friday, February 28, 2020

Problem 1 (2+2+2 points)**.**

1. *Whenever you introduce the matrix exponential, you are required to give the following problem: Show that if X and Y are matrices with $XY = YX$, then* $\exp(X+Y) = \exp(X)\exp(Y)$.

2. *More importantly, you are also required to give this problem: Find a pair of matrices X and Y so that* $\exp(X+Y) \neq \exp(X)\exp(Y)$.

3. *Suppose that A is diagonalizable, so*

$$A = V\Lambda V^{-1},$$

where Λ is a diagonal matrix. Show that $\exp(A) = VMV^{-1}$, *where M is a diagonal matrix so that* $M_{ii} = \exp(\Lambda_{ii})$.

The discrete variation of parameters formula from the proof of convergence of Euler's method for the heat equation has a continuous counterpart called Duhamel's principle. Duhamel's principle represents the solution u of the *inhomogeneous* IBVP

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2}{\partial x^2}u(x,t) + f(x,t) && \text{for } (x,t) \in (0,1)\times(0,\infty) \\ u(0,t) &= u(1,t) = 0 && \text{for } t \in [0,\infty) \\ u(x,0) &= u_0(x) && \text{for } x \in [0,1] \end{aligned}$$

in terms of solutions of the *homogeneous* IBVP

$$\begin{aligned} \frac{\partial v}{\partial t}(x,t) &= \frac{\partial^2}{\partial x^2}v(x,t) && \text{for } (x,t) \in (0,1)\times(0,\infty) \\ v(0,t) &= v(1,t) = 0 && \text{for } t \in [0,\infty) \\ v(x,0) &= z(x) && \text{for } x \in [0,1]. \end{aligned}$$

with different initial conditions z. To state Duhamel's principle, we first define the *flow map* for the IBVP. The flow Φ_t takes an initial condition to the solution of the homogeneous IBVP at time t. To be precise, $\Phi_t(z)$ is a function defined on $[0,1]$ by the formula

$$\Phi_t(z)(x) = v(x,t),$$

where $v : \Omega \times [0,\infty) \to \mathbb{R}$ is the solution of the homogeneous IBVP with initial condition z. Thus, for any fixed t, $\Phi_t : C_0([0,1]) \to C_0([0,1])$ is a mapping on the space of continuous functions

$$C_0 := \{f : [0,1] \to \mathbb{R}; f \text{ continuous with } f(0) = f(1) = 0\}.$$

Problem 2 (1+5+1 points)**.**

1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume in addition that $g(x,y)$ is continuously differentiable in its first coordinate x. Show that*

$$\frac{d}{dt}\int_{s=0}^{t} g(t,s)\,ds = g(t,t) + \int_{s=0}^{t} \frac{\partial g}{\partial x}(t,s)\,ds.$$

   Hint: *You can do this almost entirely with elementary theorems that you have known since your calculus classes. Think about the function $F(t,r) = \int_{s=0}^{t} g(r,s)\,ds$. What is its directional derivative along the vector $(1,1)$?*

2. *Define*

$$w(x,t) = \Phi_t(u_0)(x) + \int_{s=0}^{t} \Phi_{t-s}(f(\cdot,s))(x)\,ds.$$

   Show that $w(x,t)$ solves the inhomogeneous IBVP with initial condition u_0. (The notation here is the same as in class. We define $f(\cdot,s) : [0,1] \to \mathbb{R}$ by $f(\cdot,s)(x) = f(x,s)$.)

3. *There is a version of Duhamel's principle for any reasonable differential equation of the form $\frac{\partial u}{\partial t} = Lu$ where L is a linear operator, including linear ODEs and PDEs such as the diffusion equation and the advection equation. Let $A \in \mathbb{R}^{n\times n}$. What is the version of Duhamel's principle for the ODE*

$$x' = Ax?$$

   Please write your Duhamel's principle in terms of the matrix exponential $\exp(At)$*, which is the flow map for the ODE.*

Let $A, B \in \mathbb{R}^{n\times n}$. The approximation

$$\exp((A+B)t) \approx \exp\left(\frac{At}{2}\right)\exp(Bt)\exp\left(\frac{At}{2}\right)$$

is called the *Strang splitting*. It is of higher order than the Lie–Trotter splitting, as you will prove in the next problem.

Problem 3 (7 points)**.** *Prove that*

$$\exp((A+B)t) - \exp\left(\frac{At}{2}\right)\exp(Bt)\exp\left(\frac{At}{2}\right) = O(t^3).$$

The point of the following problem is to give you some experience with unstable methods. It goes without saying that you should never actually use these methods, but when faced with a complicated problem, you might not be able to determine whether a given method is or is not stable. It is then helpful to know what typically goes wrong when a method is unstable.

Problem 4 (3+3 points)**.** *Consider the 1-D heat equation on* $[0,1]$ *with zero boundary conditions and the initial value*

$$u_0(x) = \sin(\pi x).$$

This is a nice test problem, because one can show that the solution is simply

$$u(x,t) = \exp(-\pi^2 t)\sin(\pi x).$$

Now recall the finite difference method

$$u_\Delta(\ell\Delta x, (n+1)\Delta t) = u_\Delta(\ell\Delta x, n\Delta t) + \Delta t D_x^2 u_\Delta(\ell\Delta x, n\Delta t). \tag{1}$$

introduced in class. This method converges for any sequence $(\Delta x_m, \Delta t_m)$ *of discretization parameters so that* $\Delta t_m \leq \frac{1}{2}\Delta x_m^2$ *and* $\lim_{m\to\infty}(\Delta x_m, \Delta t_m) = (0,0)$*.*

1. *Compute the solution of the IBVP for the heat equation with initial condition* $u_0(x) = \sin(\pi x)$ *by the finite difference method* (1) *up to time* $T = 1/8$ *for* $L = 4, 8, 16, 32, 64$ *using the time steps*

$$\Delta t = \frac{1}{2}\Delta x^2 = \frac{1}{2L^2}.$$

   Plot $u_\Delta(\cdot, 1/8)$ *for each value of* L*.*

2. *Now compute the finite difference method up to time* $t = 1/8$ *for the same values of* L *but with*

$$\Delta t = \frac{1}{2}\Delta x = \frac{1}{2L}.$$

   Plot $u_\Delta(\cdot, 1/8)$ *for each value of* L*.*

Finally, I guess I should have you prove consistency of the Crank–Nicolson method, since I glossed over it in class.

Problem 5 (5 points)**.** *Let* $u(x,t)$ *be the solution of the initial and boundary value problem for the heat equation on* $[0,1]$ *with zero boundary conditions and initial value* $u_0(x)$*. Recall that the Crank–Nicolson method is defined by the recurrence relation*

$$u_\Delta(\cdot, (n+1)\Delta t) = \left(I - \frac{\Delta t}{2}D_x^2\right)^{-1}\left(I + \frac{\Delta t}{2}D_x^2\right)u_\Delta(\cdot, n\Delta t).$$

Prove that

$$u(\cdot, (n+1)\Delta t) = \left(I - \frac{\Delta t}{2}D_x^2\right)^{-1}\left(I + \frac{\Delta t}{2}D_x^2\right)u(\cdot, n\Delta t) + H_n,$$

where

$$\|H_n\|_\Delta x \leq C\Delta t(\Delta x^2 + \Delta t^2).$$

Give an explicit formula for C in terms of the derivatives of u. You may use any theorems from class of the homework related to the trapezoidal rule or similar methods, but if you do use such a theorem, please cite it carefully and explain why it applies.