MATH 131 Spring 2005
EXAM 2 - Solution

1. A curve is given by the equation $x^2 + xy + y^2 = 3$.

   (a) (10) Compute the derivative $\frac{dy}{dx}$ of the curve at the point $(1, 1)$.

   ANS:

   $$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$$

   or

   $$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x + y}{x + 2y} \quad \textbf{(6 pts)}$$

   and

   $$\frac{dy}{dx}\Big|_{(1,1)} = -\frac{2x + y}{x + 2y}\Big|_{(1,1)} = -\frac{1}{1} = -1 \quad \textbf{(4 pts)}$$

   (b) (10) Find the points where the tangent to the curve is horizontal.

   ANS:
   The tangent line is horizontal at a point (x, y) when

   $$\frac{dy}{dx} = 0 \quad \Rightarrow \quad -\frac{2x + y}{x + 2y} = 0 \quad \Rightarrow \quad 2x + y = 0 \quad \Rightarrow \quad y = -2x \quad \textbf{(4 pts)}$$

   Now, (x, y) must also satisfy the equation for the curve, so

   $$x^2 + x(-2x) + (-2x)^2 = 3 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1 \quad \textbf{(4 pts)}$$

   This gives the x values, so the points are $(1, -2)$ and $(-1, 2)$ (**2 pts**).

2. Differentiate the following functions

   (a) (10) $f(x) = \sqrt{\ln(\tan(x))}$

   ANS:

   $$\begin{aligned} \frac{d}{dx}\sqrt{\ln(\tan(x))} &= \frac{d}{dx}(\ln(\tan(x)))^{1/2} \\ &= \frac{1}{2}(\ln(\tan(x)))^{-1/2}\frac{1}{\tan(x)}\sec^2(x)\,. \end{aligned}$$

(b) (10) $f(x) = x^{6x}e^{x^2-1}$

ANS: There is no way to do this problem without logarithmic differentiation.

$$f(x) = x^{6x}e^{x^2-1}$$

$$\ln(f(x)) = \ln\left(x^{6x}e^{x^2-1}\right) = \ln\left(x^{6x}\right) + \ln\left(e^{x^2-1}\right) = 6x\ln(x) + x^2 - 1$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(6x\ln(x)) + \frac{d}{dx}\left(x^2-1\right)$$

$$= \frac{d}{dx}(6x\ln(x)) + 2x$$ Use the product rule on the first term.

$$= 6\cancel{x}\frac{1}{\cancel{x}} + 6\ln(x) + 2x$$

$$\frac{f'(x)}{f(x)} = 6 + 6\ln(x) + 2x$$ Multiplying both sides by $f(x)$

$$f'(x) = f(x)(6 + 6\ln(x) + 2x)$$

$$\mathbf{f'(x) = x^{6x}e^{x^2-1}(6 + 6\ln(x) + 2x)}$$

You may also use the product rule with $u = x^{6x}$ and $v = e^{x^2-1}$. But you will still need logarithmic differentiation to find $u' = x^{6x}\left(6x\frac{1}{x} + 6\ln(x)\right)$ see example 8 in section 3.8 (pg. 247) in the text book for a calculation similar to the one for u'. You will also need to find $v' = 2xe^{x^2-1}$ using the chain rule.

3. Let $f(x) = e^{3x} + \sin(x)$.

(a) (12) Compute the first three derivatives $f'(x)$, $f''(x)$, $f'''(x)$.

ANS:

$$\begin{aligned} f'(x) &= 3e^{3x} + \cos(x), \\ f''(x) &= 3(3e^{3x}) - \sin(x) = 9e^{3x} - \sin(x), \\ f'''(x) &= 9(3e^{3x}) - \cos(x) = 27e^{3x} - \cos(x). \end{aligned}$$

(b) (8) Find $f^{(37)}(0)$.

ANS:

$$\begin{aligned} f^{(37)}(x) &= 3^{37}e^{3x} + \cos(x)\,, \\ f^{(37)}(0) &= 3^{37}e^{0} + \cos(0) = 3^{37} + 1\,. \end{aligned}$$

4. In a building which is 100 ft high, a woman takes an elevator at the top of the building and moves downward at a speed of 16 ft/sec. At exactly the same time a man exits the building and travels along a straight line at a speed of 3ft/sec. Find the rate of increase of the distance between the man and woman after 5 seconds.

   ANS:

   Let x be the (horizontal) distance between the bottom of the building and the man and let y be the (vertical) distance between the bottom of the building and the woman.

   The distance z between the man and the woman is related to x and y by

   $$z^2 = x^2 + y^2.$$

   We know that at time 0 we have $x(0) = 0$ and $y(0) = 100$ and that

   $$\frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dy}{dt} = -16$$

   (Be careful about the sign!)

   After 5 seconds, we have $x(5) = 3 \times 5 = 15$ and $y(5) = 100 - 5 \times 16 = 20$. Therefore the distance between ther man and the woman after 5 seconds is

   $$\sqrt{20^2 + 15^2} = \sqrt{625} = 25\,.$$

   If we differentiate the relation $z^2 = x^2 + y^2$ with respect to t we find

   $$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}\,,$$

   or

   $$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}\,.$$

   After 5 seconds

   $$\frac{dz}{dt} = \frac{15 \times 3 + 20 \times (-16)}{25} = -11\,.$$

5. If a piece of chalk is thrown vertically upward with a velocity of 32ft/sec, then the height after the t seconds is

   $$s(t) = 32t - 16t^2\,.$$

(a) (4) Find the velocity of the piece of chalk after 2 seconds.

<u>**ANS**</u>:

$$v(t) = 32 - 32t$$

so that $v(2) = 32 - 64 = -32$. The velocity after 2 seconds is -32ft/sec

(b) (4) When is the piece of chalk at rest?

<u>**ANS**</u>:
At rest when $v(t) = 32(1 - t) = 0$, i.e., $t = 1$.

(c) (4) What is the accelaration?

<u>**ANS**</u>:

$$a(t) = -32$$

(d) (4) When is the piece of chalk speeding up/slowing down?

<u>**ANS**</u>:
Speeding up if $a < 0, v < 0$, i.e., for $1 < t$. Slowing down if $a < 0, v > 0$, i.e., for $1 > t$.

(e) (4) What is the velocity of the piece of chalk when it is 12 ft above the ground on its way up?

<u>**ANS**</u>:

$$s(t) = 32t - 16t^2 = 12$$

if $t^2 - 2t + \frac{3}{4} = 0$ which gives

$$t_1 = \frac{1}{2}, t_2 = \frac{3}{2}.$$

In the first case $v(\frac{1}{2}) = 16$, in the second $v(\frac{3}{2}) = -16$. This shows that the chalk is on its way up when $t = \frac{1}{2}$ and then the velocity is 16ft/sec.