MATH 131 Spring 2005
EXAM 1 - Solution

1. Compute the following limits.

   (a) (5) $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2}$

   ANS:

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2}\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \\ &= \lim_{x\to 0} \frac{x^2+1-1}{x^2(\sqrt{x^2+1}+1)} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{x^2+1}+1} = \frac{1}{\sqrt{1}+1} = 2\end{aligned}$$

   (b) (5) $\lim_{x\to 2} \frac{x^3-2x^2+x+2}{x+2}$

   ANS:

$$\lim_{x\to 2} \frac{x^3-2x^2+x+2}{x+2} = \frac{2^3-2(2^2)+2+2}{2+2} = \frac{8-8+2+2}{2+2} = \frac{4}{4} = 1\,.$$

   (c) (5) $\lim_{x\to\infty} \frac{4x^3-2x^2+5}{2x^3+3x^2-x-3}$

   ANS:

$$\lim_{x\to\infty} \frac{4x^3-2x^2+5}{2x^3+3x^2-x-3} = \lim_{x\to\infty} \frac{4-\frac{2}{x}+\frac{5}{x^2}}{2+\frac{3}{x}-\frac{1}{x^2}-\frac{3}{x^3}} = 2\,.$$

   (d) (5) $\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$

   ANS:

$$\lim_{x\to 3} \frac{x^2-6x+9}{x-3} = \lim_{x\to 3} \frac{(x-3)^2}{x-3} = \lim_{x\to 3} x-3 = 0\,.$$

2. (a) (6) State the definition of "The function $f(x)$ is continuous at the point $x = a$".

ANS:

$$\lim_{x \to a} f(x) = f(a)$$

(b) (8) Let $f(x)$ be the function given by

$$f(x) = \begin{cases} x^2 + c^2 & \text{if } x < 1 \\ 2cx & \text{if } x \geq 1 \end{cases}$$

Find the values of c for which the function f is continuous at $x = 1$.

ANS: Equate the two expressions at $x = 1$,

$$(1)^2 + c^2 = 2c * 1,$$

The solution of the equation is

$$0 = c^2 - 2c + 1 = (c-1)^2 \quad \Rightarrow \quad c = 1.$$

(c) (6) For the value of c found in (b), is the the function $f(x)$ differentiable? (Explain your answer)

ANS: The answer is **YES**. The piecewise derivative is

$$f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{cases},$$

and since the two expression agree at $x = 1$, the function is differentiable at $x = 1$. Differentiability at all other points is immediate.

3. For the following functions compute the derivative **using the definition of the derivative**.

(a) (10) $f(x) = x^2 - x$.

ANS: There are two ways to do this problem depending on which definition of the derivative you started with.

i. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{(x+h)^2-(x+h)-(x^2+x)}{h}$ Simplifying the top

$= \lim_{h\to 0} \dfrac{\cancel{x^2}+2xh+h^2-\cancel{x}-h-\cancel{x^2}+\cancel{x}}{h}$ additive canellation leaves us with

$= \lim_{h\to 0} \dfrac{+2x\cancel{h}+h^{\cancel{2}}-\cancel{h}^{-1}}{h}$ dividing the h through leaves us with

$= \lim_{h\to 0} 2x+h-1$

$= 2x+0-1 = \mathbf{2x-1}$

ii. $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$

$= \lim_{x\to a} \dfrac{x^2-x-(a^2-a)}{x-a}$ simplifying and rearranging terms gives us

$= \lim_{x\to a} \dfrac{x^2-a^2-x+a}{x-a}$ but $x^2-a^2=(x-a)(x+a)$ and $-x+a = -(x-a)$

$= \lim_{x\to a} \dfrac{\cancel{(x-a)}(x+a)-\cancel{(x-a)}^{-1}}{cancel x-a}$ dividing out the $x-a$

$= \lim_{x\to a} x+a-1 = a+a-1 = 2a-1$ so $\mathbf{f'(a) = 2a-1}$

(b) (10) $g(x) = \dfrac{1}{1+3x}$.

<u>ANS</u>: This one can also be done with the $x \to a$ approach but the vast majority of students chose the $h \to 0$ approach so we'll do that one.

$g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\frac{1}{1+3(x+h)}-\frac{1}{1+3x}}{h}$ for notation's sake we pull out the bottom h and bringing the fractions to a common denominator we get

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{1+3x}{1+3x}\cdot\frac{1}{1+3(x+h)} - \frac{1}{1+3x}\cdot\frac{1+3(x+h)}{1+3(x+h)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{1+3x-(1+3(x+h))}{(1+3(x+h))(1+3x)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{\cancel{1}+\cancel{3x}-\cancel{1}-\cancel{3x}-3h))}{(1+3(x+h))(1+3x)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\cancel{h}}\right)\left(\frac{-3\cancel{h}}{(1+3(x+h))(1+3x)}\right)$$

$$= \lim_{h\to 0}\frac{-3}{(1+3(x+h))(1+3x)}$$

$$= \frac{-3}{\lim_{h\to 0}((1+3(x+h))(1+3x))} \text{ But}$$

$\lim_{h\to 0}((1+3(x+h))(1+3x)) = (1+3x)(\lim_{h\to 0}(1+3(x+h))$ because $1+3x$ is a constant with respect to h so

$\lim_{h\to 0}((1+3(x+h))(1+3x)) = (1+3x)(\lim_{h\to 0}(1+3(x+\overset{0}{\cancel{h}})) = (1+3x)^2$ and finally this means that

$$\mathbf{g'(x)} = \frac{\mathbf{-3}}{\mathbf{(1+3x)^2}}$$

4. Let $f(x) = \dfrac{x^2}{x^2-1}$.

(a) (7) Find the vertical and horizontal asymptotes of $f(x)$.

ANS: $(x^2-1)=0$ implies $x=\pm 1$: Since $x^2>0$ we get

$$\lim_{x\to +1^+}\frac{x^2}{x^2-1} = +\infty, \quad \lim_{x\to +1^-}\frac{x^2}{x^2-1} = -\infty$$

and

$$\lim_{x\to -1^+}\frac{x^2}{x^2-1} = -\infty, \quad \lim_{x\to -1^-}\frac{x^2}{x^2-1} = +\infty$$

so that $x=\pm 1$ are vertical asymptotes.

Moreover,

$$\lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = 1,$$

and, similarly,

$$\lim_{x\to-\infty} \frac{x^2}{x^2-1} = 1,$$

so that $y = 1$ is the only horizontal asymptote.

(b) (8) Find the points at which the tangent line to the graph of $f(x)$ is horizontal.

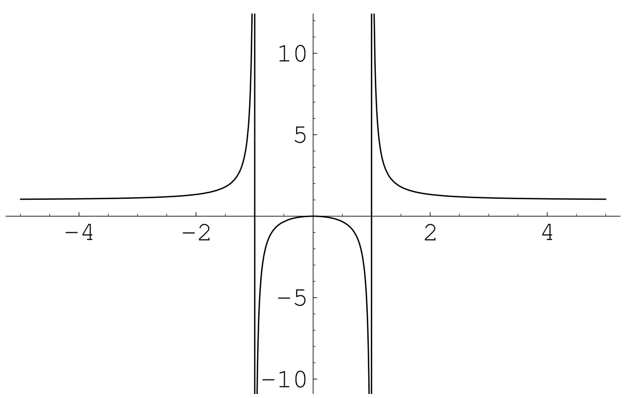
ANS: Points with horizontal tangent line have $f'(x) = 0$.

$$f'(x) = \frac{-2x}{(x^2-1)^2} = 0$$

implies that $x = 0$. The only point with horizontal tangent line is $P(0, 0)$.

(c) (5) Give a graph of $f(x)$.

ANS:



5. Let $f(x) = x\left(2\sqrt{x} - 6\right)$.

(a) (7) Find $f'(x)$.

ANS: First simplify $f(x) = 2x^{3/2} - 6x$ and then apply the power rule.

$$f'(x) = 3x^{1/2} - 6\,.$$

(b) (7) Find the equation of the tangent line to the graph of f at the point $(4, -8)$.

ANS: The equation for the tangent line to the graph of f at $x = 4$ is

$$y - f(4) = f'(4)(x - 4)\,.$$

We have $f(4) = -8$ and $f'(4) = 0$ so that the solution is

$$y + 8 = 0 \quad \text{or } y = -8\,.$$

(c) (6) Find the point at which the tangent line to the graph of f is parallel to the line $4y - 8x + 5 = 0$.

ANS: The line $4y - 8x + 5 = 0$ can be rewritten as $y = 2x - 5/4$ so that its slope is 2. Two line are parallel when their slopes are equal so the tangent line to the graph of f is parallel to $4y - 8x + 5 = 0$ if

$$2 = 3x^{1/2} - 6 \quad \text{or } x = 64/9\,.$$