

# Weyl group representations on zero weight spaces

November 30, 2014

Here we survey briefly (trying to provide reasonably complete references) the scattered work over four decades most relevant to the indicated subject. We also cite a couple of papers which illustrate the reach of the ideas discussed here: [10, 13]. While some questions have been answered, others remain open. Pramod Achar, David Ben-Zvi, Shrawan Kumar, Mark Reeder, and Eric Sommers have been helpful in providing details and references.

## 1 The basic problem

### 1.1

First we recall some familiar classical facts. Given a simple Lie algebra  $\mathfrak{g}$  over an algebraically closed field  $K$  of characteristic 0, its finite dimensional simple modules  $L(\lambda)$  are parametrized by the set of dominant integral weights in the dual  $\mathfrak{h}^*$  of a Cartan subalgebra  $\mathfrak{h}$ . Here the notion of dominance depends on fixing first a set of simple roots for the root system of  $\mathfrak{g}$  relative to  $\mathfrak{h}$  in  $\mathfrak{h}^*$ . In turn, Weyl's complete reducibility theorem ensures that any finite dimensional  $\mathfrak{g}$ -module is a direct sum of copies of some of these.

We follow Bourbaki's numbering of vertices in each Dynkin diagram [3, Chap. VI, §4]; but the reader should be alert to different conventions in some of the papers we cite. The vertices correspond to simple roots  $\alpha_1, \dots, \alpha_\ell$  (where  $\ell = \dim \mathfrak{h}$  is the rank of  $\mathfrak{g}$ ). Write  $\varpi_i$  for the fundamental weight corresponding to  $\alpha_i$ , so each dominant weight  $\lambda$  has coordinates  $c_i \in \mathbb{Z}^+$  given by  $\lambda = \sum_i c_i \varpi_i$ .

### 1.2

If  $L(\lambda)_\mu$  denotes the weight space for  $\mu$  relative to  $\mathfrak{h}$  (where  $\mu \in \mathfrak{h}^*$ ), its dimension is called the *multiplicity* of  $\mu$  in  $L(\lambda)$ . Denoting by  $W$  the Weyl group of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ , all weights of  $L(\lambda)$  in the  $W$ -orbit of each dominant weight  $\mu \leq \lambda$  occur with the same multiplicity ( $= \dim L(\lambda)_\mu$ ), necessarily positive. As usual  $\mu \leq \lambda$  means that  $\lambda - \mu$  is a nonnegative linear combination of simple roots. For example, all  $w\lambda$  have multiplicity 1.

Various methods of representation theory permit the explicit computation of these multiplicities. These sum to  $\dim L(\lambda)$ , which by itself can readily be computed from Weyl's dimension formula. Typically the methods for finding all weight multiplicities are lengthy and recursive, while it is usually not feasible to compute  $\dim L(\lambda)_0$  in isolation. (In a recent paper [11] the behavior of this dimension is studied systematically.) The theory

does predict easily that  $L(\lambda)_0$  is nonzero precisely when  $0 \leq \lambda$ , i.e., if and only if  $\lambda$  lies in the root lattice. Thus a simple algebraic group  $G$  whose Lie algebra is isomorphic to  $\mathfrak{g}$  and whose rational simple modules (also denoted  $L(\lambda)$ ) all have such highest weights is precisely a group of adjoint type.

### 1.3

We should emphasize that in the literature cited here, there is some variation in the initial set-up: sometimes one has Lie (or algebraic) groups rather than their Lie algebras, at other times a compact real Lie group. This makes no real difference for us. The underlying problems we discuss are purely algebraic in origin and involve just the root data and Weyl group. It is essential, however, to work throughout over a splitting field of characteristic 0, where the Cartan–Weyl theory of finite dimensional representations applies.

### 1.4

Since  $W$  permutes the weights in a single orbit, it follows in particular that  $L(\lambda)_0$  is a  $W$ -module. When  $\lambda = 0$ , we obviously get the trivial 1-dimensional  $W$ -module. Another familiar example is the *adjoint* module, whose highest weight is the highest root (necessarily long). Here the zero weight space of dimension  $\ell$  is a Cartan subalgebra and  $W$  acts by the reflection representation. Beyond these examples the situation is less obvious. Indeed, it is usually unclear how to determine directly whether or not the  $W$ -module  $L(\lambda)_0$  is simple, even if its dimension is compatible with simplicity. In general the following natural question still has no definitive answer:

(A) *Which simple  $W$ -modules can be realized as  $L(\lambda)_0$  for some  $\lambda$ , and how is the character of the  $W$ -module then determined by  $\lambda$ ?*

It is easy to see that  $\lambda$  can be replaced by the highest weight of the *dual* module without changing this  $W$ -module: Here  $L(\lambda)^* \cong L(\lambda^*)$ , with  $\lambda^* = -w_\circ\lambda$ , while  $w_\circ$  is the longest element of  $W$ . Recall that  $w_\circ = -1$  (so all simple modules are self-dual) unless  $\mathfrak{g}$  is of type  $A_n$  with  $n > 1$ ,  $D_n$  with  $n$  odd, or  $E_6$ . The dual action of  $\mathfrak{g}$  on  $L(\lambda^*)$  induces the dual action of  $W$  on  $L(\lambda)_0$ . This is most obvious in the group setting: here  $W$  is realized as the quotient of the normalizer of a maximal torus while the torus fixes  $L(\lambda)_0$  pointwise. But all irreducible characters of  $W$  over  $\mathbb{C}$  are  $\mathbb{Z}$ -valued, hence self-dual (and in fact the representations can be constructed over  $\mathbb{Q}$ ).

## 1.5

We should point out that not every simple  $W$ -module is realized in a zero weight space. For example, in type  $B_2$  (where  $\alpha_1$  is the long simple root), only the highest weights 0 and  $\varpi_1$  lead to 1-dimensional zero weight spaces (in the trivial module and standard 5-dimensional module for  $\mathfrak{so}_5(K)$ , respectively). This follows easily from the dimension formula for  $L(\lambda)_0$  in [3, Chap. VIII, §9, Exer. 10]. But here  $W$  is dihedral of order 8 and therefore has four irreducible characters of degree 1 (along with the reflection character of degree 2).

At least in classical Lie types, there are some combinatorial methods which permit the determination of  $W$  multiplicities, as in [2]. A considerable amount of information about the representations of  $W$  (including connections with Springer theory) can be found in the later chapters of Carter's book [5]. In the examples mentioned here we take that information for granted. A somewhat vague but more conceptual question remains unanswered in general:

(B) *Is there a predictable pattern governing the decompositions of zero weight spaces into simple modules for  $W$ ?*

Note that a detailed answer to question (A) would inevitably require case-by-case study, whereas question (B) might involve more general ideas together with study of individual cases.

## 2 Example: Type $A_n$

As usual, special (or general) linear Lie algebras and related groups are most amenable to combinatorial study.

### 2.1

When  $\mathfrak{g} = \mathfrak{sl}_2(K)$ , dominant weights are parametrized by nonnegative integers  $m$ . Here  $L(\lambda)_0 \neq 0$  precisely when  $\lambda = 2m\varpi$  (for short,  $L(2m)$ ) is *even*. It is clear that the trivial representation of  $W$  is afforded by  $L(0)$  and the sign (= reflection) representation by the adjoint module  $L(2)$ . Moreover, direct computation shows that the nontrivial element of  $W$  fixes a basis vector of weight 0 in the simple module  $L(\lambda)$ ,  $m \in \mathbb{Z}^+$  when  $\lambda = 4m\varpi$  but acts on this vector by -1 when  $\lambda = (4m + 2)\varpi$ . In this way the two irreducible characters of  $W$  alternate in their action on the 1-dimensional zero weight spaces as the even highest weight grows. (This is observed by Broer [4, §1].)

## 2.2

More generally, the simple Lie algebras  $\mathfrak{g} = \mathfrak{sl}_n(K)$  with  $n = \ell + 1$  were the first family to be systematically investigated. In 1973 a short note by E. Gutkin [7] in Russian (untranslated into English) gave what is apparently the earliest published treatment of Weyl group representations on zero weight spaces (see also the independent work of B. Kostant [9, 4.1] and D.A. Gay [6].) In this case, modules  $L(\lambda)$  with  $\dim L(\lambda)_0 \neq 0$  are parametrized by highest weights  $\lambda = c_1\varpi_1 + \cdots + c_\ell\varpi_\ell$  with  $\lambda = n\mu$  for some dominant weight  $\mu$ , since the root lattice has index  $n$  in the weight lattice.

Here (A) is completely answered, in the spirit of Weyl's classical work. If  $V = L(\varpi_1)$  is the natural  $n$ -dimensional module for  $\mathfrak{g}$ , all simple  $\mathfrak{g}$ -modules  $L(\lambda)$  can be realized systematically inside  $m$ -fold tensor powers of  $V$ ; here  $\lambda$  corresponds to a partition having at most  $m$  parts. In particular, consider  $V^{\otimes n}$ . Its highest weight  $n\varpi_1$  lies in the root lattice, so the highest weights of irreducible summands also do. To each of the  $n!$  distinct partitions  $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$  of  $n$  corresponds a dominant weight  $\lambda$  with coordinates given by  $c_i = a_i - a_{i+1}$  for  $1 \leq i \leq n - 1$ . Then each corresponding  $L(\lambda)$  occurs as a summand of  $V^{\otimes n}$ , and its zero weight space affords the simple  $W$ -module parametrized classically by the *dual* partition. Here the partitions of  $n$  simultaneously parametrize the conjugacy classes of  $S_n$  (given by disjoint cycle types) and the isomorphism types of simple  $S_n$ -modules. (In general, the partitions which yield zero weight spaces of positive dimension are the partitions of some integer divisible by  $n$ .)

As noted earlier, similar behavior is observed when  $V$  is replaced by its dual  $V^* = L(\varpi_{n-1})$ . Here the coordinates of weights are reversed.

Here is a small example: When  $n = 3$ , the irreducible characters of  $W = S_3$  are the trivial and sign characters of degree 1 along with the reflection character of degree 2. Each occurs in the regular representation of  $W$  as often as its degree. On the other hand, there are also three modules  $L(\lambda)$  for  $\mathfrak{g}$  in  $V^{\otimes 3}$  whose zero weight spaces afford these representations of  $W$ : besides  $\lambda = 0$  one has  $\lambda = \varpi_1 + \varpi_2$  (giving the 8-dimensional adjoint representation) and  $\lambda = 3\varpi_1$  (giving a 10-dimensional representation with zero weight space of dimension 1). The second of these occurs twice in  $V^{\otimes 3}$  (since 2 is the degree of the corresponding character of  $W$ ), agreeing with the dimension count  $27 = 1 + 2 \cdot 8 + 10$ .

An exercise in Bourbaki [3, Chap. VIII, §9, Exer. 9] leads more generally to an easy formula for  $\dim L(\lambda)_0$  in type  $A_2$ : here  $\lambda = c_1\varpi_1 + c_2\varpi_2$  lies in the root lattice just when  $c_1 \equiv c_2 \pmod{3}$ , and then  $\dim L(\lambda)_0 = 1 + \min(c_1, c_2)$ .

On the other hand, there seems to be no simple answer here to question (B) except in rank 1 (and possibly in rank 2).

For the other classical groups one might apply a more complicated variant of this procedure in order to deal with (A), since the Weyl groups are close relatives of  $S_n$ : for results in this direction see [2].

### 3 Small representations and Springer Correspondence

#### 3.1

When  $\mathfrak{g}$  has rank  $> 1$  and  $\lambda$  is a dominant weight in the root lattice, does  $\dim L(\lambda)_0$  exceed a specified bound  $d$  for all but finitely many dominant weights  $\lambda$ ? If so, this would reduce the first part of (A) for each Lie type to a finite (though possibly very long) list of possibilities when  $d$  is the largest degree of an irreducible character of  $W$ . It would also be one way to think about what it might mean for  $L(\lambda)$  to be “small”. But already in type  $A_2$  Bourbaki’s exercise cited above gives a negative answer to the question just asked. Leaving aside type  $A_n$ , this question still seems to be open. But a better theoretical approach would be preferred.

Broer [4] introduced a more precise notion of smallness, which is motivated by problems concerning invariants, normality of nilpotent orbits, etc. He defined *small*  $\mathfrak{g}$ -modules to be those  $L(\lambda)$  with  $\lambda$  in the root lattice for which twice a root is never a weight. The trivial and adjoint modules are obvious examples. In type  $A_n$  the modules  $L(\lambda)$  occurring as summands of  $V^{\otimes n}$  (where  $\lambda$  corresponds to a partition of  $n$ ), together with their duals, turn out to be precisely the small modules. So in this case our question (A) connects precisely with Broer’s definition; but in other Lie types the relationship is looser.

#### 3.2

Broer’s notion of smallness has been investigated extensively, especially by Reeder [14, 15, 16], cf. Sommers [17] for the exceptional types. Without going further into the technical details of Reeder’s work, we recall briefly how smallness interacts with Springer’s theory. Apart from the actions of  $W$  on zero weight spaces, there is another (considerably less direct) connection between finite dimensional modules for  $\mathfrak{g}$  (or its adjoint group  $G$ ) and simple  $W$ -modules: the *Springer Correspondence*.

Briefly, this correspondence involves the finitely many nilpotent orbits in  $\mathfrak{g}$  (or unipotent classes in  $G$ ), together with the various types of fibers  $\mathcal{B}_e$  in the flag variety  $\mathcal{B}$  lying over nilpotent elements  $e$  in Springer's resolution of singularities of the nilpotent variety. It turns out that  $W$  acts in each non-vanishing (necessarily even) cohomology degree of  $\mathcal{B}$ . Moreover, in the top degree each simple  $W$ -module is realized exactly once, though some of these may be tensored with nontrivial characters of the finite group of irreducible components in the adjoint group centralizer, say  $A(e) := C_G(e)/C_G(e)^\circ$ .

Often, but by no means always,  $A(e) = 1$  and the top cohomology becomes a simple  $W$ -module. (Case-by-case study shows that this module always fails to be simple when  $A(e) \neq 1$  even though some nontrivial character of  $A(e)$  may make no contribution. There seems to be no general explanation of this fact.)

### 3.3

What does Springer theory have to do with small representations in Broer's sense? This is most clearcut in the *simply-laced* cases. Reeder is able to verify that small  $\mathfrak{g}$ -modules (up to duality) are in bijection with certain nilpotent orbits, and then  $L(\lambda)_0$  as a  $W$ -module agrees with the top cohomology of the Springer fiber corresponding to any  $e$  in the orbit. As remarked above, this  $W$ -module is often (though not always) simple. This can be an effective though highly indirect way to show that certain simple  $W$ -modules are realized as zero weight spaces.

In case  $\mathfrak{g}$  has two root lengths, a "folding" of the Dynkin diagram via a graph automorphism  $\sigma$  relates this case to a corresponding simply-laced type, e.g.,  $G_2$  to  $D_4$  or  $F_4$  to  $E_6$ . Here the simple roots lying in an orbit of  $\sigma$  determine a single *short* simple root, while those fixed by  $\sigma$  determine the *long* simple roots in the non-simply-laced subalgebra of fixed points  $\mathfrak{g}^\sigma$ . Then those  $\mathfrak{g}$ -modules  $L(\lambda)$  for which  $\lambda$  is  $\sigma$ -invariant correspond to small representations of the latter.

The relationship with  $W$ -modules occurring in the cohomology of Springer fibers then becomes more delicate. Here the algebraic group setting is most natural. In particular, the folding  $\sigma$  is combined with passage to the *Langlands dual* group  $G^\vee$ . Thus if  $G$  is initially an adjoint group (the usual setting for investigation of small representations), then  $G^\vee$  is simply connected. Moreover, even when  $G$  and  $G^\vee$  are isomorphic as algebraic groups, as happens for instance in types  $G_2$  and  $F_4$ , the short and long roots are interchanged. This affects the Dynkin diagram numbering, to complicate further the variance of vertex numbering in the literature. Examples below

may clarify this relationship.

We remark that these ideas (including the theme of Langlands duality) have been explored geometrically at a high level in recent work of Achar–Henderson–Riche [1]. Although they do not directly address our question (A), their approach explains conceptually the surjectivity of the natural map (induced by restriction of the normalizer of a maximal torus to zero weight spaces) from the representation ring of  $G$  to that of  $W$  by comparing it to a corresponding (much more subtle) map in Springer theory already known to be surjective.

## 4 Special cases

Returning to our original question (A), we summarize briefly what the cited papers do or don't imply for Lie types beyond  $A_n$ . Much of this literature centers on the determination of small representations of  $\mathfrak{g}$  (or its adjoint group) and their applications, usually treating (A) as a side issue. We already cited papers such as [2, 11] which explore the combinatorics (including dimensions) of zero weight spaces. Here the results are explicit but usually complicated to state, while the underlying representations of  $W$  would require further work to correlate with the highest weights involved.

We consider now only the five exceptional types.

### 4.1

Reeder's work shows that there are just eight small representations of  $E_6$ : six having highest weights  $0, \varpi_2, \varpi_1 + \varpi_6, \varpi_4, \varpi_1 + \varpi_3, 3\varpi_1$ , along with duals of the latter two having highest weights  $\varpi_5 + \varpi_6, 3\varpi_6$ . In terms of the Springer Correspondence for  $E_6$ , all weights in this list except  $\varpi_4$  yield simple  $W$ -modules of the form  $L(\lambda)_0$  (because the component groups  $A(e)$  are trivial in these cases), whereas the nilpotent orbit corresponding to  $L(\varpi_4)$  has a component group of order 2 and  $L(\varpi_4)_0$  is instead a direct sum of two  $W$ -modules of dimensions 30 and 15 (the latter twisted by the nontrivial character of the component group). (See [14, p. 439].)

In this way we can indirectly realize five simple  $W$ -modules as zero weight spaces. But  $W$  has 25 distinct irreducible characters. So we see that the study of small representations relative to the Springer Correspondence falls far short of answering our question (A). This is even more obvious for types  $E_7$  and  $E_8$ , where Reeder finds only a few small representations (6 and 5, respectively) even though the Weyl groups here have 60 and 112 irreducible characters, respectively.

## 4.2

Turning to  $\mathfrak{g}$  of type  $F_4$ , we have  $|W| = 2^7 \cdot 3^2$ , while  $W$  has 25 irreducible characters; their degrees comprise the set  $\{1, 2, 4, 6, 8, 9, 12, 16\}$ . Again all  $L(\lambda)$  have nontrivial zero weight spaces and are self-dual. But explicit tables computed by Frank Lübeck [12] include only five cases in which characters of  $W$  could occur as zero weight spaces:  $\lambda = 0, \varpi_1, \varpi_4, \varpi_3, 2\varpi_4$  (in Bourbaki numbering, the first two simple roots are long). Here the spaces  $L(\lambda)_0$  have respective dimensions 1, 2, 4, 9, 12. His table currently includes only those  $L(\lambda)$  of dimension bounded by 12,000, but it would be somewhat surprising to encounter zero weight spaces of such small dimensions later on. Here only the first four highest weights belong to small representations.

## 4.3

As indicated earlier, the Dynkin diagram of  $F_4$  can be obtained by folding the diagram of  $E_6$ . Here as usual the numbering of vertices in different sources has to be treated with caution. But only the self-dual four weights at the beginning of the earlier list of eight for  $E_6$  correspond to the small representations of  $F_4$ : after passing to the Langlands dual for  $F_4$  one recovers the weights  $0, \varpi_4, \varpi_1, \varpi_3$  above, but not  $2\varpi_4$ . The zero weight spaces for  $0, \varpi_1, \varpi_4$  are in fact simple  $W$ -modules of dimensions 1, 2, 4, but  $L(\varpi_3)_0$  is instead the direct sum of two such modules having dimensions 8 and 1: see [16, Table 5.1].)

## 4.4

If  $\mathfrak{g}$  is of type  $G_2$ , its Weyl group  $W$  is dihedral of order 12 and has two irreducible characters of degree 2 along with four of degree 1. Moreover, the root lattice of  $\mathfrak{g}$  equals the weight lattice, so every  $L(\lambda)$  has a zero weight space of positive dimension (and is self-dual). But in only three of these  $L(\lambda)$  does the multiplicity of the weight 0 seem to be as small as 1 or 2 (judging from tables like [12]): while  $L(0)$  affords as usual the 1-character of  $W$ , the fundamental module  $L(\varpi_1)$  of dimension 7 has a 1-dimensional zero weight space affording the sign character of degree 1. Of course the adjoint module  $L(\varpi_2)$  of dimension 14 has zero weight space affording the reflection character of  $W$ , here of degree 2. In fact these three  $L(\lambda)$  are precisely the small representations, but that gives us no new information (cf. [16, Table 5.1]).



## References

1. P.N. Achar, A. Henderson, and S. Riche, *Geometric Satake, Springer correspondence, and small representations II*, arXiv:1205/5089 [math.RT]
2. S. Ariki, J. Matsuzawa, and I. Terada, *Representations of Weyl groups on zero weight spaces of  $\mathfrak{g}$ -modules*, Algebraic and topological theories (Kinosaki, 1984), 546–568, Kinokuniya, Tokyo, 1985.
3. N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV–VI, Chap. VII–VIII, Hermann, 1968, 1975.
4. A. Broer, *The sum of generalized exponents and Chevalley’s restriction theorem for modules of covariants*, Indag. Math. **6** (1995), 385–396.
5. R.W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, Wiley–Interscience, 1985.
6. D.A. Gay, *Characters of the Weyl group of  $SU(n)$  on zero weight spaces and centralizers of permutation representations*, Rocky Mt. J. Math. **6** (1976), 449–455.
7. E. Gutkin, *Representations of the Weyl group in the space of vectors of zero weight* [Russian], Uspeh. Mat. Nauk **28** (1973), no. 5, 237–238.
8. B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.
9. —, *On Macdonalds  $\eta$ -function formula, the Laplacian and generalized exponents*, Adv. Math. **20** (1976), 179–212.
10. —, *Clifford algebra analogue of the Hopf–Koszul–Samelson theorem, the  $\rho$ -decomposition  $C(\mathfrak{g}) = \text{End}V_\rho \otimes C(P)$ , and the  $\mathfrak{g}$ -module structure of  $\bigwedge \mathfrak{g}$* , Adv. Math. **125** (1997), 275–350.
11. S. Kumar and D. Prasad, *Dimension of zero weight space: An algebro-geometric approach*, J. Algebra **403** (2014), 324–344.
12. <http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/index.html>
13. J. Matsuzawa and M. Takahashi, *Representations of the normalizers of maximal tori of simple Lie groups*, Tsukuba J. Math. **33** (2009), no. 2, 189–237.
14. M. Reeder, *Zero weight spaces and the Springer correspondence*, Indag. Math. **9** (1998), 431–441.
15. —, *Small modules, nilpotent orbits, and motives of reductive groups*, Int. Math. Res. Not. **1998**, No. 20, 1079–1101.
16. —, *Small representations and minuscule Richardson orbits*, Int. Math. Res. Not. **2002**, No. 5, 257–275.
17. E. Sommers, *Conjectures for small representations of the exceptional groups*, Comment. Math. Univ. St. Paul. **49** (2000), 101104.