Factorization algebras in quantum field theory
Volume 2 (27 October 2020)

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&

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Introduction and overview

This is the second volume of our two-volume book on factorization algebras as they apply to quantum field theory. In volume one, we focused on the theory of factorization algebras while keeping the quantum field theory to a minimum. Indeed, we only ever discussed free theories. In volume two, we will focus on the factorization algebras associated to interacting classical and quantum field theories.

In this introduction, we will state in outline the main results that we prove in this volume. The centerpiece is a deformation quantization approach to quantum field theory, analogous to that for quantum mechanics, and the introduction to the first volume provides an extensive motivation for this perspective, which is put on solid footing here. Subsequently we explore symmetries of field theories fit into this approach, leading to classical and quantum versions of Noether's theorem in the language of factorization algebras.

Remark: Throughout the text, we refer to results from the first volume in the style "see chapter I.2" to indicate the second chapter of volume 1. $\Diamond$

1.1 The factorization algebra of classical observables

We will start with the factorization algebra associated to a classical field theory. Suppose we have a classical field theory on a manifold M, given by some action functional, possibly with some gauge symmetry. To this data we will associate a factorization algebra of *classical observables*. The construction goes as follows. First, for every open subset $U \subset M$, consider the space $\mathcal{EL}(U)$ of solutions to the Euler-Lagrange equations on U, modulo gauge. We work in perturbation theory, which means we consider solutions that live in formal neighbourhood of a fixed solution. We also work in the derived sense, which means we "impose" the Euler-Lagrange equations by a Koszul complex. In sophisticated terms, we take $\mathcal{EL}(U)$ to be the formal derived stack of solutions to the equations of motion. As U varies, the collection $\mathcal{EL}(U)$ forms a sheaf of formal derived stacks on M.

The factorization algebra Obs^{cl} of classical observables of the field theory assigns to an open U, the dg commutative algebra $\mathscr{O}(\mathcal{EL}(U))$ of functions on this formal derived stack $\mathcal{EL}(U)$. This construction is simply the derived version of functions on solutions to the Euler-Lagrange equations, and hence provides a somewhat sophisticated refinement for classical observables in the typical sense.

It takes a little work to set up a theory of formal derived geometry that can handle formal moduli spaces of solutions to non-linear partial differential equations like the Euler-Lagrange equations. In the setting of derived geometry, formal derived stacks are equivalent to homotopy Lie algebras (i.e., Lie algebras up to homotopy, often modeled by dg Lie algebras or L_∞ algebras). The theory we develop in chapter 3 takes this characterization as a definition. We define a formal elliptic moduli problem on a manifold M to be a sheaf of homotopy Lie algebras satisfying certain properties. Of course, for the field theories considered in this book, the formal moduli of solutions to the Euler-Lagrange equations always define a formal elliptic moduli problem. We develop the theory of formal elliptic moduli problems sufficiently to define the dg algebra of functions, as well as other geometric concepts.

1.2 The factorization algebra of quantum observables

In chapter 8, we give our main construction. It gives a factorization algebra Obs^q of *quantum* observables for any quantum field theory in the sense of Costello (2011b). A quantum field theory is, by that definition, something that lives over $\mathbb{C}[[\hbar]]$ and reduces modulo $\hbar$ to a classical field theory. The factorization algebra Obs^q is then a factorization algebra over $\mathbb{C}[[\hbar]]$, and modulo $\hbar$ it reduces to a factorization algebra quasi-isomorphic to the algebra Obs^{cl} of classical observables.

The construction of the factorization algebra of quantum observables is a bit technical. The techniques arise from the approach to quantum field theory developed in Costello (2011b). In that book a quantum field theory is defined to be a collection of functionals $I[L]$ on the fields that are *approximately* local. They play a role analogous to the action functional of a classical field theory. These functionals depend on a "length scale" L, and when L is close to zero the functional $I[L]$ is close to being local. The axioms of a quantum field theory are:

(i) As L varies, $I[L]$ and $I[L']$ are related by the operation of "renormalization group flow." Intuitively, if $L' > L$, then $I[L']$ is obtained from $I[L]$ by integrating out certain high-energy fluctuations of the fields.
(ii) Each $I[L]$ satisfies a scale L quantum master equation (the quantum version of a compatibility with gauge symmetry).
(iii) When we reduce modulo $\hbar$ and send $L \to 0$, then $I[L]$ becomes the interaction term in the classical Lagrangian.

The fact that $I[L]$ is never local, just close to being local as $L \to 0$, means that we have to work a bit to define the factorization algebra. The essential idea is simple, however. If $U \subset M$, we define the cochain complex $\mathrm{Obs}^q(U)$ to be the space of first-order deformations $\{I[L] + \epsilon\mathcal{O}[L]\}$ of the collection of functionals $I[L]$ that define the theory. We ask that this first-order deformation satisfies the renormalization group flow property modulo ϵ^2. This condition gives a linear expression for $\mathcal{O}[L]$ in terms of any other $\mathcal{O}[L']$. This idea reflects the familiar intuition from the path integral that observables are first-order deformations of the action functional.

The observables we are interested in do not need to be localized at a point, or indeed given by the integral over the manifold of something localized at a point. Therefore, we should not ask that $\mathcal{O}[L]$ becomes local as $L \to 0$. Instead, we ask that $\mathcal{O}[L]$ becomes supported on U as $L \to 0$.

Moreover, we do not ask that $I[L] + \epsilon\mathcal{O}[L]$ satisfies the scale L quantum master equation (modulo ϵ^2). Instead, its failure to satisfy the quantum master equation defines the differential on the cochain complex of quantum observables.

With a certain amount of work, we show that this definition defines a factorization algebra Obs^q that quantizes the factorization algebra Obs^{cl} of classical observables.

1.3 The physical importance of factorization algebras

Our key claim is that factorization algebras encode, in a mathematically clean way, the features of a quantum field theory that are important in physics.

This formalism must thus include the most important examples of quantum field theories from physics. Fortunately, the techniques developed in Costello (2011b) give a cohomological technique for constructing quantum field theories, which applies easily to many examples. For instance, Yang-Mills theory and the ϕ^4 theory on $\mathbb{R}^4$ were both constructed in Costello (2011b). (Note that we work throughout on Riemannian manifolds, not Lorentzian ones.)

As a consequence, each theory has a factorization algebra on $\mathbb{R}^4$ that encodes its observables.

1.3.1 Correlation functions from factorization algebras

In the physics literature on quantum field theory, the fundamental objects are *correlation functions* of observables. The factorization algebra of a quantum field theory contains enough data to encode the correlation functions. In this sense, its factorization algebra encodes the essential data of a quantum field theory.

Let us explain how this encoding works. Assume that we have a field theory on a compact manifold M. Suppose that we work near an *isolated* solution to the equations of motion, that is, one which admits no small deformations. (Strictly speaking, we require that the cohomology of the tangent complex to the space of solutions to the equations of motion is zero, which is a little stronger as it means that there are also no gauge symmetries preserving this solution to the equations of motion.) Some examples of theories where we have an isolated solution to the equations of motion are a massive scalar field theory, on any compact manifold, or a massless scalar field theory on the four-torus T^4 where the field has monodromy -1 around some of the cycles. In each case, we can include an interaction, such as the ϕ^4 interaction.

Since the classical observables are functions on the space of solution to the equations of motion, our assumption implies $H^*(\mathrm{Obs}^{cl}(M)) = \mathbb{C}$. A spectral sequence argument then lets us conclude that $H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]]$.

If $U_1, \ldots, U_n \subset M$ are disjoint open subsets of M, the factorization algebra structure gives a map

$$\langle - \rangle : H^*(\mathrm{Obs}^q(U_1)) \otimes \cdots \otimes H^*(\mathrm{Obs}^q(U_n)) \to H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]].$$

If $\mathcal{O}_i \in H^*(\mathrm{Obs}^q(U_i))$ are observables on the open subsets U_i, then $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ is the correlation function of these observables.

Consider again the ϕ^4 theory on T^4, where the ϕ field has monodromy -1 around one of the four circles. In the formalism of Costello (2011b), it is possible to construct the ϕ^4 theory on $\mathbb{R}^4$ so that the $\mathbb{Z}/2$ action sending ϕ to $-\phi$ is preserved. By descent, the theory — and hence the factorization algebra — exists on T^4 as well. Thus, this theory provides an example where the quantum theory can be constructed and the correlation functions defined.

1.3.2 Factorization algebras and renormalization group flow

Factorization algebras provide a satisfying geometric understanding of the RG flow, which we discuss in detail in Chapter 9 but sketch now.

In Costello (2011b), a scaling action of $\mathbb{R}_{>0}$ on the collection of theories on $\mathbb{R}^n$ was given. It provides a rigorous version of the RG flow as defined by Wilson.

There is also a natural action by scaling of the group $\mathbb{R}_{>0}$ (under multiplication) on the collection of translation-invariant factorization algebras on $\mathbb{R}^n$. Let $R_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ denote the diffeomorphism that rescales the coordinates, and let $\mathcal{F}$ be a translation-invariant factorization algebra on $\mathbb{R}^n$. Then the pull-back $R_\lambda^* \mathcal{F}$ is a new factorization algebra on $\mathbb{R}^n$.

We show that the map from theories on $\mathbb{R}^n$ to factorization algebras on $\mathbb{R}^n$ intertwines these two $\mathbb{R}_{>0}$ actions. Thus, this simple scaling action on factorization algebras is the RG flow.

In order to define have a quantum field theory with finitely many free parameters, it is generally essential to only consider *renormalizable* quantum field theories. In Costello (2011b), it was shown that for any translation-invariant field theory on $\mathbb{R}^n$, the dependence of the field theory on the scalar parameter $\lambda \in \mathbb{R}_{>0}$ is via powers of λ and of $\log \lambda$. A strictly renormalizable theory is one in which the dependence is only via $\log \lambda$, and the quantizations of Yang-Mills theory and ϕ^4 theory constructed in Costello (2011b) both have this feature.

We can translate the concept of renormalizability into the language of factorization algebras. For any translation-invariant factorization algebra $\mathcal{F}$ on $\mathbb{R}^n$, there is a family of factorization algebras $\mathcal{F}_\lambda = R_\lambda^* \mathcal{F}$ on $\mathbb{R}^n$. Because this family depends smoothly on λ, *a priori* it defines a factorization algebra over the base ring $C^\infty(\mathbb{R}_{>0})$ of smooth functions of the variable λ. We say this family is *strictly renormalizable* if it arises by extension of scalars from a factorization algebra over the base ring $\mathbb{C}[\log \lambda]$ of polynomials in $\log \lambda$. The factorization algebras associated to Yang-Mills theory and to the ϕ^4 theory both have this feature.

In this way, we formulate via factorization algebras the concept of renormalizability of a quantum field theory.

1.3.3 Factorization algebras and the operator product expansion

One disadvantage of the language of factorization algebras is that the factorization algebra structure is often very difficult to describe explicitly. The reason is that for an open set U, the space $\mathrm{Obs}^q(U)$ of quantum observables on U is a very large topological vector space and it is not obvious how one can give it a topological basis. To extract more explicit computations, we introduce the concept of a *point observable* in Chapter 10 . The space of point observables is defined to be the limit $\lim_{r\to 0} \mathrm{Obs}^q(D(0,r))$ of the space of quantum observables on a disc of radius r around the origin, as $r \to 0$. Point observables capture what physicists call *local operators*; however, we eschew the term operator as our formalism does not include a Hilbert space on which we can operate.

Given two point observables $\mathcal{O}_1$ and $\mathcal{O}_2$, we can place $\mathcal{O}_1$ at 0 and $\mathcal{O}_2$ at x and then use the factorization product on sufficiently small discs centered at 0 and x to define a product element

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(x) \in \mathrm{Obs}^q(\mathbb{R}^n).$$

The operator product is defined by expanding the product $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ as a function of x and extracting the "singular part." It is not guaranteed that such an expansion exists in general, but we prove in Chapter 10 that it does exist to order $\hbar$. This order $\hbar$ operator product expansion can be computed explicitly, and we do so in detail for several theories in Chapter 10. The methods exhibited there provide a source of concrete examples in which mathematicians can rigorously compute quantities of quantum field theory.

That chapter also contains the longest and most detailed example in this book. It has recently become clear Costello (2013b); Costello et al. (2019) that one can understand quantum groups, such as the Yangian and related algebras, using Feynman diagram computations in quantum field theory. The general idea is that one should take a quantum field theory that has one topological direction, so that the the factorization product in this direction gives us a (homotopy) associative algebra. By taking the Koszul dual of this associative algebra, one finds a new algebra which in certain examples is a quantum group. For the Yangian, the relevant Feynman diagram computations are given in Costello et al. (2019). We present in detail an example related to a different infinite-dimensional quantum group, following Costello (2017). We perform one-loop Feynman diagram computations that reproduce the commutation relations in this associative algebra. (We chose this example as the relevant Feynman diagram computations are considerably easier than those that lead to the Yangian algebra in Costello et al. (2019).)

1.4 Poisson structures and deformation quantization

In the deformation quantization approach to quantum mechanics, the associative algebra of quantum operators reduces, modulo $\hbar$, to the commutative algebra of classical operators. But this algebra of classical operators has a little more structure: it is a Poisson algebra. Deformation quantization posits that the failure of the algebra of quantum operators to be commutative is given, to first order in $\hbar$, by the Poisson bracket.

Something similar happens in our story. Classical observables are given by the algebra of functions on the derived space of solutions to the Euler-Lagrange equations. The Euler-Lagrange equations are not just any PDEs, however: they describe the critical locus of an action functional. The derived critical locus of a function on a finite-dimensional manifold carries a shifted Poisson (or P_0) structure, meaning that its dg algebra of functions has a Poisson bracket of degree 1. In the physics literature, this Poisson bracket is sometimes called the BV bracket or anti-bracket.

This feature suggests that the space of solutions to the Euler-Lagrange equations should also have a P_0 structure, and so the factorization algebra Obs^{cl} of classical observables has the structure of a P_0 algebra. We show that this guess is indeed true, as long as we use a certain homotopical version of P_0 factorization algebras.

Just as in the case of quantum mechanics, we would like the Poisson bracket on classical observables to reflect the first-order deformation into quantum observables. We find that this behavior is the case, although the statement is not as nice as that in the familiar quantum mechanical case.

Let us explain how it works. The factorization algebra of classical observables has compatible structures of dg commutative algebra and shifted Poisson bracket. The factorization algebra of quantum observables has, by contrast, no extra structure: it is simply a factorization algebra valued in cochain complexes. Modulo $\hbar^2$, the factorization algebra of quantum observables lives in an exact sequence

$$0 \to \hbar\,\mathrm{Obs}^{cl} \to \mathrm{Obs}^{q} \bmod \hbar^2 \to \mathrm{Obs}^{cl} \to 0.$$

The boundary map for this exact sequence is an operator, for every open $U \subset M$,

$$D : \mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^{cl}(U).$$

This operator is a cochain map of cohomological degree 1. Because Obs^q is not a factorization algebra valued in commutative algebras, D is not a derivation for the commutative algebra structure on $\mathrm{Obs}^{cl}(U)$.

We can measure the failure of D to be a derivation by the expression

$$D(ab) - (-1)^{|a|} a Db - (Da)b.$$

We find that this quantity is the same, up to homotopy, as the shifted Poisson bracket on classical observables.

We should view this identity as being the analog of the fact that the failure of the algebra of observables of quantum mechanics to be commutative is measured, modulo $\hbar^2$, by the Poisson bracket. Here, we find that the failure of the factorization algebra of quantum observables to have a commutative algebra structure compatible with the differential is measured by the shifted Poisson bracket on classical observables.

This analogy has been strengthened to a theorem by Safronov (2018) and Rozenbyum (unpublished). Locally constant factorization algebras on $\mathbb{R}$ are equivalent to homotopy associative algebras. Safronov and Rozenblyum show that locally constant P_0 factorization algebras on $\mathbb{R}$ are equivalent to ordinary, unshifted, Poisson algebras. Therefore a deformation quantization of a P_0 factorization algebra on $\mathbb{R}$ into a plain factorization algebra is precisely the same as a deformation of a Poisson algebra into an associative algebra; in this sense, our work recovers the usual notion of deformation quantization.

1.5 Noether's theorem

The second main theorem we prove in this volume is a factorization-algebraic version of Noether's theorem. The formulation we find of Noether's theorem is significantly more general than the traditional formulation. We will start by reminding the reader of the traditional formulation, before explaining our factorization-algebraic generalization.

1.5.1 Symmetries in classical mechanics

The simplest version of Noether's theorem applies to classical mechanics.

Suppose we have a classical-mechanical system with a continuous symmetry given by a Lie algebra $\mathfrak{g}$. Let A be the Poisson algebra of operators of the system, which is the algebra of functions on the phase space. Then, Noether's theorem, as traditionally phrased, says that there is a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ and a map of Lie algebras

$$\widehat{\mathfrak{g}} \to A$$

where A is given the Lie bracket coming from the Poisson bracket. This map sends the central element in $\widehat{\mathfrak{g}}$ to a multiple of the identity in A. Further, the image of $\widehat{\mathfrak{g}}$ in A commutes with the Hamiltonian.

From a modern point of view, this is easily understood. The phase space of the classical mechanical system is a symplectic manifold X, with a function H on it which is the Hamiltonian. The algebra of operators is the Poisson algebra of functions on X. If a Lie algebra $\mathfrak{g}$ acts as symmetries of the classical system, then it acts on X by symplectic vector fields preserving the Hamiltonian function. There is a central extension of $\mathfrak{g}$ that acts on X by Hamiltonian vector fields, assuming that $H^1(X) = 0$.

At the quantum level, the Poisson algebra of functions on X is upgraded to a non-commutative algebra (which we continue to call A), which is its deformation quantization. The quantum version of Noether's theorem says that if we have an action of a Lie algebra $\mathfrak{g}$ acting on the quantum mechanical system, there is a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ (possibly depending on $\hbar$) and a Lie algebra map $\widehat{\mathfrak{g}} \to A$. This Lie algebra map lifts canonically to a map of associative algebras

$$U\widehat{\mathfrak{g}} \to A$$

sending the central element to $1 \in A$.

1.5.2 Noether's theorem in the language of factorization algebras

Let us rewrite the quantum-mechanical Noether's theorem in terms of factorization algebras on $\mathbb{R}$. As we saw in section I.3.2, factorization algebras on $\mathbb{R}$ satisfying a certain local-constancy condition are the same as associative algebras. When translation invariant, these factorization algebras on $\mathbb{R}$ are the same as associative algebras with a derivation. A quantum mechanical system is a quantum field theory on $\mathbb{R}$, and so has as a factorization algebra Obs^q of observables. Under the equivalence between factorization algebras on $\mathbb{R}$ and associative algebra, the factorization algebra Obs^q becomes the associative algebra A of operators, and the derivation becomes the Hamiltonian.

Similarly, we can view $U_c(\mathfrak{g})$ as being a translation-invariant factorization algebra on $\mathbb{R}$, where the translation action is trivial. In section I.3.6, we give a general construction of a factorization algebra – the *factorization envelope* – associated to a sheaf of dg Lie algebras on a manifold. The associative algebra $U_c(\mathfrak{g})$ is, when interpreted as a factorization algebra on $\mathbb{R}$, the twisted factorization envelope of the sheaf $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ of dg Lie algebras on $\mathbb{R}$. We write this twisted factorization envelope as $U_c(\Omega^*_{\mathbb{R}} \otimes \mathfrak{g})$.

Noether's theorem then tells us that there is a map of translation-invariant factorization algebras

$$U_c(\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}) \to \mathrm{Obs}^q$$

on $\mathbb{R}$. We have simply reformulated Noether's theorem in factorization-algebraic language. This rewriting will become useful shortly, however, when we state a far-reaching generalization of Noether's theorem.

1.5.3 Noether's theorem in quantum field theory

Let us now phrase our general theorem. Suppose we have a quantum field theory on a manifold X, with factorization algebra Obs^q of observables. The usual formulation of Noether's theorem starts with a field theory with some Lie algebra of symmetries. We will work more generally, and ask that there is some sheaf $\mathcal{L}$ of homotopy Lie algebras on X which acts as symmetries of our QFT. (Strictly speaking, we work with sheaves of homotopy Lie algebras of a special type, which we call a local L_∞ algebras. A local L_∞ algebra is a sheaf of homotopy Lie algebras whose underlying sheaf is the smooth sections of a graded vector bundle and whose structure maps are given by multi-differential operators.) Our formulation of Noether's theorem then takes the following form.

Theorem. *In this situation, there is a canonical $\hbar$-dependent (shifted) central extension of $\mathcal{L}$, and a map*

$$U_c(\mathcal{L}) \to \mathrm{Obs}^q$$

of factorization algebras, from the twisted factorization envelope of $\mathcal{L}$ to the factorization algebra of observables of the quantum field theory.

Let us explain how a special case of this statement recovers the traditional formulation of Noether's theorem, under the assumption (merely to simplify the notation) that the central extension is trivial.

Suppose we have a theory with a Lie algebra $\mathfrak{g}$ of symmetries. One can show that this implies the sheaf $\Omega^*_X \otimes \mathfrak{g}$ of dg Lie algebras also acts on the theory. Indeed, this sheaf is simply a resolution of the constant sheaf with stalk $\mathfrak{g}$.

The factorization envelope $U(\mathcal{L})$ assigns the Chevalley-Eilenberg chain complex $C_*(\mathcal{L}_c(U))$ to an open subset $U \subset X$. This construction implies that there is a map of precosheaves $\mathcal{L}_c[1] \to U(\mathcal{L})$. Applied to $\mathcal{L} = \Omega^*_X \otimes \mathfrak{g}$, we find that a $\mathfrak{g}$-action on our theory gives a cochain map

$$\Omega^*_c(U) \otimes \mathfrak{g}[1] \to \mathrm{Obs}^q(U)$$

for every open. In degree 0, this map $\Omega^1_c \otimes \mathfrak{g} \to \mathrm{Obs}^q$ can be viewed as an $n-1$-form on X valued in observables. This $n-1$ form is the Noether current. (The other components of this map contain important homotopical information.)

If $X = M \times \mathbb{R}$ where M is compact and connected, we get a map

$$\mathfrak{g} = H^0(M) \otimes H^1_c(\mathbb{R}) \otimes \mathfrak{g} \to H^0(\mathrm{Obs}^q(U)).$$

This map is the Noether charge.

We have seen that specializing to observables of cohomological degree 0, and the sheaf $\mathcal{L} = \Omega^* \otimes \mathfrak{g}$, we recover the traditional formulation of Noether's theorem in quantum field theory. Our formulation, however, is considerably more general.

1.5.4 Noether's theorem applied to two-dimensional chiral theories

As an example of this general form of Noether's theorem, let us consider the case of two-dimensional chiral theories with symmetry group G.

In this situation, the symmetry Lie algebra is not simply the constant sheaf with values in $\mathfrak{g}$, but the sheaf $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$, the Dolbeault complex on Σ valued in $\mathfrak{g}$. In other words, it encodes the sheaf of $\mathfrak{g}$-valued holomorphic functions. This sheaf of dg Lie algebra acts on the sheaf of fields $\Omega^{1/2,*}(\Sigma, R)$ in the evident way.

Our formulation of Noether's theorem then tells us that there is some central extension of $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$ and a map of factorization algebras

$$U_c(\Omega^{0,*}_\Sigma \otimes \mathfrak{g}) \to \mathrm{Obs}^q$$

from the twisted factorization envelope to the observables of the system of free fermions.

In section I.5.5 we calculated the twisted factorization algebra of $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$, and we found that it encodes the Kac-Moody vertex algebra at level determined by the central extension.

Thus, in this example, our formulation of Noether's theorem recovers something relatively familiar: in any chiral theory with an action of G, we find a copy of the Kac-Moody algebra at an appropriate level.

1.6 Brief orienting remarks toward the literature

Since we began this project in 2008, we have been pleased to see how themes that animated our own work have gotten substantial attention from others as well:

- Encoding classical field theories, particularly in the BV formalism, using L_∞ algebras Hohm and Zwiebach (2017); Jurčo et al. (2019b,a).
- The meaning and properties of derived critical loci Vezzosi (2020); Joyce (2015); Pridham (2019).
- The role of shifted symplectic structures in derived geometry and enlarged notions of deformation quantization Pantev et al. (2013a); Calaque et al. (2017); Ben-Bassat et al. (2015); Brav et al. (2019); Pridham (2017); Melani and Safronov (2018a,b); Safronov (2017); Toën (2014).
- Factorization algebras as a natural tool in field theory, particularly for topological field theories Scheimbauer (2014); Kapranov et al. (2016); Benini et al. (2019, 2020); Beem et al. (2020).

We are grateful to take part in such a dynamic community, where we benefit from others' insights and critiques and we also have the chance to share our own. This book does not document all that activity, which is only partially represented by the published literature anyhow; we offer only a scattering of the relevant references, typically those that played a direct role in our own work or in our learning, and hence exhibit an unfortunate but hard-to-avoid bias toward close collaborators or interlocutors.

Our work builds, of course, upon the work and insights of generations of mathematicians and physicists who proceed us. As time goes on, we discover how many of our insights appear in some guise in the past. In particular, it should be clear how much Albert Schwarz and Maxim Kontsevich shaped our views and our approach by their vision and by their results, and how much we gained from engaging with work of Alberto Cattaneo, Giovanni Felder, and Andrei Losev.

There is a rich literature on BRST and BV methods in physics that we hope to help open up to mathematicians, but we do not make an attempt here to survey it, a task that is beyond us. We recommend Henneaux and Teitelboim (1992) as point for jumping into that literature, tracking who cites it and whom it cites. A nice starting point to explore current activity in Lorentzian signature is Rejzner (2016), where these BRST/BV ideas cross-fertilize with the algebraic quantum field theory approach.

1.7 Acknowledgements

The project of writing this book has stretched over more than a decade. In that course of time, we have benefited from conversations with many people, the chance to present this work at several workshops and conferences, and the feedback of various readers. The book and our understanding of this material is much better due to the interest and engagement of so many others. Thank you.

We would like to thank directly the following people, although this list is undoubtedly incomplete: David Ayala, David Ben-Zvi, Dan Berwick-Evans, Damien Calaque, Alberto Cattaneo, Lee Cohn, Ivan Contreras, Vivek Dhand, Chris Douglas, Chris Elliott, John Francis, Davide Gaiotto, Dennis Gaitsgory, Sachin Gautam, Ezra Getzler, Greg Ginot, Ryan Grady, André Henriques, Rune Haugseng, Theo Johnson-Freyd, David Kazhdan, Si Li, Jacob Lurie, Takuo Matsuoka, Aaron Mazel-Gee, Pavel Mnev, David Nadler, Thomas Nikolaus, Fred Paugam, Dmitri Pavlov, Toly Preygel, Eugene Rabinovich, Kasia Rejzner, Kolya Reshetikhin, Nick Rozenblyum, Pavel Safronov, Claudia Scheimbauer, Graeme Segal, Jackie Shadlen, Yuan Shen, Jim Stasheff, Stephan Stolz, Dennis Sullivan, Matt Szczesny, Hiro Tanaka, Peter Teichner, David Treumann, Alessandro Valentino,, Brian Williams, Philsang Yoo, nd Eric Zaslow for helpful conversations. K.C. is particularly grateful to Graeme Segal for many illuminating conversations about quantum field theory over the years. O.G. would like to thank the participants in the Spring 2014 Berkeley seminar and Fall 2014 MPIM seminar for their interest and extensive feedback. He would like to offer particular thanks to Brian Williams and Eugene Rabinovich for their work and collaborations that used the Noether machinery, and which has substantially enhanced that Part of this book.

Our work has taken place at many institutions — Northwestern University, the University of California, Berkeley, the Max Planck Institute for Mathematics, the Perimeter Institute for Theoretical Physics, and the University of Masschusetts, Amherst — which each provided a supportive environment for writing and research. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation. The Perimeter Institute is located within the Haldimand Tract, land promised to the Six Nations

(Haudonosaunee) peoples. K.C.'s research at the Perimeter Institute is supported by the Krembil Foundation and by the NSERC discovery grant program. We have also benefitted from the support of the NSF: K.C. and O.G. were partially supported by NSF grants DMS 0706945 and DMS 1007168, and O.G. was supported by NSF postdoctoral fellowship DMS 1204826 and DMS 1812049. K.C. was also partially supported by a Sloan fellowship and a Simons fellowship in the early days of this long project.

Finally, during the period of this project, we have depended upon the unrelenting support and love of our families. In fact, we have both gotten married and had children over these years! Thank you Josie, Dara, Laszlo, and Hadrian for being models of insatiable curiosity, ready sources of exuberant distraction, and endless fonts of joy. Thank you Lauren and Sophie for joining and building our lives together. Amidst all the tumult and the moves, you always encourage us, you always share your wit and wisdom and warmth. And it makes all the difference.

PART ONE

CLASSICAL FIELD THEORY

2
Introduction to classical field theory

Our goal here is to describe how the observables of a classical field theory naturally form a factorization algebra. More accurately, we are interested in what might be called classical perturbative field theory. "Classical" means that the main object of interest is the sheaf of solutions to the Euler-Lagrange equations for some local action functional. "Perturbative" means that we will only consider those solutions which are infinitesimally close to a given solution. Much of this part of the book is devoted to providing a precise mathematical definition of these ideas, with inspiration taken from deformation theory and derived geometry. In this chapter, then, we will simply sketch the essential ideas.

2.1 The Euler-Lagrange equations

The fundamental objects of a physical theory are the observables of a theory, that is, the measurements one can make in that theory. In a classical field theory, the fields that appear "in nature" are constrained to be solutions to the Euler-Lagrange equations (also called the equations of motion). Thus, the measurements one can make are the functions on the space of solutions to the Euler-Lagrange equations.

However, it is essential that we do not take the naive moduli space of solutions. Instead, we consider the *derived* moduli space of solutions. Since we are working perturbatively — that is, infinitesimally close to a given solution — this derived moduli space will be a "formal moduli problem” Lurie (2010, n.d.). In the physics literature, the procedure of taking the derived critical locus of the action functional is implemented by the BV formalism. Thus, the first step (chapter 3.1.3) in our treatment of classical field theory is to develop a language to treat formal moduli problems cut out by systems of partial differential equations on a manifold M. Since it is essential that the differential equations we consider are elliptic, we call such an object a *formal elliptic moduli problem*.

Since one can consider the solutions to a differential equation on any open subset $U \subset M$, a formal elliptic moduli problem $\mathcal{F}$ yields, in particular, a sheaf of formal moduli problems on M. This sheaf sends U to the formal moduli space $\mathcal{F}(U)$ of solutions on U.

We will use the notation $\mathcal{EL}$ to denote the formal elliptic moduli problem of solutions to the Euler-Lagrange equation on M; thus, $\mathcal{EL}(U)$ will denote the space of solutions on an open subset $U \subset M$.

2.2 Observables

In a field theory, we tend to focus on measurements that are localized in spacetime. Hence, we want a method that associates a set of observables to each region in M. If $U \subset M$ is an open subset, the observables on U are

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathcal{EL}(U)),$$

our notation for the algebra of functions on the formal moduli space $\mathcal{EL}(U)$ of solutions to the Euler-Lagrange equations on U. (We will be more precise about which class of functions we are using later.) As we are working in the derived world, $\mathrm{Obs}^{cl}(U)$ is a differential-graded commutative algebra. Using these functions, we can answer any question we might ask about the behavior of our system in the region U.

The factorization algebra structure arises naturally on the observables in a classical field theory. Let U be an open set in M, and $V_1, \ldots, V_k$ a disjoint collection of open subsets of U. Then restriction of solutions from U to each V_i induces a natural map

$$\mathcal{EL}(U) \to \mathcal{EL}(V_1) \times \cdots \times \mathcal{EL}(V_k).$$

Since functions pullback under maps of spaces, we get a natural map

$$\mathrm{Obs}^{cl}(V_1) \otimes \cdots \otimes \mathrm{Obs}^{cl}(V_k) \to \mathrm{Obs}^{cl}(U)$$

so that Obs^{cl} forms a *pre*factorization algebra. To see that Obs^{cl} is indeed a factorization algebra, it suffices to observe that the functor $\mathcal{EL}$ is a sheaf.

Since the space $\mathrm{Obs}^{cl}(U)$ of observables on a subset $U \subset M$ is a commutative algebra, and not just a vector space, we see that the observables of a classical field theory form a commutative factorization algebra (i.e., a commutative algebra object in the symmetric monoidal category of factorization algebras).

2.3 The symplectic structure

Above, we outlined a way to construct, from the elliptic moduli problem associated to the Euler-Lagrange equations, a commutative factorization algebra. This construction, however, would apply equally well to any system of differential equations. The Euler-Lagrange equations, of course, have the special property that they arise as the critical points of a functional.

In finite dimensions, a formal moduli problem which arises as the derived critical locus (section 4.1) of a function is equipped with an extra structure: a symplectic form of cohomological degree -1. For us, this symplectic form is an intrinsic way of indicating that a formal moduli problem arises as the critical locus of a functional. Indeed, any formal moduli problem with such a symplectic form can be expressed (non-uniquely) in this way.

We give (section 4.2) a definition of symplectic form on an elliptic moduli problem. We then simply *define* a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

Given a local action functional satisfying certain non-degeneracy properties, we construct (section 4.3.1) an elliptic moduli problem describing the corresponding Euler-Lagrange equations, and show that this elliptic moduli problem has a symplectic form of degree -1.

In ordinary symplectic geometry, the simplest construction of a symplectic manifold is as a cotangent bundle. In our setting, there is a similar construction: given any elliptic moduli problem $\mathcal{F}$, we construct (section 4.6) a new elliptic moduli problem $T^*[-1]\mathcal{F}$ which has a symplectic form of degree -1. It turns out that many examples of field theories of interest in mathematics and physics arise in this way.

2.4 The P_0 structure

In finite dimensions, if X is a formal moduli problem with a symplectic form of degree -1, then the dg algebra $\mathscr{O}(X)$ of functions on X is equipped with a Poisson bracket of degree 1. In other words, $\mathscr{O}(X)$ is a P_0 algebra (i.e., has a 1-shifted Poisson bracket).

In infinite dimensions, we show that something similar happens. If $\mathcal{F}$ is a classical field theory, then we show that on every open U, the commutative algebra $\mathscr{O}(\mathcal{F}(U)) = \mathrm{Obs}^{cl}(U)$ has a P_0 structure. We then show that the commutative factorization algebra Obs^{cl} forms a P_0 factorization algebra. This is not quite trivial; it is at this point that we need the assumption that our Euler-Lagrange equations are elliptic.

3
Elliptic moduli problems

The essential data of a classical field theory is the moduli space of solutions to the equations of motion of the field theory. For us, it is essential that we take not the naive moduli space of solutions, but rather the *derived* moduli space of solutions. In the physics literature, the procedure of taking the derived moduli of solutions to the Euler-Lagrange equations is known as the classical Batalin-Vilkovisky formalism.

The derived moduli space of solutions to the equations of motion of a field theory on X is a sheaf on X. In this chapter we will introduce a general language for discussing sheaves of "derived spaces" on X that are cut out by differential equations.

Our focus in this book is on perturbative field theory, so we sketch the heuristic picture from physics before we introduce a mathematical language that formalizes the picture. Suppose we have a field theory and we have found a solution to the Euler-Lagrange equations ϕ_0. We want to find the nearby solutions, and a time-honored approach is to consider a formal series expansion around ϕ_0,

$$\phi_t = \phi_0 + t\phi_1 + t^2\phi_2 + \cdots ,$$

and to solve iteratively the Euler-Lagrange equations for the higher terms ϕ_n. Of course, such an expansion is often not convergent in any reasonable sense, but this perturbative method has provided insights into many physical problems. In mathematics, particularly the deformation theory of algebraic geometry, this method has also flourished and acquired a systematic geometric interpretation. Here, though, we work in place of t with a parameter ϵ that is nilpotent, so that there is some integer n such that $\epsilon^{n+1} = 0$. Let

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots + \epsilon^n\phi_n.$$

Again, the Euler-Lagrange equation applied to ϕ becomes a system of simpler differential equations organized by each power of ϵ. As we let the order of ϵ go to infinity and find the nearby solutions, we describe the *formal neighborhood* of ϕ_0 in the space of all solutions to the Euler-Lagrange equations. (Although this procedure may seem narrow in scope, its range expands considerably by considering families of solutions, rather a single fixed solution. Our formalism is built to work in families.)

In this chapter we will introduce a mathematical formalism for this procedure, which includes derived perturbations (i.e., ϵ has nonzero cohomological degree). In mathematics, this formalism is part of derived deformation theory or formal derived geometry. Thus, before we discuss the concepts specific to classical field theory, we will explain some general techniques from deformation theory. A key role is played by a deep relationship between Lie algebras and formal moduli spaces.

3.1 Formal moduli problems and Lie algebras

In ordinary algebraic geometry, the fundamental objects are commutative algebras. In derived algebraic geometry, commutative algebras are replaced by differential graded commutative algebras concentrated in non-positive degrees (or, if one prefers, simplicial commutative algebras; over $\mathbb{Q}$, there is no difference).

We are interested in formal derived geometry, which is described by nilpotent dg commutative algebras. It is the natural mathematical framework for discussing deformations and perturbations. In this section we give a rapid overview, but appendix A.2 examines these ideas at a more leisurely pace and supplies more extensive references.

3.1.0.1 Definition. *An* Artinian dg algebra *over a field K of characteristic zero is a differential graded commutative K-algebra R, concentrated in degrees ≤ 0, such that*

(i) *each graded component R^i is finite dimensional, and $R^i = 0$ for $i \ll 0$;*
(ii) *R has a unique maximal differential ideal m such that $R/m = K$, and such that $m^N = 0$ for $N \gg 0$.*

Given the first condition, the second condition is equivalent to the statement that $H^0(R)$ is Artinian in the classical sense.

The category of Artinian dg algebras is simplicially enriched in a natural way. A map $R \to S$ is simply a map of dg algebras taking the maximal ideal m_R to that of m_S. Equivalently, such a map is a map of non-unital dg algebras $m_R \to m_S$. An n-simplex in the space $\mathrm{Maps}(R, S)$ of maps from R to S is defined to be a map of non-unital dg algebras

$$m_R \to m_S \otimes \Omega^*(\triangle^n)$$

where $\Omega^*(\triangle^n)$ is some commutative algebra model for the cochains on the n-simplex. (Normally, we will work over $\mathbb{R}$, and $\Omega^*(\triangle^n)$ will be the usual de Rham complex.)

We will (temporarily) let Art_k denote the simplicially enriched category of Artinian dg algebras over k.

3.1.0.2 Definition. *A* formal moduli problem *over a field k is a functor (of simplicially enriched categories)*

$$F : \mathrm{Art}_k \to \mathrm{sSets}$$

from Art_k *to the category* sSets *of simplicial sets, with the following additional properties.*

(i) *$F(k)$ is contractible.*
(ii) *F takes surjective maps of dg Artinian algebras to fibrations of simplicial sets.*
(iii) *Suppose that A, B, C are dg Artinian algebras, and that $B \to A$, $C \to A$ are surjective maps. Then we can form the fiber product $B \times_A C$. We require that the natural map*

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is a weak homotopy equivalence.

We remark that such a moduli problem F is *pointed*: F assigns to k a point, up to homotopy, since $F(k)$ is contractible. Since we work mostly with pointed moduli problems in this book, we will not emphasize this issue. Whenever we work with more general moduli problems, we will indicate it explicitly.

Note that, in light of the second property, the fiber product $F(B) \times_{F(A)} F(C)$ coincides with the homotopy fiber product.

The category of formal moduli problems is itself simplicially enriched, in an evident way. If F, G are formal moduli problems, and $\phi : F \to G$ is a map, we say that ϕ is a weak equivalence if for all dg Artinian algebras R, the map

$$\phi(R) : F(R) \to G(R)$$

is a weak homotopy equivalence of simplicial sets.

3.1.1 Formal moduli problems and L_∞ algebras

One very important way in which formal moduli problems arise is as the solutions to the Maurer-Cartan equation in an L_∞ algebra. As we will see later, all formal moduli problems are equivalent to formal moduli problems of this form.

If $\mathfrak{g}$ is an L_∞ algebra, and (R, m) is a dg Artinian algebra, we will let

$$\mathrm{MC}(\mathfrak{g} \otimes m)$$

denote the simplicial set of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes m$. Thus, an n-simplex in this simplicial set is an element

$$\alpha \in \mathfrak{g} \otimes m \otimes \Omega^*(\triangle^n)$$

of cohomological degree 1, which satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \sum_{n \geq 2} \tfrac{1}{n!} l_n(\alpha, \dots, \alpha) = 0.$$

It is a well-known result in derived deformation theory that sending R to $\mathrm{MC}(\mathfrak{g} \otimes m)$ defines a formal moduli problem (see Getzler (2009), Hinich (2001)). We will often use the notation $B\mathfrak{g}$ to denote this formal moduli problem.

If $\mathfrak{g}$ is finite dimensional, then a Maurer-Cartan element of $\mathfrak{g} \otimes m$ is the same thing as a map of commutative dg algebras

$$C^*(\mathfrak{g}) \to R$$

which takes the maximal ideal of $C^*(\mathfrak{g})$ to that of R.

Thus, we can think of the Chevalley-Eilenberg cochain complex $C^*(\mathfrak{g})$ as the algebra of functions on $B\mathfrak{g}$.

Under the dictionary between formal moduli problems and L_∞ algebras, a dg vector bundle on $B\mathfrak{g}$ is the same thing as a dg module over $\mathfrak{g}$. The cotangent complex to $B\mathfrak{g}$ corresponds to the $\mathfrak{g}$-module $\mathfrak{g}^\vee[-1]$, with the shifted coadjoint action. The tangent complex corresponds to the $\mathfrak{g}$-module $\mathfrak{g}[1]$, with the shifted adjoint action.

If M is a $\mathfrak{g}$-module, then sections of the corresponding vector bundle on $B\mathfrak{g}$ is the Chevalley-Eilenberg cochains with coefficients in M. Thus, we can define $\Omega^1(B\mathfrak{g})$ to be

$$\Omega^1(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]).$$

Similarly, the complex of vector fields on $B\mathfrak{g}$ is

$$\mathrm{Vect}(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}[1]).$$

Note that, if $\mathfrak{g}$ is finite dimensional, this is the same as the cochain complex of derivations of $C^*(\mathfrak{g})$. Even if $\mathfrak{g}$ is not finite dimensional, the complex $\mathrm{Vect}(B\mathfrak{g})$ is, up to a shift of one, the Lie algebra controlling deformations of the L_∞ structure on $\mathfrak{g}$.

3.1.2 The fundamental theorem of deformation theory

The following statement is at the heart of the philosophy of deformation theory:

There is an equivalence of $(\infty, 1)$-categories between the category of differential graded Lie algebras and the category of formal pointed moduli problems.

In a different guise, this statement goes back to Quillen's work Quillen (1969) on rational homotopy theory. A precise formulation of this theorem has been proved in Hinich (2001); more general theorems of this nature are considered in Lurie (n.d.); Pridham (2010). We recommend Lurie (2010) as an excellent survey of these ideas.

It would take us too far afield to describe the language in which this statement can be made precise. We will simply use this statement as motivation: we will only consider formal moduli problems described by L_∞ algebras, and this statement asserts that we lose no information in doing so.

3.1.3 Elliptic moduli problems

We are interested in formal moduli problems which describe solutions to differential equations on a manifold M. Since we can discuss solutions to a differential equation on any open subset of M, such an object will give a sheaf of derived moduli problems on M, described by a sheaf of homotopy Lie algebras. Let us give a formal definition of such a sheaf.

3.1.3.1 Definition. *Let M be a manifold. A* local L_∞ algebra *on M consists of the following data.*

(i) *A graded vector bundle L on M, whose space of smooth sections will be denoted $\mathcal{L}$.*
(ii) *A differential operator $\mathrm{d} : \mathcal{L} \to \mathcal{L}$, of cohomological degree 1 and square 0.*
(iii) *A collection of poly-differential operators*

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$$

for $n \geq 2$, which are alternating, are of cohomological degree $2-n$, and endow $\mathcal{L}$ with the structure of L_∞ algebra.

3.1.3.2 Definition. *An* elliptic L_∞ algebra *is a local L_∞ algebra $\mathcal{L}$ as above with the property that $(\mathcal{L}, \mathrm{d})$ is an elliptic complex.*

Remark: The reader who is not comfortable with the language of L_∞ algebras will lose little by only considering elliptic dg Lie algebras. Most of our examples of classical field theories will be described using dg Lie algebra rather than L_∞ algebras.

If $\mathcal{L}$ is a local L_∞ algebra on a manifold M, then it yields a presheaf

$B\mathcal{L}$ of formal moduli problems on M. This presheaf sends a dg Artinian algebra (R, m) and an open subset $U \subset M$ to the simplicial set

$$B\mathcal{L}(U)(R) = \mathrm{MC}(\mathcal{L}(U) \otimes m)$$

of Maurer-Cartan elements of the L_∞ algebra $\mathcal{L}(U) \otimes m$ (where $\mathcal{L}(U)$ refers to the sections of L on U). We will think of this as the R-points of the formal pointed moduli problem associated to $\mathcal{L}(U)$. One can show, using the fact that $\mathcal{L}$ is a fine sheaf, that this sheaf of formal moduli problems is actually a homotopy sheaf, i.e. it satisfies Čech descent. Since this point plays no role in our work, we will not elaborate further.

3.1.3.3 Definition. *A* formal pointed elliptic moduli problem *(or* elliptic moduli problem*, for brevity) is a sheaf of formal moduli problems on M that is represented by an elliptic L_∞ algebra.*

The basepoint of the moduli problem corresponds, in the setting of field theory, to the distinguished solution we are expanding around.

3.2 Examples of elliptic moduli problems related to scalar field theories

We examine the free scalar field before adding interactions. In Section 4.5 we relate this discussion to the usual formulation in terms of action functionals.

3.2.1 The free scalar field theory

Let us start with the most basic example of an elliptic moduli problem, that of harmonic functions. Let M be a Riemannian manifold. We want to consider the formal moduli problem describing functions ϕ on M that are harmonic, namely, functions that satisfy $\mathrm{D}\phi = 0$ where D is the Laplacian. The base point of this formal moduli problem is the zero function.

The elliptic L_∞ algebra describing this formal moduli problem is defined by

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

This complex is thus situated in degrees 1 and 2. The products l_n in this L_∞ algebra are all zero for $n \geq 2$.

In order to justify this definition, let us analyze the Maurer-Cartan functor of this L_∞ algebra. Let R be an ordinary (not dg) Artinian algebra, and let m be the maximal ideal of R. The set of 0-simplices of the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m \mid \mathrm{D}\phi = 0.\}$$

Indeed, because the L_∞ algebra $\mathcal{L}$ is Abelian, the set of solutions to the Maurer-Cartan equation is simply the set of closed degree 1 elements of the cochain complex $\mathcal{L} \otimes m$. All higher simplices in the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ are constant. To see this, note that if $\phi \in \mathcal{L} \otimes m \otimes \Omega^*(\triangle^n)$ is a closed element in degree 1, then ϕ must be in $C^\infty(M) \otimes m \otimes \Omega^0(\triangle^n)$. The fact that ϕ is closed amounts to the statement that $\mathrm{D}\phi = 0$ and that $\mathrm{d}_{dR}\phi = 0$, where d_{dR} is the de Rham differential on $\Omega^*(\triangle^n)$.

Let us now consider the Maurer-Cartan simplicial set associated to a differential graded Artinian algebra (R, m) with differential d_R. The the set of 0-simplices of $\mathrm{MC}_{\mathcal{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m^0, \ \ \psi \in C^\infty(M) \otimes m^{-1} \mid \mathrm{D}\phi = \mathrm{d}_R\psi.\}$$

(The superscripts on m indicate the cohomological degree.) Thus, the 0-simplices of our simplicial set can be identified with the set R-valued smooth functions ϕ on M that are harmonic up to a homotopy given by ψ and also vanish modulo the maximal ideal m.

Next, let us identify the set of 1-simplices of the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$. This is the set of closed degree 1 elements of $\mathcal{L} \otimes m \otimes \Omega^*([0,1])$. Such a closed degree 1 element has four terms:

$$\begin{aligned}
\phi_0(t) &\in C^\infty(M) \otimes m^0 \otimes \Omega^0([0,1]) \\
\phi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^1([0,1]) \\
\psi_0(t) &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^0([0,1]) \\
\psi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-2} \otimes \Omega^1([0,1]).
\end{aligned}$$

Being closed amounts to satisfying the three equations

$$\begin{aligned} \mathrm{D}\,\phi_0(t) &= \mathrm{d}_R \psi_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi_0(t) &= \mathrm{d}_R \phi_1(t) \\ \mathrm{D}\,\phi_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}\psi_0(t) &= \mathrm{d}_R \psi_1(t). \end{aligned}$$

These equations can be interpreted as follows. We think of $\phi_0(t)$ as providing a family of R-valued smooth functions on M, which are harmonic up to a homotopy specified by $\psi_0(t)$. Further, $\phi_0(t)$ is independent of t, up to a homotopy specified by $\phi_1(t)$. Finally, we have a coherence condition among our two homotopies.

The higher simplices of the simplicial set have a similar interpretation.

3.2.2 Interacting scalar field theories

Next, we will consider an elliptic moduli problem that arises as the Euler-Lagrange equation for an interacting scalar field theory. Let ϕ denote a smooth function on the Riemannian manifold M with metric g. The action functional is

$$S(\phi) = \int_M \tfrac{1}{2}\phi\,\mathrm{D}\,\phi + \tfrac{1}{4!}\phi^4 \,\mathrm{dvol}_g\,.$$

The Euler-Lagrange equation for the action functional S is

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 = 0,$$

a nonlinear PDE, whose space of solutions is hard to describe.

Instead of trying to describe the actual space of solutions to this nonlinear PDE, we will describe the formal moduli problem of solutions to this equation where ϕ is infinitesimally close to zero.

The formal moduli problem of solutions to this equation can be described as the solutions to the Maurer-Cartan equation in a certain elliptic L_∞ algebra which continue we call $\mathcal{L}$. As a cochain complex, $\mathcal{L}$ is

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

Thus, $C^\infty(M)$ is situated in degrees 1 and 2, and the differential is the Laplacian.

The L_∞ brackets l_n are all zero except for l_3. The cubic bracket l_3 is the map

$$\begin{aligned} l_3 : C^\infty(M)^{\otimes 3} &\to C^\infty(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1\phi_2\phi_3. \end{aligned}$$

Here, the copy of $C^\infty(M)$ appearing in the source of l_3 is the one situated in degree 1, whereas that appearing in the target is the one situated in degree 2.

If R is an ordinary (not dg) Artinian algebra, then the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ associated to R has for 0-simplices the set $\phi \in C^\infty(M) \otimes m$ such that $\mathrm{D}\,\phi + \frac{1}{3!}\phi^3 = 0$. This equation may look as complicated as the full nonlinear PDE, but it is substantially simpler than the original problem. For example, consider $R = \mathbb{R}[\epsilon]/(\epsilon^2)$, the "dual numbers." Then $\phi = \epsilon\phi_1$ and the Maurer-Cartan equation becomes $\mathrm{D}\,\phi_1 = 0$. For $R = \mathbb{R}[\epsilon]/(\epsilon^4)$, we have $\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3$ and the Maurer-Cartan equation becomes a triple of simpler *linear* PDE:

$$\mathrm{D}\,\phi_1 = 0, \ \mathrm{D}\,\phi_2 = 0, \ \text{and } D\phi_3 + \tfrac{1}{2}\phi_1^3 = 0.$$

We are simply reading off the ϵ^k components of the Maurer-Cartan equation. The higher simplices of this simplicial set are constant.

If R is a dg Artinian algebra, then the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ has for 0-simplices the set of pairs $\phi \in C^\infty(M) \otimes m^0$ and $\psi \in C^\infty(M) \otimes m^{-1}$ such that

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 = \mathrm{d}_R\psi.$$

We should interpret this as saying that ϕ satisfies the Euler-Lagrange equations up to a homotopy given by ψ.

The higher simplices of this simplicial set have an interpretation similar to that described for the free theory.

3.3 Examples of elliptic moduli problems related to gauge theories

We discuss three natural moduli problems: the moduli of flat connections on a smooth manifold, the moduli of self-dual connections on a smooth 4-manifold, and the moduli of holomorphic bundles on a complex manifold. In Section 4.6 we relate these directly to gauge theories.

3.3.1 Flat bundles

Next, let us discuss a more geometric example of an elliptic moduli problem: the moduli problem describing flat bundles on a manifold M. In this case, because flat bundles have automorphisms, it is more difficult to give a direct definition of the formal moduli problem.

Thus, let G be a Lie group, and let $P \to M$ be a principal G-bundle equipped with a flat connection ∇_0. Let $\mathfrak{g}_P$ be the adjoint bundle (associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$). Then $\mathfrak{g}_P$ is a bundle of Lie algebras on M, equipped with a flat connection that we will also denote ∇_0.

For each Artinian dg algebra R, we want to define the simplicial set $\mathrm{Def}_P(R)$ of R-families of flat G-bundles on M that deform P. The question is "what local L_∞ algebra yields this elliptic moduli problem?"

The *answer* is $\mathcal{L} = \Omega^*(M, \mathfrak{g}_P)$, where the differential is d_{∇_0}, the de Rham differential coupled to our connection ∇_0. But we need to explain how to find this answer so we will provide the reasoning behind our answer. This reasoning is a model for finding the local L_∞ algebras associated to field theories.

Let us start by being more precise about the formal moduli problem that we are studying. We will begin by considering only on the deformations before we examine the issue of gauge equivalence. In other words, we start by just discussing the 0-simplices of our formal moduli problem.

As the underlying topological bundle of P is rigid, we can only deform the flat connection on P. Let's consider deformations over a dg

Artinian ring R with maximal ideal m. A deformation of the connection ∇_0 on P is given by an element

$$A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0,$$

since the difference $\nabla - \nabla_0$ between any connection and our initial connection is a $\mathfrak{g}_P$-valued 1-form. The curvature of the deformed connection $\nabla_0 + A$ is

$$F(A) = \mathrm{d}_{\nabla_0} A + \tfrac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g}_P) \otimes m.$$

Note that, by the Bianchi identity, $\mathrm{d}_{\nabla_0} F(A) + [A, F(A)] = 0$.

Our first attempt to define the formal moduli functor Def_P might be that our moduli problem only returns deformations A such that $F(A) = 0$. From a homotopical perspective, it is more natural to loosen up this strict condition by requiring instead that $F(A)$ be *exact* in the cochain complex $\Omega^2(M, \mathfrak{g}_P) \otimes m$ of m-valued 2-forms on M. In other words, we ask for A to be flat up to homotopy. However, we should also ask that $F(A)$ is exact in a way compatible with the Bianchi identity, because a curvature always satisfies this condition.

Thus, as a preliminary, tentative version of the formal moduli functor Def_P, we will define the 0-simplices $\mathrm{Def}_P^{prelim}(R)[0]$ by

$$\left\{ A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0, B \in \Omega^2(M, \mathfrak{g}_P) \otimes m^{-1} \;\middle|\; \begin{array}{r} F(A) = \mathrm{d}_R B \\ \mathrm{d}_{\nabla_0} B + [A, B] = 0 \end{array} \right\}.$$

These equations say precisely that there exists a term B making $F(A)$ exact and that B satisfies a condition that enforces the Bianchi identity on $F(A)$.

This functor $\mathrm{Def}_P^{prelim}[0]$ does not behave the way that we want, though. Consider fixing our Artinian algebra to be $R = \mathbb{R}[\epsilon_n]/(\epsilon_n^2)$, where ϵ_n has degree $-n$; this is a shifted version of the "dual numbers." As a presheaf of sets on M, the functor $\mathrm{Def}_P^{prelim}[0](R)$ assigns to each open U the set

$$\{a \in \Omega^1(U, \mathfrak{g}_P), b \in \Omega^2(U, \mathfrak{g}_P) \mid d_{\nabla_0} a = 0,\ \mathrm{d}_{\nabla_0} b = 0\}.$$

In other words, we obtain the sheaf of sets $\Omega^1_{cl}(-, \mathfrak{g}_P) \times \Omega^2_{cl}(-, \mathfrak{g}_P)$, which returns closed 1-forms and closed 2-forms. This sheaf is *not*, however, a homotopy sheaf, because these sheaves are not fine and hence have higher cohomology groups.

How do we ensure that we obtain a homotopy sheaf of formal moduli problems? We will ask that B satisfy the Bianchi constraint up a sequence of higher homotopies, rather than satisfy the constraint strictly. Thus, the 0-simplices $\mathrm{Def}_P(R)[0]$ of our simplicial set of deformations are defined by

$$\left\{ \begin{array}{l} A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0 \\ B \in \bigoplus_{k \geq 2} \Omega^k(M, \mathfrak{g}_P) \otimes m^{1-k} \end{array} \middle| \; F(A) + \mathrm{d}B + [A, B] + \tfrac{1}{2}[B, B] = 0 \right\}.$$

Here, d refers to the total differential $\mathrm{d}_{\nabla_0} + \mathrm{d}_R$ on the tensor product $\Omega^{\geq 2}(M, \mathfrak{g}_P) \otimes m$ of cochain complexes.

If we let $B_i \in \Omega^i(M, \mathfrak{g}_P) \otimes m^{1-i}$, then the first few constraints on the B_i can be written as

$$\begin{aligned} \mathrm{d}_{\nabla_0} B_2 + [A, B_2] + \mathrm{d}_R B_3 &= 0 \\ \mathrm{d}_{\nabla_0} B_3 + [A, B_3] + \tfrac{1}{2}[B_2, B_2] + \mathrm{d}_R B_4 &= 0. \end{aligned}$$

Thus, B_2 satisfies the Bianchi constraint up to a homotopy defined by B_3, and so on.

The higher simplices of this simplicial set must relate gauge-equivalent solutions. If we restricted our attention to ordinary Artinian algebras — i.e., to dg algebras R concentrated in degree 0 (and so with zero differential) — then we could define the simplicial set $\mathrm{Def}_P(R)$ to be the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$, which acts on $\mathrm{Def}_P(R)[0]$ in the standard way (see, for instance, Kontsevich and Soibelman (n.d.) or Manetti (2009)).

This approach, however, does not extend well to the dg Artinian algebras. When the algebra R is not concentrated in degree 0, the higher simplices of $\mathrm{Def}_P(R)$ must also involve elements of R of negative cohomological degree. Indeed, degree 0 elements of R should be thought of as homotopies between degree 1 elements of R, and so should contribute 1-simplices to our simplicial set.

A slick way to define a simplicial set $\mathrm{Def}_P(R)[n]$ with both desiderata is as

$$\left\{ A \in \Omega^*(M, \mathfrak{g}_P) \otimes m \otimes \Omega^*(\triangle^n) \mid \mathrm{d}_{\nabla_0} A + \mathrm{d}_R A + \mathrm{d}_{\triangle^n} A + \tfrac{1}{2}[A, A] = 0 \right\},$$

where $\mathrm{d}_{\triangle^n}$ denotes the exterior derivative on $\Omega^*(\triangle^n)$.

Suppose that R is concentrated in degree 0 (so that the differential on R is zero). Then, the higher forms on M do not play any role, and

$$\mathrm{Def}_P(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m \mid \mathrm{d}_{\nabla_0} A + \tfrac{1}{2}[A, A] = 0\}.$$

One can show (see Getzler (2009)) that in this case, the simplicial set $\mathrm{Def}_P(R)$ is weakly homotopy equivalent to the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$. Indeed, a 1-simplex in the simplicial set $\mathrm{Def}_P(R)$ is given by a family of the form $A_0(t) + A_1(t)\mathrm{d}t$, where $A_0(t)$ is a smooth family of elements of $\Omega^1(M, \mathfrak{g}_P) \otimes m$ depending on $t \in [0, 1]$, and $A_1(t)$ is a smooth family of elements of $\Omega^0(M, \mathfrak{g}_P) \otimes m$. The Maurer-Cartan equation in this context says that

$$\begin{aligned} \mathrm{d}_{\nabla_0} A_0(t) + \tfrac{1}{2}[A_0(t), A_0(t)] &= 0 \\ \tfrac{\mathrm{d}}{\mathrm{d}t} A_0(t) + [A_1(t), A_0(t)] &= 0. \end{aligned}$$

The first equation says that $A_0(t)$ defines a family of flat connections. The second equation says that the gauge equivalence class of $A_0(t)$ is independent of t. In this way, gauge equivalences are represented by 1-simplices in $\mathrm{Def}_P(R)$.

It is immediate that the formal moduli problem $\mathrm{Def}_P(R)$ is represented by the elliptic dg Lie algebra

$$\mathcal{L} = \Omega^*(M, \mathfrak{g}).$$

The differential on $\mathcal{L}$ is the de Rham differential d_{∇_0} on M coupled to the flat connection on $\mathfrak{g}$. The only nontrivial bracket is l_2, which just arises by extending the bracket of $\mathfrak{g}$ over the commutative dg algebra $\Omega^*(M)$ in the appropriate way.

3.3.2 Self-dual bundles

Next, we will discuss the formal moduli problem associated to the self-duality equations on a 4-manifold. We won't go into as much detail as we did for flat connections; instead, we will simply write down the elliptic L_∞ algebra representing this formal moduli problem. (For a careful explanation, see the original article Atiyah et al. (1978).)

Let M be an oriented 4-manifold. Let G be a Lie group, and let $P \to M$ be a principal G-bundle, and let $\mathfrak{g}_P$ be the adjoint bundle of Lie algebras. Suppose we have a connection A on P with anti-self-dual curvature:

$$F(A)_+ = 0 \in \Omega^2_+(M, \mathfrak{g}_P)$$

(here $\Omega^2_+(M)$ denotes the space of self-dual two-forms).

Then, the elliptic Lie algebra controlling deformations of (P, A) is described by the diagram

$$\Omega^0(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}} \Omega^1(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_+} \Omega^2_+(M, \mathfrak{g}_P).$$

Here d_+ is the composition of the de Rham differential (coupled to the connection on $\mathfrak{g}_P$) with the projection onto $\Omega^2_+(M, \mathfrak{g}_P)$.

Note that this elliptic Lie algebra is a quotient of that describing the moduli of flat G-bundles on M.

3.3.3 Holomorphic bundles

In a similar way, if M is a complex manifold and if $P \to M$ is a holomorphic principal G-bundle, then the elliptic dg Lie algebra $\Omega^{0,*}(M, \mathfrak{g}_P)$, with differential $\overline{\partial}$, describes the formal moduli space of holomorphic G-bundles on M.

3.4 Cochains of a local L_∞ algebra

Let L be a local L_∞ algebra on M. If $U \subset M$ is an open subset, then $\mathcal{L}(U)$ denotes the L_∞ algebra of smooth sections of L on U. Let $\mathcal{L}_c(U) \subset \mathcal{L}(U)$ denote the sub-L_∞ algebra of compactly supported sections.

In appendix B.1, we define the algebra of functions on the space of sections on a vector bundle on a manifold. We are interested in the algebra

$$\mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq 0} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R} \right)_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps.

This space is naturally a graded differentiable vector space (that is, we can view it as a sheaf of graded vector spaces on the site of smooth manifolds). However, it is important that we treat this object as a differentiable pro-vector space. Basic facts about differentiable pro-vector spaces are developed in appendix I.C.4. The pro-structure comes from the filtration

$$F^i \mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq i} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R} \right)_{S_n},$$

which is the usual filtration on "power series."

The L_∞ algebra structure on $\mathcal{L}(U)$ gives, as usual, a differential on $\mathscr{O}(\mathcal{L}(U)[1])$, making $\mathscr{O}(\mathcal{L}(U)[1])$ into a differentiable pro-cochain complex.

3.4.0.1 Definition. *Define the* Lie algebra cochain complex $C^*(\mathcal{L}(U))$ *to be*

$$C^*(\mathcal{L}(U)) = \mathscr{O}(\mathcal{L}(U)[1])$$

equipped with the usual Chevalley-Eilenberg differential. Similarly, define

$$C^*_{red}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

to be the reduced Chevalley-Eilenberg complex, *that is, the kernel of the natural augmentation map* $C^*(\mathcal{L}(U)) \to \mathbb{R}$. *These are both differentiable pro-cochain complexes.*

We will think of $C^*(\mathcal{L}(U))$ as the algebra of functions on the formal moduli problem $B\mathcal{L}(U)$ associated to the L_∞ algebra $\mathcal{L}(U)$. One defines $C^*(\mathcal{L}_c(U))$ in the same way, everywhere substituting $\mathcal{L}_c$ for $\mathcal{L}$.

3.4.1 Cochains with coefficients in a module

Let L be a local L_∞ algebra on M, and let $\mathcal{L}$ denote the smooth sections. Let E be a graded vector bundle on M and equip the global smooth sections $\mathscr{E}$ with a differential that is a differential operator.

3.4.1.1 Definition. *A* local action of $\mathcal{L}$ on $\mathscr{E}$ *is an action of $\mathcal{L}$ on $\mathscr{E}$ with the property that the structure maps*

$$\mathcal{L}^{\otimes n} \otimes \mathscr{E} \to \mathscr{E}$$

(defined for $n \geq 1$) are all polydifferential operators.

Note that $\mathcal{L}$ has an action on itself, called the adjoint action, where the differential on $\mathcal{L}$ is the one coming from the L_∞ structure, and the action map

$$\mu_n : \mathcal{L}^{\otimes n} \otimes \mathcal{L} \to \mathcal{L}$$

is the L_∞ structure map l_{n+1}.

Let $L^! = L^\vee \otimes_{C^\infty_M} \mathrm{Dens}_M$. Then, $\mathcal{L}^!$ has a natural local $\mathcal{L}$-action, which we should think of as the coadjoint action. This action is defined by saying that if $\alpha_1, \ldots, \alpha_n \in \mathcal{L}$, the differential operator

$$\mu_n(\alpha_1, \ldots, \alpha_n, -) : \mathcal{L}^! \to \mathcal{L}^!$$

is the formal adjoint to the corresponding differential operator arising from the action of $\mathcal{L}$ on itself.

If E is a local module over L, then, for each $U \subset M$, we can define the Chevalley-Eilenberg cochains

$$C^*(\mathcal{L}(U), \mathscr{E}(U))$$

of $\mathcal{L}(U)$ with coefficients in $\mathscr{E}(U)$. As above, one needs to take account of the topologies on the vector spaces $\mathcal{L}(U)$ and $\mathscr{E}(U)$ when defining this Chevalley-Eilenberg cochain complex. Thus, as a graded vector space,

$$C^*(\mathcal{L}(U), \mathscr{E}(U)) = \prod_{n \geq 0} \mathrm{Hom}((\mathcal{L}(U)[1])^{\otimes n}, \mathscr{E}(U))_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps. Again, we treat this object as a differentiable pro-cochain complex.

As explained in the section on formal moduli problems (section 3.1), we should think of a local module E over L as providing, on each open subset $U \subset M$, a vector bundle on the formal moduli problem

$B\mathcal{L}(U)$ associated to $\mathcal{L}(U)$. Then the Chevalley-Eilenberg cochain complex $C^*(\mathcal{L}(U), \mathscr{E}(U))$ should be thought of as the space of sections of this vector bundle.

3.5 D-modules and local L_∞ algebras

Our definition of a local L_∞ algebra is designed to encode the derived moduli space of solutions to a system of non-linear differential equations. An alternative language for describing differential equations is the theory of D-modules. In this section we will show how our local L_∞ algebras can also be viewed as L_∞ algebras in the symmetric monoidal category of D-modules.

The main motivation for this extra layer of formalism is that local action functionals — which play a central role in classical field theory — are elegantly described using the language of D-modules.

Let C_M^∞ denote the sheaf of smooth functions on the manifold M, let Dens_M denote the sheaf of smooth densities, and let D_M the sheaf of differential operators with smooth coefficients. The ∞-jet bundle $Jet(E)$ of a vector bundle E is the vector bundle whose fiber at a point $x \in M$ is the space of jets (or formal germs) at x of sections of E. The sheaf of sections of $Jet(E)$, denoted $J(E)$, is equipped with a canonical D_M-module structure, i.e., the natural flat connection sometimes known as the Cartan distribution. This flat connection is characterized by the property that flat sections of $J(E)$ are those sections which arise by taking the jet at every point of a section of the vector bundle E. (For motivation, observe that a field ϕ (a section of E) gives a section of $Jet(E)$ that encodes all the *local* information about ϕ.)

The category of D_M modules has a symmetric monoidal structure, given by tensoring over C_M^∞. The following lemma allows us to translate our definition of local L_∞ algebra into the world of D-modules.

3.5.0.1 Lemma. *Let $E_1, \dots, E_n, F$ be vector bundles on M, and let $\mathscr{E}_i, \mathscr{F}$ denote their spaces of global sections. Then, there is a natural bijection*

$$\mathrm{PolyDiff}(\mathscr{E}_1 \times \cdots \times \mathscr{E}_n, \mathscr{F}) \cong \mathrm{Hom}_{D_M}(J(E_1) \otimes \cdots \otimes J(E_n), J(F))$$

where PolyDiff *refers to the space of polydifferential operators. On the right* *hand side, we need to consider maps which are continuous with respect to the natural adic topology on the bundle of jets.*

Further, this bijection is compatible with composition.

A more formal statement of this lemma is that the multi-category of vector bundles on M, with morphisms given by polydifferential operators, is a full subcategory of the symmetric monoidal category of D_M modules. The embedding is given by taking jets. The proof of this lemma (which is straightforward) is presented in Costello (2011b), Chapter 5.

This lemma immediately tells us how to interpret a local L_∞ algebra in the language of D-modules.

3.5.0.2 Corollary. *Let L be a local L_∞ algebra on M. Then $J(L)$ has the structure of L_∞ algebra in the category of D_M modules.*

Indeed, the lemma implies that to give a local L_∞ algebra on M is the same as to give a graded vector bundle L on M together with an L_∞ structure on the D_M module $J(L)$.

We are interested in the Chevalley-Eilenberg cochains of $J(L)$, but taken now in the category of D_M modules. Because $J(L)$ is an inverse limit of the sheaves of finite-order jets, some care needs to be taken when defining this Chevalley-Eilenberg cochain complex.

In general, if E is a vector bundle, let $J(E)^\vee$ denote the sheaf $\mathrm{Hom}_{C^\infty_M}(J(E), C^\infty_M)$, where $\mathrm{Hom}_{C^\infty_M}$ denotes continuous linear maps of C^∞_M-modules. This sheaf is naturally a D_M-module. We can form the completed symmetric algebra

$$\begin{aligned}\mathscr{O}_{red}(J(E)) &= \prod_{n>0} \mathrm{Sym}^n_{C^\infty_M}(J(E)^\vee) \\ &= \prod_{n>0} \mathrm{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)_{S_n}.\end{aligned}$$

Note that $\mathscr{O}_{red}(J(E)$ is a D_M-algebra, as it is defined by taking the completed symmetric algebra of $J(E)^\vee$ in the symmetric monoidal category of D_M-modules where the tensor product is taken over C^∞_M.

We can equivalently view $J(E)^\vee$ as an infinite-rank vector bundle with a flat connection. The symmetric power sheaf $\mathrm{Sym}^n_{C^\infty_M}(J(E)^\vee)$ is the sheaf of sections of the infinite-rank bundle whose fibre at x is the symmetric power of the fibre of $J(E)^\vee$ at x.

In the case that E is the trivial bundle $\underline{\mathbb{R}}$, the sheaf $J(\underline{\mathbb{R}})^\vee$ is naturally isomorphic to D_M as a left D_M-module. In this case, sections of the sheaf $\mathrm{Sym}^n_{C^\infty_M}(D_M)$ are objects which in local coordinates are finite sums of expressions like

$$f(x_i)\partial_{I_1}\dots\partial_{I_n}.$$

where ∂_{I_j} is the partial differentiation operator corresponding to a multi-index.

We should think of an element of $\mathscr{O}_{red}(J(E))$ as a Lagrangian on the space $\mathscr{E}$ of sections of E (a Lagrangian in the sense that an action functional is given by a Lagrangian density). Indeed, every element of $\mathscr{O}_{red}(J(E))$ has a Taylor expansion $F = \sum F_n$ where each F_n is a section

$$F_n \in \mathrm{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)^{S_n}.$$

Each such F_n is a multilinear map which takes sections $\phi_1, \dots, \phi_n \in \mathscr{E}$ and yields a smooth function $F_n(\phi_1, \dots, \phi_n) \in C^\infty(M)$, with the property that $F_n(\phi_1, \dots, \phi_n)(x)$ only depends on the ∞-jet of ϕ_i at x.

In the same way, we can interpret an element $F \in \mathscr{O}_{red}(J(E))$ as something that takes a section $\phi \in \mathscr{E}$ and yields a smooth function

$$\sum F_n(\phi, \dots, \phi) \in C^\infty(M),$$

with the property that $F(\phi)(x)$ only depends on the jet of ϕ at x.

Of course, the functional F is a formal power series in the variable ϕ. One cannot evaluate most formal power series, since the putative infinite sum makes no sense. Instead, it only makes sense to evaluate a formal power series on infinitesimal elements. In particular, one can always evaluate a formal power series on nilpotent elements of a ring.

Indeed, a formal way to characterize a formal power series is to use the functor of points perspective on Artinian algebras: if R is an auxiliary graded Artinian algebra with maximal ideal m and if $\phi \in \mathscr{E} \otimes m$, then $F(\phi)$ is an element of $C^\infty(M) \otimes m$. This assignment is functorial with respect to maps of graded Artin algebras.

3.5.1 Local functionals

We have seen that we can interpret $\mathscr{O}_{red}(J(E))$ as the sheaf of Lagrangians on a graded vector bundle E on M. Thus, the sheaf

$$\mathrm{Dens}_M \otimes_{C^\infty_M} \mathscr{O}_{red}(J(E))$$

is the sheaf of Lagrangian densities on M. A section F of this sheaf is something which takes as input a section $\phi \in \mathscr{E}$ of $\mathscr{E}$ and produces a density $F(\phi)$ on M, in such a way that $F(\phi)(x)$ only depends on the jet of ϕ at x. (As before, F is a formal power series in the variable ϕ.)

The sheaf of local action functionals is the sheaf of Lagrangian densities modulo total derivatives. Two Lagrangian densities that differ by a total derivative define the same local functional on (compactly supported) sections because the integral of total derivative vanishes. Thus, we do not want to distinguish them, as they lead to the same physics. The formal definition is as follows.

3.5.1.1 Definition. *Let E be a graded vector bundle on M, whose space of global sections is $\mathscr{E}$. Then the space of* local functionals *on $\mathscr{E}$ is*

$$\mathscr{O}_{loc}(\mathscr{E}) = \mathrm{Dens}_M \otimes_{D_M} \mathscr{O}_{red}(J(E)).$$

Here, Dens_M is the right D_M-module of densities on M.

Let $\mathscr{O}_{red}(\mathscr{E}_c)$ denote the algebra of functionals modulo constants on the space $\mathscr{E}_c$ of compactly supported sections of E. Integration induces a natural inclusion

$$\iota : \mathscr{O}_{loc}(\mathscr{E}) \to \mathscr{O}_{red}(\mathscr{E}_c),$$

where the Lagrangian density $S \in \mathscr{O}_{loc}(\mathscr{E})$ becomes the functional $\iota(S)$: $\phi \mapsto \int_M S(\phi)$. (Again, ϕ must be nilpotent and compactly supported.) From here on, we will use this inclusion without explicitly mentioning it.

3.5.2 Local Chevalley-Eilenberg complex of a local L_∞ algebra

Let L be a local L_∞ algebra. Then we can form, as above, the reduced Chevalley-Eilenberg cochain complex $C^*_{red}(J(L))$ of L. This is the D_M-

algebra $\mathscr{O}_{red}(J(L)[1])$ equipped with a differential encoding the L_∞ structure on L.

3.5.2.1 Definition. *For $\mathcal{L}$ a local L_∞-algebra, the local Chevalley-Eilenberg complex is*

$$C^*_{red,loc}(\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$

This is the space of local action functionals on $\mathcal{L}[1]$, equipped with the Chevalley-Eilenberg differential. In general, if $\mathfrak{g}$ is an L_∞ algebra, we think of the Lie algebra cochain complex $C^*(\mathfrak{g})$ as being the algebra of functions on $B\mathfrak{g}$. In this spirit, we sometimes use the notation $\mathscr{O}_{loc}(B\mathcal{L})$ for the complex $C^*_{red,loc}(\mathcal{L})$.

Note that $C^*_{red,loc}(\mathcal{L})$ is *not* a commutative algebra. Although the D_M-module $C^*_{red}(J(L))$ is a commutative D_M-module, the functor $\mathrm{Dens}_M \otimes_{D_M} -$ is not a symmetric monoidal functor from D_M-modules to cochain complexes, so it does not take commutative algebras to commutative algebras.

Note that there is a natural inclusion of cochain complexes

$$C^*_{red,loc}(\mathcal{L}) \to C^*_{red}(\mathcal{L}_c(M)),$$

where $\mathcal{L}_c(M)$ denotes the L_∞ algebra of compactly supported sections of L. The complex on the right hand side was defined earlier (see definition 3.4.0.1) and includes *non*local functionals.

3.5.3 Central extensions and local cochains

In this section we will explain how local cochains are in bijection with certain central extensions of a local L_∞ algebra. To avoid some minor analytical difficulties, we will only consider central extensions that are split as precosheaves of graded vector spaces.

3.5.3.1 Definition. *Let $\mathcal{L}$ be a local L_∞ algebra on M. A k-*shifted local central extension of* $\mathcal{L}$ *is an L_∞ structure on the precosheaf $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$, where $\underline{\mathbb{C}}$ is the constant precosheaf which takes value $\mathbb{C}$ on any open subset. We use the notation $\widetilde{\mathcal{L}}_c$ for the precosheaf $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$. We require that this L_∞ structure has the following properties.*

(i) *The sequence*

$$0 \to \underline{\mathbb{C}}[k] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$

is an exact sequence of precosheaves of L_∞ algebras, where $\underline{\mathbb{C}}[k]$ is given the abelian structure and $\mathcal{L}_c$ is given its original structure.

(ii) *This implies that the L_∞ structure on $\widetilde{\mathcal{L}}_c$ is determined from that on $\mathcal{L}_c$ by L_∞ structure maps*

$$\widetilde{l}_n : \mathcal{L}_c \to \underline{\mathbb{C}}[k]$$

for $n \geq 1$. We require that these structure maps are given by local *action functionals.*

Two such central extensions, say $\widetilde{\mathcal{L}}_c$ and $\widetilde{\mathcal{L}}'_c$, are isomorphic *if there is an L_∞-isomorphism*

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathcal{L}}'_c$$

that is the identity on $\underline{\mathbb{C}[k]}$ and on the quotient $\mathcal{L}_c$. This L_∞ isomorphism must satisfy an additional property: the terms in this L_∞ -isomorphism, which are given (using the decomposition of $\widetilde{\mathcal{L}}_c$ and $\widetilde{\mathcal{L}}'_c$ as $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$) by functionals

$$\mathcal{L}_c^{\otimes n} \to \underline{\mathbb{C}}[k],$$

must be local.

This definition refines the definition of central extension given in section I.3.6 to include an extra locality property.

Example: Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a Lie algebra with an invariant pairing. Let $\mathcal{L} = \Omega^{0,*}_\Sigma \otimes \mathfrak{g}$. Consider the Kac-Moody central extension, as defined in section I.3.6. We let

$$\widetilde{\mathcal{L}}_c = \underline{\mathbb{C}} \cdot c \oplus \mathcal{L}_c,$$

where the central parameter c is of degree 1 and the Lie bracket is defined by

$$[\alpha, \beta]_{\widetilde{\mathcal{L}}_c} = [\alpha, \beta]_{\mathcal{L}_c} + c \int \alpha \partial \beta.$$

This is a local central extension. As shown in section I.5.5, the factorization envelope of this extension recovers the vertex algebra of an associated affine Kac-Moody algebra. $\Diamond$

3.5.3.2 Lemma. *Let $\mathcal{L}$ be a local L_∞ algebra on a manifold M. There is a bijection between isomorphism classes of k-shifted local central extensions of $\mathcal{L}$ and classes in $H^{k+2}(\mathscr{O}_{loc}(B\mathcal{L}))$.*

Proof This result is almost immediate. Indeed, any closed degree $k+2$ element of $\mathscr{O}_{loc}(B\mathcal{L})$ give a local L_∞ structure on $\underline{\mathbb{C}}[k] \oplus \mathcal{L}_c$, where the L_∞ structure maps

$$\widetilde{l}_n : \mathcal{L}_c(U) \to \mathbb{C}[k]$$

arise from the natural cochain map $\mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{red}(\mathcal{L}_c(U))$. The fact that we start with a closed element of $\mathscr{O}_{loc}(B\mathcal{L})$ corresponds to the fact that the L_∞ axioms hold. Isomorphisms of local central extensions correspond to adding an exact cocycle to a closed degree $k+2$ element in $\mathscr{O}_{loc}(B\mathcal{L})$. □

Particularly important is the case when we have a -1-shifted central extension. As explained in section I.3.6.3, in this situation we can form the twisted factorization envelope, which is a factorization algebra over $\mathbb{C}[t]$ (where t is of degree 0) defined by sending an open subset U to the Chevalley-Eilenberg chain complex $C_*(\widetilde{\mathcal{L}}_c(U))$. We think of $\mathbb{C}[t]$ as the Chevalley-Eilenberg chains of the Abelian Lie algebra $\mathbb{C}[-1]$. In this situation, we can set t to be a particular value, leading to a *twisted* factorization envelope of $\mathcal{L}$. Twisted factorization envelopes will play a central role in our formulation of Noether's theorem at the quantum level in chapter 13.

3.5.4

Calculations of local L_∞ algebra cohomology play an important role in quantum field theory. Indeed, the obstruction-deformation complex describing quantizations of a classical field theory are local L_∞ algebra cohomology groups. Thus, it will be helpful to be able to compute some examples.

Before we start, let us describe a general result that will facilitate computation.

3.5.4.1 Lemma. *Let M be an oriented manifold and let $\mathcal{L}$ be a local L_∞-*

algebra on M. Then, there is a natural quasi-isomorphism

$$\Omega^*(M, C^*_{red}(J(L)))[\mathrm{dimM}] \cong C^*_{red,loc}(\mathcal{L}).$$

Proof By definition,

$$\mathscr{O}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red} J(\mathcal{L})$$

where D_M is the sheaf of C^∞ differential operators. The D_M-module $C^*_{red}(J(\mathcal{L}))$ is flat (this was checked in Costello (2011b)), so we can replace the tensor product over D_M with the left-derived tensor product.

Since M is oriented, we can replace Dens_M by Ω^d_M where $d = \dim M$. The right D_M-module Ω^d_M has a free resolution of the form

$$\cdots \to \Omega^{d-1}_M \otimes_{C^\infty_M} D_M \to \Omega^d M \otimes_{C^\infty_M} D_M$$

where $\Omega^i_M \otimes_{C^\infty_M} D_M$ is in cohomological degree $-i$, and the differential in this complex is the de Rham differential coupled to the left D_M-module structure on D_M. (This is sometimes called the Spencer resolution.)

It follows that we the derived tensor product can be represented as

$$\Omega^d_M \otimes^{mbbL}_{D_M} C^*_{red}(J(\mathcal{L})) = \Omega^*(M, C^*_{red}(J(L)))[d]$$

as desired. □

3.5.4.2 Lemma. *Let Σ be a Riemann surface. Let $\mathcal{L}$ be the local L_∞ algebra on Σ defined by $\mathcal{L}(U) = \Omega^{0,*}(U, TU)$. In other words, $\mathcal{L}$ is the Dolbeault resolution of the sheaf of holomorphic vector fields on Σ.*

Then,

$$H^i(\mathscr{O}(B\mathcal{L})) = H^*(\Sigma)[-1].$$

Remark: The class in $H^1(\mathscr{O}(B\mathcal{L}))$ corresponding to the class $1 \in H^0(\Sigma)$ leads to a local central extension of $\mathcal{L}$. One can check that the corresponding twisted factorization envelope corresponds to the Virasoro vertex algebra, in the same way that we showed in section I.5.5 that the Kac-Moody extension above leads to the Kac-Moody vertex algebra. ♢

Proof The previous lemma tells us that we need to compute the de Rham cohomology with coefficients in the D_Σ-module $C^*_{red}(J(L))[2]$.

Suppose we want to compute the de Rham cohomology with coefficients in any complex M of D_Σ-modules. There is a spectral sequence converging to this cohomology, associated to the filtration on $\Omega^*(\Sigma, M)$ by form degree. The E_2 page of this spectral sequence is the de Rham complex $\Omega^*(\Sigma, \mathcal{H}^*(M))$ with coefficients in the cohomology D_Σ-module $\mathcal{H}^*(M)$.

We will use this spectral sequence in our example. The first step is to compute the cohomology of the D_Σ-module $C^*_{red}(J(\mathcal{L}))$. We will compute the cohomology of the fibres of this sheaf at an arbitrary point $x \in \Sigma$. Let us choose a holomorphic coordinate z at x. The fibre $J_x(\mathcal{L})$ at x is the dg Lie algebra $\mathbb{C}[[z, \overline{z}, \mathrm{d}\overline{z}]]\partial_z$ with differential $\overline{\partial}$. This dg Lie algebra is quasi-isomorphic to the Lie algebra of formal vector fields $\mathbb{C}[[z]]\partial_z$.

A calculation performed by Gelfand-Fuchs shows that the reduced Lie algebra cohomology of $\mathbb{C}[[z]]\partial_z$ is concentrated in degree 3, where it is one-dimensional. A cochain representative for the unique non-zero cohomology class is $\partial_z^\vee (z\partial_z)^\vee (z^2\partial_z)^\vee$ where $(z^k\partial_z)^\vee$ refers to the element in $(\mathbb{C}[[z]]\partial_z)^\vee$ in the dual basis.

Thus, we find that the cohomology of $C^*_{red}(J(L))$ is a rank one local system situated in cohomological degree 3. Choosing a formal coordinate at a point in a Riemann surface trivializes the fibre of this line bundle. The trivialization is independent of the coordinate choice, and compatible with the flat connection. From this we deduce that

$$\mathcal{H}^*(C^*_{red}(J(\mathcal{L}))) = C^\infty_\Sigma[-3]$$

is the trivial rank one local system, situated in cohomological degree 3.

Therefore, the cohomology of $\mathscr{O}_{loc}(B\mathcal{L})$ is a shift by -1 of the de Rham cohomology of this trivial flat line bundle, completing the result. □

3.5.5 Cochains with coefficients in a local module for a local L_∞ algebras

Let L be a local L_∞ algebra on M, and let E be a local module for L. Then $J(E)$ has an action of the L_∞ algebra $J(L)$, in a way compatible with the D_M-module on both $J(E)$ and $J(L)$.

3.5.5.1 Definition. *Suppose that E has a local action of L. Then the* local cochains $C^*_{loc}(\mathcal{L}, \mathscr{E})$ *of $\mathcal{L}$ with coefficients in $\mathscr{E}$ is defined to be the flat sections of the D_M-module of cochains of $J(L)$ with coefficients in $J(E)$.*

More explicitly, the D_M-module $C^*(J(L), J(E)$ is

$$\prod_{n \geq 0} \mathrm{Hom}_{C^\infty_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n},$$

equipped with the usual Chevalley-Eilenberg differential. The sheaf of flat sections of this D_M module is the subsheaf

$$\prod_{n \geq 0} \mathrm{Hom}_{D_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n},$$

where the maps must be D_M-linear. In light of the fact that

$$\mathrm{Hom}_{D_M}\left(J(L)^{\otimes n}, J(E)\right) = \mathrm{PolyDiff}(\mathcal{L}^{\otimes n}, \mathscr{E}),$$

we see that $C^*_{loc}(\mathcal{L}, \mathscr{E})$ is precisely the subcomplex of the Chevalley-Eilenberg cochain complex

$$C^*(\mathcal{L}, \mathscr{E}) = \prod_{n \geq 0} \mathrm{Hom}_{\mathbb{R}}((\mathcal{L}[1])^{\otimes n}, \mathscr{E})_{S_n}$$

consisting of those cochains built up from polydifferential operators.

4
The classical Batalin-Vilkovisky formalism

In the preceding chapter we explained how to encode the formal neighborhood of a solution to the Euler-Lagrange equations — a formal elliptic moduli problem — by an elliptic L_∞ algebra. As we explain in this chapter, the elliptic moduli problems arising from action functionals possess even more structure: a shifted symplectic form, so that the formal moduli problem is a derived symplectic space.

Our starting point is the finite-dimensional model that motivates the Batalin-Vilkovisky formalism for classical field theory. With this model in mind, we then develop the relevant definitions in the language of elliptic L_∞ algebras. The end of the chapter is devoted to several examples of classical BV theories, notably *cotangent* field theories, which are the analogs of cotangent bundles in ordinary symplectic geometry.

4.1 The classical BV formalism in finite dimensions

Before we discuss the Batalin-Vilkovisky formalism for classical field theory, we will discuss a finite-dimensional toy model (which we can think of as a 0-dimensional classical field theory). Our model for the space of fields is a finite-dimensional smooth manifold manifold M. The "action functional" is given by a smooth function $S \in C^\infty(M)$. Classical field theory is concerned with solutions to the equations of motion. In our setting, the equations of motion are given by the subspace $\mathrm{Crit}(S) \subset M$. Our toy model will not change if M is a smooth algebraic variety or a complex manifold, or indeed a smooth formal scheme. Thus we will write $\mathscr{O}(M)$ to indicate whatever class of functions (smooth, polynomial, holomorphic, power series) we are considering on M.

If S is not a nice function, then this critical set can by highly singular. The classical Batalin-Vilkovisky formalism tells us to take, instead the *derived* critical locus of S. (Of course, this is exactly what a derived algebraic geometer — see Lurie (n.d.), Toën (2009) — would tell us to do as well.) We will explain the essential idea without formulating it precisely inside any particular formalism for derived geometry. For such a treatment, see Vezzosi (2020).

The critical locus of S is the intersection of the graph

$$\Gamma(\mathrm{d}S) \subset T^*M$$

with the zero-section of the cotangent bundle of M. Algebraically, this means that we can write the algebra $\mathscr{O}(\mathrm{Crit}(S))$ of functions on $\mathrm{Crit}(S)$ as a tensor product

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

Derived algebraic geometry tells us that the derived critical locus is obtained by replacing this tensor product with a derived tensor product. Thus, the derived critical locus of S, which we denote $\mathrm{Crit}^h(S)$, is an object whose ring of functions is the commutative dg algebra

$$\mathscr{O}(\mathrm{Crit}^h(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

In derived algebraic geometry, as in ordinary algebraic geometry, spaces are determined by their algebras of functions. In derived geometry, however, one allows differential-graded algebras as algebras of functions (normally one restricts attention to differential-graded algebras concentrated in non-positive cohomological degrees).

We will take this derived tensor product as a definition of $\mathscr{O}(\mathrm{Crit}^h(S))$.

4.1.1 An explicit model

It is convenient to consider an explicit model for the derived tensor product. By taking a standard Koszul resolution of $\mathscr{O}(M)$ as a module over $\mathscr{O}(T^*M)$, one sees that $\mathscr{O}(\mathrm{Crit}^h(S))$ can be realized as the complex

$$\mathscr{O}(\mathrm{Crit}^h(S)) \simeq \ldots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} \mathscr{O}(M).$$

In other words, we can identify $\mathscr{O}(\mathrm{Crit}^h(S))$ with functions on the "graded manifold" $T^*[-1]M$, equipped with the differential given by contracting with the 1-form $\mathrm{d}S$. This notation $T^*[-1]M$ denotes the ordinary smooth manifold M equipped with the graded-commutative algebra $\mathrm{Sym}_{C^\infty_M}(\Gamma(M, TM)[1])$ as its ring of functions.

Note that

$$\mathscr{O}(T^*[-1]M) = \Gamma(M, \wedge^* TM)$$

has a Poisson bracket of cohomological degree 1, called the Schouten-Nijenhuis bracket. This Poisson bracket is characterized by the fact that if $f, g \in \mathscr{O}(M)$ and $X, Y \in \Gamma(M, TM)$, then

$$\begin{aligned} \{X, Y\} &= [X, Y] \\ \{X, f\} &= Xf \\ \{f, g\} &= 0 \end{aligned}$$

and the Poisson bracket between other elements of $\mathscr{O}(T^*[-1]M)$ is inferred from the Leibniz rule.

The differential on $\mathscr{O}(T^*[-1]M)$ corresponding to that on $\mathscr{O}(\mathrm{Crit}^h(S))$ is given by

$$\mathrm{d}\phi = \{S, \phi\}$$

for $\phi \in \mathscr{O}(T^*[-1]M)$.

The derived critical locus of any function thus has a symplectic form of cohomological degree -1. It is manifest in this model and hence can be found in others. In the Batalin-Vilkovisky formalism, the space of fields always has such a symplectic structure. However, one does not require that the space of fields arises as the derived critical locus of a function.

4.2 The classical BV formalism in infinite dimensions

We would like to consider classical field theories in the BV formalism. We have already explained how the language of elliptic moduli problems captures the formal geometry of solutions to a system of PDE. Now we need to discuss the shifted symplectic structures possessed by a derived critical locus. For us, a classical field theory will be specified by an elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

We defined the notion of formal elliptic moduli problem on a manifold M using the language of L_∞ algebras. Thus, in order to give the definition of a classical field theory, we need to understand the following question: what extra structure on an L_∞ algebra $\mathfrak{g}$ endows the corresponding formal moduli problem with a symplectic form?

In order to answer this question, we first need to understand a little about what it means to put a shifted symplectic form on a (formal) derived stack.

In the seminal work of Schwarz (1993); Alexandrov et al. (1997), a definition of a shifted symplectic form on a dg manifold is given. Dg manifolds where an early attempt to develop a theory of derived geometry. It turns out that dg manifolds are sufficient to capture some aspects of the modern theory of derived geometry, including formal derived geometry.

In the world of dg manifolds, as in any model of derived geometry, all spaces of tensors are cochain complexes. In particular, the space of i-forms $\Omega^i(\mathcal{M})$ on a dg manifold is a cochain complex. The differential on this cochain complex is called the internal differential on i-forms. In addition to the internal differential, there is also a de Rham differential $d_{dR} : \Omega^i(\mathcal{M}) \to \Omega^{i+1}(\mathcal{M})$ which is a cochain map. Schwarz defined a symplectic form on a dg manifold $\mathcal{M}$ to be a two-form ω which is both closed in the differential on the complex of two-forms, and which is also closed under the de Rham differential mapping two-forms to three-forms. A symplectic form is also required to be non-degenerate. The symplectic two-form ω will have some cohomological degree, which for the case relevant to the BV formalism is -1.

Following these ideas, Pantev et al. (2013a) give a definition of shifted symplectic structure in the more recent language of derived stacks. In this approach, instead of asking that the two-form defining the symplectic structure be closed both in the internal differential on two-forms and closed under the de Rham differential, one constructs a double complex

$$\Omega^{\geq 2} = \Omega^2 \to \Omega^3[-1] \to \dots$$

as the subcomplex of the de Rham complex consisting of 2 and higher forms. One then looks for an element of this double complex which is closed under the total differential (the sum of the de Rham differential and the internal differential on each space of k-forms) and whose 2-form component is non-degenerate in a suitable sense.

However, it turns out that, in the case of formal derived stacks, the definition given by Schwarz and that given by Pantev et al. coincides. One can also show that in this situation there is a Darboux lemma, showing that we can take the symplectic form to have constant coefficients. In order to explain what we mean by this, let us explain how to understand forms on a formal derived stack in terms of the associated L_∞-algebra.

Given a pointed formal moduli problem $\mathcal{M}$, the associated L_∞ algebra $\mathfrak{g}_{\mathcal{M}}$ has the property that

$$\mathfrak{g}_{\mathcal{M}} = T_p\mathcal{M}[-1].$$

Further, we can identify geometric objects on $\mathcal{M}$ in terms of $\mathfrak{g}_{\mathcal{M}}$ as follows.

$C^*(\mathfrak{g}_{\mathcal{M}})$	the algebra $\mathscr{O}(\mathcal{M})$ of functions on $\mathcal{M}$
$\mathfrak{g}_{\mathcal{M}}$-modules	$\mathscr{O}_{\mathcal{M}}$-modules
the $\mathfrak{g}_{\mathcal{M}}$-module $g_{\mathcal{M}}[1]$	the tangent bundle $T\mathcal{M}$
the $\mathfrak{g}_{\mathcal{M}}$-module $g^*_{\mathcal{M}}[1]$	the cotangent bundle $T^*\mathcal{M}$
$C^*(\mathfrak{g}_{\mathcal{M}}, V)$	the $\mathscr{O}_{\mathcal{M}}$-module for the $\mathfrak{g}_{\mathcal{M}}$-module V

Following this logic, we see that the complex of 2-forms on $\mathcal{M}$ is identified with $C^*(\mathfrak{g}_{\mathcal{M}}, \wedge^2(\mathfrak{g}_{\mathcal{M}}^\vee[-1]))$.

As we have seen, according to Schwarz, a symplectic form on $\mathcal{M}$ is a two-form on $\mathcal{M}$ which is closed for both the internal and de Rham differentials. Any constant-coefficient two-form is automatically closed under the de Rham differential. A constant-coefficient two-form of degree k is an element of $\mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}})^\vee$ of cohomological degree $k-2$, i.e. a symmetric pairing on $\mathfrak{g}_{\mathcal{M}}$ of this degree. Such a two-form is closed for the internal differential if and only if it is invariant.

To give a formal pointed moduli problem with a symplectic form of cohomological degree k is the same as to give an L_∞ algebra with an invariant and non-degenerate pairing of cohomological degre $k-2$.

Thus, we find that constant coefficient symplectic two-forms of degree k on $\mathcal{M}$ are precisely the same as non-degenerate symmetric invariant pairings on $\mathfrak{g}_{\mathcal{M}}$. The relation between derived symplectic geometry and invariant pairings on Lie algebras was first developed in Kontsevich (1993).

The following formal Darboux lemma makes this relationship into an equivalence.

4.2.0.1 Lemma. *Let $\mathfrak{g}$ be a finite-dimensional L_∞ algebra. Then, k-shifted symplectic structures on the formal derived stack $B\mathfrak{g}$ (in the sense of Pantev et al. (2013a)) are the same as symmetric invariant non-degenerate pairings on $\mathfrak{g}$ of cohomological degree $k-2$.*

The proof is a little technical, and appears in Appendix C. The proof of a closely related statement in a non-commutative setting was given in Kontsevich and Soibelman (2009). In the statement of the lemma, "the same" means that simplicial sets parametrizing the two objects are canonically equivalent.

Following this idea, we will define a classical field theory to be an elliptic L_∞ algebra equipped with a non-degenerate invariant pairing of cohomological degree -3. Let us first define what it means to have an invariant pairing on an elliptic L_∞ algebra.

4.2.0.2 Definition. *Let M be a manifold, and let E be an elliptic L_∞ algebra* *on M. An* invariant pairing on *E* of cohomological degree *k is a symmetric vector bundle map*

$$\langle -, - \rangle_E : E \otimes E \to \mathrm{Dens}(M)[k]$$

satisfying some additional conditions:

(i) Non-degeneracy*: we require that this pairing induces a vector bundle isomorphism*

$$E \to E^\vee \otimes \mathrm{Dens}(M)[-3].$$

(ii) Invariance*: let $\mathscr{E}_c$ denotes the space of compactly supported sections of E. The pairing on E induces an inner product on $\mathscr{E}_c$, defined by*

$$\begin{aligned} \langle -, - \rangle : \mathscr{E}_c \otimes \mathscr{E}_c &\to \mathbb{R} \\ \alpha \otimes \beta &\to \int_M \langle \alpha, \beta \rangle . \end{aligned}$$

We require it to be an invariant pairing on the L_∞ algebra $\mathscr{E}_c$.

Recall that a symmetric pairing on an L_∞ algebra $\mathfrak{g}$ is called *invariant* if, for all n, the linear map

$$\begin{aligned} \mathfrak{g}^{\otimes n+1} &\to \mathbb{R} \\ \alpha_1 \otimes \cdots \otimes \alpha_{n+1} &\mapsto \langle l_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle \end{aligned}$$

is graded anti-symmetric in the α_i.

4.2.0.3 Definition. *A* formal pointed elliptic moduli problem with a symplectic form of cohomological degree *k on a manifold M is an elliptic L_∞ algebra on M with an invariant pairing of cohomological degree $k-2$.*

4.2.0.4 Definition. *In the BV formalism, a* (perturbative) classical field theory *on M is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1.*

4.3 The derived critical locus of an action functional

The critical locus of a function f is, of course, the zero locus of the 1-form $\mathrm{d}f$. We are interested in constructing the derived critical locus of a local functional $S \in \mathscr{O}_{loc}(B\mathcal{L})$ on the formal moduli problem associated to a local L_∞ algebra $\mathcal{L}$ on a manifold M. Thus, we need to understand what kind of object the exterior derivative $\mathrm{d}S$ of such an action functional S is.

If $\mathfrak{g}$ is an L_∞ algebra, then we should think of $C^*_{red}(\mathfrak{g})$ as the algebra of functions on the formal moduli problem $B\mathfrak{g}$ that vanish at the base point. Similarly, $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$ should be the thought of as the space of 1-forms on $B\mathfrak{g}$. The exterior derivative is thus a map

$$\mathrm{d} : C^*_{red}(\mathfrak{g}) \to C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]),$$

namely the universal derivation.

We will define a similar exterior derivative for a local Lie algebra $\mathcal{L}$ on M. The analog of $\mathfrak{g}^\vee$ is the $\mathcal{L}$-module $\mathcal{L}^!$, whose sections are (up to completion) the Verdier dual of the sheaf $\mathcal{L}$. Thus, our exterior derivative will be a map

$$\mathrm{d} : \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Recall that $\mathscr{O}_{loc}(B\mathcal{L})$ denotes the subcomplex of $C^*_{red}(\mathcal{L}_c(M))$ consisting of local functionals. The exterior derivative for the L_∞ algebra $\mathcal{L}_c(M)$ is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \mathcal{L}_c(M)^\vee[-1]).$$

Note that the dual $\mathcal{L}_c(M)^\vee$ of $\mathcal{L}_c(M)$ is the space $\overline{\mathcal{L}}^!(M)$ of distributional sections of the bundle $L^!$ on M. Thus, the exterior derivative is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)[-1]).$$

Note that

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \subset C^*(\mathcal{L}_c(M), \mathcal{L}^!(M)) \subset C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)).$$

We will now show that d preserves locality and more.

4.3.0.1 Lemma. *The exterior derivative takes the subcomplex $\mathscr{O}_{loc}(B\mathcal{L})$ of $C^*_{red}(\mathcal{L}_c(M))$ to the subcomplex $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ of $C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M))$.*

Proof The content of this lemma is the familiar statement that the Euler-Lagrange equations associated to a local action functional are differential equations. We will give a formal proof, but the reader will see that we only use integration by parts.

Any functional
$$F \in \mathscr{O}_{loc}(B\mathcal{L})$$
can be written as a sum $F = \sum F_n$ where
$$F_n \in \mathrm{Dens}_M \otimes_{D_M} \mathrm{Hom}_{C^\infty_M}\left(J(L)^{\otimes n}, C^\infty_M\right)_{S_n}.$$
Any such F_n can be written as a finite sum
$$F_n = \sum_i \omega D_1^i \dots D_n^i$$
where ω is a section of Dens_M and D_j^i are differential operators from $\mathcal{L}$ to C^∞_M. (The notation $\omega D_1^i \dots D_n^i$ means simply to multiply the density ω by the outputs of the differential operators, which are smooth functions.)

If we view $F \in \mathscr{O}(\mathcal{L}_c(M))$, then the nth Taylor component of F is the linear map
$$\mathcal{L}_c(M)^{\otimes n} \to \mathbb{R}$$
defined by
$$\phi_1 \otimes \dots \otimes \phi_n \to \sum_i \int_M \omega(D_1^i \phi_1) \dots (D_n^i \phi_n).$$
Thus, the $(n-1)$th Taylor component of $\mathrm{d}F$ is given by the linear map
$$\begin{array}{cccc} \mathrm{d}F_n : & \mathcal{L}_c(M)^{\otimes n-1} & \to & \overline{\mathcal{L}}^!(M) = \mathcal{L}_c(M)^\vee \\ & \phi_1 \otimes \dots \otimes \phi_{n-1} & \mapsto & \sum_i \omega(D_1^i\phi_1) \cdots (D_{n-1}^i \phi_{n-1}) D_n^i(-) \\ & & & + \text{ symmetric terms} \end{array}$$
where the right hand side is viewed as a linear map from $\mathcal{L}_c(M)$ to $\mathbb{R}$. Now, using integration by parts, we see that
$$(\mathrm{d}F_n)(\phi_1, \dots, \phi_{n-1})$$
is in the subspace $\mathcal{L}^!(M) \subset \overline{L}^!(M)$ of smooth sections of the bundle $L^!(M)$, inside the space of distributional sections.

It is clear from the explicit expressions that the map
$$\mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} \to \mathcal{L}^!(M)$$

is a polydifferential operator, and so it defines an element of $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ as desired. □

4.3.1 Field theories from action functionals

Physicists normally think of a classical field theory as being associated to an action functional. In this section we will show how to construct a classical field theory in our sense from an action functional.

We will work in a very general setting. Recall (section 3.1.3) that we defined a local L_∞ algebra on a manifold M to be a sheaf of L_∞ algebras where the structure maps are given by differential operators. We will think of a local L_∞ algebra $\mathcal{L}$ on M as defining a formal moduli problem cut out by some differential equations. We will use the notation $B\mathcal{L}$ to denote this formal moduli problem.

We want to take the derived critical locus of a local action functional

$$S \in \mathscr{O}_{loc}(B\mathcal{L})$$

of cohomological degree 0. (We also need to assume that S is at least quadratic: this condition insures that the base-point of our formal moduli problem $B\mathcal{L}$ is a critical point of S). We have seen (section 4.3) how to apply the exterior derivative to a local action functional S yields an element

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]),$$

which we think of as being a local 1-form on $B\mathcal{L}$.

The critical locus of S is the zero locus of $\mathrm{d}S$. We thus need to explain how to construct a new local L_∞ algebra that we interpret as being the derived zero locus of $\mathrm{d}S$.

4.3.2 Finite dimensional model

We will first describe the analogous construction in finite dimensions. Let $\mathfrak{g}$ be an L_∞ algebra, M be a $\mathfrak{g}$-module of finite total dimension, and

α be a closed, degree zero element of $C^*_{red}(\mathfrak{g}, M)$. The subscript *red* indicates that we are taking the reduced cochain complex, so that α is in the kernel of the augmentation map $C^*(\mathfrak{g}, M) \to M$.

We think of M as a dg vector bundle on the formal moduli problem $B\mathfrak{g}$, and so α is a section of this vector bundle. The condition that α is in the reduced cochain complex translates into the statement that α vanishes at the basepoint of $B\mathfrak{g}$. We are interested in constructing the L_∞ algebra representing the zero locus of α.

We start by writing down the usual Koszul complex associated to a section of a vector bundle. In our context, the commutative dg algebra representing this zero locus of α is given by the total complex of the double complex

$$\cdots \to C^*(\mathfrak{g}, \wedge^2 M^\vee) \xrightarrow{\vee\alpha} C^*(\mathfrak{g}, M^\vee) \xrightarrow{\vee\alpha} C^*(\mathfrak{g}).$$

In words, we have written down the symmetric algebra on the dual of $\mathfrak{g}[1] \oplus M[-1]$. It follows that this commutative dg algebra is the Chevalley-Eilenberg cochain complex of $\mathfrak{g} \oplus M[-2]$, equipped with an L_∞ structure arising from the differential on this complex.

Note that the direct sum $\mathfrak{g} \oplus M[-2]$ (without a differential depending on α) has a natural semi-direct product L_∞ structure, arising from the L_∞ structure on $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $M[-2]$. This L_∞ structure corresponds to the case $\alpha = 0$.

4.3.2.1 Lemma. *The L_∞ structure on $\mathfrak{g} \oplus M[-2]$ describing the zero locus of α is a deformation of the semidirect product L_∞ structure, obtained by adding to the structure maps l_n the maps*

$$\begin{aligned} D_n\alpha : \mathfrak{g}^{\otimes n} &\to M \\ X_1 \otimes \cdots \otimes X_n &\mapsto \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n}\alpha. \end{aligned}$$

This is a curved L_∞ algebra unless the section α vanishes at $0 \in \mathfrak{g}$.

Proof The proof is a straightforward computation. □

Note that the maps $D_n\alpha$ in the statement of the lemma are simply the homogeneous components of the cochain α.

We will let $Z(\alpha)$ denote $\mathfrak{g} \oplus M[-2]$, equipped with this L_∞ structure arising from α.

Recall that the formal moduli problem $B\mathfrak{g}$ is the functor from dg Artin rings (R, m) to simplicial sets, sending (R, m) to the simplicial set of Maurer-Cartan elements of $\mathfrak{g} \otimes m$. In order to check that we have constructed the correct derived zero locus for α, we should describe the formal moduli problem associated $Z(\alpha)$.

Thus, let (R, m) be a dg Artin ring, and $x \in \mathfrak{g} \otimes m$ be an element of degree 1, and $y \in M \otimes m$ be an element of degree -1. Then (x, y) satisfies the Maurer-Cartan equation in $Z(\alpha)$ if and only if

(i) x satisfies the Maurer-Cartan equation in $\mathfrak{g} \otimes m$ and
(ii) $\alpha(x) = \mathrm{d}_x y \in M$, where

$$\mathrm{d}_x = \mathrm{d}y + \mu_1(x, y) + \tfrac{1}{2!}\mu_2(x, x, y) + \cdots : M \to M$$

is the differential obtained by deforming the original differential by that arising from the Maurer-Cartan element x. (Here $\mu_n : \mathfrak{g}^{\otimes n} \otimes M \to M$ are the action maps.)

In other words, we see that an R-point of $BZ(\alpha)$ is both an R-point x of $B\mathfrak{g}$ and a homotopy between $\alpha(x)$ and 0 in the fiber M_x of the bundle M at $x \in B\mathfrak{g}$. The fibre M_x is the cochain complex M with differential d_x arising from the solution x to the Maurer-Cartan equation. Thus, we are described the homotopy fiber product between the section α and the zero section in the bundle M, as desired.

4.3.3 The derived critical locus of a local functional

Let us now return to the situation where $\mathcal{L}$ is a local L_∞ algebra on a manifold M and $S \in \mathscr{O}(B\mathcal{L})$ is a local functional that is at least quadratic. Let

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$$

denote the exterior derivative of S. Note that $\mathrm{d}S$ is in the reduced cochain complex, i.e. the kernel of the augmentation map $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \to \mathcal{L}^![-1]$.

Let

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

be the nth Taylor component of $\mathrm{d}S$. The fact that $\mathrm{d}S$ is a local cochain means that $\mathrm{d}_n S$ is a polydifferential operator.

4.3.3.1 Definition. *The* derived critical locus of S *is the local L_∞ algebra obtained by adding the maps*

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

to the structure maps l_n of the semi-direct product L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![-3]$. We denote this local L_∞ algebra by $\mathrm{Crit}(S)$.

If (R, m) is an auxiliary Artinian dg ring, then a solution to the Maurer-Cartan equation in $\mathrm{Crit}(S) \otimes m$ consists of the following data:

(i) a Maurer-Cartan element $x \in \mathcal{L} \otimes m$ and
(ii) an element $y \in \mathcal{L}^! \otimes m$ such that

$$(\mathrm{d}S)(x) = \mathrm{d}_x y.$$

Here $\mathrm{d}_x y$ is the differential on $\mathcal{L}^! \otimes m$ induced by the Maurer-Cartan element x. These two equations say that x is an R-point of $B\mathcal{L}$ that satisfies the Euler-Lagrange equations up to a homotopy specified by y.

4.3.4 Symplectic structure on the derived critical locus

Recall that a classical field theory is given by a local L_∞ algebra that is elliptic and has an invariant pairing of degree -3. The pairing on the local L_∞ algebra $\mathrm{Crit}(S)$ constructed above is evident: it is given by the natural bundle isomorphism

$$(L \oplus L^![-3])^![-3] \cong L^![-3] \oplus L.$$

In other words, the pairing arises, by a shift, from the natural bundle map $L \otimes L^! \to \mathrm{Dens}_M$.

4.3.4.1 Lemma. *This pairing on* $\mathrm{Crit}(S)$ *is invariant.*

Proof The original L_∞ structure on $\mathcal{L} \oplus \mathcal{L}^![-3]$ (that is, the L_∞ structure not involving S) is easily seen to be invariant. We will verify that the deformation of this structure coming from S is also invariant.

We need to show that if

$$\alpha_1, \dots, \alpha_{n+1} \in \mathcal{L}_c \oplus \mathcal{L}_c^![-3]$$

are compactly supported sections of $L \oplus L^1[-3]$, then

$$\langle l_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle$$

is totally antisymmetric in the variables α_i. Now, the part of this expression that comes from S is just

$$\left(\frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_{n+1}} \right) S(0).$$

The fact that partial derivatives commute — combined with the shift in grading due to $C^*(\mathcal{L}_c) = \mathscr{O}(\mathcal{L}_c[1])$ — immediately implies that this term is totally antisymmetric. □

Note that, although the local L_∞ algebra $\mathrm{Crit}(S)$ always has a symplectic form, it does not always define a classical field theory, in our sense. To be a classical field theory, we also require that the local L_∞ algebra $\mathrm{Crit}(S)$ is elliptic.

4.4 A succinct definition of a classical field theory

We defined a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In this section we will rewrite this definition in a more concise (but less conceptual) way. This version is included largely for consistency with Costello (2011b) — where the language of elliptic moduli problems is not used — and for ease of reference when we discuss the quantum theory.

4.4.0.1 Definition. *Let E be a graded vector bundle on a manifold M. A* degree -1 symplectic structure on E *is an isomorphism of graded vector bundles*

$$\phi : E \cong E^![-1]$$

that is anti-symmetric, in the sense that $\phi^* = -\phi$ where ϕ^* is the formal adjoint of ϕ.

Note that if L is an elliptic L_∞ algebra on M with an invariant pairing of degree -3, then the graded vector bundle $L[1]$ on M has a -1 symplectic form. Indeed, by definition, L is equipped with a symmetric isomorphism $L \cong L^![-3]$, which becomes an antisymmetric isomorphism $L[1] \cong (L[1])^![-1]$.

Note also that the tangent space at the basepoint to the formal moduli problem $B\mathcal{L}$ associated to $\mathcal{L}$ is $\mathcal{L}[1]$ (equipped with the differential induced from that on $\mathcal{L}$). Thus, the algebra $C^*(\mathcal{L})$ of cochains of $\mathcal{L}$ is isomorphic, as a graded algebra without the differential, to the algebra $\mathscr{O}(\mathcal{L}[1])$ of functionals on $\mathcal{L}[1]$.

Now suppose that E is a graded vector bundle equipped with a -1 symplectic form. Let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$, as defined in section 3.5.1.

4.4.0.2 Proposition. *For E a graded vector bundle equipped with a -1 symplectic form, let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$. Then we have the following.*

(i) *The symplectic form on $\mathscr{E}$ induces a Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$, of degree* 1.
(ii) *Equipping $E[-1]$ with a local L_∞ algebra structure compatible with the given pairing on $E[-1]$ is equivalent to picking an element $S \in \mathscr{O}_{loc}(\mathscr{E})$ that has cohomological degree* 0*, is at least quadratic, and satisfies*

$$\{S, S\} = 0,$$

the classical master equation.

Proof Let $L = E[-1]$. Note that L is a local L_∞ algebra, with the zero differential and zero higher brackets (i.e., a totally abelian L_∞ algebra). We write $\mathscr{O}_{loc}(B\mathcal{L})$ or $C^*_{red,loc}(\mathcal{L})$ for the reduced local cochains of $\mathcal{L}$. This is a complex with zero differential that coincides with $\mathscr{O}_{loc}(\mathscr{E})$.

We have seen that the exterior derivative (section 4.3) gives a map

$$\mathrm{d} : \mathscr{O}_{loc}(\mathscr{E}) = \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Note that the isomorphism

$$\mathcal{L} \cong \mathcal{L}^![-3]$$

gives an isomorphism

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \cong C^*_{loc}(\mathcal{L}, \mathcal{L}[2]).$$

Finally, $C^*_{loc}(\mathcal{L}, \mathcal{L}[2])$ is the L_∞ algebra controlling deformations of $\mathcal{L}$ as a local L_∞ algebra. It thus remains to verify that $\mathscr{O}_{loc}(B\mathcal{L}) \subset C^*_{loc}(\mathcal{L}, \mathcal{L}[2])$ is a sub L_∞ algebra, which is straightforward. □

Note that the finite-dimensional analog of this statement is simply the fact that on a formal symplectic manifold, all symplectic derivations (which correspond, after a shift, to deformations of the formal symplectic manifold) are given by Hamiltonian functions, defined up to the addition of an additive constant. The additive constant is not mentioned in our formulation because $\mathscr{O}_{loc}(\mathscr{E})$, by definition, consists of functionals without a constant term.

Thus, we can make a concise definition of a field theory.

4.4.0.3 Definition. *A* pre-classical field theory *on a manifold M consists of a graded vector bundle E on M, equipped with a symplectic pairing of degree* -1*, and a local functional*

$$S \in \mathscr{O}_{loc}(\mathscr{E}_c(M))$$

of cohomological degree 0*, satisfying the following properties.*

(i) *S satisfies the classical master equation $\{S, S\} = 0$.*
(ii) *S is at least quadratic (so that $0 \in \mathscr{E}_c(M)$ is a critical point of S).*

In this situation, we can write S as a sum (in a unique way)

$$S(e) = \langle e, Qe \rangle + I(e)$$

where $Q : \mathscr{E} \to \mathscr{E}$ is a skew self-adjoint differential operator of cohomological degree 1 and square zero.

4.4.0.4 Definition. *A pre-classical field is a* classical field theory *if the complex $(\mathscr{E}, Q)$ is elliptic.*

There is one more property we need of a classical field theories in order to be apply the quantization machinery of Costello (2011b).

4.4.0.5 Definition. *A* gauge fixing operator *is a map*

$$Q^{GF} : \mathscr{E}(M) \to \mathscr{E}(M)$$

that is a differential operator of cohomological degree -1 *such that* $(Q^{GF})^2 = 0$ *and*

$$[Q, Q^{GF}] : \mathscr{E}(M) \to \mathscr{E}(M)$$

is a generalized Laplacian in the sense of Berline et al. (1992).

The only classical field theories we will try to quantize are those that admit a gauge fixing operator. Thus, we will only consider classical field theories which have a gauge fixing operator. An important point which will be discussed at length in the chapter on quantum field theory is the fact that the observables of the quantum field theory are independent (up to homotopy) of the choice of gauge fixing condition.

4.5 Examples of scalar field theories from action functionals

Let us now give some basic examples of field theories arising as the derived critical locus of an action functional. We will only discuss scalar field theories in this section.

4.5.1

Let (M, g) be a Riemannian manifold. Let $\underline{\mathbb{R}}$ be the trivial line bundle on M and $Dens_M$ the density line bundle. Note that the volume form $\mathrm{d}Vol_g$ provides an isomorphism between these line bundles. Let

$$S(\phi) = \frac{1}{2} \int_M \phi \, \mathrm{D} \, \phi$$

denote the action functional for the free massless field theory on M. Here D is the Laplacian on M, viewed as a differential operator from $C^\infty(M)$ to $\mathrm{Dens}(M)$, so $D\phi = (\Delta_g \phi) \mathrm{d}Vol_g$.

The derived critical locus of S is described by the elliptic L_∞ algebra

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2]$$

where $\mathrm{Dens}(M)$ is the global sections of the bundle of densities on M. Thus, $C^\infty(M)$ is situated in degree 1, and the space $\mathrm{Dens}(M)$ is situated in degree 2. The pairing between $\mathrm{Dens}(M)$ and $C^\infty(M)$ gives the invariant pairing on $\mathcal{L}$, which is symmetric of degree -3 as desired.

4.5.2 Interacting scalar field theories

Next, let us write down the derived critical locus for a basic interacting scalar field theory, given by the action functional

$$S(\phi) = \tfrac{1}{2} \int_M \phi \, \mathrm{D} \phi + \tfrac{1}{4!} \int_M \phi^4.$$

The cochain complex underlying our elliptic L_∞ algebra is, as before,

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2].$$

The interacting term $\frac{1}{4!} \int_M \phi^4$ gives rise to a higher bracket l_3 on $\mathcal{L}$, defined by the map

$$\begin{aligned} C^\infty(M)^{\otimes 3} &\to \mathrm{Dens}(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1 \phi_2 \phi_3 \mathrm{d}Vol_g. \end{aligned}$$

Let (R, m) be a nilpotent Artinian ring, concentrated in degree 0. Then a section of $\phi \in C^\infty(M) \otimes m$ satisfies the Maurer-Cartan equation in this L_∞ algebra if and only if

$$\mathrm{D} \phi + \tfrac{1}{3!} \phi^3 \mathrm{d}Vol = 0.$$

Note that this is precisely the Euler-Lagrange equation for S. Thus, the formal moduli problem associated to $\mathcal{L}$ is, as desired, the derived version of the moduli of solutions to the Euler-Lagrange equations for S.

4.6 Cotangent field theories

We have defined a field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In geometry, cotangent bundles are the basic examples of symplectic manifolds. We can apply this construction in our setting: given any elliptic moduli problem, we will produce a new elliptic moduli problem – its shifted cotangent bundle – that has a symplectic form of degree -1. We call the field theories that arise by this construction *cotangent field theories*. It turns out that a surprising number of field theories of interest in mathematics and physics arise as cotangent theories, including, for example, both the A- and the B-models of mirror symmetry and their half-twisted versions.

We should regard cotangent field theories as the simplest and most basic class of non-linear field theories, just as cotangent bundles are the simplest class of symplectic manifolds. One can show, for example, that the phase space of a cotangent field theory is always an (infinite-dimensional) cotangent bundle, whose classical Hamiltonian function is linear on the cotangent fibers.

4.6.1 The cotangent bundle to an elliptic moduli problem

Let $\mathcal{L}$ be an elliptic L_∞ algebra on a manifold X, and let $\mathcal{M}_{\mathcal{L}}$ be the associated elliptic moduli problem.

Let $L^!$ be the bundle $L^\vee \otimes \mathrm{Dens}(X)$. Note that there is a natural pairing between compactly supported sections of L and compactly supported sections of $L^!$.

Recall that we use the notation $\mathcal{L}$ to denote the space of sections of L. Likewise, we will let $\mathcal{L}^!$ denote the space of sections of $L^!$.

4.6.1.1 Definition. *Let $T^*[k]B\mathcal{L}$ denote the elliptic moduli problem associated to the elliptic L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![k-2]$.*

This elliptic L_∞ algebra has a pairing of cohomological degree $k-2$.

The L_∞ structure on the space $\mathcal{L} \oplus \mathcal{L}^![k-2]$ of sections of the direct sum bundle $L \oplus L^![k-2]$ arises from the natural $\mathcal{L}$-module structure on $\mathcal{L}^!$.

4.6.1.2 Definition. *Let $\mathcal{M} = B\mathcal{L}$ be an elliptic moduli problem corresponding to an elliptic L_∞ algebra $\mathcal{L}$. Then the* cotangent field theory *associated* *to $\mathcal{M}$ is the -1-symplectic elliptic moduli problem $T^*[-1]\mathcal{M}$, whose elliptic L_∞ algebra is $\mathcal{L} \oplus \mathcal{L}^![-3]$.*

To animate these definition, we will now list some basic examples of cotangent theories, both gauge theories and nonlinear sigma models.

In order to make the discussion more transparent, we will not explicitly describe the elliptic L_∞ algebra related to every elliptic moduli problem we mention. Instead, we may simply define the elliptic moduli problem in terms of the geometric objects it classifies. In all examples, it is straightforward, using the techniques we have discussed so far, to write down the elliptic L_∞ algebra describing the formal neighborhood of a point in the elliptic moduli problems we will consider.

4.6.2 Self-dual Yang-Mills theory

Let X be an oriented 4-manifold equipped with a conformal class of a metric. Let G be a compact Lie group. Let $\mathcal{M}(X, G)$ denote the elliptic moduli problem parametrizing principal G-bundles on X with a connection whose curvature is self-dual.

Then we can consider the cotangent theory $T^*[-1]\mathcal{M}(X, G)$. This theory is known in the physics literature as *self-dual Yang-Mills theory*.

Let us describe the L_∞ algebra of this theory explicitly. Observe that the elliptic L_∞ algebra describing the completion of $\mathcal{M}(X, G)$ near a point (P, ∇) is

$$\Omega^0(X, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_\nabla} \Omega^1(X, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_-} \Omega^2_-(X, \mathfrak{g}_P)$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to the principal G-bundle P. Here d_- denotes the connection followed by projection onto the anti-self-dual 2-forms.

Thus, the elliptic L_∞ algebra describing $T^*[-1]\mathcal{M}$ is given by the diagram

$$\begin{array}{ccccccc} \Omega^0(X,\mathfrak{g}_P) & \xrightarrow{d_\nabla} & \Omega^1(X,\mathfrak{g}_P) & \xrightarrow{d_-} & \Omega^2_-(X,\mathfrak{g}_P) & & \\ & & \oplus & & \oplus & & \\ & & \Omega^2_-(X,\mathfrak{g}_P) & \xrightarrow{d_\nabla} & \Omega^3(X,\mathfrak{g}_P) & \xrightarrow{d_\nabla} & \Omega^4(X,\mathfrak{g}_P) \end{array}$$

This is a standard presentation of the fields of self-dual Yang-Mills theory in the BV formalism (see Cattaneo et al. (1998b) and Costello (2011b)). Note that it is, in fact, a dg Lie algebra, so there are no nontrivial higher brackets.

Ordinary Yang-Mills theory arises as a deformation of the self-dual theory. One simply deforms the differential in the diagram above by including a term that is the identity from $\Omega^2_-(X,\mathfrak{g}_P)$ in degree 1 to the copy of $\Omega^2_-(X,\mathfrak{g}_P)$ situated in degree 2.

4.6.3 The holomorphic σ-model

Let E be an elliptic curve and let X be a complex manifold. Let $\mathcal{M}(E,X)$ denote the elliptic moduli problem parametrizing holomorphic maps from $E \to X$. As before, there is an associated cotangent field theory $T^*[-1]\mathcal{M}(E,X)$. (In Costello (2011a) it is explained how to describe the formal neighborhood of any point in this mapping space in terms of an elliptic L_∞ algebra on E.)

In Costello (2010), this field theory was called a holomorphic Chern-Simons theory, because of the formal similarities between the action functional of this theory and that of the holomorphic Chern-Simons gauge theory. In the physics literature (Witten (2007); Nekrasov (2005)) this theory is known as the twisted $(0,2)$ supersymmetric sigma model, or as the curved $\beta\gamma$ system.

This theory has an interesting role in both mathematics and physics. For instance, it was shown in Costello (2010, 2011a) that the partition function of this theory (at least, the part which discards the contributions of non-constant maps to X) is the Witten genus of X.

4.6.4 Twisted supersymmetric gauge theories

Of course, there are many more examples of cotangent theories, as there are very many elliptic moduli problems. In Costello (2013a), it is shown how twisted versions of supersymmetric gauge theories can be written as cotangent theories. We will focus on holomorphic (or minimal) twists. Holomorphic twists are richer than the more well-studied topological twists, but contain less information than the full untwisted supersymmetric theory. As explained in Costello (2013a), one can obtain topological twists from holomorphic twists by applying a further twist.

The most basic example is the twisted $\mathscr{N} = 1$ field theory. If X is a complex surface and G is a complex Lie group, then the $\mathscr{N} = 1$ twisted theory is simply the cotangent theory to the elliptic moduli problem of holomorphic principal G-bundles on X. If we fix a principal G-bundle $P \to X$, then the elliptic L_∞ algebra describing this formal moduli problem near P is

$$\Omega^{0,*}(X, \mathfrak{g}_P),$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to P. It is a classic result of Kodaira and Spencer that this dg Lie algebra describes deformations of the holomorphic principal bundle P.

The cotangent theory to this elliptic moduli problem is thus described by the elliptic L_∞ algebra

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus \mathfrak{g}_P^\vee \otimes K_X[-1].).$$

Note that K_X denotes the canonical line bundle, which is the appropriate holomorphic substitute for the smooth density line bundle.

4.6.5 The twisted $\mathscr{N} = 2$ theory

Twisted versions of gauge theories with more supersymmetry have similar descriptions, as is explained in Costello (2013a). The $\mathscr{N} = 2$ theory is the cotangent theory to the elliptic moduli problem for holomorphic G-bundles $P \to X$ together with a holomorphic section of the adjoint bundle $\mathfrak{g}_P$. The underlying elliptic L_∞ algebra describing this moduli problem is

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1]).$$

Thus, the cotangent theory has

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1] \oplus \mathfrak{g}_P^\vee \otimes K_X \oplus \mathfrak{g}_P^\vee \otimes K_X[-1])$$

for its elliptic L_∞ algebra.

4.6.6 The twisted $\mathscr{N} = 4$ theory

Finally, we will describe the twisted $\mathscr{N} = 4$ theory. There are two versions of this twisted theory: one used in the work of Vafa-Witten Vafa and Witten (1994) on S-duality, and another by Kapustin-Witten Kapustin and Witten (2007) in their work on geometric Langlands. Here we will describe only the latter.

Let X again be a complex surface and G a complex Lie group. Then the twisted $\mathscr{N} = 4$ theory is the cotangent theory to the elliptic moduli problem describing principal G-bundles $P \to X$, together with a holomorphic section $\phi \in H^0(X, T^*X \otimes \mathfrak{g}_P)$ satisfying

$$[\phi, \phi] = 0 \in H^0(X, K_X \otimes g_P).$$

Here T^*X is the holomorphic cotangent bundle of X.

The elliptic L_∞ algebra describing this is

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus T^*X \otimes \mathfrak{g}_P[-1] \oplus K_X \otimes \mathfrak{g}_P[-2]).$$

Of course, this elliptic L_∞ algebra can be rewritten as

$$(\Omega^{*,*}(X, \mathfrak{g}_P), \overline{\partial}),$$

where the differential is just $\overline{\partial}$ and does not involve ∂. The Lie bracket arises from extending the Lie bracket on $\mathfrak{g}_P$ by tensoring with the commutative algebra structure on the algebra $\Omega^{*,*}(X)$ of forms on X.

Thus, the corresponding cotangent theory has

$$\Omega^{*,*}(X, \mathfrak{g}_P) \oplus \Omega^{*,*}(X, \mathfrak{g}_P)[1]$$

for its elliptic Lie algebra.

5
The observables of a classical field theory

So far we have given a definition of a classical field theory, combining the ideas of derived deformation theory and the classical BV formalism. Our goal in this chapter is to show that the observables for such a theory do indeed form a commutative factorization algebra, denoted Obs^{cl}, and to explain how to equip it with a shifted Poisson bracket. The first part is straightforward — implicitly, we have already done it! — but the Poisson bracket is somewhat subtle, due to complications that arise when working with infinite-dimensional vector spaces. We will exhibit a sub-factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$ of Obs^{cl} which is equipped with a commutative product and Poisson bracket, and such that the inclusion map $\widetilde{\mathrm{Obs}}^{cl} \to \mathrm{Obs}^{cl}$ is a quasi-isomorphism.

5.1 The factorization algebra of classical observables

We have given two descriptions of a classical field theory, and so we provide the two descriptions of the associated observables.

Let $\mathcal{L}$ be the elliptic L_∞ algebra of a classical field theory on a manifold M. Thus, the associated elliptic moduli problem is equipped with a symplectic form of cohomological degree -1.

5.1.0.1 Definition. *The* observables with support in the open subset U

is the commutative dg algebra

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

The factorization algebra of observables *for this classical field theory, denoted* Obs^{cl}*, assigns the cochain complex* $\mathrm{Obs}^{cl}(U)$ *to the open set* U*.*

The interpretation of this definition should be clear from the preceding chapters. The elliptic L_∞ algebra $\mathcal{L}$ encodes the space of solutions to the Euler-Lagrange equations for the theory (more accurately, the formal neighborhood of the solution given by the basepoint of the formal moduli problem). Its Chevalley-Eilenberg cochains $C^*(\mathcal{L}(U))$ on the open U are interpreted as the algebra of functions on the space of solutions over the open U.

By the results of section I.6.6, we know that this construction is in fact a factorization algebra.

We often call Obs^{cl} simply the *classical observables*, in contrast to the factorization algebras of some quantization, which we will call the quantum observables.

Alternatively, let E be a graded vector bundle on M, equipped with a symplectic pairing of degree -1 and a local action functional S which satisfies the classical master equation. As we explained in section 4.4 this data is an alternative way of describing a classical field theory. The bundle L whose sections are the local L_∞ algebra $\mathcal{L}$ is $E[-1]$.

5.1.0.2 Definition. *The* observables with support in the open subset U *is the commutative dg algebra*

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

equipped with the differential $\{S, -\}$*.*

The factorization algebra of observables *for this classical field theory, denoted* Obs^{cl}*, assigns the cochain complex* $\mathrm{Obs}^{cl}(U)$ *to the open* U*.*

Recall that the operator $\{S, -\}$ is well-defined because the bracket with the local functional is always well-defined.

The underlying graded-commutative algebra of $\mathrm{Obs}^{cl}(U)$ is manifestly the functions on the fields $\mathscr{E}(U)$ over the open set U. The differential imposes the relations between observables arising from the Euler-Lagrange equations for S. In physical language, we are giving a cochain complex whose cohomology is the "functions on the fields that are on-shell."

It is easy to check that this definition of classical observables coincides with the one in terms of cochains of the sheaf of L_∞-algebras $\mathcal{L}(U)$.

5.2 The graded Poisson structure on classical observables

Recall the following definition.

5.2.0.1 Definition. *A P_0 algebra (in the category of cochain complexes) is a commutative differential graded algebra together with a Poisson bracket $\{-,-\}$ of cohomological degree 1, which satisfies the Jacobi identity and the Leibniz rule.*

The main result of this chapter is the following.

5.2.0.2 Theorem. *For any classical field theory (Section 4.4) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a weak equivalence of commutative factorization algebras $\widetilde{\mathrm{Obs}}^{cl} \simeq \mathrm{Obs}^{cl}$.*

Concretely, $\widetilde{\mathrm{Obs}}^{cl}(U)$ is built from functionals on the space of solutions to the Euler-Lagrange equations that have more regularity than the functionals in $\mathrm{Obs}^{cl}(U)$.

The idea of the definition of the P_0 structure is very simple. Let us start with a finite-dimensional model. Let $\mathfrak{g}$ be an L_∞ algebra equipped with an invariant antisymmetric element $P \in \mathfrak{g} \otimes \mathfrak{g}$ of cohomological degree 3. This element can be viewed (according to the correspondence between formal moduli problems and Lie algebras given in section 3.1) as a bivector on $B\mathfrak{g}$, and so it defines a Poisson bracket on $\mathscr{O}(B\mathfrak{g}) = C^*(\mathfrak{g})$. Concretely, this Poisson bracket is defined, on the generators $\mathfrak{g}^\vee[-1]$ of $C^*(\mathfrak{g})$, as the map

$$\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathbb{R}$$

determined by the tensor P.

Now let $\mathcal{L}$ be an elliptic L_∞ algebra describing a classical field theory. Then the kernel for the isomorphism $\mathcal{L}(U) \cong \mathcal{L}^!(U)[-3]$ is an element $P \in \overline{\mathcal{L}}(U) \otimes \overline{\mathcal{L}}(U)$, which is symmetric, invariant, and of degree 3.

We would like to use this idea to define the Poisson bracket on

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

As in the finite dimensional case, in order to define such a Poisson bracket, we would need an invariant tensor in $\mathcal{L}(U)^{\otimes 2}$. The tensor representing our pairing is instead in $\overline{\mathcal{L}}(U)^{\otimes 2}$, which contains $\mathcal{L}(U)^{\otimes 2}$ as a dense subspace. In other words, we run into a standard problem in analysis: our construction in finite-dimensional vector spaces does not port immediately to infinite-dimensional vector spaces.

We solve this problem by finding a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

such that the Poisson bracket *is* well-defined on the subcomplex and the inclusion is a weak equivalence. Up to quasi-isomorphism, then, we have the desired Poisson structure.

5.3 The Poisson structure for free field theories

In this section, we will construct a P_0 structure on the factorization algebra of observables of a free field theory. More precisely, we will construct for every open subset U, a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

of the complex of classical observables such that

(i) $\widetilde{\mathrm{Obs}}^{cl}$ forms a sub-commutative factorization algebra of Obs^{cl};

(ii) the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$ is a weak equivalence of differentiable pro-cochain complexes for every open set U; and
(iii) $\widetilde{\mathrm{Obs}}^{cl}$ has the structure of P_0 factorization algebra.

The complex $\mathrm{Obs}^{cl}(U)$ consists of a product over all n of certain distributional sections of a vector bundle on U^n. The complex $\widetilde{\mathrm{Obs}}^{cl}$ is defined by considering instead smooth sections on U^n of the same vector bundle.

Let us now make this definition more precise. Recall that a free field theory is a classical field theory associated to an elliptic L_∞ algebra $\mathcal{L}$ that is abelian, i.e., where all the brackets $\{l_n \mid n \geq 2\}$ vanish.

Thus, let L be the graded vector bundle associated to an abelian elliptic L_∞ algebra, and let $\mathcal{L}(U)$ be the elliptic complex of sections of L on U. To say that L defines a field theory means we have a symmetric isomorphism $\mathcal{L} \cong \mathcal{L}^![-3]$.

Recall (see appendix I.B.7.2) that we use the notation $\overline{\mathcal{L}}(U)$ to denote the space of distributional sections of L on U. A lemma of Atiyah-Bott (see appendix I.D) shows that the inclusion

$$\mathcal{L}(U) \hookrightarrow \overline{\mathcal{L}}(U)$$

is a continuous homotopy equivalence of topological cochain complexes.

It follows that the natural map

$$C^*(\overline{\mathcal{L}}(U)) \hookrightarrow C^*(\mathcal{L}(U))$$

is a cochain homotopy equivalence. Indeed, because we are dealing with an abelian L_∞ algebra, the Chevalley-Eilenberg cochains become quite simple:

$$\begin{aligned} C^*(\mathcal{L}(U)) &= \widehat{\mathrm{Sym}}(\mathcal{L}(U)^\vee[-1]), \\ C^*(\overline{\mathcal{L}}(U)) &= \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}(U)^\vee[-1]), \end{aligned}$$

where, as always, the symmetric algebra is defined using the completed tensor product. The differential is simply the differential on, for instance, $\mathcal{L}(U)^\vee$ extended as a derivation, so that we are simply taking the completed symmetric algebra of a complex. The complex $C^*(\mathcal{L}(U))$ is built from distributional sections of the bundle $(L^!)^{\boxtimes n}[-n]$ on U^n, and the complex $C^*(\overline{\mathcal{L}}(U))$ is built from smooth sections of the same bundle.

Note that

$$\mathcal{L}(U)^\vee = \overline{\mathcal{L}}_c^!(U) = \overline{\mathcal{L}}_c(U)[3].$$

Thus,

$$C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}_c(U)[2]),$$
$$C^*(\overline{\mathcal{L}}(U)) = \widehat{\mathrm{Sym}}(\mathcal{L}_c(U)[2]).$$

We can define a Poisson bracket of degree 1 on $C^*(\overline{\mathcal{L}}(U))$ as follows. On the generators $\mathcal{L}_c(U)[2]$, it is defined to be the given pairing

$$\langle -, - \rangle : \mathcal{L}_c(U) \times \mathcal{L}_c(U) \to \mathbb{R},$$

since we *can* pair smooth sections. This pairing extends uniquely, by the Leibniz rule, to continuous bilinear map

$$C^*(\overline{\mathcal{L}}(U)) \times C^*(\overline{\mathcal{L}}(U)) \to C^*(\overline{\mathcal{L}}(U)).$$

In particular, we see that $C^*(\overline{\mathcal{L}}(U))$ has the structure of a P_0 algebra in the multicategory of differentiable cochain complexes.

Let us define the modified observables in this theory by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*(\overline{\mathcal{L}}(U)).$$

We have seen that $\widetilde{\mathrm{Obs}}^{cl}(U)$ is homotopy equivalent to $\mathrm{Obs}^{cl}(U)$ and that $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a P_0 structure.

5.3.0.1 Lemma. $\mathrm{Obs}^{cl}(U)$ *has the structure of a* P_0 *factorization algebra.*

Proof It remains to verify that if $U_1, \ldots, U_n$ are disjoint open subsets of M, each contained in an open subset W, then the map

$$\widetilde{\mathrm{Obs}}^{cl}(U_1) \times \cdots \times \widetilde{\mathrm{Obs}}^{cl}(U_n) \to \widetilde{\mathrm{Obs}}^{cl}(W)$$

is compatible with the P_0 structures. This map automatically respects the commutative structure, so it suffices to verify that for $\alpha \in \widetilde{\mathrm{Obs}}^{cl}(U_i)$ and $\beta \in \widetilde{\mathrm{Obs}}^{cl}(U_j)$, where $i \neq j$, then

$$\{\alpha, \beta\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

That this bracket vanishes follows from the fact that if two "linear observables" $\phi, \psi \in \mathcal{L}_c(W)$ have disjoint support, then

$$\langle \phi, \psi \rangle = 0.$$

Every Poisson bracket reduces to a sum of brackets between linear terms by applying the Leibniz rule repeatedly. □

5.4 The Poisson structure for a general classical field theory

In this section we will prove the following.

5.4.0.1 Theorem. *For any classical field theory (section 4.4) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a quasi-isomorphism*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}$$

of commutative factorization algebras.

5.4.1 Functionals with smooth first derivative

For a free field theory, we defined a subcomplex $\widetilde{\mathrm{Obs}}^{cl}$ of observables which are built from smooth sections of a vector bundle on U^n, instead of distributional sections as in the definition of Obs^{cl}. It turns out that, for an interacting field theory, this subcomplex of Obs^{cl} is not preserved by the differential. Instead, we have to find a subcomplex built from distributions on U^n which are not smooth but which satisfy a mild regularity condition. We will call also this complex $\widetilde{\mathrm{Obs}}^{cl}$ (thus introducing a conflict with the terminology introduced in the case of free field theories).

Let $\mathcal{L}$ be an elliptic L_∞ algebra on M that defines a classical field theory. Recall that the cochain complex of observables is

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)),$$

where $\mathcal{L}(U)$ is the L_∞ algebra of sections of L on U.

Recall that as a graded vector space, $C^*(\mathcal{L}(U))$ is the algebra of functionals $\mathscr{O}(\mathcal{L}(U)[1])$ on the graded vector space $\mathcal{L}(U)[1]$. In appendix B.1, given any graded vector bundle E on M, we define a subspace

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}(\mathscr{E}(U))$$

of functionals that have "smooth first derivative". A function $\Phi \in \mathscr{O}(\mathscr{E}(U))$ is in $\mathscr{O}^{sm}(\mathscr{E}(U))$ precisely if

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c^!(U).$$

(The exterior derivative of a general function in $\mathscr{O}(\mathscr{E}(U))$ will lie *a priori* in the larger space $\mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}_c^!(U)$.) The space $\mathscr{O}^{sm}(\mathscr{E}(U))$ is a differentiable pro-vector space.

Recall that if $\mathfrak{g}$ is an L_∞ algebra, the exterior derivative maps $C^*(\mathfrak{g})$ to $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$. The complex $C^*_{sm}(\mathcal{L}(U))$ of cochains with smooth first derivative is thus defined to be the subcomplex of $C^*(\mathcal{L}(U))$ consisting of those cochains whose first derivative lies in $C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1])$, which is a subcomplex of $C^*(\mathcal{L}(U), \mathcal{L}(U)^\vee[-1])$.

In other words, $C^*_{sm}(\mathcal{L}(U))$ is defined by the fiber diagram

$$\begin{array}{ccc} C^*_{sm}(\mathcal{L}(U)) & \xrightarrow{\mathrm{d}} & C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1]) \\ \downarrow & & \downarrow \\ C^*(\mathcal{L}(U)) & \xrightarrow{\mathrm{d}} & C^*(\mathcal{L}(U), \overline{\mathcal{L}_c}^!(U)[-1]). \end{array}$$

(Note that differentiable pro-cochain complexes are closed under taking limits, so that this fiber product is again a differentiable pro-cochain complex; more details are provided in appendix B.1.)

Note that

$$C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

is a sub-commutative dg algebra for every open U. Furthermore, as U varies, $C^*_{sm}(\mathcal{L}(U))$ defines a sub-commutative prefactorization algebra of the prefactorization algebra defined by $C^*(\mathcal{L}(U))$.

We define

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U)) = \mathrm{Obs}^{cl}(U).$$

The next step is to construct the Poisson bracket.

5.4.2 The Poisson bracket

Because the elliptic L_∞ algebra L defines a classical field theory, it is equipped with an isomorphism $L \cong L^![-3]$. Thus, we have an isomorphism

$$\Phi : C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1]) \cong C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]).$$

In appendix B.2, we show that $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ — which we think of as vector fields on the formal manifold $B\mathcal{L}(U)$ — has a natural structure of a dg Lie algebra in the multicategory of differentiable pro-cochain complexes. The bracket is, of course, a version of the bracket of vector fields. Further, $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ acts on $C^*(\mathcal{L}(U))$ by derivations. This action is in the multicategory of differentiable pro-cochain complexes: the map

$$C^*(\mathcal{L}(U), \mathcal{L}(U)[1]) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is a smooth bilinear cochain map. We will write $\mathrm{Der}(C^*(\mathcal{L}(U)))$ for this dg Lie algebra $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$.

Thus, composing the map Φ above with the exterior derivative d and with the inclusion $\mathcal{L}_c(U) \hookrightarrow \mathcal{L}(U)$, we find a cochain map

$$C^*_{sm}(\mathcal{L}(U)) \to C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]) \to \mathrm{Der}(C^*(\mathcal{L}(U)))[1].$$

If $f \in C^*_{sm}(\mathcal{L}(U))$, we will let $X_f \in \mathrm{Der}(C^*(\mathcal{L}(U)))$ denote the corresponding derivation. If f has cohomological degree k, then X_f has cohomological degree $k+1$.

If $f, g \in C^*_{sm}(\mathcal{L}(U)) = \widetilde{\mathrm{Obs}}^{cl}(U)$, we define

$$\{f, g\} = X_f g \in \widetilde{\mathrm{Obs}}^{cl}(U).$$

This bracket defines a bilinear map

$$\widetilde{\mathrm{Obs}}^{cl}(U) \times \widetilde{\mathrm{Obs}}^{cl}(U) \to \widetilde{\mathrm{Obs}}^{cl}(U).$$

Note that we are simply adopting the usual formulas to our setting.

5.4.2.1 Lemma. *This map is smooth, i.e., a bilinear map in the multicategory of differentiable pro-cochain complexes.*

Proof This follows from the fact that the map

$$\mathrm{d} : \widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Der}(C^*(\mathcal{L}(U)))[1]$$

is smooth, which is immediate from the definitions, and from the fact that the map

$$\mathrm{Der}(C^*(\mathcal{L}(U)) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is smooth (which is proved in appendix B.2). $\square$

5.4.2.2 Lemma. *This bracket satisfies the Jacobi rule and the Leibniz rule. Further, for U, V disjoint subsets of M, both contained in W, and for any $f \in \widetilde{\mathrm{Obs}}^{cl}(U)$, $g \in \widetilde{\mathrm{Obs}}^{cl}(V)$, we have*

$$\{f, g\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

Proof The proof is straightforward. $\square$

Following the argument for lemma 5.3.0.1, we obtain a P_0 factorization algebra.

5.4.2.3 Corollary. $\widetilde{\mathrm{Obs}}^{cl}$ *defines a P_0 factorization algebra in the valued in the multicategory of differentiable pro-cochain complexes.*

The final thing we need to verify is the following.

5.4.2.4 Proposition. *For all open subset $U \subset M$, the map*

$$\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$$

is a weak equivalence.

Proof It suffices to show that it is a weak equivalence on the associated graded for the natural filtration on both sides. Now, $\mathrm{Gr}^n\,\widetilde{\mathrm{Obs}}^{cl}(U)$ fits into a fiber diagram

$$\begin{tikzcd}
\mathrm{Gr}^n\,\widetilde{\mathrm{Obs}}^{cl}(U) \arrow[r] \arrow[d] & \mathrm{Sym}^n(\overline{\mathcal{L}}_c^!(U)[-1]) \otimes \mathcal{L}_c^!(U) \arrow[d] \\
\mathrm{Gr}^n\,\mathrm{Obs}^{cl}(U) \arrow[r] & \mathrm{Sym}^n(\overline{\mathcal{L}}_c^!(U)[-1]) \otimes \overline{\mathcal{L}}_c^!(U).
\end{tikzcd}$$

Note also that

$$\mathrm{Gr}^n\,\mathrm{Obs}^{cl}(U) = \mathrm{Sym}^n\,\overline{\mathcal{L}}_c^!(U).$$

The Atiyah-Bott lemma of appendix I.D shows that the inclusion

$$\mathcal{L}_c^!(U) \hookrightarrow \overline{\mathcal{L}}_c^!(U)$$

is a continuous cochain homotopy equivalence. We can thus choose a homotopy inverse

$$P : \overline{\mathcal{L}}_c^!(U) \to \mathcal{L}_c^!(U)$$

and a homotopy

$$H : \overline{\mathcal{L}}_c^!(U) \to \overline{\mathcal{L}}_c^!(U)$$

such that $[\mathrm{d}, H] = P - \mathrm{Id}$ and such that H preserves the subspace $\mathcal{L}_c^!(U)$.

Now,

$$\mathrm{Sym}^n \mathcal{L}_c^!(U) \subset \mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Sym}^n \overline{\mathcal{L}}_c^!(U).$$

Using the projector P and the homotopy H, one can construct a projector

$$P_n = P^{\otimes n} : \overline{\mathcal{L}}_c^!(U)^{\otimes n} \to \mathcal{L}_c^!(U)^{\otimes n}.$$

We can also construct a homotopy

$$H_n : \overline{\mathcal{L}}_c^!(U)^{\otimes n} \to \mathcal{L}_c^!(U)^{\otimes n}.$$

The homotopy H_n is defined inductively by the formula

$$H_n = H \otimes P_{n-1} + 1 \otimes H_{n-1}.$$

This formula defines a homotopy because

$$[\mathrm{d}, H_n] = P \otimes P_{n-1} - 1 \otimes P_{n-1} + 1 \otimes P_{n-1} - 1 \otimes 1.$$

Notice that the homotopy H_n preserves all the subspaces of the form

$$\overline{\mathcal{L}}_c^!(U)^{\otimes k} \otimes \mathcal{L}_c^!(U) \otimes \overline{\mathcal{L}}_c^!(U)^{\otimes n-k-1}.$$

This will be important momentarily.

Next, let

$$\pi : \overline{\mathcal{L}}_c^!(U)^{\otimes n}[-n] \to \mathrm{Sym}^n(\overline{\mathcal{L}}_c^!(U)[-1])$$

be the projection, and let

$$\Gamma_n = \pi^{-1} \mathrm{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U).$$

Then Γ_n is acted on by the symmetric group S_n, and the S_n invariants are $\widetilde{\mathrm{Obs}}^{cl}(U)$.

Thus, it suffices to show that the inclusion

$$\Gamma_n \hookrightarrow \overline{\mathcal{L}}_c(U)^{\otimes n}$$

is a weak equivalence of differentiable spaces. We will show that it is continuous homotopy equivalence.

The definition of $\widetilde{\mathrm{Obs}}^{cl}(U)$ allows one to identify

$$\Gamma_n = \cap_{k=0}^{n-1} \overline{\mathcal{L}}_c^!(U)^{\otimes k} \otimes \mathcal{L}_c^!(U) \otimes \overline{\mathcal{L}}_c^!(U)^{\otimes n-k-1}.$$

The homotopy H_n preserves Γ_n, and the projector P_n maps

$$\overline{\mathcal{L}}_c^!(U)^{\otimes n} \to \mathcal{L}_c(U)^{\otimes n} \subset \Gamma_n.$$

Thus, P_n and H_n provide a continuous homotopy equivalence between $\overline{\mathcal{L}}_c^!(U)^{\otimes n}$ and Γ_n, as desired. □

PART TWO

QUANTUM FIELD THEORY

6

Introduction to quantum field theory

As explained in the introduction, this book develops a version of deformation quantization for field theories, rather than mechanics. In the chapters on classical field theory, we showed that the observables of a classical BV theory naturally form a commutative factorization algebra, with a homotopical P_0 structure. In the following chapters, we will show that every quantization of a classical BV theory produces a factorization algebra that we call the quantum observables of the quantum field theory. To be precise, the main theorem of this part is the following.

6.0.0.1 Theorem. *Any quantum field theory on a manifold M, in the sense of Costello (2011b), gives rise to a factorization algebra Obs^q on M of* quantum observables*. This is a factorization algebra over $\mathbb{C}[[\hbar]]$, valued in differentiable pro-cochain complexes, and it quantizes the homotopy P_0 factorization algebra of classical observables of the corresponding classical field theory.*

For free field theories, this factorization algebra of quantum observables is essentially the same as the one discussed in Chapter I.4. The only difference is that, when discussing free field theories, we normally set $\hbar = 1$ and took our observables to be polynomial functions of the fields. When we discuss interacting theories, we take our observables to be power series on the space of fields, and we take $\hbar$ to be a formal parameter.

Chapter 7 is thus devoted to reviewing the formalism of Costello (2011b), stated in a form most suitable to our purposes here. It's important to note that, in contrast to the deformation quantization of Poisson manifolds, a classical BV theory may not possess any quantizations (i.e., quantization may be *obstructed*) or it may have many quantizations. A central result of Costello (2011b), stated in section 7.5, is that there is a space of BV quantizations. Moreover, this space can be constructed as a tower of fibrations, where the fiber between any pair of successive layers is described by certain cohomology groups of local functionals. These cohomology groups can be computed just from the classical theory.

The machinery of Costello (2011b) allows one to construct many examples of quantum field theories, by calculating the appropriate cohomology groups. For example, in Costello (2011b), the quantum Yang-Mills gauge theory is constructed. Theorem 6.0.0.1, together with the results of Costello (2011b), thus produces many interesting examples of factorization algebras.

Remark: We forewarn the reader that our definitions and constructions involve a heavy use of functional analysis and (perhaps more surprisingly) simplicial sets, which is our preferred way of describing a space of field theories. Making a quantum field theory typically requires many choices, and as mathematicians, we wish to pin down precisely how the quantum field theory depends on these choices. The machinery we use gives us very precise statements, but statements that can be forbidding at first sight. We encourage the reader, on a first pass through this material, to simply make all necessary choices (such as a parametrix) and focus on the output of our machine, namely the factorization algebra of quantum observables. Keeping track of the dependence on choices requires careful bookkeeping (aided by the machinery of simplicial sets) but is straightforward once the primary construction is understood. $\Diamond$

The remainder of this chapter consists of an introduction to the quantum BV formalism, building on our motivation for the classical BV formalism in section 4.1.

6.1 The quantum BV formalism in finite dimensions

In section 4.1, we motivated the classical BV formalism with a finite-dimensional toy model. To summarize, we described the *derived* critical locus of a function S on a smooth manifold M of dimension n. The functions on this derived space $\mathscr{O}(\mathrm{Crit}^h(S))$ form a commutative dg algebra,

$$\Gamma(M, \wedge^n TM) \xrightarrow{\vee \mathrm{d}S} \dots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} C^\infty(M),$$

the polyvector fields $PV(M)$ on M with the differential given by contraction with $\mathrm{d}S$. This complex remembers how $\mathrm{d}S$ vanishes and not just where it vanishes.

The quantum BV formalism uses a deformation of this *classical BV complex* to encode, in a homological way, oscillating integrals.

In finite dimensions, there already exists a homological approach to integration: the de Rham complex. For instance, on a compact, oriented n-manifold without boundary, M, we have the commuting diagram

$$\begin{tikzcd} \Omega^n(M) \arrow[rr, "\int_M"] \arrow[dr, "{[-]}"'] & & \mathbb{R} \\ & H^n(M) \arrow[ur, "{\langle [M], - \rangle}"'] & \end{tikzcd}$$

where $[\mu]$ denotes the cohomology class of the top form μ and $\langle [M], -\rangle$ denotes pairing the class with the fundamental class of M. Thus, integration factors through the de Rham cohomology.

Suppose μ is a smooth probability measure, so that $\int_M \mu = 1$ and μ is everywhere nonnegative (which depends on the choice of orientation). Then we can interpret the expected value of a function f on M — an "observable on the space of fields M" — as the cohomology class $[f\mu] \in H^n(M)$.

The BV formalism in finite dimensions secretly exploits this use of the de Rham complex, as we explain momentarily. For an infinite-dimensional manifold, though, the de Rham complex ceases to encode integration over the whole manifold because there are no top forms. In contrast, the BV version scales to the infinite-dimensional setting. Infinite dimensions, of course, introduces extra difficulties to do with the fact that integration in infinite dimensions is not well-defined. These difficulties manifest themselves as ultraviolte divergences of quantum field theory, and we deal with them using the techniques developed in Costello (2011b).

In the classical BV formalism, we work with the polyvector fields rather than de Rham forms. A choice of probability measure μ, however, produces a map between these graded vector spaces

$$\begin{tikzcd} \Gamma(M, \wedge^n TM) \arrow[d, "\vee\mu"'] & \dots & \Gamma(M, \wedge^2 TM) \arrow[d, "\vee\mu"'] & \Gamma(M, TM) \arrow[d, "\vee\mu"'] & C^\infty(M) \arrow[d, "\vee\mu"'] \\ C^\infty(M) & \dots & \Omega^{n-2}(M) & \Omega^{n-1}(M) & \Omega^n(M) \end{tikzcd}$$

where $\vee\mu$ simply contracts a k-polyvector field with μ to get a $n-k$-form. When μ is nowhere-vanishing (i.e., when μ is a *volume form*), this map is an isomorphism and so we can "pull back" the exterior derivative to equip the polyvector fields with a differential. This differential is usually called the *divergence operator for* μ, so we denote it div_μ.

By the *divergence complex for* μ, we mean the polyvector fields (concentrated in nonpositive degrees) with differential div_μ. Its cohomology is isomorphic, by construction, to $H^*_{dR}(M)[n]$. In particular, given a function f on M, viewed as living in degree zero and providing an "observable," we see that its cohomology class $[f]$ in the divergence complex corresponds to the expected value of f against μ. More precisely, we can define the ratio $[f]/[1]$ as the expected value of f. Under the map $\vee\mu$, it goes to the usual expected value.

What we've done above is provide an alternative homological approach to integration. More accurately, we've shown how "integration against a volume form" can be encoded by an appropriate choice of differential on the polyvector fields. Cohomology classes in this divergence complex encode the expected values of functions against this measure. Of course, this is what we want from the path integral! The divergence complex is the motivating example for the quantum BV formalism, and so it is also called a *quantum BV complex*.

We can now explain why this approach to homological integration is more suitable to extension to infinite dimensions than the usual de

Rham picture. Even for an infinite-dimensional manifold M, the polyvector fields are well-defined (although one must make choices in how to define them, depending on one's preferences with functional analysis). One can still try to construct a "divergence-type operator" and view it as the effective replacement for the probability measure. By taking cohomology classes, we compute the expected values of observables. The difficult part is making sense of the divergence operator; this is achieved through renormalization.

This vein of thought leads to a question: how to characterize, in an abstract fashion, the nature of a divergence operator? An answer leads, as we've shown, to a process for defining a homological path integral. Below, we'll describe one approach, but first we examine a simple case.

Remark: The cohomology of the complex (both in the finite and infinite dimensional settings) always makes sense, but H^0 is not always one-dimensional. For example, on a manifold X that is not closed, the de Rham cohomology often vanishes at the top. If the manifold is disconnected but closed, the top de Rham cohomology has dimension equal to the number of components of the manifold. In general, one must choose what class of functions to integrate against the volume form, and the cohomology depends on this choice (e.g., consider compactly supported de Rham cohomology).

Instead of computing expected values, the cohomology provides relations between expected values of observables. We will see how the cohomology encodes relations in the example below. In the setting of conformal field theory, for instance, one often uses such relations to obtain formulas for the operator product expansion. ◊

6.2 The "free scalar field" in finite dimensions

A concrete example is in order. We will work with a simple manifold, the real line, equipped with the Gaussian measure and recover the baby case of Wick's lemma. The generalization to a finite-dimensional vector space will be clear.

Remark: This example is especially pertinent to us because in this book we are working with perturbative quantum field theories. Hence, for us, there is always a free field theory — whose space of fields is a vector space equipped with some kind of Gaussian measure — that we've modified by adding an interaction to the action functional. The underlying vector space is equipped with a linear pairing that yields the BV Laplacian, as we work with it. As we will see in this example, the usual BV formalism relies upon the underlying "manifold" being linear in nature. To extend to a global nonlinear situation, on e needs to develop new techniques (see, for instance, Costello (2011a)). $\Diamond$

Before we undertake the Gaussian measure, let's begin with the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$. This is not a probability measure, but it is nowhere-vanishing, which is the only property necessary to construct a divergence operator. In this case, we compute

$$\mathrm{div}_{Leb} : f\frac{\partial}{\partial x} \mapsto \frac{\partial f}{\partial x}.$$

In one popular notion, we use ξ to denote the vector field $\partial/\partial x$, and the polyvector fields are then $C^\infty(\mathbb{R})[\xi]$, where ξ has cohomological degree -1. The divergence operator becomes

$$\mathrm{div}_{Leb} = \frac{\partial}{\partial x}\frac{\partial}{\partial \xi},$$

which is also the standard example of the BV Laplacian $\triangle$. (In short, the usual BV Laplacian on $\mathbb{R}^n$ is simply the divergence operator for the Lebesgue measure.) We will use $\triangle$ for it, as this notation will continue throughout the book.

It is easy to see, by direct computation or the Poincaré lemma, that the cohomology of the divergence complex for the Lebesgue measure is simply $H^{-1} \cong \mathbb{R}$ and $H^0 \cong \mathbb{R}$.

Let μ_b be the usual Gaussian probability measure on $\mathbb{R}$ with variance b:

$$\mu_b = \sqrt{\frac{1}{2\pi b}}e^{-x^2/2b}\mathrm{d}x.$$

As μ is a nowhere-vanishing probability measure, we obtain a divergence operator

$$\mathrm{div}_b : f\frac{\partial}{\partial x} \mapsto \frac{\partial f}{\partial x} - \frac{x}{b}f.$$

We have

$$\mathrm{div}_b = \triangle + \vee \mathrm{d}S$$

where $S = -x^2/2b$. Note that this complex is a deformation of the classical BV complex for S by adding the BV Laplacian $\triangle$.

This divergence operator preserves the subcomplex of polynomial polyvector fields. That is, a vector field with polynomial coefficient goes to a polynomial function.

Explicitly, we see

$$\mathrm{div}_b\left(x^n \frac{\partial}{\partial x}\right) = nx^{n-1} - \frac{1}{b}x^{n+1}.$$

Hence, at the level of cohomology, we see $[x^{n+1}] = bn[x^{n-1}]$. We have just obtained the following, by a purely cohomological process.

6.2.0.1 Lemma (Baby case of Wick's lemma). *The expected value of x^n with respect to the Gaussian measure is zero if n odd and $b^k(2k-1)(2k-3)\cdots 5\cdot 3$ if $n = 2k$.*

Since Wick's lemma appears by this method, it should be clear that one can recover the usual Feynman diagrammatic expansion. Indeed, the usual arguments with integration by parts are encoded here by the relations between cohomology classes.

Note that for any function $S : \mathbb{R} \to \mathbb{R}$, the volume form $e^S \mathrm{d}x$ has divergence operator

$$\mathrm{div}_S = \triangle + \frac{\partial S}{\partial x}\frac{\partial}{\partial x},$$

and using the Schouten bracket $\{-,-\}$ on polyvector fields, we can write it as

$$\mathrm{div}_S = \triangle + \{S, -\}.$$

The *quantum master equation* (QME) is the equation $\mathrm{div}_S^2 = 0$. The *classical master equation* (CME) is the equation $\{S,S\} = 0$, which just encodes the fact that the differential of the classical BV complex is square-zero. (In the examples we've discussed so far, this property is immediate, but in many contexts, such as gauge theories, finding such a function S can be a nontrivial process.)

6.3 An operadic description

Before we provide abstract properties that characterize a divergence operator, we should recall properties that characterize the classical BV complex. Of course, functions on the derived critical locus are a commutative dg algebra. Polyvector fields, however, also have the Schouten bracket — the natural extension of the Lie bracket of vector fields and functions — which is a Poisson bracket of cohomological degree 1 and which is compatible with the differential $\vee S = \{S, -\}$. Thus, we introduced the notion of a P_0 algebra, where P_0 stands for "Poisson-zero," in appendix I.A.3.2. In chapter 5, we showed that the factorization algebra of observables for a classical BV theory have a lax P_0 structure.

Examining the divergence complex for a measure of the form $e^S \mathrm{d}x$ in the preceding section, we saw that the divergence operator was a deformation of $\{S, -\}$, the differential for the classical BV complex. Moreover, a simple computation shows that a divergence operator satisfies

$$\operatorname{div}(ab) = (\operatorname{div} a)b + (-1)^{|a|}a(\operatorname{div} b) + (-1)^{|a|}\{a, b\}$$

for any polyvector fields a and b. (This relation follows, under the polyvector-de Rham isomorphism given by the measure, from the fact that the exterior derivative is a derivation for the wedge product.) Axiomatizing these two properties, we obtain the notion of a Beilinson-Drinfeld algebra, discussed in appendix I.A.3.2. The differential of a BD algebra possesses many of the essential properties of a divergence operator, and so we view a BD algebra as a homological way to encode integration on (a certain class of) derived spaces.

In short, the quantum BV formalism aims to find, for a P_0 algebra A^{cl}, a BD algebra A^q such that $A^{cl} = A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/(\hbar)$. We view it as moving from studying functions on the derived critical locus of some action functional S to the divergence complex for $e^S \mathscr{D}\phi$.

This motivation for the definition of a BD algebra is complementary to our earlier motivation, which emphasizes the idea that we simply want to deform from a commutative factorization algebra to a "plain," or E_0, factorization algebra. It grows out of the path integral approach to quantum field theory, rather than extending to field theory the deformation quantization approach to mechanics.

For us, the basic situation is a formal moduli space $\mathcal{M}$ with -1-symplectic pairing. Its algebra of functions is a P_0 algebra. By a version of the Darboux lemma for formal moduli spaces, we can identify $\mathcal{M}$ with an L_∞ algebra $\mathfrak{g}$ equipped with an invariant symmetric pairing. Geometrically, this means the symplectic pairing is translation-invariant and all the nonlinearity is pushed into the brackets. As the differential d on $\mathscr{O}(\mathcal{M})$ respects the Poisson bracket, we view it as a symplectic vector field of cohomological degree 1, and in this formal situation, we can find a Hamiltonian function S such that $d = \{S, -\}$.

Comparing to our finite-dimensional example above, we are seeing the analog of the fact that any nowhere-vanishing volume form on $\mathbb{R}^n$ can be written as $e^S \mathrm{d}x_1 \cdots \mathrm{d}x_n$. The associated divergence operator looks like $\triangle + \{S, -\}$, where the BV Laplacian $\triangle$ is the divergence operator for Lebesgue measure.

The translation-invariant Poisson bracket on $\mathscr{O}(\mathcal{M})$ also produces a translation-invariant BV Laplacian $\triangle$. Quantizing then amounts to finding a function $I \in \hbar\mathscr{O}(\mathcal{M})[[\hbar]]$ such that

$$\{S, -\} + \{I, -\} + \hbar\triangle$$

is square-zero. In the BV formalism, we call I a "solution to the quantum master equation for the action S." As shown in chapter 6 of Costello (2011b), we have the following relationship.

6.3.0.1 Proposition. *Let $\mathcal{M}$ be a formal moduli space with -1-symplectic structure. There is an equivalence of spaces*

$$\{\textit{solutions of the QME}\} \simeq \{\textit{BD quantizations}\}.$$

6.4 Equivariant BD quantization and volume forms

We now return to our discussion of volume forms and formulate a precise relationship with BD quantization. This relationship, first noted in Koszul (1985), generalizes naturally to the setting of cotangent field theories. In section 9.4, we explain how cotangent quantizations provide volume forms on elliptic moduli problems.

For a smooth manifold M, there is a special feature of a divergence complex that we have not yet discussed. Polyvector fields have a natural action of the multiplicative group $\mathbb{G}_m$, where functions have weight zero, vector fields have weight -1, and k-vector fields have weight $-k$. This action arises because polyvector fields are functions on the shifted cotangent bundle $T^*[-1]M$, and there is always a scaling action on the cotangent fibers.

We can make the classical BV complex into a $\mathbb{G}_m$-equivariant P_0 algebra, as follows. Simply equip the Schouten bracket with weight 1 and the commutative product with weight zero. We now ask for a $\mathbb{G}_m$-equivariant BD quantization.

To make this question precise, we rephrase our observations operadically. Equip the operad P_0 with the $\mathbb{G}_m$ action where the commutative product is weight zero and the Poisson bracket is weight 1. An equivariant P_0 algebra is then a P_0 algebra with a $\mathbb{G}_m$ action such that the bracket has weight 1 and the product has weight zero. Similarly, equip the operad BD with the $\mathbb{G}_m$ action where $\hbar$ has weight -1, the product has weight zero, and the bracket has weight 1. A filtered BD algebra is a BD algebra with a $\mathbb{G}_m$ action with the same weights.

Given a volume form μ on M, the $\hbar$-weighted divergence complex $(PV(M)[[\hbar]], \hbar\, \mathrm{div}_\mu)$ is a filtered BD algebra.

On an smooth manifold, we saw that each volume form μ produced a divergence operator div_μ, via "conjugating" the exterior derivative d by the isomorphism $\vee\mu$. In fact, any rescaling $c\mu$, with $c \in \mathbb{R}^\times$, produces the same divergence operator. Since we want to work with probability measures, this fact meshes well with our objectives: we would always divide by the integral $\int_X \mu$ anyway. In fact, one can show that every filtered BD quantization of the P_0 algebra $PV(M)$ arises in this way.

6.4.0.1 Proposition. *There is a bijection between projective volume forms on M, and filtered BV quantizations of $PV(M)$.*

See Costello (2011a) for more details on this point.

6.5 How renormalization group flow interlocks with the BV formalism

So far, we have introduced the quantum BV formalism in the finite dimensional setting and extracted the essential algebraic structures. Applying these ideas in the setting of field theories requires nontrivial work. Much of this work is similar in flavor to our construction of a lax P_0 structure on Obs^{cl}: issues with functional analysis block the most naive approach, but there are alternative approaches, often well-known in physics, that accomplish our goal, once suitably reinterpreted.

Here, we build on the approach of Costello (2011b). The book uses exact renormalization group flow to define the notion of effective field theory and develops an effective version of the BV formalism. In chapter 7, we review these ideas in detail. We will sketch how to apply the BV formalism to formal elliptic moduli problems $\mathcal{M}$ with -1-symplectic pairing.

The main problem here is the same as in defining a shifted Poisson structure on the classical observables: the putative Poisson bracket $\{-,-\}$, arising from the symplectic structure, is well-defined only on a subspace of all observables. As a result, the associated BV Laplacian $\triangle$ is also only partially-defined.

To work around this problem, we use the fact that every parametrix Φ for the elliptic complex underlying $\mathcal{M}$ yields a mollified version $\triangle_\Phi$ of the BV Laplacian, and hence a mollified bracket $\{-,-\}_\Phi$. An *effective field theory* consists of a BD algebra Obs_Φ for every parametrix and a homotopy equivalence for any two parametrices, $\mathrm{Obs}_\Phi \simeq \mathrm{Obs}_\Psi$, satisfying coherence relations. In other words, we get a family of BD algebras over the space of parametrices. The renormalization group (RG) flow provides the homotopy equivalences for any pair of parametrices. Modulo $\hbar$, we also get a family Obs^{cl}_Φ of P_0 algebras over the space of parametrices. The tree-level RG flow produces the homotopy equivalences modulo $\hbar$.

An effective field theory is a quantization of $\mathcal{M}$ if, in the limit as $\triangle_\Phi$ goes to $\triangle$, the P_0 algebra goes to the functions $\mathscr{O}(\mathcal{M})$ on the formal moduli problem.

The space of parametrices is contractible, so an effective field theory describes just one BD algebra, up to homotopy equivalence. From the perspective developed thus far, we interpret this BD algebra as encoding integration over $\mathcal{M}$.

There is another way to interpret this definition, though, that may be attractive. The RG flow amounts to a Feynman diagram expansion, and hence we can see it as a definition of functional integration (in particular, flowing from energy scale Λ to Λ' integrates over the space of functions with energies between those scales). In Costello (2011b), the RG flow is extended to the setting where the underlying free theory is an elliptic complex, not just given by an elliptic operator.

6.6 Overview of the rest of this Part

Here is a detailed summary of the chapters on quantum field theory.

(i) In sections 7.1 to 7.5 we give an overview of the definition of QFT developed in Costello (2011b).

(ii) In section 8.1 we recall the definition of a free theory in the BV formalism and construct the factorization algebra of quantum observables of a general free theory, using the factorization envelope construction of section I.3.3. It generalizes the discussion in chapter I.4.

(iii) In section 8.2 we show how the definition of a QFT leads immediately to a construction of a BD algebra of "global observables" on the manifold M, which we denote $\mathrm{Obs}^q_{\mathscr{P}}(M)$.

(iv) In section 8.3 we start the construction of the factorization algebra associated to a QFT. We construct a cochain complex $\mathrm{Obs}^q(M)$ of global observables, which is quasi-isomorphic to (but much smaller than) the BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$.

(v) In section 8.5 we construct, for every open subset $U \subset M$, the subspace $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U.

(vi) Section 8.6 accomplishes the primary aim of the chapter. In it, we prove that the cochain complexes $\mathrm{Obs}^q(U)$ form a factorization algebra. The proof of this result is the most technical part of the chapter.

(vii) In section 9.1 we show that translation-invariant theories have translation-invariant factorization algebras of observables, and we treat the holomorphic situation as well.

(viii) In section 9.4 we explain how to interpret our definition of a QFT in the special case of a cotangent theory: roughly speaking, a quantization of the cotangent theory to an elliptic moduli problem yields a locally-defined volume form on the moduli problem we start with.

7
Effective field theories and Batalin-Vilkovisky quantization

In this chapter, we will give a summary of the definition of a QFT as developed in Costello (2011b). We will emphasize the aspects used in our construction of the factorization algebra associated to a QFT. This means that important aspects of the story there — such as the concept of renormalizability — will not be mentioned. The introductory chapter of Costello (2011b) is a leisurely exposition of the main physical and mathematical ideas, and we encourage the reader to examine it before delving into what follows. The approach there is perturbative and hence has the flavor of formal geometry (that is, geometry with formal manifolds).

A perturbative field theory is defined to be a family of effective field theories parametrized by some notion of "scale." The notion of scale can be quite flexible; the simplest version is where the scale is a positive real number, the length. In this case, the effective theory at a length scale L is obtained from the effective theory at scale ϵ by integrating out over fields with length scale between ϵ and L. In order to construct factorization algebras, we need a more refined notion of "scale," where there is a scale for every parametrix Φ of a certain elliptic operator. We denote such a family of effective field theories by $\{I[\Phi]\}$, where $I[\Phi]$ is the "interaction term" in the action functional $S[\Phi]$ at "scale" Φ. We always study families with respect to a fixed free theory.

A local action functional (see section 7.1) S is a real-valued function on the space of fields such that $S(\phi)$ is given by integrating some function of the field and its derivatives over the base manifold (the "space-time”). The main result of Costello (2011b) states that the space of perturbative QFTs is the “same size” as the space of local action functionals. More precisely, the space of perturbative QFTs defined modulo $\hbar^{n+1}$ is a torsor over the space of QFTs defined modulo $\hbar^n$ for the abelian group of local action functionals. In consequence, the space of perturbative QFTs is non-canonically isomorphic to local action functionals with values in $\mathbb{R}[[\hbar]]$ (where the choice of isomorphism amounts to choosing a way to construct counterterms).

The starting point for many physical constructions — such as the path integral — is a local action functional. However, a naive application of these constructions to such an action functional yields a nonsensical answer. Many of these constructions *do* work if, instead of applying them to a local action functional, they are applied to a family $\{I[\Phi]\}$ of effective action functionals. Thus, one can view the family of effective action functionals $\{I[\Phi]\}$ as a quantum version of the local action functional defining classical field theory. The results of Costello (2011b) allow one to construct such families of action functionals. Many formal manipulations with path integrals in the physics literature apply rigorously to families $\{I[\Phi]\}$ of effective actions. Our strategy for constructing the factorization algebra of observables is to mimic path-integral definitions of observables one can find in the physics literature, but replacing local functionals by families of effective actions.

7.1 Local action functionals

In studying field theory, there is a special class of functions on the fields, known as local action functionals, that parametrize the possible classical physical systems. Let M be a smooth manifold. Let $\mathscr{E} = C^\infty(M, E)$ denote the smooth sections of a $\mathbb{Z}$-graded super vector bundle E on M, which has finite rank when all the graded components are included. We call $\mathscr{E}$ the *fields*.

Various spaces of functions on the space of fields are defined in the Appendix B.1.

7.1.0.1 Definition. *A* functional F *is an element of*

$$\mathscr{O}(\mathscr{E}) = \prod_{n=0}^{\infty} \mathrm{Hom}_{DVS}(\mathscr{E}^{\times n}, \mathbb{R})^{S_n}.$$

This is also the completed symmetric algebra of $\mathscr{E}^{\vee}$, where the tensor product is the completed projective one.

Let $\mathscr{O}_{red}(\mathscr{E}) = \mathscr{O}(\mathscr{E})/\mathbb{C}$ be the space of functionals on $\mathscr{E}$ modulo constants.

Note that every element of $\mathscr{O}(\mathscr{E})$ has a Taylor expansion whose terms are smooth multilinear maps

$$\mathscr{E}^{\times n} \to \mathbb{C}.$$

Such smooth mulitilinear maps are the same as compactly-supported distributional sections of the bundle $(E^!)^{\boxtimes n}$ on M^n. Concretely, a functional is then an infinite sequence of vector-valued distributions on powers of M.

The local functionals depend only on the local behavior of a field, so that at each point of M, a local functional should only depend on the jet of the field at that point. In the Lagrangian formalism for field theory, their role is to describe the permitted actions, so we call them *local action functionals*. A local action functional is the essential datum of a *classical* field theory.

7.1.0.2 Definition. *A functional F is* local *if each homogeneous component F_n is a finite sum of terms of the form*

$$F_n(\phi) = \int_M (D_1\phi) \cdots (D_n\phi)\, d\mu,$$

where each D_i is a differential operator from $\mathscr{E}$ to $C^\infty(M)$ and $d\mu$ is a density on M.

We let

$$\mathscr{O}_{loc}(\mathscr{E}) \subset \mathscr{O}_{red}(\mathscr{E})$$

denote denote the space of local action functionals modulo constants.

As explained in section 4.4, a classical BV theory is a choice of local action functional S of cohomological degree 0 such that $\{S, S\} = 0$. That is, S must satisfy the classical master equation.

7.2 The definition of a quantum field theory

In this section, we will give the formal definition of a quantum field theory. The definition is a little long and somewhat technical. The reader should consult the first chapter of Costello (2011b) for physical motivations for this definition. We will provide some justification for the definition from the point of view of homological algebra shortly (section 8.2).

7.2.1

7.2.1.1 Definition. *A* free BV theory *on a manifold M consists of the following data:*

(i) *a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that is of finite rank;*
(ii) *a graded antisymmetric map of vector bundles $\langle -,-\rangle_{loc} : E \otimes E \to$ $\mathrm{Dens}(M)$ of cohomological degree -1 that is fiberwise nondegenerate. It induces a graded antisymmetric pairing of degree -1 on compactly supported smooth sections $\mathscr{E}_c$ of E:*

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(iii) *a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.*

In our constructions, we require the existence of a *gauge-fixing operator* $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the following properties:

(i) it is a square-zero differential operator of cohomological degree -1 ;
(ii) it is self adjoint for the symplectic pairing;
(iii) $D = [Q, Q^{GF}]$ is a generalized Laplacian on M, in the sense of Berline et al. (1992). This means that D is an order 2 differential operator whose symbol $\sigma(D)$, which is an endomorphism of the pullback bundle p^*E on the cotangent bundle $p : T^*M \to M$, is

$$\sigma(D) = g \,\mathrm{Id}_{p^*E}$$

where g is some Riemannian metric on M, viewed as a function on T^*M.

All our constructions vary homotopically with the choice of gauge fixing operator. In practice, there is a natural contractible space of gauge fixing operators, so that our constructions are independent (up to contractible choice) of the choice of gauge fixing operator. (As an example of contractibility, if the complex $\mathscr{E}$ is simply the de Rham complex, each metric gives a gauge fixing operator d^*. The space of metrics is contractible.)

7.2.2 Operators and kernels

Let us recall the relationship between kernels and operators on $\mathscr{E}$. Any continuous linear map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ can be represented by a kernel

$$K_F \in \mathcal{D}(M^2, E \boxtimes E^!).$$

Here $\mathcal{D}(M, -)$ denotes distributional sections. We can also identify this space as

$$\begin{aligned}\mathcal{D}(M^2, E \boxtimes E^!) &= \operatorname{Hom}_{DVS}(\mathscr{E}_c^! \times \mathscr{E}_c, \mathbb{C}) \\ &= \operatorname{Hom}_{DVS}(\mathscr{E}_c, \overline{\mathscr{E}}) \\ &= \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}^!.\end{aligned}$$

Here $\widehat{\otimes}_\pi$ denotes the completed projective tensor product.

The symplectic pairing on $\mathscr{E}$ gives an isomorphism between $\overline{\mathscr{E}}$ and $\overline{\mathscr{E}}^![-1]$. This allows us to view the kernel for any continuous linear map F as an element

$$K_F \in \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}} = \operatorname{Hom}_{DVS}(\mathscr{E}_c^! \times \mathscr{E}_c^!, \mathbb{C})$$

. If F is of cohomological degree k, then the kernel K_F is of cohomological degree $k+1$.

If the map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ has image in $\overline{\mathscr{E}}_c$ and extends to a continuous linear map $\mathscr{E} \to \overline{\mathscr{E}}_c$, then the kernel K_F has compact support. If F has image in $\mathscr{E}$ and extends to a continuous linear map $\overline{\mathscr{E}}_c \to \mathscr{E}$, then the kernel K_F is smooth.

Our conventions are such that the following hold.

(i) $K_{[Q,F]} = QK_F$, where Q is the total differential on $\overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}$.

(ii) Suppose that $F : \mathscr{E}_c \to \mathscr{E}_c$ is skew-symmetric with respect to the degree -1 pairing on $\mathscr{E}_c$. Then K_F is symmetric. Similarly, if F is symmetric, then K_F is anti-symmetric.

7.2.3 The heat kernel

In this section we will discuss heat kernels associated to the generalized Laplacian $D = [Q, Q^{GF}]$. These generalized heat kernels will not be essential to our story; most of our constructions will work with a general parametrix for the operator D, and the heat kernel simply provides a convenient example.

Suppose that we have a free BV theory with a gauge fixing operator Q^{GF}. As above, let $D = [Q, Q^{GF}]$. If our manifold M is compact, then this leads to a heat operator e^{-tD} acting on sections $\mathscr{E}$. The heat kernel K_t is the corresponding kernel, which is an element of $\overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}\widehat{\otimes}_\pi C^\infty(\mathbb{R}_{\geq 0})$. Further, if $t > 0$, the operator e^{-tD} is a smoothing operator, so that the kernel K_t is in $\mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$. Since the operator e^{-tD} is skew symmetric for the symplectic pairing on $\mathscr{E}$, the kernel K_t is symmetric.

The kernel K_t is uniquely characterized by the following properties:

(i) The heat equation:
$$\frac{\mathrm{d}}{\mathrm{d}t}K_t + (D \otimes 1)K_t = 0.$$

(ii) The initial condition that $K_0 \in \overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}$ is the kernel for the identity operator.

On a non-compact manifold M, there is more than one heat kernel satisfying these properties.

7.2.4 Parametrices

In Costello (2011b), two equivalent definitions of a field theory are given: one based on the heat kernel, and one based on a general parametrix. We will use exclusively the parametrix version in this book.

Before we define the notion of parametrix, we need a technical definition.

7.2.4.1 Definition. *If M is a manifold, a subset $V \subset M^n$ is* proper *if all of the projection maps $\pi_1, \dots, \pi_n : V \to M$ are proper. We say that a function, distribution, etc. on M^n has* proper support *if its support is a proper subset of M^n.*

7.2.4.2 Definition. *A* parametrix Φ *is a distributional section*

$$\Phi \in \overline{\mathscr{E}}(M) \widehat{\otimes}_\pi \overline{\mathscr{E}}(M)$$

of the bundle $E \boxtimes E$ on $M \times M$ with the following properties.

(i) Φ *is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}}(M) \widehat{\otimes}_\pi \overline{\mathscr{E}}(M)$.*
(ii) Φ *is of cohomological degree* 1.
(iii) Φ *has proper support.*
(iv) *Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that*

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is a smooth section of $E \boxtimes E$ on $M \times M$. Thus,

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}} \in \mathscr{E}(M) \widehat{\otimes}_\pi \mathscr{E}(M).$$

(Here K_{Id} is the kernel corresponding to the identity operator).

Remark: For clarity's sake, note that our definition depends on a choice of Q^{GF}. Thus, we are defining here parametrices for the generalized Laplacian $[Q, Q^{GF}]$, *not* general parametrices for the elliptic complex $\mathscr{E}$. $\Diamond$

Note that the parametrix Φ can be viewed (using the correspondence between kernels and operators described above) as a linear map $A_\Phi : \mathscr{E} \to \mathscr{E}$. This operator is of cohomological degree 0, and has the property that

$$A_\Phi [Q, Q^{GF}] = \mathrm{Id} + \text{a smoothing operator}$$
$$[Q, Q^{GF}] A_\Phi = \mathrm{Id} + \text{a smoothing operator}.$$

This property – being both a left and right inverse to the operator $[Q, Q^{GF}]$, up to a smoothing operator – is the standard definition of a parametrix.

An example of a parametrix is the following. For M compact, let $K_t \in$

$\mathscr{E}\widehat{\otimes}_{\pi}\mathscr{E}$ be the heat kernel. Then, the kernel $\int_0^L K_t \mathrm{d}t$ is a parametrix, for any $L > 0$.

It is a standard result in the theory of pseudodifferential operators (see e.g. Tar (n.d.)) that every elliptic operator admits a parametrix. Normally a parametrix is not assumed to have proper support; however, if Φ is a parametrix satisfying all conditions except that of proper support, and if $f \in C^{\infty}(M \times M)$ is a smooth function with proper support that is 1 in a neighborhood of the diagonal, then $f\Phi$ is a parametrix with proper support. This shows that parametrices with proper support always exist.

Let us now list some key properties of parametrices, all of which are consequences of elliptic regularity.

7.2.4.3 Lemma. *Parametrices satisfy the following:*

(i) *If Φ, Ψ are parametrices, then the section $\Phi - \Psi$ of the bundle $E \boxtimes E$ on $M \times M$ is smooth.*
(ii) *Any parametrix Φ is smooth away from the diagonal in $M \times M$.*
(iii) *Any parametrix Φ is such that $(Q \otimes 1 + 1 \otimes Q)\Phi$ is smooth on all of $M \times M$.*

(Note that $Q \otimes 1 + 1 \otimes Q$ is the natural differential on the space $\overline{\mathscr{E}}\widehat{\otimes}_{\beta}\overline{\mathscr{E}}$.)

Proof We will let Q denote $Q \otimes 1 + 1 \otimes Q$, and similarly $Q^{GF} = Q^{GF} \otimes 1 + 1 \otimes Q^{GF}$, acting on the space $\overline{\mathscr{E}}\widehat{\otimes}_{\beta}\overline{\mathscr{E}}$. Note that

$$[Q, Q^{GF}] = [Q, Q^{GF}] \otimes 1 + 1 \otimes [Q, Q^{GF}].$$

We now verify the assertions:

(i) Since $[Q, Q^{GF}](\Phi - \Psi)$ is smooth, and the operator $[Q, Q^{GF}]$ is elliptic, this step follows from elliptic regularity.
(ii) Away from the diagonal, Φ is annihilated by the elliptic operator $[Q, Q^{GF}]$, and so is smooth.
(iii) Note that

$$[Q, Q^{GF}]Q\Phi = Q[Q, Q^{GF}]\Phi$$

and that $[Q, Q^{GF}]\Phi - 2K_{Id}$ is smooth, where K_{Id} is the kernel for the identity operator. Since $QK_{Id} = 0$, the statement follows.

□

If Φ, Ψ are parametrices, we say that $\Phi < \Psi$ if the support of Φ is contained in the support of Ψ. In this way, parametrices acquire a partial order.

7.2.5 The propagator for a parametrix

In what follows, we will use the notation $Q, Q^{GF}, [Q, Q^{GF}]$ for the operators $Q \otimes 1 + 1 \otimes Q$, etc.

If Φ is a parametrix, we let

$$P(\Phi) = \tfrac{1}{2} Q^{GF} \Phi \in \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}.$$

This is the propagator associated to Φ. We let

$$K_\Phi = K_{\mathrm{Id}} - QP(\Phi)..$$

Note that

$$\begin{aligned} QP(\Phi)) &= \tfrac{1}{2}[Q, Q^{GF}]\Phi - Q\Phi \\ &= K_{id} + \text{ smooth kernels }. \end{aligned}$$

Thus, K_Φ is smooth.

We remark that

$$K_\Phi - K_\Psi = QP(\Psi) - QP(\Phi),$$

which is an important identity we will use repeatedly.

To relate to Section 7.2.3 and Costello (2011b), we note that if M is a compact manifold and if

$$\Phi = \int_0^L K_t \mathrm{d}t$$

is the parametrix associated to the heat kernel, then

$$P(\Phi) = P(0, L) = \int_0^L (Q^{GF} \otimes 1) K_t \mathrm{d}t$$

and $K_\Phi = K_L$.

7.2.6 Classes of functionals

In the appendix B.1 we define various classes of functions on the space $\mathscr{E}_c$ of compactly-supported fields. Here we give an overview of those classes. Many of the conditions seem somewhat technical at first, but they arise naturally as one attempts both to discuss the support of an observable and to extend the algebraic ideas of the BV formalism in this infinite-dimensional setting.

We are interested, firstly, in functions modulo constants, which we call $\mathscr{O}_{red}(\mathscr{E}_c)$. Every functional $F \in \mathscr{O}_{red}(\mathscr{E}_c)$ has a Taylor expansion in terms of symmetric smooth linear maps

$$F_k : \mathscr{E}_c^{\times k} \to \mathbb{C}$$

(for $k > 0$). Such linear maps are the same as distributional sections of the bundle $(E^!)^{\boxtimes k}$ on M^k. We say that F has *proper support* if the support of each F_k (as defined above) is a proper subset of M^k. The space of functionals with proper support is denoted $\mathscr{O}_P(\mathscr{E}_c)$ (as always in this section, we work with functionals modulo constants). This condition equivalently means that, when we think of F_k as an operator

$$\mathscr{E}_c^{\times k-1} \to \overline{\mathscr{E}}^!,$$

it extends to a smooth multilinear map

$$F_k : \mathscr{E}^{\times k-1} \to \overline{\mathscr{E}}^!.$$

At various points in this book, we will need to consider *functionals with smooth first derivative*, which are functionals satisfying a certain technical regularity constraint. Functionals with smooth first derivative are needed in two places in the text: when we define the Poisson bracket on classical observables, and when we give the definition of a quantum field theory. In terms of the Taylor components F_k, viewed as multilinear operators $\mathscr{E}_c^{\times k-1} \to \overline{\mathscr{E}}^!$, this condition means that the F_k has image in $\mathscr{E}^!$. (For more detail, see Appendix B.1.)

We are interested in the functionals with smooth first derivative and with proper support. We denote this space by $\mathscr{O}_{P,sm}(\mathscr{E})$. These are the functionals with the property that the Taylor components F_k, when viewed as operators, give continuous linear maps

$$\mathscr{E}^{\times k-1} \to \mathscr{E}^!.$$

7.2.7 The renormalization group flow

Let Φ and Ψ be parametrices. Then $P(\Phi) - P(\Psi)$ is a smooth kernel with proper support.

Given any element

$$\alpha \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E} = C^\infty(M \times M, E \boxtimes E)$$

of cohomological degree 0, we define an operator

$$\partial_\alpha : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

This map is an order 2 differential operator, which, on components, is the map given by contraction with α:

$$\alpha \vee - : \operatorname{Sym}^n \mathscr{E}^\vee \to \operatorname{Sym}^{n-2} \mathscr{E}^\vee.$$

The operator ∂_α is the unique order 2 differential operator that is given by pairing with α on $\operatorname{Sym}^2 \mathscr{E}^\vee$ and that is zero on $\operatorname{Sym}^{\leq 1} \mathscr{E}^\vee$.

We define a map

$$\begin{aligned} W(\alpha, -) : \mathscr{O}^+(\mathscr{E})[[\hbar]] &\to \mathscr{O}^+(\mathscr{E})[[\hbar]] \\ F &\mapsto \hbar \log \left(e^{\hbar \partial_\alpha} e^{F/\hbar} \right), \end{aligned}$$

known as the *renormalization group flow* with respect to α. (When $\alpha = P(\Phi) - P(\Psi)$, we call it the RG flow from Ψ to Φ.) This formula is a succinct way of summarizing a Feynman diagram expansion. In particular, $W(\alpha, F)$ can be written as a sum over Feynman diagrams with the Taylor components F_k of F labelling vertices of valence k, and with α as propagator. (All of this, and indeed everything else in this section, is explained in far greater detail in chapter 2 of Costello (2011b).) For this map to be well-defined, the functional F must have only cubic and higher terms modulo $\hbar$. The notation $\mathscr{O}^+(\mathscr{E})[[\hbar]]$ denotes this restricted class of functionals.

If $\alpha \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E}$ has proper support, then the operator $W(\alpha, -)$ extends (uniquely, of course) to a continuous (or equivalently, smooth) operator

$$W(\alpha, -) : \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]] \to \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]].$$

Our philosophy is that a parametrix Φ is like a choice of "scale" for our field theory. The renormalization group flow relating the scale given by Φ and that given by Ψ is $W(P(\Phi) - P(\Psi), -)$.

Because $P(\Phi)$ is not a smooth kernel, the operator $W(P(\Phi), -)$ is not well-defined. This is just because the definition of $W(P(\Phi), -)$ involves multiplying distributions. In physics terms, the singularities that appear when one tries to define $W(P(\Phi), -)$ are called ultraviolet divergences.

However, if $I \in \mathscr{O}^+_{P,sm}(\mathscr{E})$, the tree level part

$$W_0(P(\Phi), I) = W((P(\Phi), I) \mod \hbar$$

is a well-defined element of $\mathscr{O}^+_{P,sm}(\mathscr{E})$. The $\hbar \to 0$ limit of $W(P(\Phi), I)$ is called the tree-level part because, whereas the whole object $W(P(\Phi), I)$ is defined as a sum over graphs, the $\hbar \to 0$ limit $W_0(P(\Phi), I)$ is defined as a sum over trees. It is straightforward to see that $W_0(P(\Phi), I)$ only involves multiplication of distributions with transverse singular support, and so is well defined.

7.2.8 The BD algebra structure associated to a parametrix

A parametrix also leads to a BV operator

$$\triangle_\Phi = \partial_{K_\Phi} : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Again, this operator preserves the subspace $\mathscr{O}_{P,sm}(\mathscr{E})$ of functions with proper support and smooth first derivative. The operator $\triangle_\Phi$ commutes with Q, and it satisfies $(\triangle_\Phi)^2 = 0$. In a standard way, we can use the BV operator $\triangle_\Phi$ to define a bracket

$$\{I, J\}_\Phi = \triangle_\Phi(IJ) - (\triangle_\Phi I)J - (-1)^{|I|} I \triangle_\Phi J$$

on the space $\mathscr{O}(\mathscr{E})$.

This bracket is a Poisson bracket of cohomological degree 1. If we give the graded-commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$ the standard product, the Poisson bracket $\{-,-\}_\Phi$, and the differential $Q + \hbar\triangle_\Phi$, then it becomes a BD algebra.

The bracket $\{-,-\}_\Phi$ extends uniquely to a continuous linear map

$$\mathscr{O}_P(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Further, the space $\mathscr{O}_{P,sm}(\mathscr{E})$ is closed under this bracket. (Note, however, that $\mathscr{O}_{P,sm}(\mathscr{E})$ is *not* a commutative algebra if M is not compact: the product of two functionals with proper support no longer has proper support.)

A functional $F \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is said to satisfy the Φ-quantum master equation if

$$QF + \hbar \triangle_\Phi F + \tfrac{1}{2}\{F, F\}_\Phi = 0.$$

It is shown in Costello (2011b) that if F satisfies the Φ-QME, and if Ψ is another parametrix, then $W(P(\Psi) - P(\Phi), F)$ satisfies the Ψ-QME. This follows from the identity

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = \triangle_\Psi - \triangle_\Phi$$

of order 2 differential operators on $\mathscr{O}(\mathscr{E})$. This relationship between the renormalization group flow and the quantum master equation is a key part of the approach to QFT of Costello (2011b).

7.2.9 The definition of a field theory

Our definition of a field theory is as follows.

7.2.9.1 Definition. *Let $(\mathscr{E}, Q, \langle -, - \rangle)$ be a free BV theory. Fix a gauge fixing condition Q^{GF}. Then a* quantum field theory *(with this space of fields) consists of the following data.*

(i) *For all parametrices Φ, a functional*

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]]$$

that we call the scale Φ effective interaction. *As we explained above, the subscripts indicate that $I[\Phi]$ must have smooth first derivative and proper support. The superscript $+$ indicates that, modulo $\hbar$, $I[\Phi]$ must be at least cubic. Note that we work with functions modulo constants.*

(ii) *For two parametrices Φ, Ψ, $I[\Phi]$ must be related by the renormalization group flow:*

$$I[\Phi] = W(P(\Phi) - P(\Psi), I[\Psi]).$$

(iii) *Each $I[\Phi]$ must satisfy the* Φ-quantum master equation

$$(Q + \hbar \triangle_\Phi) e^{I[\Phi]/\hbar} = 0.$$

Equivalently,

$$Q I[\Phi] + \hbar \triangle_\Phi I[\Phi] + \tfrac{1}{2}\{I[\Phi], I[\Phi]\}_\Phi.$$

(iv) *Finally, we require that $I[\Phi]$ satisfies a locality axiom. Let*

$$I_{i,k}[\Phi] : \mathscr{E}_c^{\times k} \to \mathbb{C}$$

be the kth Taylor component of the coefficient of $\hbar^i$ in $I[\Phi]$. We can view this as a distributional section of the bundle $(E^!)^{\boxtimes k}$ on M^k. Our locality axiom says that, as Φ tends to zero, the support of

$$I_{i,k}[\Phi]$$

becomes closer and closer to the small diagonal in M^k.

For the constructions in this book, it turns out to be useful to have precise bounds on the support of $I_{i,k}[\Phi]$. To give these bounds, we need some notation. Let $\operatorname{Supp}(\Phi) \subset M^2$ be the support of the parametrix Φ, and let $\operatorname{Supp}(\Phi)^n \subset M^2$ be the subset obtained by convolving $\operatorname{Supp}(\Phi)$ with itself n times. (Thus, $(x, y) \in \operatorname{Supp}(\Phi)^n$ if there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in \operatorname{Supp}(\Phi)$.)

Our support condition is that, if $e_j \in \mathscr{E}_c$, then

$$I_{i,k}(e_1, \ldots, e_k) = 0$$

unless, for all $1 \le r < s \le k$,

$$\operatorname{Supp}(e_r) \times \operatorname{Supp}(e_s) \subset \operatorname{Supp}(\Phi)^{3i+k}.$$

Remark: (i) The locality axiom condition as presented here is a little unappealing. An equivalent axiom is that for all open subsets $U \subset M^k$ containing the small diagonal $M \subset M^k$, there exists a parametrix Φ_U such that

$$\operatorname{Supp} I_{i,k}[\Phi] \subset U \text{ for all } \Phi < \Phi_U.$$

In other words, by choosing a small parametrix Φ, we can make the support of $I_{i,k}[\Phi]$ as close as we like to the small diagonal on M^k.

We present the definition with a precise bound on the size of the support of $I_{i,k}[\Phi]$ because this bound will be important later in the construction of the factorization algebra. Note, however, that the precise exponent $3i + k$ which appears in the definition (in $\operatorname{Supp}(\Phi)^{3i+k}$) is not important. What is important is that we have some bound of this form.

(ii) It is important to emphasize that the notion of quantum field theory is only defined once we have chosen a gauge fixing operator. Later, we will explain in detail how to understand the dependence on this choice. More precisely, we will construct a simplicial set of QFTs and show how this simplicial set only depends on the homotopy class of gauge fixing operator (in most examples, the space of natural gauge fixing operators is contractible).

$\Diamond$

Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional (defined modulo constants) that satisfies the classical master equation

$$QI_0 + \tfrac{1}{2}\{I_0, I_0\} = 0.$$

Suppose that I_0 is at least cubic.

Then, as we have seen above, we can define a family of functionals

$$I_0[\Phi] = W_0\left(P(\Phi), I_0\right) \in \mathscr{O}_{P,sm}(\mathscr{E})$$

as the tree-level part of the renormalization group flow operator from scale 0 to the scale given by the parametrix Φ. The compatibility between this classical renormalization group flow and the classical master equation tells us that $I_0[\Phi]$ satisfies the Φ-classical master equation

$$QI_0[\Phi] + \tfrac{1}{2}\{I_0[\Phi], I_0[\Phi]\}_\Phi = 0.$$

7.2.9.2 Definition. *Let $I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$ be the collection of effective interactions defining a quantum field theory. Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional satisfying the classical master equation, and so defining a classical field theory. We say that the quantum field theory $\{I[\Phi]\}$ is a* quantization *of the classical field theory defined by I_0 if*

$$I[\Phi] = I_0[\Phi] \ mod\ \hbar,$$

or, equivalently, if

$$\lim_{\Phi \to 0} I[\Phi] - I_0 \ mod\ \hbar = 0.$$

7.3 Families of theories over nilpotent dg manifolds

Before discussing the interpretation of these axioms and also explaining the results of Costello (2011b) that allow one to construct such quantum field theories, we will explain how to define families of quantum field theories over some base dg algebra. The fact that we can work in families in this way means that the moduli space of quantum field theories is something like a derived stack. For instance, by considering families over the base dg algebra of forms on the n-simplex, we see that the set of quantizations of a given classical field theory is a simplicial set.

One particularly important use of the families version of the theory is that it allows us to show that our constructions and results are independent, up to homotopy, of the choice of gauge fixing condition (provided one has a contractible — or at least connected — space of gauge fixing conditions, which happens in most examples).

In later sections, we will work implicitly over some base dg ring in the sense described here, although we will normally not mention this base ring explicitly.

7.3.0.1 Definition. *A* nilpotent dg manifold *is a manifold X (possibly with corners), equipped with a sheaf $\mathscr{A}$ of commutative differential graded algebras over the sheaf Ω^*_X, with the following properties.*

(i) *$\mathscr{A}$ is concentrated in finitely many degrees.*

(ii) *Each $\mathscr{A}^i$ is a locally free sheaf of Ω^0_X-modules of finite rank. This means that $\mathscr{A}^i$ is the sheaf of sections of some finite rank vector bundle A^i on X.*

(iii) *We are given a map of dg Ω^*_X-algebras $\mathscr{A} \to C^\infty_X$.*

We will let $\mathscr{I} \subset \mathscr{A}$ be the ideal which is the kernel of the map $\mathscr{A} \to C^\infty_X$: we require that $\mathscr{I}$, its powers $\mathscr{I}^k$, and each $\mathscr{A}/\mathscr{I}^k$ are locally free sheaves of C^∞_X-modules. Also, we require that $\mathscr{I}^k = 0$ for k sufficiently large.

Note that the differential d on $\mathscr{A}$ is necessary a differential operator.

We will use the notation $\mathscr{A}^\sharp$ to refer to the bundle of graded algebras on X whose smooth sections are $\mathscr{A}^\sharp$, the graded algebra underlying the dg algebra $\mathscr{A}$.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are nilpotent dg manifolds, a map $(Y, \mathscr{B}) \to (X, \mathscr{A})$

is a smooth map $f : Y \to X$ together with a map of dg $\Omega^*(X)$-algebras $\mathscr{A} \to \mathscr{B}$.

Here are some basic examples.

(i) $\mathscr{A} = C^\infty(X)$ and $\mathscr{I} = 0$. This describes the smooth manifold X.
(ii) $\mathscr{A} = \Omega^*(X)$ and $\mathscr{I} = \Omega^{>0}(X)$. This equips X with its de Rham complex as a structure sheaf. (Informally, we can say that "constant functions are the only functions on a small open" so that this dg manifold is sensitive to topological rather than smooth structure.)
(iii) If R is a dg Artinian $\mathbb{C}$-algebra with maximal ideal m, then R can be viewed as giving the structure of nilpotent graded manifold on a point.
(iv) If again R is a dg Artinian algebra, then for any manifold $(X, R \otimes \Omega^*(X))$ is a nilpotent dg manifold.
(v) If X is a complex manifold, then $\mathscr{A} = (\Omega^{0,*}(X), \overline{\partial})$ is a nilpotent dg manifold.

Remark: We study field theories in families over nilpotent dg manifolds for both practical and structural reasons. First, we certainly wish to discuss familes of field theories over smooth manifolds. However, we would also like to access a "derived moduli space" of field theories.

In derived algebraic geometry, one says that a derived stack is a functor from the category of non-positively graded dg rings to that of simplicial sets. Thus, such non-positively graded dg rings are the "test objects" one uses to define derived algebraic geometry. Our use of nilpotent dg manifolds mimics this story: we could say that a C^∞ derived stack is a functor from nilpotent dg manifolds to simplicial sets. The nilpotence hypothesis is not a great restriction, as the test objects used in derived algebraic geometry are naturally pro-nilpotent, where the pro-nilpotent ideal consists of the elements in degrees < 0.

Second, from a practical point of view, our arguments are tractable when working over nilpotent dg manifolds. This is related to the fact that we choose to encode the analytic structure on the vector spaces we consider using the language of differentiable vector spaces. Differentiable vector spaces are, by definition, objects where one can talk about smooth families of maps depending on a smooth manifold. In fact, the definition of differentiable vector space is strong enough that one can talk about smooth families of maps depending on nilpotent dg manifolds. ◊

We can now give a precise notion of "family of field theories." We will start with the case of a family of field theories parameterized by the nilpotent dg manifold $X = (X, C^\infty_X)$, i.e. the sheaf of dg rings on X is just the sheaf of smooth functions.

7.3.0.2 Definition. *Let M be a manifold and let $(X, \mathscr{A})$ be a nilpotent dg manifold. A family over $(X, \mathscr{A})$ of free BV theories is the following data.*

(i) *A graded bundle E on $M \times X$ of locally free $A^\sharp$-modules. We will refer to global sections of E as $\mathscr{E}$. The space of those sections $s \in \Gamma(M \times X, E)$ with the property that the map $\operatorname{Supp} s \to X$ is proper will be denoted $\mathscr{E}_c$. Similarly, we let $\overline{\mathscr{E}}$ denote the space of sections which are distributional on M and smooth on X, that is,*

$$\overline{\mathscr{E}} = \mathscr{E} \otimes_{C^\infty(M\times X)} \left(\mathcal{D}(M)\widehat{\otimes}_\pi C^\infty(X)\right).$$

(This is just the algebraic tensor product, which is reasonable as $\mathscr{E}$ is a finitely generated projective $C^\infty(M \times X)$-module).

As above, we let

$$E^! = \operatorname{Hom}_{A^\sharp}(E, A^\sharp) \otimes \operatorname{Dens}_M$$

denote the "dual" bundle. There is a natural $\mathscr{A}^\sharp$-valued pairing between $\mathscr{E}$ and $\mathscr{E}_c^!$.

(ii) *A differential operator $Q : \mathscr{E} \to \mathscr{E}$, of cohomological degree 1 and square-zero, making $\mathscr{E}$ into a dg module over the dg algebra $\mathscr{A}$.*

(iii) *A map*

$$E \otimes_{A^\sharp} E \to \operatorname{Dens}_M \otimes A^\sharp$$

which is of degree -1, anti-symmetric, and leads to an isomorphism

$$\operatorname{Hom}_{A^\sharp}(E, A^\sharp) \otimes \operatorname{Dens}_M \to E$$

of sheaves of $A^\sharp$-modules on $M \times X$.

This pairing leads to a degree -1 anti-symmetric $\mathscr{A}$-linear pairing

$$\langle -, - \rangle : \mathscr{E}_c \widehat{\otimes}_\pi \mathscr{E}_c \to \mathscr{A}.$$

We require it to be a cochain map. In other words, if $e, e' \in \mathscr{E}_c$,

$$\mathrm{d}_{\mathscr{A}} \langle e, e' \rangle = \langle Qe, e' \rangle + (-1)^{|e|} \langle e, Qe' \rangle.$$

7.3.0.3 Definition. *Let $(E, Q, \langle -, - \rangle)$ be a family of free BV theories on M parameterized by $\mathscr{A}$. A* gauge fixing condition *on $\mathscr{E}$ is an $\mathscr{A}$-linear differential operator*

$$Q^{GF} : \mathscr{E} \to \mathscr{E}$$

such that

$$\mathrm{D} = [Q, Q^{GF}] : \mathscr{E} \to \mathscr{E}$$

is a generalized Laplacian, in the following sense.

Note that D is an $\mathscr{A}$-linear cochain map. Thus, we can form

$$\mathrm{D}_0 : \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X) \to \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X)$$

by reducing modulo the maximal ideal $\mathscr{I}$ of $\mathscr{A}$.

Let $E_0 = E/I$ be the bundle on $M \times X$ obtained by reducing modulo the ideal I in the bundle of algebras A. Let

$$\sigma(\mathrm{D}_0) : \pi^* E_0 \to \pi^* E_0$$

be the symbol of the $C^\infty(X)$-linear operator D_0. Thus, $\sigma(\mathrm{D}_0)$ is an endomorphism of the bundle of $\pi^* E_0$ on $(T^* M) \times X$.

We require that $\sigma(\mathrm{D}_0)$ is the product of the identity on E_0 with a smooth family of metrics on M parameterized by X.

Throughout this section, we will fix a family of free theories on M, parameterized by $\mathscr{A}$. We will take $\mathscr{A}$ to be our base ring throughout, so that everything will be $\mathscr{A}$-linear. We would also like to take tensor products over $\mathscr{A}$. Since $\mathscr{A}$ is a topological dg ring and we are dealing with topological modules, the issue of tensor products is a little fraught. Instead of trying to define such things, we will use the following shorthand notations:

(i) $\mathscr{E} \otimes_{\mathscr{A}} \mathscr{E}$ is defined to be sections of the bundle

$$E \boxtimes_{A^\sharp} E = \pi_1^* E \otimes_{A^\sharp} \pi_2^* E$$

on $M \times M \times X$, with its natural differential which is a differential operator induced from the differentials on each copy of $\mathscr{E}$.

(ii) $\overline{\mathscr{E}}$ is the space of sections of the bundle E on $M \times X$ which are smooth in the X-direction and distributional in the M-direction. Similarly for $\overline{\mathscr{E}}_c$, $\overline{\mathscr{E}}^!$, etc.

(iii) $\overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$ is defined to be sections of the bundle $E \boxtimes_{A^\sharp} E$ on $M \times M \times X$, which are distributions in the M-directions and smooth as functions of X.

(iv) If $x \in X$, let $\mathscr{E}_x$ denote the sections on M of the restriction of the bundle E on $M \times X$ to $M \times x$. Note that $\mathscr{E}_x$ is an $A^\sharp_x$-module. Then, we define $\mathscr{O}(\mathscr{E})$ to be the space of smooth sections of the bundle of topological (or differentiable) vector spaces on X whose fibre at x is

$$\mathscr{O}(\mathscr{E})_x = \prod_n \mathrm{Hom}_{DVS/A^\sharp_x}(\mathscr{E}_x^{\times n}, \mathscr{A}^\sharp_x)_{S_n}.$$

That is an element of $\mathscr{O}(\mathscr{E}_x)$ is something whose Taylor expansion is given by smooth $A^\sharp_x$-multilinear maps to $A^\sharp_x$.

If $F \in \mathscr{O}(\mathscr{E})$ is a smooth section of this bundle, then the Taylor terms of F are sections of the bundle $(E^!)^{\boxtimes_{A^\sharp} n}$ on $M^n \times X$ which are distributional in the M^n-directions, smooth in the X-directions, and whose support maps properly to X.

In other words: when we want to discuss spaces of functionals on $\mathscr{E}$, or tensor powers of $\mathscr{E}$ or its distributional completions, we just to everything we did before fibrewise on X and linear over the bundle of algebras $A^\sharp$. Then, we take sections of this bundle on X.

7.3.1

Now that we have defined free theories over a base ring $\mathscr{A}$, the definition of an interacting theory over $\mathscr{A}$ is very similar to the definition given when $\mathscr{A} = \mathbb{C}$. First, one defines a parametrix to be an element

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$$

with the same properties as before, but where now we take all tensor products (and so on) over $\mathscr{A}$. More precisely,

(i) Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(ii) Φ is of cohomological degree 1.
(iii) Φ is closed under the differential on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.

(iv) Φ has proper support: this means that the map $\operatorname{Supp}\Phi \to M \times X$ is proper.
(v) Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is an element of $\mathscr{E} \otimes \mathscr{E}$ (where, as before, $K_{\mathrm{Id}} \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ is the kernel for the identity map).

An interacting field theory is then defined to be a family of $\mathscr{A}$-linear functionals

$$I[\Phi] \in \mathscr{O}_{red}(\mathscr{E})[[\hbar]] = \prod_{n \geq 1} \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}^{\otimes_{\mathscr{A}} n}, \mathscr{A})_{S_n}[[\hbar]]$$

satisfying the renormalization group flow equation, quantum master equation, and locality condition, just as before. In order for the RG flow to make sense, we require that each $I[\Phi]$ has proper support and smooth first derivative. In this context, this means the following. Let $I_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathscr{A}$ be the kth Taylor component of the coefficient of $\hbar^i$ in $I_{i,k}[\Phi]$. Proper support means that any projection map

$$\operatorname{Supp} I_{i,k}[\Phi] \subset M^k \times X \to M \times X$$

is proper. Smooth first derivative means, as usual, that when we think of $I_{i,k}[\Phi]$ as an operator $\mathscr{E}^{\otimes k-1} \to \overline{\mathscr{E}}$, the image lies in $\mathscr{E}$.

If we have a family of theories over $(X, \mathscr{A})$, and a map

$$f : (Y, \mathscr{B}) \to (X, \mathscr{A})$$

of dg manifolds, then we can base change to get a family over $(Y, \mathscr{B})$. The bundle on Y of $B_x^\sharp$-modules of fields is defined, fibre by fibre, by

$$(f^*\mathscr{E})_y = \mathscr{E}_{f(y)} \otimes_{A^\sharp_{f(y)}} B^\sharp_y.$$

The gauge fixing operator

$$Q^{GF} : f^*\mathscr{E} \to f^*\mathscr{E}$$

is the $\mathscr{B}$-linear extension of the gauge fixing condition for the family of theories over $\mathscr{A}$.

If

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}} \subset f^*\overline{\mathscr{E}} \otimes_{\mathscr{B}} f^*\overline{\mathscr{E}}$$

is a parametrix for the family of free theories $\mathscr{E}$ over $\mathscr{A}$, then it defines a parametrix $f^*\Phi$ for the family of free theories $f^*\mathscr{E}$ over $\mathscr{B}$. For parametrices of this form, the effective action functionals

$$f^* I[f^*\Phi] \in \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]] = \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes_{\mathscr{A}} \mathscr{B}$$

is simply the image of the original effective action functional

$$I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \subset \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]].$$

For a general parametrix Ψ for $f^*\mathscr{E}$, the effective action functional is defined by the renormalization group equation

$$f^* I[\Psi] = W\left(P(\Psi) - P(f^*\Phi), f^* I[f^*\Phi]\right).$$

This is well-defined because

$$P(\Psi) - P(f^*\Phi) \in f^*\mathscr{E} \otimes_{\mathscr{B}} f^*\mathscr{E}$$

has no singularities.

The compatibility between the renormalization group equation and the quantum master equation guarantees that the effective action functionals $f^* I[\Psi]$ satisfy the QME for every parametrix Ψ. The locality axiom for the original family of effective action functionals $I[\Phi]$ guarantees that the pulled-back family $f^* I[\Psi]$ satisfy the locality axiom necessary to define a family of theories over $\mathscr{B}$.

7.4 The simplicial set of theories

One of the main reasons for introducing theories over a nilpotent dg manifold $(X, \mathscr{A})$ is that this allows us to talk about the simplicial set of theories. This is essential, because the main result we will use from Costello (2011b) is homotopical in nature: it relates the simplicial set of theories to the simplicial set of local functionals.

We introduce some useful notation. Let us fix a family of classical field theories on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$. As above, the fields of such a theory are a dg $\mathscr{A}$-module $\mathscr{E}$ equipped with an $\mathscr{A}$-linear local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation $QI + \frac{1}{2}\{I, I\} = 0$.

By pulling back along the projection map

$$(X \times \triangle^n, \mathscr{A} \otimes C^\infty(\triangle^n)) \to (X, \mathscr{A}),$$

we get a new family of classical theories over the dg base ring $\mathscr{A} \otimes C^\infty(\triangle^n)$, whose fields are $\mathscr{E} \otimes C^\infty(\triangle^n)$. We can then ask for a gauge fixing operator

$$Q^{GF} : \mathscr{E} \otimes C^\infty(\triangle^n) \to \mathscr{E} \otimes C^\infty(\triangle^n).$$

for this family of theories. This is the same thing as a smooth family of gauge fixing operators for the original theory depending on a point in the n-simplex.

7.4.0.1 Definition. *Let $(\mathscr{E}, I)$ denote the classical theory we start with over $\mathscr{A}$. Let $\mathscr{GF}(\mathscr{E}, I)$ denote the simplicial set whose n-simplices are such families of gauge fixing operators over $\mathscr{A} \otimes C^\infty(\triangle^n)$. If there is no ambiguity as to what classical theory we are considering, we will denote this simplicial set by $\mathscr{GF}$.*

Any such gauge fixing operator extends, by $\Omega^*(\triangle^n)$-linearity, to a linear map $\mathscr{E} \otimes \Omega^*(\triangle^n) \to \mathscr{E} \otimes \Omega^*(\triangle^n)$, which thus defines a gauge fixing operator for the family of theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ pulled back via the projection

$$(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A}).$$

(Note that $\Omega^*(\triangle^n)$ is equipped with the de Rham differential.)

Example: Suppose that $\mathscr{A} = \mathbb{C}$, and the classical theory we are considering is Chern-Simons theory on a 3-manifold M, where we perturb around the trivial bundle. Then, the space of fields is $\mathscr{E} = \Omega^*(M) \otimes \mathfrak{g}[1]$ and $Q = \mathrm{d}_{dR}$. For every Riemannian metric on M, we find a gauge fixing operator $Q^{GF} = \mathrm{d}^*$. More generally, if we have a smooth family

$$\{g_\sigma \mid \sigma \in \triangle^n\}$$

of Riemannian metrics on M, depending on the point σ in the n-simplex, we get an n-simplex of the simplicial set $\mathscr{GF}$ of gauge fixing operators.

Thus, if $\mathrm{Met}(M)$ denotes the simplicial set whose n-simplices are the set of Riemannian metrics on the fibers of the submersion $M \times \triangle^n \to \triangle^n$, then we have a map of simplicial sets

$$\mathrm{Met}(M) \to \mathscr{GF}.$$

Note that the simplicial set $\mathrm{Met}(M)$ is (weakly) contractible (which follows from the familiar fact that, as a topological space, the space of metrics on M is contractible).

A similar remark holds for almost all theories we consider. For example, suppose we have a theory where the space of fields

$$\mathscr{E} = \Omega^{0,*}(M, V)$$

is the Dolbeault complex on some complex manifold M with coefficients in some holomorphic vector bundle V. Suppose that the linear operator $Q : \mathscr{E} \to \mathscr{E}$ is the $\overline{\partial}$-operator. The natural gauge fixing operators are of the form $\overline{\partial}^*$. Thus, we get a gauge fixing operator for each choice of Hermitian metric on M together with a Hermitian metric on the fibers of V. This simplicial set is again contractible.

It is in this sense that we mean that, in most examples, there is a natural contractible space of gauge fixing operators. $\Diamond$

7.4.1

We will use the shorthand notation $(\mathscr{E}, I)$ to denote the classical field theory over $\mathscr{A}$ that we start with; and we will use the notation $(\mathscr{E}_{\triangle^n}, I_{\triangle^n})$ to refer to the family of classical field theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ obtained by base-change along the projection $(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A})$.

7.4.1.1 Definition. *We let $\mathscr{T}^{(n)}$ denote the simplicial set whose k-simplices consist of the following data.*

(i) *A k-simplex $Q^{GF}_{\triangle^k} \in \mathscr{GF}[k]$, defining a gauge-fixing operator for the family of theories $(\mathscr{E}_{\triangle^k}, I_{\triangle^k})$ over $\mathscr{A} \otimes \Omega^*(\triangle^k)$.*
(ii) *A quantization of the family of classical theories with gauge fixing operator $(\mathscr{E}_{\triangle^k}, I_{\triangle^k}, Q^{GF}_{\triangle^k})$, defined modulo $\hbar^{n+1}$.*

We let $\mathscr{T}^{(\infty)}$ denote the corresponding simplicial set where the quantizations are defined to all orders in $\hbar$.

Note that there are natural maps of simplicial sets $\mathscr{T}^{(n)} \to \mathscr{T}^{(m)}$, and that $\mathscr{T}^{(\infty)} = \varprojlim \mathscr{T}^{(n)}$. Further, there are natural maps $\mathscr{T}^{(n)} \to \mathscr{GF}$.

Note further that $\mathscr{T}^{(0)} = \mathscr{G}\mathscr{F}$.

This definition describes the most sophisticated version of the set of theories we will consider. Let us briefly explain how to interpret this simplicial set of theories.

Suppose for simplicity that our base ring $\mathscr{A}$ is just $\mathbb{C}$. Then, a 0-simplex of $\mathscr{T}^{(0)}$ is simply a gauge-fixing operator for our theory. A 0-simplex of $\mathscr{T}^{(n)}$ is a gauge fixing operator, together with a quantization (defined with respect to that gauge-fixing operator) to order n in $\hbar$.

A 1-simplex of $\mathscr{T}^{(0)}$ is a homotopy between two gauge fixing operators. Suppose that we fix a 0-simplex of $\mathscr{T}^{(0)}$, and consider a 1-simplex of $\mathscr{T}^{(\infty)}$ in the fiber over this 0-simplex. Such a 1-simplex is given by a collection of effective action functionals

$$I[\Phi] \in \mathscr{O}^{+}_{P,sm}(\mathscr{E}) \otimes \Omega^*([0,1])[[\hbar]]$$

one for each parametrix Φ, which satisfy a version of the QME and the RG flow, as explained above.

We explain in some more detail how one should interpret such a 1-simplex in the space of theories. Let us fix a parametrix Φ on $\mathscr{E}$ and extend it to a parametrix for the family of theories over $\Omega^*([0,1])$. We can then expand our effective interaction $I[\Phi]$ as

$$I[\Phi] = J[\Phi](t) + J'[\Phi](t)\mathrm{d}t$$

where $J[\Phi](t), J'[\Phi](t)$ are elements

$$J[\Phi](t), J'[\Phi](t) \in \mathscr{O}^{+}_{P,sm}(\mathscr{E}) \otimes C^{\infty}([0,1])[[\hbar]].$$

Here t is the coordinate on the interval $[0,1]$.

The quantum master equation implies that the following two equations hold, for each value of $t \in [0,1]$,

$$QJ[\Phi](t) + \tfrac{1}{2}\{J[\Phi](t), J[\Phi](t)\}_{\Phi} + \hbar \triangle_{\Phi} J[\Phi](t) = 0,$$
$$\frac{\partial}{\partial t} J[\Phi](t) + QJ'[\Phi](t) + \{J[\Phi](t), J'[\Phi](t)\}_{\Phi} + \hbar \triangle_{\Phi} J'[\Phi](t) = 0.$$

The first equation tells us that for each value of t, $J[\Phi](t)$ is a solution of the quantum master equation. The second equation tells us that the

t-derivative of $J[\Phi](t)$ is homotopically trivial as a deformation of the solution to the QME $J[\Phi](t)$.

In general, if I is a solution to some quantum master equation, a transformation of the form

$$I \mapsto I + \epsilon J = I + \epsilon Q I' + \{I, I'\} + \hbar \triangle I'$$

is often called a "BV canonical transformation" in the physics literature. In the physics literature, solutions of the QME related by a canonical transformation are regarded as equivalent: the canonical transformation can be viewed as a change of coordinates on the space of fields.

For us, this interpretation is not so important. If we have a family of theories over $\Omega^*([0,1])$, given by a 1-simplex in $\mathscr{T}^{(\infty)}$, then the factorization algebra we will construct from this family of theories will be defined over the dg base ring $\Omega^*([0,1])$. This implies that the factorization algebras obtained by restricting to 0 and 1 are quasi-isomorphic.

7.4.2 Generalizations

We will shortly state the theorem that allows us to construct such quantum field theories. Let us first, however, briefly introduce a slightly more general notion of "theory."

We work over a nilpotent dg manifold $(X, \mathscr{A})$. Recall that part of the data of such a manifold is a differential ideal $I \subset \mathscr{A}$ whose quotient is $C^\infty(X)$. In the above discussion, we assumed that our classical action functional S was at least quadratic; we then split S as

$$S = \langle e, Qe \rangle + I(e)$$

into kinetic and interacting terms.

We can generalize this to the situation where S contains linear terms, as long as they are accompanied by elements of the ideal $\mathscr{I} \subset \mathscr{A}$. In this situation, we also have some freedom in the splitting of S into kinetic and interacting terms; we require only that linear and quadratic terms in the interaction I are weighted by elements of the nilpotent ideal $\mathscr{I}$.

In this more general situation, the classical master equation $\{S, S\} =$

0 does not imply that $Q^2 = 0$, only that $Q^2 = 0$ modulo the ideal $\mathscr{I}$. However, this does not lead to any problems; the definition of quantum theory given above can be easily modified to deal with this more general situation.

In the L_∞ language used in Chapter 3, this more general situation describes a family of curved L_∞ algebras over the base dg ring $\mathscr{A}$ with the property that the curving vanishes modulo the nilpotent ideal $\mathscr{I}$.

Recall that ordinary (not curved) L_∞ algebras correspond to formal pointed moduli problems. These curved L_∞ algebras correspond to families of formal moduli problems over $\mathscr{A}$ which are pointed modulo $\mathscr{I}$.

7.5 The theorem on quantization

Let M be a manifold, and suppose we have a family of classical BV theories on M over a nilpotent dg manifold $(X, \mathscr{A})$. Suppose that the space of fields on M is the $\mathscr{A}$-module $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})$ be the dg $\mathscr{A}$-module of local functionals with differential $Q + \{I, -\}$.

Given a cochain complex C, we denote the Dold-Kan simplicial set associated to C by $\mathrm{DK}(C)$. Its n-simplices are the closed, degree 0 elements of $C \otimes \Omega^*(\triangle^n)$.

7.5.0.1 Theorem. *All of the simplicial sets $\mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan complexes and $\mathscr{T}^{(\infty)}(\mathscr{E}, I)$. The maps $p : \mathscr{T}^{(n+1)}(\mathscr{E}, I) \to \mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan fibrations.*

Further, there is a homotopy fiber diagram of simplicial sets

$$\begin{tikzcd}
\mathscr{T}^{(n+1)}(\mathscr{E}, I) \arrow[r] \arrow[d, "p"'] & 0 \arrow[d] \\
\mathscr{T}^{(n)}(\mathscr{E}, I) \arrow[r, "O"] & \mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E})[1], Q + \{I, -\})
\end{tikzcd}$$

where O is the "obstruction map."

In more prosaic terms, the second part of the theorem says the following. If $\alpha \in \mathscr{T}^{(n)}(\mathscr{E}, I)[0]$ is a zero-simplex of $\mathscr{T}^{(n)}(\mathscr{E}, I)$, then there is an obstruction $O(\alpha) \in \mathscr{O}_{loc}(\mathscr{E})$. This obstruction is a closed degree 1 element. The simplicial set $p^{-1}(\alpha) \in \mathscr{T}^{(n+1)}(\mathscr{E}, I)$ of extensions of α to the next order in $\hbar$ is homotopy equivalent to the simplicial set of ways of making $O(\alpha)$ exact. In particular, if the cohomology class $[O(\alpha)] \in H^1(\mathscr{O}_{loc}(E), Q + \{I, -\})$ is non-zero, then α does not admit a lift to the next order in $\hbar$. If this cohomology class is zero, then the simplicial set of possible lifts is a torsor for the simplicial Abelian group $\mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E}))[1]$.

Note also that a first order deformation of the classical field theory $(\mathscr{E}, Q, I)$ is given by a closed degree 0 element of $\mathscr{O}_{loc}(\mathscr{E})$. Further, two such first order deformations are equivalent if they are cohomologous. Thus, this theorem tells us that the moduli space of QFTs is "the same size" as the moduli space of classical field theories: at each order in $\hbar$, the data needed to describe a QFT is a local action functional.

The first part of the theorem says can be interpreted as follows. A Kan simplicial set can be thought of as an "infinity-groupoid." Since we can consider families of theories over arbitrary nilpotent dg manifolds, we can consider $\mathscr{T}^{\infty}(\mathscr{E}, I)$ as a functor from the category of nilpotent dg manifolds to that of Kan complexes, or infinity-groupoids. Thus, the space of theories forms something like a "derived stack," following Toën (2009); Lurie (n.d.).

This theorem also tells us in what sense the notion of "theory" is independent of the choice of gauge fixing operator. The simplicial set $\mathscr{T}^{(0)}(\mathscr{E}, I)$ is the simplicial set $\mathscr{GF}$ of gauge fixing operators. Since the map

$$\mathscr{T}^{(\infty)}(\mathscr{E}, I) \to \mathscr{T}^{(0)}(\mathscr{E}, I) = \mathscr{GF}$$

is a fibration, a path between two gauge fixing conditions Q_0^{GF} and Q_1^{GF} leads to a homotopy between the corresponding fibers, and thus to an equivalence between the ∞-groupoids of theories defined using Q_0^{GF} and Q_1^{GF}.

As we mentioned several times, there is often a natural contractible simplicial set mapping to the simplicial set $\mathscr{GF}$ of gauge fixing operators. Thus, $\mathscr{GF}$ often has a canonical "homotopy point". From the homotopical point of view, having a homotopy point is just as good as having an actual point: if $S \to \mathscr{GF}$ is a map out of a contractible simplicial set, then the fibers in $\mathscr{T}^{(\infty)}$ above any point in S are canonically homotopy equivalent.

8

The observables of a quantum field theory

8.1 Free fields

Before we give our general construction of the factorization algebra associated to a quantum field theory, we will give the much easier construction of the factorization algebra for a free field theory.

Let us recall the definition of a free BV theory.

8.1.0.1 Definition. *A free BV theory on a manifold M consists of the following data:*

(i) *a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that has finite rank;*
(ii) *an antisymmetric map of vector bundles $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ of degree -1 that is fiberwise nondegenerate. It induces a symplectic pairing on compactly supported smooth sections $\mathscr{E}_c$ of E:*

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(iii) *a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.*

Remark: When we consider deforming free theories into interacting theories, we will need to assume the existence of a "gauge fixing operator": this is a degree -1 operator $Q^{GF} : \mathscr{E} \to E$ such that $[Q, Q^{GF}]$ is a generalized Laplacian in the sense of Berline et al. (1992). $\Diamond$

On any open set $U \subset M$, the commutative dg algebra of classical observables supported in U is

$$\mathrm{Obs}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}^\vee(U)), Q),$$

where

$$\mathscr{E}^\vee(U) = \overline{\mathscr{E}}^!_c(U)$$

denotes the distributions dual to $\mathscr{E}$ with compact support in U and Q is the derivation given by extending the natural action of Q on the distributions.

In section 5.3 we constructed a sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}^!_c(U)), Q)$$

defined as the symmetric algebra on the compactly-supported smooth (rather than distributional) sections of the bundle $E^!$. We showed that the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$ is a weak equivalence of factorization algebras. Further, $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a Poisson bracket of cohomological degree 1, defined on the generators by the natural pairing

$$\mathscr{E}^!_c(U) \widehat{\otimes}_\pi \mathscr{E}^!_c(U) \to \mathbb{R},$$

which arises from the dual pairing on $\mathscr{E}_c(U)$. In this section we will show how to construct a quantization of the P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$.

8.1.1 The Heisenberg algebra construction

Our quantum observables on an open set U will be built from a certain Heisenberg Lie algebra.

Recall the usual construction of a Heisenberg algebra. If V is a symplectic vector space, viewed as an abelian Lie algebra, then the Heisenberg algebra $\mathrm{Heis}(V)$ is the central extension

$$0 \to \mathbb{C} \cdot \hbar \to \mathrm{Heis}(V) \to V$$

whose bracket is $[x, y] = \hbar \langle x, y \rangle$.

Since the element $\hbar \in \mathrm{Heis}(V)$ is central, the algebra $\widehat{U}(\mathrm{Heis}(V))$ is an algebra over $\mathbb{C}[[\hbar]]$, the completed universal enveloping algebra of the Abelian Lie algebra $\mathbb{C} \cdot \hbar$.

In quantum mechanics, this Heisenberg construction typically appears in the study of systems with quadratic Hamiltonians. In this context, the space V can be viewed in two ways. Either it is the space of solutions to the equations of motion, which is a linear space because we are dealing with a free field theory; or it is the space of linear observables dual to the space of solutions to the equations of motion. The natural symplectic pairing on V gives an isomorphism between these descriptions. The algebra $\widehat{U}(\mathrm{Heis}(V))$ is then the algebra of non-linear observables.

Our construction of the quantum observables of a free field theory will be formally very similar. We will start with a space of linear observables, which (after a shift) is a cochain complex with a symplectic pairing of cohomological degree 1. Then, instead of applying the usual universal enveloping algebra construction, we will take Chevalley-Eilenberg chain complex, whose cohomology is the Lie algebra homology.[1] This fits with our operadic philosophy: Chevalley-Eilenberg chains are the E_0 analog of the universal enveloping algebra.

8.1.2 The basic homological construction

Let us start with a 0-dimensional free field theory. Thus, let V be a cochain complex equipped with a symplectic pairing of cohomological degree -1. We will think of V as the space of fields of our theory. The space of linear observables of our theory is $V^\vee$; the Poisson bracket on $\mathscr{O}(V)$ induces a symmetric pairing of degree 1 on $V^\vee$. We will construct the space of all observables from a Heisenberg Lie algebra built on $V^\vee[-1]$, which has a symplectic pairing $\langle -, - \rangle$ of degree -1. Note that there is an isomorphism $V \cong V^\vee[-1]$ compatible with the pairings on both sides.

8.1.2.1 Definition. *The Heisenberg algebra* $\mathrm{Heis}(V)$ *is the Lie algebra central extension*

$$0 \to \mathbb{C} \cdot \hbar[-1] \to \mathrm{Heis}(V) \to V^\vee[-1] \to 0$$

[1] As usual, we always use gradings such that the differential has degree $+1$.

whose bracket is

$$[v + \hbar a, w + \hbar b] = \hbar \langle v, w \rangle$$

The element $\hbar$ labels the basis element of the center $\mathbb{C}[-1]$.

Putting the center in degree 1 may look strange, but it is necessary to do this in order to get a Lie bracket of cohomological degree 0.

Let $\widehat{C}_*(\mathrm{Heis}(V))$ denote the completion[2] of the Lie algebra chain complex of $\mathrm{Heis}(V)$, defined by the product of the spaces $\mathrm{Sym}^n \mathrm{Heis}(V)$, instead of their sum.

In this zero-dimensional toy model, the classical observables are

$$\mathrm{Obs}^{cl} = \mathscr{O}(V) = \prod_n \mathrm{Sym}^n(V^\vee).$$

This is a commutative dg algebra equipped with the Poisson bracket of degree 1 arising from the pairing on V. Thus, $\mathscr{O}(V)$ is a P_0 algebra.

8.1.2.2 Lemma. *The completed Chevalley-Eilenberg chain complex $\widehat{C}_*(\mathrm{Heis}(V))$ is a BD algebra (recall appendix I.A.3.2) that quantizes the P_0 algebra $\mathscr{O}(V)$.*

Proof The completed Chevalley-Eilenberg complex for $\mathrm{Heis}(V)$ has the completed symmetric algebra $\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1])$ as its underlying graded vector space. Note that

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1]) = \mathrm{Sym}(V^\vee \oplus \mathbb{C} \cdot \hbar) = \widehat{\mathrm{Sym}}(V^\vee)[[\hbar]],$$

so that $\widehat{C}_*(\mathrm{Heis}(V))$ is a flat $\mathbb{C}[[\hbar]]$ module which reduces to $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$. The Chevalley-Eilenberg chain complex $\widehat{C}_*(\mathrm{Heis}(V))$ inherits a product, corresponding to the natural product on the symmetric algebra $\widehat{\mathrm{Sym}}(\mathrm{Heis}(V)[1])$. Further, it has a natural Poisson bracket of cohomological degree 1 arising from the Lie bracket on $\mathrm{Heis}(V)$, extended to be a derivation of $\widehat{C}_*(\mathrm{Heis}(V))$. Note that, since $\mathbb{C} \cdot \hbar[-1]$ is central in $\mathrm{Heis}(V)$, this Poisson bracket reduces to the given Poisson bracket on $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$.

In order to prove that we have a BD quantization, it remains to verify

[2] One doesn't need to take the completed Lie algebra chain complex. We do this to be consistent with our discussion of the observables of interacting field theories, where it is essential to complete.

that, although the commutative product on $\widehat{C}_*(\mathrm{Heis}(V))$ is not compatible with the product, it satisfies the BD axiom:

$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{|a|} a \cdot (\mathrm{d}b) + \hbar\{a, b\}.$$

This follows by definition. □

8.1.3 Cosheaves of Heisenberg algebras

Next, let us give the analog of this construction for a general free BV theory E on a manifold M. As above, our classical observables are defined by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = \widehat{\mathrm{Sym}}\, \mathscr{E}_c^!(U)$$

which has a Poisson bracket arising from the pairing on $\mathscr{E}_c^!(U)$. Recall that this is a factorization algebra.

To construct the quantum theory, we define, as above, a Heisenberg algebra $\mathrm{Heis}(U)$ as a central extension

$$0 \to \mathbb{C}[-1] \cdot \hbar \to \mathrm{Heis}(U) \to \mathscr{E}_c^!(U)[-1] \to 0.$$

Note that $\mathrm{Heis}(U)$ is a pre-cosheaf of Lie algebras. The bracket in this Heisenberg algebra arises from the pairing on $\mathscr{E}_c^!(U)$.

We then define the quantum observables by

$$\mathrm{Obs}^q(U) = \widehat{C}_*(\mathrm{Heis}(U)).$$

The underlying cochain complex is, as before,

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(U)[1])$$

where the completed symmetric algebra is defined (as always) using the completed tensor product.

8.1.3.1 Proposition. *Sending U to $\mathrm{Obs}^q(U)$ defines a BD factorization algebra in the category of differentiable pro-cochain complexes over $\mathbb{R}[[\hbar]]$, which quantizes $\mathrm{Obs}^{cl}(U)$.*

Proof First, we need to define the filtration on $\mathrm{Obs}^q(U)$ making it into a differentiable pro-cochain complex. The filtration is defined, in the identification

$$\mathrm{Obs}^q(U) = \widehat{\mathrm{Sym}}\,\mathscr{E}_c^!(U)[[\hbar]]$$

by saying

$$F^n \mathrm{Obs}^q(U) = \prod_k \hbar^k \mathrm{Sym}^{\geq n-2k} \mathscr{E}_c^!(U).$$

This filtration is engineered so that the $F^n \mathrm{Obs}^q(U)$ is a subcomplex of $\mathrm{Obs}^q(U)$.

It is immediate that Obs^q is a BD pre-factorization algebra quantizing $\mathrm{Obs}^{cl}(U)$. The fact that it is a factorization algebra follows from the fact that $\mathrm{Obs}^{cl}(U)$ is a factorization algebra, and then a simple spectral sequence argument. (A more sophisticated version of this spectral sequence argument, for interacting theories, is given in section 8.6.) $\square$

8.2 The BD algebra of global observables

In this section, we will try to motivate our definition of a quantum field theory from the point of view of homological algebra. All of the constructions we will explain will work over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, but to keep the notation simple we will not normally mention the base ring $\mathscr{A}$.

Thus, suppose that $(\mathscr{E}, I, Q, \langle -, - \rangle)$ is a classical field theory on a manifold M. We have seen (Chapter 5, section 5.2) how such a classical field theory gives immediately a commutative factorization algebra whose value on an open subset is

$$\mathrm{Obs}^{cl}(U) = (\mathscr{O}(\mathscr{E}(U)), Q + \{I, -\}).$$

Further, we saw that there is a P_0 sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\mathscr{O}_{sm}(\mathscr{E}(U)), Q + \{I, -\}).$$

In particular, we have a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ of global sections of this P_0 algebra. We can think of $\widetilde{\mathrm{Obs}}^{cl}(M)$ as the algebra of functions on the derived space of solutions to the Euler-Lagrange equations.

In this section we will explain how a quantization of this classical field theory will give a quantization (in a homotopical sense) of the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ into a BD algebra $\mathrm{Obs}^q(M)$ of global observables. This BD algebra has some locality properties, which we will exploit later to show that $\mathrm{Obs}^q(M)$ is indeed the global sections of a factorization algebra of quantum observables.

In the case when the classical theory is the cotangent theory to some formal elliptic moduli problem $B\mathcal{L}$ on M (encoded in an elliptic L_∞ algebra $\mathcal{L}$ on M), there is a particularly nice class of quantizations, which we call cotangent quantizations. Cotangent quantizations have a very clear geometric interpretation: they are locally-defined volume forms on the sheaf of formal moduli problems defined by $\mathcal{L}$.

8.2.1 The BD algebra associated to a parametrix

Suppose we have a quantization of our classical field theory (defined with respect to some gauge fixing condition, or family of gauge fixing conditions). Then, for every parametrix Φ, we have seen how to construct a cohomological degree 1 operator

$$\triangle_\Phi : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

and a Poisson bracket

$$\{-,-\}_\Phi : \mathscr{O}(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

such that $\mathscr{O}(\mathscr{E})[[\hbar]]$, with the usual product, with bracket $\{-,-\}_\Phi$ and with differential $Q + \hbar\triangle_\Phi$, forms a BD algebra.

Further, since the effective interaction $I[\Phi]$ satisfies the quantum master equation, we can form a new BD algebra by adding $\{I[\Phi],-\}_\Phi$ to the differential of $\mathscr{O}(\mathscr{E})[[\hbar]]$.

8.2.1.1 Definition. *Let* $\mathrm{Obs}^q_\Phi(M)$ *denote the BD algebra*

$$\mathrm{Obs}^q_\Phi(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi\right),$$

with bracket $\{-,-\}_\Phi$ *and the usual product.*

Remark: Note that $I[\Phi]$ is not in $\mathscr{O}(\mathscr{E})[[\hbar]]$, but rather in $\mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$.

However, as we remarked earlier in 7.2.8, the bracket

$$\{I[\Phi],-\}_\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

is well-defined. $\Diamond$

Remark: Note that we consider $\mathrm{Obs}^q_\Phi(M)$ as a BD algebra valued in the multicategory of differentiable pro-cochain complexes (see appendix I.C). This structure includes a filtration on $\mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]]$. The filtration is defined by saying that

$$F^n \mathscr{O}(\mathscr{E})[[\hbar]] = \prod_i \hbar^i \operatorname{Sym}^{\geq (n-2i)}(\mathscr{E}^\vee);$$

it is easily seen that the differential $Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi$ preserves this filtration. $\Diamond$

We will show that for varying Φ, the BD algebras $\mathrm{Obs}^q_\Phi(M)$ are canonically weakly equivalent. Moreover, we will show that there is a canonical weak equivalence of P_0 algebras

$$\mathrm{Obs}^q_\Phi(M) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \simeq \widetilde{\mathrm{Obs}}^{cl}(M).$$

To show this, we will construct a family of BD algebras over the dg base ring of forms on a certain contractible simplicial set of parametrices that restricts to $\mathrm{Obs}^q_\Phi(M)$ at each vertex.

Before we get into the details of the construction, however, let us say something about how this result allows us to interpret the definition of a quantum field theory.

A quantum field theory gives a BD algebra for each parametrix. These BD algebras are all canonically equivalent. Thus, at first glance, one might think that the data of a QFT is entirely encoded in the BD algebra for a single parametrix. However, this does not take account of a key part of our definition of a field theory, that of *locality*.

The BD algebra associated to a parametrix Φ has underlying commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$, equipped with a differential which we temporarily denote

$$\mathrm{d}_\Phi = Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi.$$

If $K \subset M$ is a closed subset, we have a restriction map

$$\mathscr{E} = \mathscr{E}(M) \to \mathscr{E}(K),$$

where $\mathscr{E}(K)$ denotes germs of smooth sections of the bundle E on K. There is a dual map on functionals $\mathscr{O}(\mathscr{E}(K)) \to \mathscr{O}(\mathscr{E})$. We say a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is *supported on K* if it is in the image of this map.

As $\Phi \to 0$, the effective interaction $I[\Phi]$ and the BV Laplacian $\triangle_\Phi$ become more and more local (i.e., their support gets closer to the small diagonal). This tells us that, for very small Φ, the operator d_Φ only increases the support of a functional in $\mathscr{O}(\mathscr{E})[[\hbar]]$ by a small amount. Further, by choosing Φ to be small enough, we can increase the support by an arbitrarily small amount.

Thus, a quantum field theory is

(i) A family of BD algebra structures on $\mathscr{O}(\mathscr{E})[[\hbar]]$, one for each parametrix, which are all homotopic (and which all have the same underlying graded commutative algebra).
(ii) The differential d_Φ defining the BD structure for a parametrix Φ increases support by a small amount if Φ is small.

This property of d_Φ for small Φ is what will allow us to construct a factorization algebra of quantum observables. If d_Φ did not increase the support of a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ at all, the factorization algebra would be easy to define: we would just set $\mathrm{Obs}^q(U) = \mathscr{O}(\mathscr{E}(U))[[\hbar]]$, with differential d_Φ. However, because d_Φ does increase support by some amount (which we can take to be arbitrarily small), it takes a little work to push this idea through.

Remark: The precise meaning of the statement that d_Φ increases support by an arbitrarily small amount is a little delicate. Let us explain what we mean. A functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ has an infinite Taylor expansion of the form $f = \sum \hbar^i f_{i,k}$, where $f_{i,k} : \mathscr{E}^{\widehat{\otimes}_\pi k} \to \mathbb{C}$ is a symmetric linear map. We let $\mathrm{Supp}_{\leq(i,k)} f$ be the unions of the supports of $f_{r,s}$ where $(r,s) \leq (i,k)$ in the lexicographical ordering. If $K \subset M$ is a subset, let $\Phi^n(K)$ denote the subset obtained by convolving n times with $\mathrm{Supp}\,\Phi \subset M^2$. The differential d_Φ has the following property: there are constants $c_{i,k} \in \mathbb{Z}_{>0}$ of a purely combinatorial nature (independent of the theory we are considering) such that, for all $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$,

$$\mathrm{Supp}_{\leq(i,k)}\, \mathrm{d}_\Phi f \subset \Phi^{c_{i,k}}(\mathrm{Supp}_{\leq(i,k)}\, f).$$

Thus, we could say that d_Φ increase support by an amount linear in $\operatorname{Supp}\Phi$. We will use this concept in the main theorem of this chapter. $\Diamond$

8.2.2

Let us now turn to the construction of the equivalences between $\mathrm{Obs}^q_\Phi(M)$ for varying parametrices Φ. The first step is to construct the simplicial set $\mathscr{P}$ of parametrices; we will then construct a BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$ over the base dg ring $\Omega^*(\mathscr{P})$, which we define below.

Let

$$V \subset C^\infty(M \times M, E \boxtimes E) = \mathscr{E}\widehat{\otimes}_\pi \mathscr{E}$$

denote the subspace of those elements which are cohomologically closed and of degree 1, symmetric, and have proper support.

Note that the set of parametrices has the structure of an affine space for V: if Φ, Ψ are parametrices, then

$$\Phi - \Psi \in V$$

and, conversely, if Φ is a parametrix and $A \in V$, then $\Phi + A$ is a new parametrix.

Let $\mathscr{P}$ denote the simplicial set whose n-simplices are affine-linear maps from $\triangle^n$ to the affine space of parametrices. It is clear that $\mathscr{P}$ is contractible.

For any vector space V, let $V_\triangle$ denote the simplicial set whose k-simplices are affine linear maps $\triangle^k \to V$. For any convex subset $U \subset V$, there is a sub-simplicial set $U_\triangle \subset V_\triangle$ whose k-simplices are affine linear maps $\triangle^k \to U$. Note that $\mathscr{P}$ is a sub-simplicial set of $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}_\triangle$, corresponding to the convex subset of parametrices inside $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$.

Let $\mathscr{C}\mathscr{P}[0] \subset \overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$ denote the cone on the affine subspace of parametrices, with vertex the origin $\overline{0}$. An element of $\mathscr{C}\mathscr{P}[0]$ is an element of $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$ of the form $t\Phi$, where Φ is a parametrix and $t \in [0,1]$. Let $\mathscr{C}\mathscr{P}$ denote the simplicial set whose k-simplices are affine linear maps to $\mathscr{C}\mathscr{P}[0]$.

Recall that the simplicial de Rham algebra $\Omega^*_\triangle(S)$ of a simplicial set S is defined as follows. Any element $\omega \in \Omega^i_\triangle(S)$ consists of an i-form

$$\omega(\phi) \in \Omega^i(\triangle^k)$$

for each k-simplex $\phi : \triangle^k \to S$. If $f : \triangle^k \to \triangle^l$ is a face or degeneracy map, then we require that

$$f^*\omega(\phi) = \omega(\phi \circ f).$$

The main results of this section are as follows.

8.2.2.1 Theorem. *There is a BD algebra* $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *over* $\Omega^*(\mathscr{P})$ *which, at each* 0*-simplex* Φ*, is the BD algebra* $\mathrm{Obs}^q_\Phi(M)$ *discussed above.*

The underlying graded commutative algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *is* $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\mathscr{P})[[\hbar]]$.

For every open subset $U \subset M \times M$*, let* $\mathscr{P}_U$ *denote the parametrices whose support is in* U*. Let* $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *denote the restriction of* $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *to* U*. The differential on* $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *increases support by an amount linear in* U *(in the sense explained precisely in the remark above).*

The bracket $\{-,-\}_{\mathscr{P}_U}$ *on* $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *is also approximately local, in the following sense. If* $O_1, O_2 \in \mathrm{Obs}^q_{\mathscr{P}_U}(M)$ *have the property that*

$$\mathrm{Supp}\, O_1 \times \mathrm{Supp}\, O_2 \cap U = \emptyset \in M \times M,$$

then $\{O_1, O_2\}_{\mathscr{P}_U} = 0$.

Further, there is a P_0 *algebra* $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ *over* $\Omega^*(\mathscr{C}\mathscr{P})$ *equipped with a quasi-isomorphism of* P_0 *algebras over* $\Omega^*(\mathscr{P})$,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\mathscr{P}} \simeq \mathrm{Obs}^q_{\mathscr{P}}(M) \text{ modulo } \hbar,$$

and with an isomorphism of P_0 *algebras,*

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\overline{0}} \cong \widetilde{\mathrm{Obs}}^{cl}(M),$$

where $\widetilde{\mathrm{Obs}}^{cl}(M)$ *is the* P_0 *algebra constructed in Chapter 5.*

The underlying commutative algebra of $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ *is* $\widetilde{\mathrm{Obs}}^{cl}(M) \otimes \Omega^*(\mathscr{C}\mathscr{P})$*, the differential on* $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ *increases support by an arbitrarily small amount, and the Poisson bracket on* $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ *is approximately local in the same sense as above.*

Proof We need to construct, for each k-simplex $\phi : \triangle^k \to \mathscr{P}$, a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$. We view the k-simplex as a subset of $\mathbb{R}^{k+1}$ by

$$\triangle^k := \left\{ (\lambda_0, \dots, \lambda_k) \subset [0,1]^{k+1} \ : \ \sum_i \lambda_i = 1 \right\}.$$

Since simplices in $\mathscr{P}$ are affine linear maps to the space of parametrices, the simplex ϕ is determined by $k+1$ parametrices $\Phi_0, \dots, \Phi_k$, with

$$\phi(\lambda_0, \dots, \lambda_k) = \sum_i \lambda_i \Phi_i$$

for $\lambda_i \in [0,1]$ and $\sum \lambda_i = 1$.

The graded vector space underlying our BD algebra is

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k).$$

The structure as a BD algebra will be encoded by an order two, $\Omega^*(\triangle^k)$-linear differential operator

$$\triangle_\phi : \mathrm{Obs}^q_\phi(M) \to \mathrm{Obs}^q_\phi(M).$$

We need to recall some notation in order to define this operator. Each parametrix Φ provides an order two differential operator $\triangle_\Phi$ on $\mathscr{O}(\mathscr{E})$, the BV Laplacian corresponding to Φ. Further, if Φ, Ψ are two parametrices, then the difference between the propagators $P(\Phi) - P(\Psi)$ is an element of $\mathscr{E} \otimes \mathscr{E}$, so that contracting with $P(\Phi) - P(\Psi)$ defines an order two differential operator $\partial_{P(\Phi)} - \partial_{P(\Psi)}$ on $\mathscr{O}(\mathscr{E})$. (This operator defines the infinitesimal version of the renormalization group flow from Ψ to Φ.) We have the equation

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = -\triangle_\Phi + \triangle_\Psi.$$

Note that although the operator $\partial_{P(\Phi)}$ is only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$, because $P(\Phi) \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$, the difference $\partial_{P(\Phi)}$ and $\partial_{P(\Psi)}$ is nonetheless well-defined on $\mathscr{O}(\mathscr{E})$ because $P(\Phi) - P(\Psi) \in \mathscr{E} \otimes \mathscr{E}$.

The BV Laplacian $\triangle_\phi$ associated to the k-simplex $\phi : \triangle^k \to \mathscr{P}$ is defined by the formula

$$\triangle_\phi = \sum_{i=0}^k \lambda_i \triangle_{\Phi_i} - \sum_{i=0}^k \mathrm{d}\lambda_i \partial_{P(\Phi_i)},$$

where the $\lambda_i \in [0,1]$ are the coordinates on the simplex $\triangle^k$ and, as above, the Φ_i are the parametrices associated to the vertices of the simplex ϕ.

It is not entirely obvious that this operator makes sense as a linear map $\mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$, because the operators $\partial_{P(\Phi)}$ are only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$. However, since $\sum \mathrm{d}\lambda_i = 0$, we have

$$\sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} = \sum \mathrm{d}\lambda_i (\partial_{P(\Phi_i)} - \partial_{P(\Phi_0)}),$$

and the right hand side is well defined.

It is immediate that $\triangle_\phi^2 = 0$. If we denote the differential on the classical observables $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^n)$ by $Q + \mathrm{d}_{dR}$, we have

$$[Q + \mathrm{d}_{dR}, \triangle_\phi] = 0.$$

To see this, note that

$$\begin{aligned}
[Q + \mathrm{d}_{dR}, \triangle_\phi] &= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} + \sum \mathrm{d}\lambda_i [Q, \partial_{\Phi_i} - \partial_{\Phi_0}] \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} - \sum \mathrm{d}\lambda_i (\triangle_{\Phi_i} - \triangle_{\Phi_0}) \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_0} \\
&= 0,
\end{aligned}$$

where we use various identities from earlier.

The operator $\triangle_\phi$ defines, in the usual way, an $\Omega^*(\triangle^k)$-linear Poisson bracket $\{-,-\}_\phi$ on $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$.

We have effective action functionals $I[\Psi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Ψ. Let

$$I[\phi] = I[\sum \lambda_i \Phi_i] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes C^\infty(\triangle^k).$$

The renormalization group equation tells us that $I[\sum \lambda_i \Phi_i]$ is smooth (actually polynomial) in the λ_i.

We define the structure of BD algebra on the graded vector space

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k)$$

as follows. The product is the usual one; the bracket is $\{-,-\}_\phi$, as above; and the differential is

$$Q + \mathrm{d}_{dR} + \hbar \triangle_\phi + \{I[\phi], -\}_\phi.$$

We need to check that this differential squares to zero. This is equivalent to the quantum master equation

$$(Q + \mathrm{d}_{dR} + \hbar \triangle_\phi) e^{I[\phi]/\hbar} = 0.$$

This holds as a consequence of the quantum master equation and renormalization group equation satisfied by $I[\phi]$. Indeed, the renormalization group equation tells us that

$$e^{I[\phi]/\hbar} = \exp\left(\hbar \sum \lambda_i \left(\partial_{P\Phi_i)} - \partial_{P(\Phi_0)}\right)\right) e^{I[\Phi_0]/\hbar}.$$

Thus,

$$\mathrm{d}_{dR} e^{I[\phi]/\hbar} = \hbar \sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} e^{I[\phi]/\hbar}$$

The QME for each $I[\sum \lambda_i \Phi_i]$ tells us that

$$(Q + \hbar \sum \lambda_i \triangle_{\Phi_i}) e^{I[\phi]/\hbar} = 0.$$

Putting these equations together with the definition of $\triangle_\phi$ shows that $I[\phi]$ satisfies the QME.

Thus, we have constructed a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$ for every simplex $\phi : \triangle^k \to \mathscr{P}$. It is evident that these BD algebras are compatible with face and degeneracy maps, and so glue together to define a BD algebra over the simplicial de Rham complex $\Omega^*_\triangle(\mathscr{P})$ of $\mathscr{P}$.

Let ϕ be a k-simplex of $\mathscr{P}$, and let

$$\mathrm{Supp}(\phi) = \cup_{\lambda \in \triangle^k} \mathrm{Supp}(\sum \lambda_i \Phi_i).$$

We need to check that the bracket $\{O_1, O_2\}_\phi$ vanishes for observables O_1, O_2 such that $(\mathrm{Supp}\, O_1 \times \mathrm{Supp}) O_2 \cap \mathrm{Supp}\, \phi = \emptyset$. This is immediate, because the bracket is defined by contracting with tensors in $\mathscr{E} \otimes \mathscr{E}$ whose supports sit inside $\mathrm{Supp}\, \phi$.

Next, we need to verify that, on a k-simplex ϕ of $\mathscr{P}$, the differential $Q + \{I[\phi], -\}_\phi$ increases support by an amount linear in $\mathrm{Supp}(\phi)$. This follows from the support properties satisfied by $I[\Phi]$ (which are detailed in the definition of a quantum field theory, definition 7.2.9.1).

It remains to construct the P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$. The construction is almost identical, so we will not give all details. A zero-simplex of $\mathscr{C}\mathscr{P}$ is an element of $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ of the form $\Psi = t\Phi$, where Φ is a parametrix. We can use the same formulae we used for parametrices to construct a propagator $P(\Psi)$ and Poisson bracket $\{-,-\}_\Psi$ for each $\Psi \in \mathscr{C}\mathscr{P}$. The kernel defining the Poisson bracket $\{-,-\}_\Psi$ need not be smooth. This means that the bracket $\{-,-\}_\Psi$ is only defined on the subspace $\mathscr{O}_{sm}(\mathscr{E})$ of functionals with smooth first derivative. In particular, if $\Psi = 0$ is the vertex of the cone $\mathscr{C}\mathscr{P}$, then $\{-,-\}_0$ is the Poisson bracket defined in Chapter 5 on $\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E})$.

For each $\Psi \in \mathscr{C}\mathscr{P}$, we can form a tree-level effective interaction

$$I_0[\Psi] = W_0\left(P(\Psi), I\right) \in \mathscr{O}_{sm,P}(\mathscr{E}),$$

where $I \in \mathscr{O}_{loc}(\mathscr{E})$ is the classical action functional we start with. There are no difficulties defining this expression because we are working at tree-level and using functionals with smooth first derivative. If $\Psi = 0$, then $I_0[0] = I$.

The P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$ is defined in almost exactly the same way as we defined the BD algebra over $\Omega^*_{\mathscr{P}}$. The underlying commutative algebra is $\mathscr{O}_{sm}(\mathscr{E}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. On a k-simplex ψ with vertices $\Psi_0, \ldots, \Psi_k$, the Poisson bracket is

$$\{-,-\}_\psi = \sum \lambda_i \{-,-\}_{\Psi_i} + \sum \mathrm{d}\lambda_i \{-,-\}_{P(\Psi_i)},$$

where $\{-,-\}_{P(\Psi_i)}$ is the Poisson bracket of cohomological degree 0 defined using the propagator $P(\Psi_i) \in \overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}$ as a kernel. If we let $I_0[\psi] = I_0[\sum \lambda_i \Psi_i]$, then the differential is

$$\mathrm{d}_\psi = Q + \{I_0[\psi], -\}_\psi.$$

The renormalization group equation and classical master equation satisfied by the $I_0[\Psi]$ imply that $\mathrm{d}_\psi^2 = 0$. If $\Psi = 0$, this P_0 algebra is clearly the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ constructed in Chapter 5. When restricted to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$, this P_0 algebra is the sub P_0 algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$ obtained by restricting to functionals with smooth first derivative; the inclusion

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M) \mid_{\mathscr{P}} \hookrightarrow \mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$$

is thus a quasi-isomorphism, using proposition 5.4.2.4 of Chapter 5. □

8.3 Global observables

In the next few sections, we will prove our quantization theorem. Our proof is by construction, associating a factorization algebra on M to a quantum field theory on M, in the sense of Costello (2011b). This construction is a quantization of the P_0 factorization algebra associated to the corresponding classical field theory.

More precisely, we will show the following.

8.3.0.1 Theorem. *For any quantum field theory on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$, there is a factorization algebra Obs^q on M, valued in the multicategory of differentiable pro-cochain complexes flat over $\mathscr{A}[[\hbar]]$.*

There is an isomorphism of factorization algebras

$$\mathrm{Obs}^q \otimes_{\mathscr{A}[[\hbar]]} \mathscr{A} \cong \mathrm{Obs}^{cl}$$

between Obs^q modulo $\hbar$ and the commutative factorization algebra Obs^{cl}.

8.3.1

So far we have constructed a BD algebra $\mathrm{Obs}^q_\Phi(M)$ for each parametrix Φ; these BD algebras are all weakly equivalent to each other. In this section we will define a cochain complex $\mathrm{Obs}^q(M)$ of global observables which is independent of the choice of parametrix. For every open subset $U \subset M$, we will construct a subcomplex $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U. The complexes $\mathrm{Obs}^q(U)$ will form our factorization algebra.

Thus, suppose we have a quantum field theory on M, with space of fields $\mathscr{E}$ and effective action functionals $\{I[\Phi]\}$, one for each parametrix (as explained in section 7.2).

An *observable* for a quantum field theory (that is, an element of the cochain complex $\mathrm{Obs}^q(M)$) is simply a first-order deformation $\{I[\Phi] + \delta O[\Phi]\}$ of the family of effective action functionals $I[\Phi]$, which satisfies a renormalization group equation but does not necessarily satisfy the locality axiom in the definition of a quantum field theory. Definition 8.3.1.3 makes this idea precise.

Remark: This definition is motivated by a formal argument with the path integral. Let $S(\phi)$ be the action functional for a field ϕ, and let $O(\phi)$ be another function of the field, describing a measurement that one could make. Heuristically, the expectation value of the observable is

$$\langle O \rangle = \frac{1}{Z_S} \int O(\phi) e^{-S(\phi)/\hbar} \mathscr{D}\phi,$$

where Z_S denotes the partition function, simply the integral without O. A formal manipulation shows that

$$\langle O \rangle = \frac{d}{d\delta} \frac{1}{Z_S} \int e^{(-S(\phi)+\hbar\delta O(\phi))/\hbar} \mathscr{D}\phi.$$

In other words, we can view O as a first-order deformation of the action functional S and compute the expectation value as the change in the partition function. Because the book Costello (2011b) gives an approach to the path integral that incorporates the BV formalism, we can define and compute expectation values of observables by exploiting the second description of $\langle O \rangle$ given above. ◊

Earlier we defined cochain complexes $\mathrm{Obs}^q_\Phi(M)$ for each parametrix. The underlying graded vector space of $\mathrm{Obs}^q_\Phi(M)$ is $\mathscr{O}(\mathscr{E})[[\hbar]]$; the differential on $\mathrm{Obs}^q_\Phi(M)$ is

$$\widehat{Q}_\Phi = Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi.$$

8.3.1.1 Definition. *Define a linear map*

$$W^\Phi_\Psi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

by requiring that, for an element $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ of cohomological degree i,

$$I[\Phi] + \delta W^\Phi_\Psi(f) = W\left(P(\Phi) - P(\Psi), I[\Psi] + \delta f\right)$$

where δ is a square-zero parameter of cohomological degree $-i$.

8.3.1.2 Lemma. *The linear map*

$$W^\Phi_\Psi : \mathrm{Obs}^q_\Psi(M) \to \mathrm{Obs}^q_\Phi(M)$$

is an isomorphism of differentiable pro-cochain complexes.

Proof The fact that W_Ψ^Φ intertwines the differentials $\widehat{Q}_\Phi$ and $\widehat{Q}_\Psi$ follows from the compatibility between the quantum master equation and the renormalization group equation described in Costello (2011b), Chapter 5 and summarized in section 7.2. It is not hard to verify that W_Ψ^Φ is a map of differentiable pro-cochain complexes. The inverse to W_Ψ^Φ is W_Φ^Ψ. □

8.3.1.3 Definition. *A* global observable O *of cohomological degree* i *is an assignment to every parametrix* Φ *of an element*

$$O[\Phi] \in \mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathcal{E})[[\hbar]]$$

of cohomological degree i *such that*

$$W_\Psi^\Phi O[\Psi] = O[\Phi].$$

If O *is an observable of cohomological degree* i*, we let* $\widehat{Q}O$ *be defined by*

$$\widehat{Q}(O)[\Phi] = \widehat{Q}_\Phi(O[\Phi]) = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar\triangle_\Phi O[\Phi].$$

This makes the space of observables into a differentiable pro-cochain complex, which we call $\mathrm{Obs}^q(M)$.

Thus, if $O \in \mathrm{Obs}^q(M)$ is an observable of cohomological degree i, and if δ is a square-zero parameter of cohomological degree $-i$, then the collection of effective interactions $\{I[\Phi] + \delta O[\Phi]\}$ satisfy most of the axioms needed to define a family of quantum field theories over the base ring $\mathbb{C}[\delta]/\delta^2$. The only axiom which is not satisfied is the locality axiom: we have not imposed any constraints on the behavior of the $O[\Phi]$ as $\Phi \to 0$.

8.4 Local observables

So far, we have defined a cochain complex $\mathrm{Obs}^q(M)$ of global observables on the whole manifold M. If $U \subset M$ is an open subset of M, we would like to isolate those observables which are "supported on U".

The idea is to say that an observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for sufficiently small parametrices, $O[\Phi]$ is supported on U. The precise definition is as follows.

8.4.0.1 Definition. *An observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for each $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, there exists a compact subset $K \subset U^k$ and a parametrix Φ, such that*

$$\operatorname{Supp} O_{i,k}[\Psi] \subset K$$

for all parametrices $\Psi \leq \Phi$.

Remark: Recall that $O_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathbb{C}$ is the kth term in the Taylor expansion of the coefficient of $\hbar^i$ of the functional $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$. $\Diamond$

Remark: As always, the definition works over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, even though we suppress this from the notation. In this generality, instead of a compact subset $K \subset U^k$ we require $K \subset U^k \times X$ to be a set such that the map $K \to X$ is proper. $\Diamond$

We let $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ be the sub-graded vector space of observables supported on U.

8.4.0.2 Lemma. $\mathrm{Obs}^q(U)$ *is a sub-cochain complex of* $\mathrm{Obs}^q(M)$*. In other words, if $O \in \mathrm{Obs}^q(U)$, then so is $\widehat{Q}O$.*

Proof The only thing that needs to be checked is the support condition. We need to check that, for each (i,k), there exists a compact subset K of U^k such that, for all sufficiently small Φ, $\widehat{Q}O_{i,k}[\Phi]$ is supported on K.

Note that we can write

$$\widehat{Q}O_{i,k}[\Phi] = QO_{i,k}[\Phi] + \sum_{\substack{a+b=i \\ r+s=k+2}} \{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_{\Phi} + \Delta_{\Phi} O_{i-1,k+2}[\Phi].$$

We now find a compact subset K for $\widehat{Q}O_{i,k}[\Phi]$. We know that, for each (i,k) and for all sufficiently small Φ, $O_{i,k}[\Phi]$ is supported on $\tilde{K}$, where $\tilde{K}$ is some compact subset of U^k. It follows that $QO_{i,k}[\Phi]$ is supported on $\tilde{K}$.

By making $\tilde{K}$ bigger, we can assume that for sufficiently small Φ, $O_{i-1,k+2}[\Phi]$ is supported on L, where L is a compact subset of U^{k+2} whose image in U^k, under every projection map, is in $\tilde{K}$. This implies that $\Delta_{\Phi} O_{i-1,k+2}[\Phi]$ is supported on $\tilde{K}$.

The locality condition for the effective actions $I[\Phi]$ implies that, by choosing Φ to be sufficiently small, we can make $I_{i,k}[\Phi]$ supported as close as we like to the small diagonal in M^k. It follows that, by choosing Φ to be sufficiently small, the support of $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ can be taken to be a compact subset of U^k. Since there are only a finite number of terms like $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ in the expression for $(\widehat{Q}O)_{i,k}[\Phi]$, we see that for Φ sufficiently small, $(\widehat{Q}O)_{i,k}[\Phi]$ is supported on a compact subset K of U^k, as desired. □

8.4.0.3 Lemma. $\mathrm{Obs}^q(U)$ *has a natural structure of differentiable pro-cochain complex space.*

Proof Our general strategy for showing that something is a differentiable vector space is to ensure that everything works in families over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. Thus, suppose that the theory we are working with is defined over $(X, \mathscr{A})$. If Y is a smooth manifold, we say a smooth map $Y \to \mathrm{Obs}^q(U)$ is an observable for the family of theories over $(X \times Y, \mathscr{A} \widehat{\otimes}_\pi C^\infty(Y))$ obtained by base-change along the map $X \times Y \to X$ (so this family of theories is constant over Y).

The filtration on $\mathrm{Obs}^q(U)$ (giving it the structure of pro-differentiable vector space) is inherited from that on $\mathrm{Obs}^q(M)$. Precisely, an observable $O \in \mathrm{Obs}^q(U)$ is in $F^k \mathrm{Obs}^q(U)$ if, for all parametrices Φ,

$$O[\Phi] \in \prod \hbar^i \,\mathrm{Sym}^{\geq (2k-i)} \,\mathscr{E}^\vee.$$

The renormalization group flow W_Φ^Ψ preserves this filtration.

So far we have verified that $\mathrm{Obs}^q(U)$ is a pro-object in the category of pre-differentiable cochain complexes. The remaining structure we need is a flat connection

$$\nabla : C^\infty(Y, \mathrm{Obs}^q(U)) \to \Omega^1(Y, \mathrm{Obs}^q(U))$$

for each manifold Y, where $C^\infty(Y, \mathrm{Obs}^q(U))$ is the space of smooth maps $Y \to \mathrm{Obs}^q(U)$.

This flat connection is equivalent to giving a differential on

$$\Omega^*(Y, \mathrm{Obs}^q(U)) = C^\infty(Y, \mathrm{Obs}^q(U)) \otimes_{C^\infty(Y)} \Omega^*(Y)$$

making it into a dg module for the dg algebra $\Omega^*(Y)$. Such a differential is provided by considering observables for the family of theories over the nilpotent dg manifold $(X \times Y, \mathscr{A} \widehat{\otimes}_\pi \Omega^*(Y))$, pulled back via the projection map $X \times Y \to Y$. □

8.5 Local observables form a prefactorization algebra

At this point, we have constructed the cochain complex $\mathrm{Obs}^q(M)$ of global observables of our factorization algebra. We have also constructed, for every open subset $U \subset M$, a sub-cochain complex $\mathrm{Obs}^q(U)$ of observables supported on U.

In this section we will see that the local quantum observables $\mathrm{Obs}^q(U)$ of a quantum field on a manifold M form a prefactorization algebra.

The definition of local observables makes it clear that they form a pre-cosheaf: there are natural injective maps of cochain complexes

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$$

if $U \subset U'$ is an open subset.

Let U, V be disjoint open subsets of M. The structure of prefactorization algebra on the local observables is specified by the pre-cosheaf structure mentioned above, and a bilinear cochain map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

These product maps need to be maps of cochain complexes which are compatible with the pre-cosheaf structure and with reordering of the disjoint opens. Further, they need to satisfy a certain associativity condition which we will verify.

8.5.1 Defining the product map

Suppose that $O \in \mathrm{Obs}^q(U)$ and $O' \in \mathrm{Obs}^q(V)$ are observables on U and V respectively. Note that $O[\Phi]$ and $O'[\Phi]$ are elements of the cochain complex

$$\mathrm{Obs}^q_\Phi(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]], \widehat{Q}_\Phi\right)$$

which is a BD algebra and so a commutative algebra (ignoring the differential, of course). (The commutative product is simply the usual product of functions on $\mathscr{E}$.) In the definition of the prefactorization product, we will use the product of $O[\Phi]$ and $O'[\Phi]$ taken in the commutative algebra $\mathcal{O}(\mathcal{E})$. This product will be denoted $O[\Phi] * O'[\Phi] \in \mathcal{O}(\mathcal{E})$.

Recall (see definition 8.3.1.1) that we defined a linear renormalization group flow operator W_{Φ}^{Ψ}, which is an isomorphism between the cochain complexes $\mathrm{Obs}^q_{\Phi}(M)$ and $\mathrm{Obs}^q_{\Psi}(M)$.

The main result of this section is the following.

8.5.1.1 Theorem. *For all observables $O \in \mathrm{Obs}^q(U)$, $O' \in \mathrm{Obs}^q(V)$, where U and V are disjoint, the limit*

$$\lim_{\Psi \to 0} W_{\Psi}^{\Phi}\left(O[\Psi] * O'[\Psi]\right) \in \mathscr{O}(\mathscr{E})[[\hbar]]$$

exists. Further, this limit satisfies the renormalization group equation, so that we can define an observable $m(O, O')$ by

$$m(O, O')[\Phi] = \lim_{\Psi \to 0} W_{\Psi}^{\Phi}\left(O[\Psi] * O'[\Psi]\right).$$

The map

$$\begin{aligned} \mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) &\mapsto \mathrm{Obs}^q(U \amalg V) \\ O \times O' &\mapsto m(O, O') \end{aligned}$$

is a smooth bilinear cochain map, and it makes Obs^q into a prefactorization algebra in the multicategory of differentiable pro-cochain complexes.

Proof We will show that, for each i, k, the Taylor term

$$W_{\Phi}^{\Psi}(O[\Phi] * O'[\Phi])_{i,k} : \mathscr{E}^{\otimes k} \to \mathbb{C}$$

is independent of Φ for Φ sufficiently small. This will show that the limit exists.

Note that

$$W_{\Gamma}^{\Psi}\left(W_{\Phi}^{\Gamma}\left(O[\Phi] * O'[\Phi]\right)\right) = W_{\Phi}^{\Psi}\left(O[\Phi] * O'[\Phi]\right).$$

Thus, to show that the limit $\lim_{\Phi \to 0} W_{\Phi}^{\Psi}\left(O[\Phi] * O'[\Phi]\right)$ is eventually constant, it suffices to show that, for all sufficiently small Φ, Γ satisfying $\Phi < \Gamma$,

$$W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi])_{i,k} = (O[\Gamma] * O'[\Gamma])_{i,k}.$$

This turns out to be an exercise in the manipulation of Feynman diagrams. In order to prove this, we need to recall a little about the Feynman diagram expansion of $W_{\Phi}^{\Gamma}(O[\Phi])$. (Feynman diagram expansions of the renormalization group flow are discussed extensively in Costello (2011b).)

We have a sum of the form

$$W_{\Phi}^{\Gamma}(O[\Phi])_{i,k} = \sum_{G} \frac{1}{|\mathrm{Aut}(G)|} w_G \left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi) \right).$$

The sum is over all connected graphs G with the following decorations and properties.

(i) The vertices v of G are labelled by an integer $g(v) \in \mathbb{Z}_{\geq 0}$, which we call the genus of the vertex.
(ii) The first Betti number of G, plus the sum of over all vertices of the genus $g(v)$, must be i (the "total genus").
(iii) G has one special vertex.
(iv) G has k tails (or external edges).

The weight $w_G \left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi) \right)$ is computed by the contraction of a collection of symmetric tensors. One places $O[\Phi]_{r,s}$ at the special vertex, when that vertex has genus r and valency s; places $I[\Phi]_{g,v}$ at every other vertex of genus g and valency v; and puts the propagator $P(\Gamma) - P(\Phi)$ on each edge.

Let us now consider $W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi])$. Here, we a sum over graphs with one special vertex, labelled by $O[\Phi] * O'[\Phi]$. This is the same as having two special vertices, one of which is labelled by $O[\Phi]$ and the other by $O'[\Phi]$. Diagrammatically, it looks like we have split the special vertex into two pieces. When we make this maneuver, we introduce possibly disconnected graphs; however, each connected component must contain at least one of the two special vertices.

Let us now compare this to the graphical expansion of

$$O[\Gamma] * O'[\Gamma] = W_{\Phi}^{\Gamma}(O[\Phi]) * W_{\Phi}^{\Gamma}(O'[\Phi]).$$

The Feynman diagram expansion of the right hand side of this expression consists of graphs with two special vertices, labelled by $O[\Phi]$ and $O'[\Phi]$ respectively (and, of course, any number of other vertices, labelled by $I[\Phi]$, and the propagator $P(\Gamma) - P(\Phi)$ labelling each edge). Further, the relevant graphs have precisely two connected components, each of which contains one of the special vertices.

Thus, we see that

$$W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi]) - W_{\Phi}^{\Gamma}(O[\Phi]) * W_{\Phi}^{\Gamma}(O'[\Phi]).$$

is a sum over *connected* graphs, with two special vertices, one labelled by $O[\Phi]$ and the other by $O'[\Phi]$. We need to show that the weight of such graphs vanish for Φ, Γ sufficiently small, with $\Phi < \Gamma$.

Graphs with one connected component must have a chain of edges connecting the two special vertices. (A chain is a path in the graph with no repeated vertices or edges.) For a graph G with "total genus" i and k tails, the length of any such chain is bounded by $2i + k$.

It is important to note here that we require a non-special vertex of genus zero to have valence at least three and a vertex of genus one to have valence at least one. See Costello (2011b) for more discussion. If we are considering a family of theories over some dg ring, we do allow bivalent vertices to be accompanied by nilpotent parameters in the base ring; nilpotence of the parameter forces there to be a global upper bound on the number of bivalent vertices that can appear. The argument we are presenting works with minor modifications in this case too.

Each step along a chain of edges involves a tensor with some support that depends on the choice of parametrices *Phi* and Γ. As we move from the special vertex O toward the other O', we extend the support, and our aim is to show that we can choose Φ and Γ to be small enough so that the support of the chain, excluding $O'[\Phi]$, is disjoint from the support of $O'[\Phi]$. The contraction of a distribution and function with disjoint supports is zero, so that the weight will vanish. We now make this idea precise.

Let us choose arbitrarily a metric on M. By taking Φ and Γ to be sufficiently small, we can assume that the support of the propagator on each edge is within ϵ of the diagonal in this metric, and ϵ can be taken to be as small as we like. Similarly, the support of the $I_{r,s}[\Gamma]$ labelling a vertex of genus r and valency s can be taken to be within $c_{r,s}\epsilon$ of the diagonal, where $c_{r,s}$ is a combinatorial constant depending only on r and s. In addition, by choosing Φ to be small enough we can ensure that the supports of $O[\Phi]$ and $O'[\Phi]$ are disjoint.

Now let G' denote the graph G with the special vertex for O' removed. This graph corresponds to a symmetric tensor whose support is within some distance $C_G\epsilon$ of the small diagonal, where C_G is a combinatorial constant depending on the graph G'. As the supports K and

K' (of O and O', respectively) have a finite distance d between them, we can choose ϵ small enough that $C_G\epsilon < d$. It follows that, by choosing Φ and Γ to be sufficiently small, the weight of any connected graph is obtained by contracting a distribution and a function which have disjoint support. The graph hence has weight zero.

As there are finitely many such graphs with total genus i and k tails, we see that we can choose Γ small enough that for any $\Phi < \Gamma$, the weight of all such graphs vanishes.

Thus we have proved the first part of the theorem and have produced a bilinear map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

It is a straightforward to show that this is a cochain map and satisfies the associativity and commutativity properties necessary to define a prefactorization algebra. The fact that this is a smooth map of differentiable pro-vector spaces follows from the fact that this construction works for families of theories over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. □

8.6 Local observables form a factorization algebra

We have seen how to define a prefactorization algebra Obs^q of observables for our quantum field theory. In this section we will show that this prefactorization algebra is in fact a factorization algebra. In the course of the proof, we show that modulo $\hbar$, this factorization algebra is isomorphic to Obs^{cl}.

8.6.0.1 Theorem. *The prefactorization algebra* Obs^q *of quantum observables has the following properties.*

(i) *It is, in fact, a factorization algebra.*
(ii) *There is an isomorphism*

$$\mathrm{Obs}^q \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathrm{Obs}^{cl}$$

between the reduction of the factorization algebra of quantum observables modulo $\hbar$*, and the factorization algebra of classical observables.*

8.6.1 Proof of the theorem

This theorem will be a corollary of a more technical proposition.

8.6.1.1 Proposition. *For any open subset $U \subset M$, filter $\mathrm{Obs}^q(U)$ by saying that the k-th filtered piece $G^k \mathrm{Obs}^q(U)$ is the sub $\mathbb{C}[[\hbar]]$-module consisting of those observables that are zero modulo $\hbar^k$. Note that this is a filtration by sub prefactorization algebras over the ring $\mathbb{C}[[\hbar]]$.*

Then, there is an isomorphism of prefactorization algebras (in differentiable pro-cochain complexes)

$$\mathrm{Gr}\,\mathrm{Obs}^q \simeq \mathrm{Obs}^{cl} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]].$$

This isomorphism makes $\mathrm{Gr}\,\mathrm{Obs}^q$ into a factorization algebra.

Remark: We can give $G^k \mathrm{Obs}^q(U)$ the structure of a pro-differentiable cochain complex, as follows. The filtration on $G^k \mathrm{Obs}^q(U)$ that defines the pro-structure is obtained by intersecting $G^k \mathrm{Obs}^q(U)$ with the filtration on $\mathrm{Obs}^q(U)$ defining the pro-structure. Then the inclusion $G^k \mathrm{Obs}^q(U) \hookrightarrow \mathrm{Obs}^q(U)$ is a cofibration of differentiable pro-vector spaces (see appendix I.C.4). $\Diamond$

Proof of the theorem, assuming the proposition We need to show that for every open U and for every Weiss cover $\mathfrak{U}$, the natural map

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^q) \to \mathrm{Obs}^q(U) \tag{†}$$

is a quasi-isomorphism of differentiable pro-cochain complexes.

The basic idea is that the filtration induces a spectral sequence for both $\check{C}(\mathfrak{U}, \mathrm{Obs}^q)$ and $\mathrm{Obs}^q(U)$, and we will show that the induced map of spectral sequences is an isomorphism on the first page. Because we are working with differentiable pro-cochain complexes, this is a little subtle. The relevant statements about spectral sequences in this context are developed in this context in Appendix I.C.4.

Note that $\check{C}(\mathfrak{U}, \mathrm{Obs}^q)$ is filtered by $\check{C}(\mathfrak{U}, G^k \mathrm{Obs}^q)$. The map (†) preserves the filtrations. Thus, we have a maps of inverse systems

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^q / G^k \mathrm{Obs}^q) \to \mathrm{Obs}^q(U) / G^k \mathrm{Obs}^q(U).$$

These inverse systems satisfy the properties of lemma I.C.4.4. Further, it is clear that

$$\mathrm{Obs}^q(U) = \varprojlim \mathrm{Obs}^q(U)/G^k \mathrm{Obs}^q(U).$$

We also have

$$\check{C}(\mathfrak{U}, \mathrm{Obs}^q) = \varprojlim \check{C}(\mathfrak{U}, \mathrm{Obs}^q / G^k \mathrm{Obs}^q).$$

This equality is less obvious, and uses the fact that the Čech complex is defined using the completed direct sum as described in Appendix I.C.4.

Using lemma I.C.4.4, we need to verify that the map

$$\check{C}(\mathfrak{U}, \mathrm{Gr}\,\mathrm{Obs}^q) \to \mathrm{Gr}\,\mathrm{Obs}^q(U)$$

is an equivalence. This follows from the proposition because $\mathrm{Gr}\,\mathrm{Obs}^q$ is a factorization algebra. □

Proof of the proposition The first step in the proof of the proposition is the following lemma.

8.6.1.2 Lemma. *Let* $\mathrm{Obs}^q_{(0)}$ *denote the prefactorization algebra of observables that are only defined modulo* $\hbar$. *Then there is an isomorphism*

$$\mathrm{Obs}^q_{(0)} \simeq \mathrm{Obs}^{cl}$$

of differential graded prefactorization algebras.

Proof of lemma Let $O \in \mathrm{Obs}^{cl}(U)$ be a classical observable. Thus, O is an element of the cochain complex $\mathscr{O}(\mathscr{E}(U))$ of functionals on the space of fields on U. We need to produce an element of $\mathrm{Obs}^q_{(0)}$ from O. An element of $\mathrm{Obs}^q_{(0)}$ is a collection of functionals $O[\Phi] \in \mathscr{O}(\mathscr{E})$, one for every parametrix Φ, satisfying a classical version of the renormalization group equation and an axiom saying that $O[\Phi]$ is supported on U for sufficiently small Φ.

Given an element

$$O \in \mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

we define an element

$$\{O[\Phi]\} \in \mathrm{Obs}^q_{(0)}$$

by the formula

$$O[\Phi] = \lim_{\Gamma \to 0} W_\Gamma^\Phi(O) \text{ modulo } \hbar.$$

The Feynman diagram expansion of the right hand side only involves trees, since we are working modulo $\hbar$. As we are only using trees, the limit exists. The limit is defined by a sum over trees with one special vertex, where each edge is labelled by the propagator $P(\Phi)$, the special vertex is labelled by O, and every other vertex is labelled by the classical interaction $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ of our theory.

The map

$$\mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^q_{(0)}(U)$$

we have constructed is easily seen to be a map of cochain complexes, compatible with the structure of prefactorization algebra present on both sides. (The proof is a variation on the argument in section 11, chapter 5 of Costello (2011b), about the scale 0 limit of a deformation of the effective interaction I modulo $\hbar$.)

A simple inductive argument on the degree shows this map is an isomorphism.

Because the construction works over an arbitrary nilpotent dg manifold, it is clear that these maps are maps of differentiable cochain complexes. □

The next (and most difficult) step in the proof of the proposition is the following lemma. We use it to work inductively with the filtration of quantum observables.

Let $\mathrm{Obs}^q_{(k)}$ denote the prefactorization algebra of observables defined modulo $\hbar^{k+1}$.

8.6.1.3 Lemma. *For all open subsets $U \subset M$, the natural quotient map of differentiable pro-cochain complexes*

$$\mathrm{Obs}^q_{(k+1)}(U) \to \mathrm{Obs}^q_{(k)}(U)$$

is a fibration of differentiable pro-cochain complexes (see appendix I.C.5.6 for the definition of a fibration). The fiber is isomorphic to $\mathrm{Obs}^{cl}(U)$.

Proof of lemma We give the set $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the lexicographical ordering, so that $(i,k) > (r,s)$ if $i > r$ or if $i = r$ and $k > s$.

We will let $\mathrm{Obs}^q_{\leq (i,k)}(U)$ be the quotient of $\mathrm{Obs}^q_{(i)}$ consisting of functionals

$$O[\Phi] = \sum_{(r,s) \leq (i,k)} \hbar^r O_{(r,s)}[\Phi]$$

satisfying the renormalization group equation and locality axiom as before, but where $O_{(r,s)}[\Phi]$ is only defined for $(r,s) \leq (i,k)$. Similarly, we will let $\mathrm{Obs}^q_{<(i,k)}(U)$ be the quotient where the $O_{(r,s)}[\Phi]$ are only defined for $(r,s) < (i,k)$.

We will show that the quotient map

$$\mathsf{q} : \mathrm{Obs}^q_{\leq (i,k)}(U) \to \mathrm{Obs}^q_{<(i,k)}(U)$$

is a fibration. The result will follow.

Recall what it means for a map $f : V \to W$ of differentiable cochain complexes to be a fibration. For X a manifold, let $C^\infty_X(V)$ denote the sheaf of cochain complexes on X of smooth maps to V. We say f is a fibration if for every manifold X, the induced map of sheaves $C^\infty_X(V) \to C^\infty_X(W)$ is surjective in each degree. Equivalently, we require that for all smooth manifolds X, every smooth map $X \to W$ lifts locally on X to a map to V.

Now, by definition, a smooth map from X to $\mathrm{Obs}^q(U)$ is an observable for the constant family of theories over the nilpotent dg manifold $(X, C^\infty(X))$. Thus, in order to show q is a fibration, it suffices to show the following. For any family of theories over a nilpotent dg manifold $(X, \mathscr{A})$, any open subset $U \subset M$, and any observable α in the $\mathscr{A}$-module $\mathrm{Obs}^q_{<(i,k)}(U)$, we can lift α to an element of $\mathrm{Obs}^q_{\leq (i,k)}(U)$ locally on X.

To prove this, we will first define, for every parametrix Φ, a map

$$L_\Phi : \mathrm{Obs}^q_{<(i,k)}(U) \to \mathrm{Obs}^q_{\leq (i,k)}(M)$$

with the property that the composed map

$$\mathrm{Obs}^q_{<(i,k)}(U) \xrightarrow{L_\Phi} \mathrm{Obs}^q_{\leq (i,k)}(M) \to \mathrm{Obs}^q_{<(i,k)}(M)$$

is the natural inclusion map. Then, for every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we will show that $L_\Phi(O)$ is supported on U, for sufficiently small parametrices Φ, so that $L_\Phi(O)$ provides the desired lift.

For

$$O \in \mathrm{Obs}^q_{<(i,k)}(U),$$

we define

$$L_\Phi(O) \in \mathrm{Obs}^q_{\leq(i,k)}(M)$$

by

$$L_\Phi(O)_{r,s}[\Phi] = \begin{cases} O_{r,s}[\Phi] & \text{if } (r,s) < (i,k) \\ 0 & \text{if } (r,s) = (i,k) \end{cases}.$$

For $\Psi \neq \Phi$, we obtain $L_\Phi(O)_{r,s}[\Psi]$ by the renormalization group flow from $L_\Phi(O)_{r,s}[\Phi]$. The RG flow equation tells us that if $(r,s) < (i,k)$, then

$$L_\Phi(O)_{r,s}[\Psi] = O_{r,s}[\Psi].$$

However, the RG equation for $L_\Phi(O)_{r,s}$ is non-trivial and tells us that

$$I_{i,k}[\Psi] + \delta\left(L_\Phi(O)_{i,k}[\Psi]\right) = W_{i,k}\left(P(\Psi) - P(\Phi), I[\Phi] + \delta O[\Phi]\right)$$

for δ a square-zero parameter of cohomological degree opposite to that of O.

To complete the proof of this lemma, we prove the required local lifting property in the sublemma below. □

8.6.1.4 Sub-lemma. *For each $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we can find a parametrix Φ — locally over the parametrizing manifold X — so that $L_\Phi O$ lies in $\mathrm{Obs}^q_{\leq(i,k)}(U) \subset \mathrm{Obs}^q_{\leq(i,k)}(M)$.*

Proof Although the observables Obs^q form a factorization algebra on the manifold M, they also form a sheaf on the parametrizing base manifold X. That is, for every open subset $V \subset X$, let $\mathrm{Obs}^q(U)\mid_V$ denote the observables for our family of theories restricted to V. In other words, $\mathrm{Obs}^q(U)\mid_V$ denotes the sections of this sheaf $\mathrm{Obs}^q(U)$ on V.

The map L_Φ constructed above is then a map of sheaves on X.

For every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we need to find an open cover

$$X = \bigcup_\alpha Y_\alpha$$

of X, and on each Y_α a parametrix Φ_α (for the restriction of the family of theories to Y_α) such that

$$L_{\Phi_\alpha}(O\mid_{Y_\alpha}) \in \mathrm{Obs}^q_{\leq(i,k)}(U)\mid_{Y_\alpha}.$$

More informally, we need to show that locally in X, we can find a parametrix Φ such that for all sufficiently small Ψ, the support of $L_\Phi(O)_{(i,k)}[\Psi]$ is in a subset of $U^k \times X$ which maps properly to X.

This argument resembles previous support arguments (e.g., the product lemma from section 8.5). The proof involves an analysis of the Feynman diagrams appearing in the expression

$$L_\Phi(O)_{i,k}[\Psi] = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} w_\gamma\left(O[\Phi]; I[\Phi]; P(\Psi) - P(\Phi)\right). \qquad (\star)$$

The sum is over all connected Feynman diagrams of genus i with k tails. The edges are labelled by $P(\Psi) - P(\Phi)$. Each graph has one special vertex, where $O[\Phi]$ appears. More explicitly, if this vertex is of genus r and valency s, it is labelled by $O_{r,s}[\Phi]$. Each non-special vertex is labelled by $I_{a,b}[\Phi]$, where a is the genus and b the valency of the vertex. Note that only a finite number of graphs appear in this sum.

By assumption, O is supported on U. This means that there exists some parametrix Φ_0 and a subset $K \subset U \times X$ mapping properly to X such that for all $\Phi < \Phi_0$, $O_{r,s}[\Phi]$ is supported on K^s. (Here by $K^s \subset U^s \times X$ we mean the fibre product over X.)

Further, each $I_{a,b}[\Phi]$ is supported as close as we like to the small diagonal $M \times X$ in $M^k \times X$. We can find precise bounds on the support of $I_{a,b}[\Phi]$, as explained in section 7.2. To describe these bounds, let us choose metrics for X and M. For a parametrix Φ supported within ϵ of the diagonal $M \times X$ in $M \times M \times X$, the effective interaction $I_{a,b}[\Phi]$ is supported within $(2a+b)\epsilon$ of the diagonal.

(In general, if $A \subset M^n \times X$, the ball of radius ϵ around A is defined to be the union of the balls of radius ϵ around each fibre A_x of $A \to X$.

It is in this sense that we mean that $I_{a,b}[\Phi]$ is supported within $(2a+b)\epsilon$ of the diagonal.)

Similarly, for every parametrix Ψ with $\Psi < \Phi$, the propagator $P(\Psi) - P(\Phi)$ is supported within ϵ of the diagonal.

In sum, there exists a set $K \subset U \times X$, mapping properly to X, such that for all $\epsilon > 0$, there exists a parametrix Φ_ϵ, such that

(i) $O[\Phi_\epsilon]_{r,s}$ is supported on K^s for all $(r,s) < (i,k)$.
(ii) $I_{a,b}[\Phi_\epsilon]$ is supported within $(2a+b)\epsilon$ of the small diagonal.
(iii) For all $\Psi < \Phi_\epsilon$, $P(\Psi) - P(\Phi_\epsilon)$ is supported within ϵ of the small diagonal.

The weight w_γ of a graph in the graphical expansion of the expression $(\star)$ above (using the parametrices Φ_ϵ and any $\Psi < \Phi_\epsilon$) is thus supported in the ball of radius $c\epsilon$ around K^k (where c is some combinatorial constant, depending on the number of edges and vertices in γ). There are a finite number of such graphs in the sum, so we can choose the combinatorial constant c uniformly over the graphs.

Since $K \subset U \times X$ maps properly to X, locally on X, we can find an ϵ so that the closed ball of radius $c\epsilon$ is still inside $U^k \times X$. This completes the proof. □

□

8.7 The map from theories to factorization algebras is a map of presheaves

In Costello (2011b), it is shown how to restrict a quantum field theory on a manifold M to any open subset U of M. Factorization algebras also form a presheaf in an obvious way. In this section, we will prove the following result.

8.7.0.1 Theorem. *The map from the simplicial set of theories on a manifold M to the ∞-groupoid of factorization algebras on M extends to a map of simplicial presheaves.*

The proof of this will rely on the results we have already proved, and in particular on thefact that observables form a factorization algebra.

As a corollary, we have the following very useful result.

8.7.0.2 Corollary. *For every open subset $U \subset M$, there is an isomorphism of graded differentiable vector spaces*

$$\mathrm{Obs}^q(U) \cong \mathrm{Obs}^{cl}(U)[[\hbar]].$$

Note that what we have proved already is that there is a filtration on $\mathrm{Obs}^q(U)$ whose associated graded is $\mathrm{Obs}^{cl}(U)[[\hbar]]$. This result shows that this filtration is split as a filtration of differentiable vector spaces.

Proof By the theorem, $\mathrm{Obs}^q(U)$ can be viewed as global observables for the field theory obtained by restricting our field theory on M to one on U. Choosing a parametrix on U allows one to identify global observables with $\mathrm{Obs}^{cl}(U)[[\hbar]]$, with differential $\mathrm{d} + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi$. This is an isomorphism of differentiable vector spaces. □

The proof of this theorem is a little technical, and uses the same techniques we have discussed so far. Before we explain the proof of the theorem, we need to explain how to restrict theories to open subsets.

Let $\mathscr{E}(M)$ denote the space of fields for a field theory on M. In order to relate field theories on U and on M, we need to relate parametrices on U and on M. If

$$\Phi \in \overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\overline{\mathscr{E}}(M)$$

is a parametrix on M (with proper support as always), then the restriction

$$\Phi\mid_U \in \overline{\mathscr{E}}(U)\widehat{\otimes}_\beta\overline{\mathscr{E}}(U)$$

of Φ to U may no longer be a parametrix. It will satisfy all the conditions required to be a parametrix except that it will typically not have proper support.

We can modify $\Phi\mid_U$ so that it has proper support, as follows. Let $K \subset U$ be a compact set, and let f be a smooth function on $U \times U$ with the following properties:

(i) f is 1 on $K \times K$.
(ii) f is 1 on a neighbourhood of the diagonal.
(iii) f has proper support.

Then, $f\Phi\mid_U$ does have proper support, and further, $f\Phi\mid_U$ is equal to Φ on $K \times K$.

Conversely, given any parametrix Φ on U, there exists a parametrix $\widetilde{\Phi}$ on M such that Φ and $\widetilde{\Phi}$ agree on K. One can construct $\widetilde{\Phi}$ by taking any parametrix Ψ on M, and observing that, when restricted to U, Ψ and Φ differ by a smooth section of the bundle $E \boxtimes E$ on $U \times U$.

We can then choose a smooth section σ of this bundle on $U \times U$ such that f has compact support and $\sigma = \Psi - \Phi$ on $K \times K$. Then, we let $\widetilde{\Phi} = \Psi - f$.

Let us now explain what it means to restrict a theory on M to one on U. Then we will state the theorem that there exists a unique such restriction.

Fix a parametrix Φ on U. Let $K \subset U$ be a compact set, and consider the compact set

$$L_n = (\operatorname{Supp}\Phi *)^n K \subset U.$$

Here we are using the convolution construction discussed earlier, whereby the collection of proper subsets of $U \times U$ acts on that of compact sets in U by convolution. Thus, L_n is the set of those $x \in U$ such that there exists a sequence $(y_0, \dots, y_n)$ where (y_i, y_{i+1}) is in $\operatorname{Supp}\Phi$, $y_n \in K$ and $y_0 = x$.

8.7.0.3 Definition. *Fix a theory on M, specified by a collection $\{I[\Psi]\}$ of effective interactions. Then a restriction of $\{I[\Psi]\}$ to U consists of a collection of effective interactions $\{I^U[\Phi]\}$ with the following propery. For every parametrix Φ on U, and for all compact sets $K \subset U$, let $L_n \subset U$ be as above.*

Let $\widetilde{\Phi}_n$ be a parametrix on M with the property that

$$\widetilde{\Phi}_n = \Phi \text{ on } L_n \times L_n.$$

Then we require that

$$I^U_{i,k}[\Phi](e_1, \dots, e_k) = I_{i,k}[\widetilde{\Phi}_n](e_1, \dots, e_k)$$

where $e_i \in \mathscr{E}_c(U)$ have support on K, and where $n \geq 2i + k$.

This definition makes sense in families with obvious modifications.

8.7.0.4 Theorem. *Any theory $\{I[\Psi]\}$ on M has a unique restriction on U.*

This restriction map works in families, and so defines a map of simplicial sets from the simplicial set of theories on M to that on U.

In this way, we have a simplicial presheaf $\mathcal{T}$ on M whose value on U is the simplicial set of theories on U (quantizing a given classical theory). This simplicial presheaf is a homotopy sheaf, meaning that it satisfies Čech descent.

Proof It is obvious that the restriction, it if exists, is unique. Indeed, we have specified each $I^U_{i,k}[\Phi]$ for every Φ and for every compact subset $K \subset U$. Since each $I^U_{i,k}[\Phi]$ must have compact support on U^k, it is determined by its behaviour on compact sets of the form K^k.

In Costello (2011b), a different definition of restriction was given, defined not in terms of general parametrices but in terms of those defined by the heat kernel. One therefore needs to check that the notion of restriction defined in Costello (2011b) coincides with the one discussed in this theorem. This is easy to see by a Feynman diagram argument similar to the ones we discussed earlier. The statement that the simplicial presheaf of theories satisfies Čech descent is proved in Costello (2011b). □

Now here is the main theorem in this section.

8.7.0.5 Theorem. *The map which assigns to a field theory the corresponding factorization algebra is a map of presheaves. Further, the map which assigns to an n-simplex in the simplicial set of theories, a factorization algebra over $\Omega^*(\triangle^n)$, is also a map of presheaves.*

Let us explain what this means concretely. Consider a theory on M and let Obs^q_M denote the corresponding factorization algebra. Let Obs^q_U denote the factorization algebra for the theory restricted to U, and let $Obs^q_M\,|_U$ denote the factorization algebra Obs^q_M restricted to U (that is, we only consider open subsets contained in M). Then there is a canonical isomorphism of factorization algebras on U,

$$\mathrm{Obs}^q_U \cong \mathrm{Obs}^q_M \mid_U .$$

In addition, this construction works in families, and in particular in families over $\Omega^*(\triangle^n)$.

Proof Let $V \subset U$ be an open set whose closure in U is compact. We will first construct an isomorphism of differentiable cochain complexes

$$\mathrm{Obs}^q_M(V) \cong \mathrm{Obs}^q_U(V).$$

Later we will check that this isomorphism is compatible with the product structures. Finally, we will use the codescent properties for factorization algebras to extend to an isomorphism of factorization algebras defined on all open subsets $V \subset U$, and not just those whose closure is compact.

Thus, let $V \subset U$ have compact closure, and let $O \in \mathrm{Obs}^q_M(V)$. Thus, O is something which assigns to every parametrix Φ on M a collection of functionals $O_{i,k}[\Phi]$ satisfying the renormalization group equation and a locality axiom stating that for each i, k, there exists a parametrix Φ_0 such that $O_{i,k}[\Phi]$ is supported on V for $\Phi \leq \Phi_0$.

We want to construct from such an observable a collection of functionals $\rho(O)_{i,k}[\Psi]$, one for each parametrix Ψ on U, satisfying the RG flow on U and the same locality axiom. It suffices to do this for a collection of parametrices which include parametrices which are arbitrarily small (that is, with support contained in an arbitrarily small neighbourhood of the diagonal in $U \times U$).

Let $L \subset U$ be a compact subset with the property that $\overline{V} \subset \mathrm{Int}\, L$. Choose a function f on $U \times U$ which is 1 on a neighbourhood of the diagonal, 1 on $L \times L$, and has proper support. If Ψ is a parametrix on M, we let Ψ^f be the parametrix on U obtained by multiplying the restriction of Ψ to $U \times U$ by f. Note that the support of Ψ^f is a subset of that of Ψ.

The construction is as follows. Choose (i, k). We define

$$\rho(O)_{r,s}[\Psi^f] = O_{r,s}[\Psi]$$

for all $(r, s) \leq (i, k)$ and all Ψ sufficiently small. We will not spell out what we mean by sufficiently small, except that it in particular means it is small enough so that $O_{r,s}[\Psi]$ is supported on V for all $(r,s) \leq (i,k)$. The value of $\rho(O)_{r,s}$ for other parametrices is determined by the RG flow.

To check that this construction is well-defined, we need to check that if we take some parametrix $\widetilde{\Psi}$ on M which is also sufficiently small, then the $\rho(O)_{r,s}[\Psi^f]$ and $\rho(O)_{r,s}[\widetilde{\Psi}^f]$ are related by the RG flow for observables for the theory on U. This RG flow equation relating these two quantities is a sum over connected graphs, with one vertex labelled by $\rho(O)[\Psi^f]$, all other vertices labelled by $I^U[\Psi^f]$, and all internal edges labelled by $P(\widetilde{\Psi}^f) - P(\Psi^f)$. Since we are only considering $(r,s) \leq (i,k)$ only finitely many graphs can appear, and the number of internal edges of these graphs is bounded by $2i+k$. We are assuming that both Ψ and $\widetilde{\Psi}$ are sufficiently small so that $O_{r,s}[\Psi]$ and $O_{r,s}[\widetilde{\Psi}]$ have compact support on V. Also, by taking Ψ sufficiently small, we can assume that $I^U[\Psi]$ has support arbitrarily close to the diagonal. It follows that, if we choose both Ψ and $\widetilde{\Psi}$ to be sufficiently small, there is a compact set $L' \subset U$ containing V such that the weight of each graph appearing in the RG flow is zero if one of the inputs (attached to the tails) has support on the complement of L'. Further, by taking Ψ and $\widetilde{\Psi}$ sufficiently small, we can arrange so that L' is as small as we like, and in particular, we can assume that $L' \subset \operatorname{Int} L$ (where L is the compact set chosen above).

Recall that the weight of a Feynman diagram involves pairing quantities attached to edges with multilinear functionals attached to vertices. A similar combinatorial analysis tells us that, for each vertex in each graph appearing in this sum, the inputs to the multilinear functional attached to the vertex are all supported in L'.

Now, for Ψ sufficiently small, we have

$$I^U_{r,s}[\Psi^f](e_1,\dots,e_s) = I_{r,s}[\Psi](e_1,\dots,e_s)$$

if all of the e_i are supported in L'. (This follows from the definition of the restriction of a theory. Recall that I^U indicates the theory on U and I indicates the theory on M).

It follows that, in the sum over diagrams computing the RG flow, we get the same answer if we label the vertices by $I[\Psi]$ instead of $I^U[\Psi^f]$.

The RG flow equation now follows from that for the original observable $O[\Psi]$ on M.

The same kind of argument tells us that if we change the choice of compact set $L \subset U$ with $\overline{V} \subset \operatorname{Int} L$, and if we change the bumb function f we chose, the map

$$\rho : \mathrm{Obs}^q_M(V) \to \mathrm{Obs}^q_U(V)$$

does not change.

A very similar argument also tells us that this map is a cochain map. It is immediate that ρ is an isomorphism, and that it commutes with the maps arising from inclusions $V \subset V'$.

We next need to verify that this map respects the product structure. Recall that the product of two observables O, O' in V, V' is defined by saying that $([\Psi]O'[\Psi])_{r,s}$ is simply the naive product in the symmetric algebra $\mathrm{Sym}^* \mathscr{E}^!_c(V \amalg V')$ for $(r,s) \leq (i,k)$ (some fixed (i,k)) and for Ψ sufficiently small.

Since, for $(r,s) \leq (i,k)$ and for Ψ sufficiently small, we defined

$$\rho(O)_{r,s}[\Psi^f] = O_{r,s}[\Psi],$$

we see immediately that ρ respects products.

Thus, we have constructed an isomorphism

$$\mathrm{Obs}^q_M \mid_U \cong \mathrm{Obs}^q_U$$

of prefactorization algebras on U, where we consider open subsets in U with compact closure. We need to extend this to an isomorphism of factorization algebras. To do this, we use the following property: for any open subset $W \subset U$,

$$\mathrm{Obs}^q_U(W) = \operatorname*{colim}_{V \subset W} \mathrm{Obs}^q_U(V)$$

where the colimit is over all open subsets with compact closure. (The colimit is taken, of course, in the category of filtered differentiable cochain complexes, and is simply the naive and not homotopy colimit). The same holds if we replace Obs^q_U by Obs^q_M. Thus we have constructed an isomorphism

$$\mathrm{Obs}^q_U(W) \cong \mathrm{Obs}^q_M(W)$$

for all open subsets W. The associativity axioms of prefactorization algebras, combined with the fact that $\mathrm{Obs}^q(W)$ is a colimit of $\mathrm{Obs}^q(V)$ for V with compact closure and the fact that the isomorphisms we have constructed respect the product structure for such open subsets V, implies that we have constructed an isomorphism of factorization algebras on U. $\square$

9
Further aspects of quantum observables

In this chapter we examine how factorization algebras and BV quantization interact with notions like translation invariance, renormalizability, and correlation functions.

9.1 Translation invariance for field theories and observables

In this section, we will show that a translation-invariant quantum field theory on $\mathbb{R}^n$ gives rise to a smoothly translation-invariant factorization algebra on $\mathbb{R}^n$ (see section I.4.8). We will also show that a holomorphically translation-invariant field theory on $\mathbb{C}^n$ gives rise to a holomorphically translation-invariant factorization algebra.

9.1.1

First, we need to define what it means for a field theory to be translation-invariant. Let us consider a classical field theory on $\mathbb{R}^n$. Recall that such a theory is given by

(i) A graded vector bundle E whose sections are $\mathscr{E}$;
(ii) An antisymmetric pairing $E \otimes E \to \mathrm{Dens}_{\mathbb{R}^n}$;

(iii) A differential operator $Q : \mathscr{E} \to \mathscr{E}$ making $\mathscr{E}$ into an elliptic complex, which is skew-self adjoint;
(iv) A local action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

A classical field theory is translation-invariant if

(i) The graded bundle E is translation-invariant, so that we are given an isomorphism between E and the trivial bundle with fibre E_0.
(ii) The pairing, differential Q, and local functional I are all translation-invariant.

It takes a little more work to say what it means for a quantum field theory to be translation-invariant. Suppose we have a translation-invariant classical field theory, equipped with a translation-invariant gauge fixing operator Q^{GF}. As before, a quantization of such a field theory is given by a family of interactions $I[\Phi] \in \mathscr{O}_{sm,P}(\mathscr{E})$, one for each parametrix Φ.

9.1.1.1 Definition. *A* translation-invariant quantization *of a translation-invariant classical field theory is a quantization with the property that, for every translation-invariant parametrix Φ, the effective interaction $I[\Phi]$ is translation-invariant.*

Remark: In general, in order to give a quantum field theory on a manifold M, we do not need to give an effective interaction $I[\Phi]$ for all parametrices. We only need to specify $I[\Phi]$ for a collection of parametrices such that the intersection of the supports of Φ is the small diagonal $M \subset M^2$. The functional $I[\Psi]$ for all other parametrices Ψ is defined by the renormalization group flow. It is easy to construct a collection of translation-invariant parametrices satisfying this condition. ♢

9.1.1.2 Proposition. *The factorization algebra associated to a translation-invariant quantum field theory is smoothly translation-invariant.*

See section I.4.8 for the definition of smoothly translation-invariant.

Proof Let Obs^q denote the factorization algebra of quantum observables for our translation-invariant theory. An observable supported on $U \subset \mathbb{R}^n$ is defined by a family $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$, one for each translation-invariant parametrix, which satisfies the RG flow and (in the sense we explained in section 8.4) is supported on U for sufficiently small parametrices. The renormalization group flow

$$W_\Psi^\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

for translation-invariant parametrices Ψ, Φ commutes with the action of $\mathbb{R}^n$ on $\mathscr{O}(\mathscr{E})$ by translation, and therefore acts on $\mathrm{Obs}^q(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$, let $T_x U$ denote the x-translate of U. It is immediate that the action of $x \in \mathbb{R}^n$ on $\mathrm{Obs}^q(\mathbb{R}^n)$ takes $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(\mathbb{R}^n)$ to $\mathrm{Obs}^q(T_x U)$. It is not difficult to verify that the resulting map

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(T_x U)$$

is an isomorphism of differentiable pro-cochain complexes and that it is compatible with the structure of a factorization algebra.

We need to verify the smoothness hypothesis of a smoothly translation-invariant factorization algebra. This is the following. Suppose that $U_1, \dots, U_k$ are disjoint open subsets of $\mathbb{R}^n$, all contained in an open subset V. Let $A' \subset \mathbb{R}^{nk}$ be the subset consisting of those $x_1, \dots, x_k$ such that the closures of $T_{x_i} U_i$ remain disjoint and in V. Let A be the connected component of 0 in A'. We need only examine the case where A is non-empty.

We need to show that the composed map

$$\begin{aligned} m_{x_1,\dots,x_k} : \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to \\ \mathrm{Obs}^q(T_{x_1} U_1) \times \cdots \times \mathrm{Obs}^q(T_{x_k} U_k) \to \mathrm{Obs}^q(V) \end{aligned}$$

varies smoothly with $(x_1, \dots, x_k) \in A$. In this diagram, the first map is the product of the translation isomorphisms $\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i)$, and the second map is the product map of the factorization algebra.

The smoothness property we need to check says that the map $m_{x_1,\dots,x_k}$ lifts to a multilinear map of differentiable pro-cochain complexes

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to C^\infty(A, \mathrm{Obs}^q(V)),$$

where on the right hand side the notation $C^\infty(A, \mathrm{Obs}^q(V))$ refers to the smooth maps from A to $\mathrm{Obs}^q(V)$.

This property is local on A, so we can replace A by a smaller open subset if necessary.

Let us assume (replacing A by a smaller subset if necessary) that there exist open subsets U_i' containing U_i, which are disjoint and contained in V and which have the property that for each $(x_1, \dots, x_k) \in A$, $T_{x_i} U_i \subset U_i'$.

Then, we can factor the map $m_{x_1,\dots,x_k}$ as a composition

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \xrightarrow{i_{x_1} \times \cdots \times i_{x_k}} \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V). \tag{†}$$

Here, the map $i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$ is the composition

$$\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$$

of the translation isomorphism with the natural inclusion map $\mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$. The second map in equation (†) is the product map associated to the disjoint subsets $U_1', \dots, U_k' \subset V$.

By possibly replacing A by a smaller open subset, let us assume that $A = A_1 \times \cdots \times A_k$, where the A_i are open subsets of $\mathbb{R}^n$ containing the origin. It remains to show that the map

$$i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$$

is smooth in x_i, that is, extends to a smooth map

$$\mathrm{Obs}^q(U_i) \to C^\infty(A_i, \mathrm{Obs}^q(U_i')).$$

Indeed, the fact that the product map

$$m : \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V)$$

is a smooth multilinear map implies that, for every collection of smooth maps $\alpha_i : Y_i \to \mathrm{Obs}^q(U_i')$ from smooth manifolds Y_i, the resulting map

$$\begin{aligned} Y_1 \times \cdots \times Y_k &\to \mathrm{Obs}^q(V) \\ (y_1, \dots y_k) &\mapsto m(\alpha_1(y), \dots, \alpha_k(y)) \end{aligned}$$

is smooth.

Thus, we have reduced the result to the following statement: for all open subsets $A \subset \mathbb{R}^n$ and for all $U \subset U'$ such that $T_x U \subset U'$ for all $x \in A$, the map $i_x : \mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$ is smooth in $x \in A$.

But this statement is tractable. Let

$$O \in \mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(U') \subset \mathrm{Obs}^q(\mathbb{R}^n)$$

be an observable. It is obvious that the family of observables $T_x O$, when viewed as elements of $\mathrm{Obs}^q(\mathbb{R}^n)$, depends smoothly on x. We need to verify that it depends smoothly on x when viewed as an element of $\mathrm{Obs}^q(U')$.

This amounts to showing that the support conditions which ensure an observable is in $\mathrm{Obs}^q(U')$ hold uniformly for x in compact sets in A.

The fact that O is in $\mathrm{Obs}^q(U)$ means the following. For each (i,k), there exists a compact subset $K \subset U$ and $\epsilon > 0$ such that for all translation-invariant parametrices Φ supported within ϵ of the diagonal and for all $(r,s) \leq (i,k)$ in the lexicographical ordering, the Taylor coefficient $O_{r,s}[\Phi]$ is supported on K^s.

We need to enlarge K to a subset $L \subset U' \times A$, mapping properly to A, such that $T_x O$ is supported on L in this sense (again, for $(r,s) \leq (i,k)$). Taking $L = K \times A$, embedded in $U' \times A$ by

$$(k,x) \mapsto (T_x k, x)$$

suffices. $\square$

Remark: Essentially the same proof will give us the somewhat stronger result that for any manifold M with a smooth action of a Lie group G, the factorization algebra corresponding to a G-equivariant field theory on M is smoothly G-equivariant. $\Diamond$

9.2 Holomorphically translation-invariant theories and observables

We can similarly talk about holomorphically translation-invariant classical and quantum field theories on $\mathbb{C}^n$. In this context, we will take our space of fields to be $\Omega^{0,*}(\mathbb{C}^n, V)$, where V is some translation-invariant holomorphic vector bundle on $\mathbb{C}^n$. The pairing must arise from a translation-invariant map of holomorphic vector bundles

$$V \otimes V \to K_{\mathbb{C}^n}$$

of cohomological degree $n-1$, where $K_{\mathbb{C}^n}$ denotes the canonical bundle. This means that the composed map

$$\Omega_c^{0,*}(\mathbb{C}^n, V)^{\otimes 2} \to \Omega_c^{0,*}(\mathbb{C}^n, K_{\mathbb{C}^n}) \xrightarrow{\int} \mathbb{C}$$

is of cohomological degree -1.

Let

$$\eta_i = \frac{\partial}{\partial \bar{z}_i} \vee - : \Omega^{0,k}(\mathbb{C}^n, V) \to \Omega^{0,k-1}(\mathbb{C}^n, V)$$

be the contraction operator. The cohomological differential operator Q on $\Omega^{0,*}(V)$ must be of the form

$$Q = \bar{\partial} + Q_0$$

where Q_0 is translation-invariant and satisfies the following conditions:

(i) Q_0 (and hence Q) must be skew self-adjoint with respect to the pairing on $\Omega_c^{0,*}(\mathbb{C}^n, V)$.
(ii) We assume that Q_0 is a purely holomorphic differential operator, so that we can write Q_0 as a finite sum

$$Q_0 = \sum \frac{\partial}{\partial z^I} \mu_I$$

where $\mu_I : V \to V$ are linear maps of cohomological degree 1. (Here we are using multi-index notation). Note that this implies that

$$[Q_0, \eta_i] = 0,$$

for $i = 1, \dots, n$. In terms of the μ_I, the adjointness condition says that μ_I is skew-symmetric if $|I|$ is even and symmetric if $|I|$ is odd.

The other piece of data of a classical field theory is the local action functional $I \in \mathscr{O}_{loc}(\Omega^{0,*}(\mathbb{C}^n, V))$. We assume that I is translation-invariant, of course, but also that

$$\eta_i I = 0$$

for $i = 1 \dots n$, where the linear map η_i on $\Omega^{0,*}(\mathbb{C}^n, V)$ is extended in the natural way to a derivation of the algebra $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$ preserving the subspace of local functionals.

Any local functional I on $\Omega^{0,*}(\mathbb{C}^n, V)$ can be written as a sum of functionals of the form

$$\phi \mapsto \int_{\mathbb{C}^n} \mathrm{d}z_1 \dots \mathrm{d}z_n A(D_1\phi \dots D_k\phi)$$

where $A : V^{\otimes k} \to \mathbb{C}$ is a linear map, and each D_i is in the space

$$\mathbb{C}\left[\mathrm{d}\overline{z}_i, \eta_i, \frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial z_i}\right].$$

(Recall that η_i indicates $\frac{\partial}{\partial \mathrm{d}\overline{z}_i}$). The condition that $\eta_i I = 0$ for each i means that we only consider those D_i which are in the subspace

$$\mathbb{C}\left[\eta_i, \frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial z_i}\right].$$

In other words, as a differential operator on the graded algebra $\Omega^{0,*}(\mathbb{C}^n)$, each D_i has constant coefficients.

It turns out that, under some mild hypothesis, any such action functional I is equivalent (in the sense of the BV formalism) to one which has only z_i derivatives, and no $\overline{z}_i$ or $\mathrm{d}\overline{z}_i$ derivatives.

9.2.0.1 Lemma. *Suppose that $Q = \overline{\partial}$, so that $Q_0 = 0$. Then, any interaction I satisfying the classical master equation and the condition that $\eta_i I = 0$ for $i = 1, \dots, n$ is equivalent to one only involving derivatives in the z_i.*

Proof Let $\mathscr{E} = \Omega^{0,*}(\mathbb{C}^n, V)$ denote the space of fields of our theory, and let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})^{hol}$ denote those functions which are translation-invariant and are in the kernel of the operators η_i, and let $\mathscr{O}_{loc}(\mathscr{E})^{hol'}$ denote those which in addition have only z_i derivatives. We will show that the inclusion map

$$\mathscr{O}_{loc}(\mathscr{E})^{hol'} \to \mathscr{O}_{loc}(\mathscr{E})^{hol}$$

is a quasi-isomorphism, where both are equipped with just the $\overline{\partial}$ differential. Both sides are graded by polynomial degree of the local functional, so it suffices to show this for local functionals of a fixed degree.

Note that the space V is filtered, by saying that F^i consists of those elements of degrees $\geq i$. This induces a filtration on $\mathscr{E}$ by the subspaces $\Omega^{0,*}(\mathbb{C}^n, F^i V)$. After passing to the associated graded, the operator Q becomes $\overline{\partial}$. By considering a spectral sequence with respect to this filtration, we see that it suffices to show we have a quasi-isomorphism in the case $Q = \overline{\partial}$.

But this follows immediately from the fact that the inclusion

$$\mathbb{C}\left[\frac{\partial}{\partial z_i}\right] \hookrightarrow \mathbb{C}\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right]$$

is a quasi-isomorphism, where the right hand side is equipped with the differential $[\overline{\partial}, -]$. To see that this map is a quasi-isomorphism, note that the $\overline{\partial}$ operator sends η_i to $\frac{\partial}{\partial \overline{z}_i}$. □

Recall that the action functional I induces the structure of L_∞ algebra on $\Omega^{0,*}(\mathbb{C}^n, V)[-1]$ whose differential is Q, and whose L_∞ structure maps are encoded by the Taylor components of I. Under the hypothesis of the previous lemma, this L_∞ algebra is L_∞ equivalent to one which is the Dolbeault complex with coefficients in a translation-invariant local L_∞ algebra whose structure maps only involve z_i derivatives.

There are many natural examples of holomorphically translation-invariant classical field theories. Geometrically, they arise from holomorphic moduli problems. For instance, one could take the cotangent theory to the derived moduli of holomorphic G bundles on $\mathbb{C}^n$, or the cotangent theory to the derived moduli space of such bundles equipped with holomorphic sections of some associated bundles, or the cotangent theory to the moduli of holomorphic maps from C^n to some complex manifold.

As is explained in great detail in , holomorphically translation-invariant field theories arise very naturally in physics as holomorphic (or minimal) twists of supersymmetric field theories in even dimensions.

9.2.1

A holomorphically translation invariant classical theory on $\mathbb{C}^n$ has a natural gauge fixing operator, namely

$$\overline{\partial}^* = -\sum \eta_i \frac{\partial}{\partial \overline{z}_i}.$$

Since $[\eta_i, Q_0] = 0$, we see that $[Q, \overline{\partial}^*] = [\overline{\partial}, \overline{\partial}^*]$ is the Laplacian. (More generally, we can consider a family of gauge fixing operators coming from the $\overline{\partial}^*$ operator for a family of flat Hermitian metrics on $\mathbb{C}^n$. Since the space of such metrics is $GL(n, \mathbb{C})/U(n)$ and thus contractible, we see that everything is independent up to homotopy of the choice of gauge fixing operator.)

We say a translation-invariant parametrix

$$\Phi \in \overline{\Omega}^{0,*}(\mathbb{C}^n, V)^{\otimes 2}$$

is *holomorphically translation-invariant* if

$$(\eta_i \otimes 1 + 1 \otimes \eta_i)\Phi = 0$$

for $i = 1, \ldots, n$. For example, if Φ_0 is a parametrix for the scalar Laplacian

$$\triangle = -\sum \frac{\partial}{\partial z_i} \frac{\partial}{\partial \overline{z}_i}$$

then

$$\Phi_0 \prod_{i=1}^{n} \mathrm{d}(\overline{z}_i - \overline{w}_i) c$$

defines such a parametrix. Here z_i and w_i indicate the coordinates on the two copies of $\mathbb{C}^n$, and $c \in V \otimes V$ is the inverse of the pairing on v. Clearly, we can find holomorphically translation-invariant parametrices which are supported arbitrarily close to the diagonal. This means that we can define a field theory by only considering $I[\Phi]$ for holomorphically translation-invariant parametrices Φ.

9.2.1.1 Definition. *A* holomorphically translation-invariant quantization *of a holomorphically translation-invariant classical field theory as above is a translation-invariant quantization such that for each holomorphically translation-invariant parametrix Φ, the effective interaction $I[\Phi]$ satisfies*

$$\eta_i I[\Phi] = 0$$

for $i = 1, \ldots, n$. Here η_i abusively denotes the natural extension of the contraction η_i to a derivation on $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$.

The usual obstruction theory arguments hold for constructing holomorphically-translation invariant quantizations. At each order in $\hbar$, the obstruction-deformation complex is the subcomplex of the complex $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^n}$ of translation-invariant local functionals which are also in the kernel of the operators η_i.

9.2.1.2 Proposition. *A holomorphically translation-invariant quantum field* *theory on $\mathbb{C}^n$ leads to a holomorphically translation-invariant factorization algebra.*

Proof This follows immediately from proposition 9.1.1.2. Indeed, quantum observables form a smoothly translation-invariant factorization algebra. Such an observable O on U is specified by a family $O[\Phi] \in \mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ of functionals defined for each holomorphically translation-invariant parametrix Φ, which are supported on U for Φ sufficiently small. The operators $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i$ act in a natural way on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ by derivations, and each commutes with the renormalization group flow W_Ψ^Φ for holomorphically translation-invariant parametrices Ψ, Φ. Thus, $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}$ and η_i define derivations of the factorization algebra Obs^q. Explicitly, if $O \in \mathrm{Obs}^q(U)$ is an observable, then for each holomorphically translation-invariant parametrix Φ,

$$\left(\frac{\partial}{\partial z_i} O\right)[\Phi] = \frac{\partial}{\partial z_i}(O[\Phi]),$$

and similarly for $\frac{\partial}{\partial \overline{z}_i}$ and η_i.

By Definition I.5.2.1, a holomorphically translation-invariant factorization algebra is a translation-invariant factorization algebra where the derivation operator $\frac{\partial}{\partial \overline{z}_i}$ on observables is homotopically trivialized.

Note that, for a holomorphically translation-invariant parametrix Φ, $[\eta_i, \triangle_\Phi] = 0$ and η_i is a derivation for the Poisson bracket $\{-,-\}_\Phi$. It follows that

$$[Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi, \eta_i] = [Q, \eta_i]$$

as operators on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$. Since we wrote $Q = \overline{\partial} + Q_0$ and required that $[Q_0, \eta_i] = 0$, we have

$$[Q, \eta_i] = [\overline{\partial}, \eta_i] = \frac{\partial}{\partial \overline{z}_i}.$$

Since the differential on $\mathrm{Obs}^q(U)$ is defined by

$$(\widehat{Q}O)[\Phi] = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar\triangle_\Phi O[\Phi],$$

we see that $[\widehat{Q}, \eta_i] = \frac{\partial}{\partial \overline{z}_i}$, as desired. □

As we showed in Chapter I.5, a holomorphically translation invariant factorization algebra in one complex dimension, with some mild additional conditions, gives rise to a vertex algebra. Let us verify that these conditions hold in the examples of interest. We first need a definition.

9.2.1.3 Definition. *A holomorphically translation-invariant field theory on $\mathbb{C}$ is S^1-*invariant *if the following holds. First, we have an S^1 action on the vector space V, inducing an action of S^1 on the space of fields $\mathscr{E} = \Omega^{0,*}(\mathbb{C}, V)$ of fields, by combining the S^1 action on V with the natural one on $\Omega^{0,*}(\mathbb{C})$ coming from rotation on $\mathbb{C}$. We suppose that all the structures of the field theory are S^1-invariant. More precisely, the symplectic pairing on $\mathscr{E}$ and the differential Q on $\mathscr{E}$ must be S^1-invariant. Further, for every S^1-invariant parametrix Φ, the effective interaction $I[\Phi]$ is S^1-invariant.*

9.2.1.4 Lemma. *Suppose we have a holomorphically translation invariant field theory on $\mathbb{C}$ that is also S^1-invariant. The corresponding factorization algebra then satisfies the conditions stated in theorem I.5.3.3 of Chapter I.5, allowing us to construct a vertex algebra structure on the cohomology.*

Proof Let $\mathcal{F}$ denote the factorization algebra of observables of our theory. Note that if $U \subset \mathbb{C}$ is an S^1-invariant subset, then S^1 acts on $\mathcal{F}(U)$.

Recall that $\mathcal{F}$ is equipped with a complete decreasing filtration, and is viewed as a factorization algebra valued in pro-differentiable cochain complexes. Recall that we need to check the following properties.

(i) The S^1 action on $\mathcal{F}(D(0,r))$ extends to a smooth action of the algebra $\mathcal{D}(S^1)$ of distributions on S^1.

(ii) Let $\operatorname{Gr}^m \mathcal{F}(D(0,r))$ denote the associated graded with respect to the filtration on $\mathcal{F}(D(0,r))$. Let $\operatorname{Gr}^m_k \mathcal{F}(D(0,r))$ refer to the kth S^1-eigenspace in $\operatorname{Gr}^m \mathcal{F}(D(0,r))$. Then, we require that the map

$$\operatorname{Gr}^m_k \mathcal{F}(D(0,r)) \to \operatorname{Gr}^m_k \mathcal{F}(D(0,r'))$$

is a quasi-isomorphism of differentiable vector spaces.

(iii) The differentiable vector space $H^*(\operatorname{Gr}^m_k \mathcal{F}(D(0,r)))$ is finite-dimensional for all k and is zero for $k \gg 0$.

Let us first check that the S^1 action extends to a $\mathcal{D}(S^1)$-action. If $\lambda \in S^1$ let ρ^*_λ denote this action. We need to check that for any observable

$\{O[\Phi]\}$ and for every distribution $D(\lambda)$ on $\mathcal{S}^1$ the expression

$$\int_{\lambda \in S^1} D(\lambda)\rho_\lambda^* O[\Phi]$$

makes sense and defines another observable. Further, this construction must be smooth in both $D(\lambda)$ and the observable $O[\Phi]$, meaning that it must work families.

For fixed Φ, each $O_{i,k}[\Phi]$ is simply a distribution on $\mathbb{C}^k$ with some coefficients. For any distribution α on $\mathbb{C}^k$, the expression $\int_\lambda D(\lambda)\rho_\lambda^* \alpha$ makes sense and is continuous in both α and the distribution$D$. Indeed, $\int_\lambda D(\lambda)\rho_\lambda^* \alpha$ is simply the push-forward map in distributions applied to the action map $S^1 \times \mathbb{C}^k \to \mathbb{C}^k$.

It follows that, for each distribution D on S^1, we can define

$$D * O_{i,k}[\Phi] := \int_{\lambda \in S^1} D(\lambda)\rho_\lambda^* O_{i,k}[\Phi].$$

As a function of D and $O_{i,k}[\Phi]$, this construction is smooth. Further, sending an observable $O[\Phi]$ to $D * O[\Phi]$ commutes with the renormalization group flow (between S^1-equivariant parametrices). It follows that we can define a new observable $D * O$ by

$$(D * O)_{i,k}[\Phi] = D * (O_{i,k}[\Phi]).$$

Now, a family of observables O^x (parametrized by $x \in M$, a smooth manifold) is smooth if and only if the family of functionals $O^x_{i,k}[\Phi]$ are smooth for all i, k and all Φ. In fact one need not check this for all Φ, but for any collection of parametrices which includes arbitrarily small parametrices. If follows that the map sending D and O to $D * O$ is smooth, that is, takes smooth families to smooth families.

Let us now check the remaining assumptions of theorem I.5.3.3. Let $\mathcal{F}$ denote the factorization algebra of quantum observables of the theory and let $\mathcal{F}_k$ denote the kth eigenspace of the S^1 action. We first need to check that the inclusion

$$\mathrm{Gr}_k^m \mathcal{F}(D(0,r)) \to \mathrm{Gr}_k^m \mathcal{F}(D(0,r'))$$

is a quasi-isomorphism for $r < r'$. We need it to be a quasi-isomorphism of completed filtered differentiable vector spaces. The space $\mathrm{Gr}^m \mathcal{F}(D(0,r))$ is a finite direct sum of spaces of the form

$$\overline{\Omega}_c^{0,*}(D(0,r)^l, V^{\boxtimes l})_{S_l}.$$

It thus suffices to check that for the map

$$\overline{\Omega}_c^{0,*}(D(0,r)^m) \to \overline{\Omega}_c^{0,*}(D(0,r')^m)$$

is a quasi-isomorphism on each S^1-eigenspace. This claim is immediate.

The same holds to check that the cohomology of $\mathrm{Gr}_k^m \mathcal{F}(D(0,r))$ is zero for $k \gg 0$ and that it is finite-dimensional as a differentiable vector space. □

We have seen that any S^1-equivariant and holmorphically translation-invariant factorization algebra on $\mathbb{C}$ gives rise to a vertex algebra. We have also seen that the obstruction-theory method applies in this situation to construct holomorphically translation invariant factorization algebras from appropriate Lagrangians. In this way, we have a very general method for constructing vertex algebras.

9.3 Renormalizability and factorization algebras

A central concept in field theory is that of *renormalizability*. This notion is discussed in detail in Chapter 4 of Costello (2011b). The basic idea is the following.

The group $\mathbb{R}_{>0}$ acts on the collection of field theories on $\mathbb{R}^n$, where the action is induced from the scaling action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$. This action is implemented differently in different models for field theories. In the language if factorization algebras it is very simple, because any factorization algebra on $\mathbb{R}^n$ can be pushed forward under any diffeomorphism of $\mathbb{R}^n$ to yield a new factorization algebra on $\mathbb{R}^n$. Push-forward of factorization algebras under the map $x \mapsto \lambda^{-1}x$ (for $\lambda \in \mathbb{R}_{>0}$) defines the renormalization group flow on factorization algebra.

We will discuss how to implement this rescaling in the definition of field theory given in Costello (2011b) shortly. The main result of this section is the statement that the map which assigns to a field theory the corresponding factorization algebra of observables intertwines the action of $\mathbb{R}_{>0}$.

Acting by elements $\lambda \in \mathbb{R}_{>0}$ on a fixed quantum field theory produces a one-parameter family of theories, depending on λ. Let F denote a fixed theory, either in the language of factorization algebras, the language of Costello (2011b), or any other approach to quantum field theory. We will call this family of theories $\rho_\lambda(F)$. We will view the theory $\rho_\lambda(F)$ as being obtained from F by "zooming in" on $\mathbb{R}^n$ by an amount dicated by λ, if $\lambda < 1$, or by zooming out if $\lambda > 1$.

We should imagine the theory F as having some number of continuous parameters, called coupling constants. Classically, the coupling constants are simply constants appearing next to various terms in the Lagrangian. At the quantum level, we could think of the structure constants of the factorization algebra as being functions of the coupling constants (we will discuss this more precisely below).

Roughly speaking, a theory is *renormalizable* if, as $\lambda \to 0$, the family of theories $\rho_\lambda(F)$ converges to a limit. While this definition is a good one non-pertubatively, in perturbation theory it is not ideal. The reason is that often the coupling constants depend on the scale through quantities like $\lambda^\hbar$. If $\hbar$ was an actual real number, we could analyze the behaviour of $\lambda^\hbar$ for λ small. In perturbation theory, however, $\hbar$ is a formal parameter, and we must expand $\lambda^\hbar$ in a series of the form $1 + \hbar \log \lambda + \dots$. The coefficients of this series always grow as $\lambda \to 0$.

In other words, from a perturbative point of view, one can't tell the difference between a theory that has a limit as $\lambda \to 0$ and a theory whose coupling constants have logarithmic growth in λ.

This motivates us to define a theory to be *perturbatively renormalizable* if it has logarithmic growth as $\lambda \to 0$. We will introduce a formal definition of perturbative renormalizability shortly. Let us first indicate why this definition is important.

It is commonly stated (especially in older books) that perturbative renormalizability is a necessary condition for a theory to exist (in perturbation theory) at the quantum level. This is *not* the case. Instead, renormalizability is a criterion which allows one to select a finite-dimensional space of well-behaved quantizations of a given classical field theory, from a possibly infinite dimensional space of all possible quantizations.

There are other criteria which one wants to impose on a quantum theory and which also help select a small space of quantizations: for instance, symmetry criteria. (In addition, one also requires that the quantum master equation holds, which is a strong constraint. This, however, is part of the definition of a field theory that we use). There are examples of non-renormalizable field theories for which nevertheless a unique quantization can be selected by other criteria. (An example of this nature is the Kodaira-Spencer theory of gravity, often called BCOV theory.)

9.3.1 The renormalization group action on factorization algebras

Let us now discuss the concept of renormalizability more formally. We will define the action of the group $\mathbb{R}_{>0}$ on the set of theories in the definition used in Costello (2011b), and on the set of factorization algebras on $\mathbb{R}^n$. We will see that the map which assigns a factorization algebra to a theory is $\mathbb{R}_{>0}$-equivariant.

Let us first define the action of $\mathbb{R}_{>0}$ on the set of factorization algebras on $\mathbb{R}^n$.

9.3.1.1 Definition. *If $\mathcal{F}$ is a factorization algebra on $\mathbb{R}^n$, and $\lambda \in \mathbb{R}_{>0}$, let $\rho_\lambda(\mathcal{F})$ denote the factorization algebra on $\mathbb{R}^n$ that is the pushforward of $\mathcal{F}$ under the diffeomorphism $\lambda^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given by multiplying by λ^{-1}. Thus,*

$$\rho_\lambda(\mathcal{F})(U) = \mathcal{F}(\lambda(U))$$

and the product maps in $\rho_\lambda(\mathcal{F})$ arise from those in $\mathcal{F}$. We call this action of $\mathbb{R}_{>0}$ on the collection of factorization algebras on $\mathbb{R}^n$ the local renormalization group *action.*

Thus, the action of $\mathbb{R}_{>0}$ on factorization algebras on $\mathbb{R}^n$ is simply the obvious action of diffeomorphisms on $\mathbb{R}^n$ on factorization algebras on $\mathbb{R}^n$.

9.3.2 The renormalization group flow on classical theories

The action on field theories as defined in Costello (2011b) is more subtle. Let us start by describing the action of $\mathbb{R}_{>0}$ on classical field theories. Suppose we have a translation-invariant classical field on $\mathbb{R}^n$, with space of fields $\mathscr{E}$. The space $\mathscr{E}$ is the space of sections of a trivial vector bundle on $\mathbb{R}^n$ with fibre E_0. The vector space E_0 is equipped with a degree -1 symplectic pairing valued in the line ω_0, the fibre of the bundle of top forms on $\mathbb{R}^n$ at 0. We also, of course, have a translation-invariant local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

Let us choose an action ρ^0_λ of the group $\mathbb{R}_{>0}$ on the vector space E_0 with the property that the symplectic pairing on E_0 is $\mathbb{R}_{>0}$-equivariant, where the action of $\mathbb{R}_{>0}$ acts on the line ω_0 with weight $-n$. Let us further assume that this action is diagonalizable, and that the eigenvalues of ρ^0_λ are rational integer powers of λ. (In practise, only integer or half-integer powers appear).

The choice of such an action, together with the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$ by rescaling, induces an action of $\mathbb{R}_{>0}$ on

$$\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E_0$$

which sends

$$\phi \otimes e_0 \mapsto \phi(\lambda^{-1}x)\rho^0_\lambda(e_0),$$

where $\phi \in C^\infty(\mathbb{R}^n)$ and $e_0 \in E_0$. The convention that $x \mapsto \lambda^{-1}x$ means that for small λ, we are looking at small scales (for instance, as $\lambda \to 0$ the metric becomes large).

This action therefore induces an action on spaces associated to $\mathscr{E}$, such as the spaces $\mathscr{O}(\mathscr{E})$ of functionals and $\mathscr{O}_{loc}(\mathscr{E})$ of local functionals. The compatibility between the action of $\mathbb{R}_{>0}$ and the symplectic pairing on E_0 implies that the Poisson bracket on the space $\mathscr{O}_{loc}(\mathscr{E})$ of local functionals on $\mathscr{E}$ is preserved by the $\mathbb{R}_{>0}$ action. Let us denote the action of $\mathbb{R}_{>0}$ on $\mathscr{O}_{loc}(\mathscr{E})$ by ρ_λ.

9.3.2.1 Definition. *The* local renormalization group flow *on the space of translation-invariant classical field theories sends a classical action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ to $\rho_\lambda(I)$.*

This definition makes sense because ρ_λ preserves the Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$. Note also that, if the action of ρ_λ^0 on E_0 has eigenvalues in $\frac{1}{n}\mathbb{Z}$, then the action of ρ_λ on the space $\mathscr{O}_{loc}(\mathscr{E})$ is diagonal and has eigenvalues again in $\frac{1}{n}\mathbb{Z}$.

The action of $\mathbb{R}_{>0}$ on the space of classical field theories up to isomorphism is independent of the choice of action of $\mathbb{R}_{>0}$ on E_0. If we choose a different action, inducing a different action ρ'_λ of $\mathbb{R}_{>0}$ on everything, then $\rho_\lambda I$ and $\rho'_\lambda I$ are related by a linear and symplectic change of coordinates on the space of fields which covers the identity on $\mathbb{R}^n$. Field theories related by such a change of coordinates are equivalent.

It is often convenient to choose the action of $\mathbb{R}_{>0}$ on the space E_0 so that the quadratic part of the action is invariant. When we can do this, the local renormalization group flow acts only on the interactions (and on any small deformations of the quadratic part that one considers). Let us give some examples of the local renormalization group flow on classical field theories. Many more details are given in Costello (2011b).

Consider the free massless scalar field theory on $\mathbb{R}^n$. The complex of fields is

$$C^\infty(\mathbb{R}^n) \xrightarrow{D} C^\infty(\mathbb{R}^n).$$

We would like to choose an action of $\mathbb{R}_{>0}$ so that both the symplectic pairing and the action functional $\int \phi D \phi$ are invariant. This action must, of course, cover the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$ by rescaling. If ϕ, ψ denote fields in the copies of $C^\infty(\mathbb{R}^n)$ in degrees 0 and 1 respectively, the desired action sends

$$\begin{aligned} \rho_\lambda(\phi(x)) &= \lambda^{\frac{2-n}{2}} \phi(\lambda^{-1}x) \\ \rho_\lambda(\psi(x)) &= \lambda^{\frac{-n-2}{2}} \psi(\lambda^{-1}x). \end{aligned}$$

Let us then consider how ρ_λ acts on possible interactions. We find, for example, that if

$$I_k(\phi) = \int \phi^k$$

then

$$\rho_\lambda(I_k) = \lambda^{n - \frac{k(n-2)}{2}} I_k.$$

9.3.2.2 Definition. *A classical theory is* renormalizable *if, as $\lambda \to 0$, it flows to a fixed point under the local renormalization group flow.*

For instance, we see that in dimension 4, the most general renormalizable classical action for a scalar field theory which is invariant under the symmetry $\phi \mapsto -\phi$ is

$$\int \phi \, \mathrm{D} \, \phi + m^2 \phi^2 + c\phi^4.$$

Indeed, the ϕ^4 term is fixed by the local renormalization group flow, whereas the ϕ^2 term is sent to zero as $\lambda \to 0$.

9.3.2.3 Definition. *A classical theory is* strictly renormalizable *if it is a fixed point under the local renormalization group flow.*

A theory that is renormalizable has good small-scale behaviour, in that the coupling constants (classically) become small at small scales. (At the quantum level there may also be logarithmic terms which we will discuss shortly). A renormalizable theory may, however, have bad large-scale behaviour: for instance, in four dimensions, a mass term $\int \phi^2$ becomes large at large scales. A strictly renormalizable theory is one that is classically scale-invariant. At the quantum level, we will define a strictly renormalizable theory to be one which is scale-invariant up to logarithmic corrections.

Again in four dimensions, the only strictly renormalizable interaction for the scalar field theory which is invariant under $\phi \mapsto -\phi$ is the ϕ^4 interaction. In six dimensions, the ϕ^3 interaction is strictly renormalizable, and in three dimensions the ϕ^6 interaction (together with finitely many other interactions involving derivatives) are strictly renormalizable.

As another example, recall that the graded vector space of fields of pure Yang-Mills theory (in the first order formalism) is

$$\left(\Omega^0[1] \oplus \Omega^1 \oplus \Omega^2_+ \oplus \Omega^2_+[-1] \oplus \Omega^3[-1] \oplus \Omega^4[-2]\right) \otimes \mathfrak{g}.$$

(Here Ω^i indicates forms on $\mathbb{R}^4$). The action of $\mathbb{R}_{>0}$ is the natural one, coming from pull-back of forms under the map $x \mapsto \lambda^{-1}x$. The Yang-Mills action functional

$$S(A, B) = \int F(A) \wedge B + B \wedge B$$

is obviously invariant under the action of $\mathbb{R}_{>0}$, since it only involves wedge product and integration, as well as projection to Ω^2_+. (Here $A \in$

$\Omega^1 \otimes \mathfrak{g}$) and $B \in \Omega^2_+ \otimes \mathfrak{g}$). The other terms in the full BV action functional are also invariant, because the symplectic pairing on the space of fields and the action of the gauge group are both scale-invariant.

Something similar holds for Chern-Simons theory on $\mathbb{R}^3$, where the space of fields is $\Omega^*(\mathbb{R}^3) \otimes \mathfrak{g}[1]$. The action of $\mathbb{R}> 0$ is by pull-back by the map $x \mapsto \lambda^{-1}x$, and the Chern-Simons functional is obviously invariant.

9.3.2.4 Lemma. *The map assigning to a translation-invariant classical field theory on $\mathbb{R}^n$, its associated P_0 factorization algebra, commutes with the action of the local renormalization group flow.*

Proof The action of $\mathbb{R}_{>0}$ on the space of fields of the theory induces an action on the space $\mathrm{Obs}^{cl}(\mathbb{R}^n)$ of classical observables on $\mathbb{R}^n$, by sending an observable O (which is a functon on the space $\mathscr{E}(\mathbb{R}^n)$ of fields) to the observable

$$\rho_\lambda O : \phi \mapsto O(\rho_\lambda(\phi)).$$

This preserves the Poisson bracket on the subspace $\widetilde{\mathrm{Obs}}^{cl}(\mathbb{R}^n)$ of functionals with smooth first derivative, because by assumption the symplectic pairing on the space of fields is scale invariant. Further, it is immediate from the definition of the local renormalization group flow on classical field theories that

$$\rho_\lambda\{S, O\} = \{\rho_\lambda(I), \rho_\lambda(O)\}$$

where $S \in \mathscr{O}_{loc}(\mathscr{E})$ is a translation-invariant solution of the classical master equation (whose quadratic part is elliptic).

Let $\mathrm{Obs}^{cl}_\lambda$ denote the factorization algebra on $\mathbb{R}^n$ coming from the theory $\rho_\lambda(S)$ (where S is some fixed classical action). Then, we see that we have an isomorphism of cochain complexes

$$\rho_\lambda : \mathrm{Obs}^{cl}(\mathbb{R}^n) \cong \mathrm{Obs}^{cl}_\lambda(\mathbb{R}^n).$$

We next need to check what this isomorphism does to the support conditions. Let $U \subset \mathbb{R}^n$ and let $O \in \mathrm{Obs}^{cl}(U)$ be an observable supported on U. Then, one can check easily that $\rho_\lambda(O)$ is supported on $\lambda^{-1}(U)$. Thus, ρ_λ gives an isomorphism

$$\mathrm{Obs}^{cl}(U) \cong \mathrm{Obs}^{cl}_\lambda(\lambda^{-1}(U)).$$

and so,

$$\mathrm{Obs}^{cl}(\lambda U) \cong \mathrm{Obs}^{cl}_{\lambda}(U).$$

The factorization algebra

$$\rho_\lambda \mathrm{Obs}^{cl} = \lambda_* \mathrm{Obs}^{cl}$$

assigns to an open set $U \subset \mathbb{R}^n$ the value of Obs^{cl} on $\lambda(U)$. Thus, we have constructed an isomorphism of precosheaves on $\mathbb{R}^n$,

$$\rho_\lambda \mathrm{Obs}^{cl} \cong \mathrm{Obs}^{cl}_{\lambda}.$$

This isomorphism compatible with the commutative product and the (homotopy) Poisson bracket on both side, as well as the factorization product maps. □

9.3.3 The renormalization group flow on quantum field theories.

The most interesting version of the renormalization group flow is, of course, that on quantum field theories. Let us fix a classical field theory on $\mathbb{R}^n$, with space of fields as above $\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E_0$ where E_0 is a graded vector space. In this section we will define an action of the group $\mathbb{R}_{>0}$ on the simplicial set of quantum field theories with space of fields $\mathscr{E}$, quantizing the action on classical field theories that we constructed above. We will show that the map which assigns to a quantum field theory the corresponding factorization algebra commutes with this action.

Let us assume, for simplicity, that we have chosen the linear action of $\mathbb{R}_{>0}$ on E_0 so that it leaves invariant a quadratic action functional on $\mathscr{E}$ defining a free theory. Let $Q : \mathscr{E} \to \mathscr{E}$ be the corresponding cohomological differential, which, by assumption, is invariant under the $\mathbb{R}_{>0}$ action. (This step is not necessary, but will make the exposition simpler).

Let us also assume (again for simplicity) that there exists a gauge fixing operator $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the property that

$$\rho_\lambda Q^{GF} \rho_{-\lambda} = \lambda^k Q^{GF}$$

for some $k \in \mathbb{Q}$. For example, for a massless scalar field theory on $\mathbb{R}^n$, we have seen that the action of $\mathbb{R}_{>0}$ on the space $C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n)[-1]$ of fields sends ϕ to $\lambda^{\frac{2-n}{2}} \phi(\lambda^{-1}x)$ and ψ to $\lambda^{\frac{-2-n}{2}} \psi(\lambda^{-1}x)$ (where ϕ is the field of cohomological degree 0 and ψ is the field of cohomological degree 1). The gauge fixing operator is the identity operator from $C^\infty(\mathbb{R}^n)[-1]$ to $C^\infty(\mathbb{R}^n)[0]$. In this case, we have $\rho_\lambda Q^{GF} \rho_{-\lambda} = \lambda^2 Q^{GF}$.

As another example, consider pure Yang-Mills theory on $\mathbb{R}^4$. The fields, as we have described above, are built from forms on $\mathbb{R}^4$, equipped with the natural action of $\mathbb{R}_{>0}$. The gauge fixing operator is d^*. It is easy to see that $\rho_\lambda \mathrm{d}^* \rho_{-\lambda} = \lambda^2 \mathrm{d}^*$. The same holds for Chern-Simons theory, which also has a gauge fixing operator defined by d^* on forms.

A translation-invariant quantum field theory is defined by a family

$$\{I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})^{\mathbb{R}^n}[[\hbar]] \mid \Phi \text{ a translation-invariant parametrix}\}$$

that satisfies the renormalization group equation, quantum master equation, and the locality condition. We need to explain how scaling of $\mathbb{R}^n$ by $\mathbb{R}_{>0}$ acts on the (simplicial) set of quantum field theories. To do this, we first need to explain how this scaling action acts on the set of parametrices.

9.3.3.1 Lemma. *If Φ is a translation-invariant parametrix, then $\lambda^k \rho_\lambda(\Phi)$ is also a parametrix, where as above k measures the failure of Q^{GF} to commute with ρ_λ.*

Proof All of the axioms characterizing a parametrix are scale invariant, except the statement that

$$([Q, Q^{GF}] \otimes 1)\Phi = K_{id} - \text{ something smooth.}$$

We need to check that $\lambda^k \rho_\lambda \Phi$ also satisfies this. Note that

$$\rho_\lambda([Q, Q^{GF}] \otimes 1)\Phi = \lambda^k([Q, Q^{GF}] \otimes 1)\rho_\lambda(\Phi)$$

since ρ_λ commutes with Q but not with Q^{GF}. Also, ρ_λ preserves K_{id} and smooth kernels, so the desired identity holds. □

This lemma suggests a way to define the action of the group $\mathbb{R}_{>0}$ on the set of quantum field theories.

9.3.3.2 Lemma. *If $\{I[\Phi]\}$ is a theory, define $I_\lambda[\Phi]$ by*

$$I_\lambda[\Phi] = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}(\Phi)]).$$

Then, the collection of functionals $\{I_\lambda[\Phi]\}$ defines a new theory.

On the right hand side of the equation in the lemma, we are using the natural action of ρ_λ on all spaces associated to $\mathscr{E}$, such as the space $\mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$ (to define $\rho_{-\lambda}(\Phi)$) and the space of functions on $\mathscr{E}$ (to define how ρ_λ acts on the function $I[\lambda^{-k}\rho_{-\lambda}(\Phi)]$).

Note that this lemma is discussed in more detail in Costello (2011b), except that there the language of heat kernels is used. We will prove the lemma here anyway, because the proof is quite simple.

Proof We need to check that $I_\lambda[\Phi]$ satisfies the renormalization group equation, locality action, and quantum master equation. Let us first check the renormalization group flow. As a shorthand notation, let us write Φ_λ for the parametrix $\lambda^k\rho_\lambda(\Phi)$. Then, note that the propagator $P(\Phi_\lambda)$ is

$$P(\Phi_\lambda) = \rho_\lambda P(\Phi).$$

Indeed,

$$\begin{aligned}\rho_\lambda \tfrac{1}{2}(Q^{GF}\otimes 1 + 1\otimes Q^{GF})\Phi &= \lambda^k \tfrac{1}{2}(Q^{GF}\otimes 1 + 1\otimes Q^{GF})\rho_\lambda(\Phi)\\ &= P(\Phi_\lambda).\end{aligned}$$

It follows from this that, for all functionals $I \in \mathscr{O}_P^+(\mathscr{E})[[\hbar]]$,

$$\rho_\lambda(W(P(\Phi)-P(\Psi),I) = W(P(\Phi_\lambda)-P(\Psi_\lambda),\rho_\lambda(I).)$$

We need to verify the renormalization group equation, which states that

$$W(P(\Phi)-P(\Psi),I_\lambda[\Psi]) = I_\lambda[\Phi].$$

Because $I_\lambda[\Phi] = \rho_\lambda I[\Phi_{-\lambda}]$, this is equivalent to

$$\rho_{-\lambda}W(P(\Phi)-P(\Psi),\rho_\lambda(I[\Psi_{-\lambda}]) = I[\Phi_{-\lambda}].$$

Bringing $\rho_{-\lambda}$ inside the W reduces us to proving the identity

$$W(P(\Phi_{-\lambda}-P(\Psi_{-\lambda},I[\Psi_{-\lambda}]) = I[\Phi_{-\lambda}]$$

which is the renormalization group identity for the functionals $I[\Phi]$.

The fact that $I_\lambda[\Phi]$ satisfies the quantum master equation is proved in a similar way, using the fact that

$$\rho_\lambda(\triangle_\Phi I) = \triangle_{\Phi_\lambda}\rho_\lambda(I)$$

where $\triangle_\Phi$ denotes the BV Laplacian associated to Φ and I is any functional.

Finally, the locality axiom is an immediate consequence of that for the original functionals $I[\Phi]$. □

9.3.3.3 Definition. *The* local renormalization group flow *is the action of $\mathbb{R}_{>0}$ on the set of theories that sends, as in the previous lemma, a theory $\{I[\Phi]\}$ to the theory*

$$\{I_\lambda[\Phi]\} = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}\Phi]).$$

Note that this works in families, and so defines an action of $\mathbb{R}_{>0}$ on the simplicial set of theories.

Note that this definition simply means that we act by $\mathbb{R}_{>0}$ on everything involved in the definition of a theory, including the parametrices.

Let us now quote some results from Costello (2011b), concerning the behaviour of this action. Let us recall that to begin with, we chose an action of $\mathbb{R}_{>0}$ on the space $\mathscr{E} = C^\infty(\mathbb{R}^4) \otimes E_0$ of fields, which arose from the natural rescaling action on $C^\infty(\mathbb{R}^4)$ and an action on the finite-dimensional vector space E_0. We assumed that the action on E_0 is diagonalizable, where on each eigenspace ρ_λ acts by λ^a for some $a \in \mathbb{Q}$. Let $m \in \mathbb{Z}$ be such that the exponents of each eigenvalue are in $\frac{1}{m}\mathbb{Z}$.

9.3.3.4 Theorem. *For any theory $\{I[\Phi]\}$ and any parametrix Φ, the family of functionals $I_\lambda[\Phi]$ depending on λ live in*

$$\mathscr{O}^+_{sm,P}(\mathscr{E})\left[\log\lambda, \lambda^{\frac{1}{m}}, \lambda^{-\frac{1}{m}}\right][[\hbar]].$$

In other words, the functionals $I_\lambda[\Phi]$ depend on λ only through polynomials in $\log\lambda$ and $\lambda^{\pm\frac{1}{m}}$. (More precisely, each functional $I_{\lambda,i,k}[\Phi]$ in the Taylor expansion of $I_\lambda[\Phi]$ has such polynomial dependence, but as we quantify over all i and k the degree of the polynomials may be arbitrarily large).

In Costello (2011b), this result is only stated under the hypothesis that $m = 2$, which is the case that arises in most examples, but the proof in Costello (2011b) works in general.

9.3.3.5 Lemma. *The action of* $\mathbb{R}_{>0}$ *on quantum field theories lifts that on classical field theories described earlier.*

This basic point is also discussed in Costello (2011b); it follows from the fact that at the classical level, the limit of $I[\Phi]$ as $\Phi \to 0$ exists and is the original classical interaction.

9.3.3.6 Definition. *A quantum theory is* renormalizable *if the functionals* $I_\lambda[\Phi]$ *depend on* λ *only by polynomials in* $\log \lambda$ *and* $\lambda^{\frac{1}{m}}$ *(where we assume that* $m > 0$*). A quantum theory is* strictly renormalizable *if it only depends on* λ *through polynomials in* $\log \lambda$.

Note that at the classical level, a strictly renormalizable theory must be scale-invariant, because logarithmic contributions to the dependence on λ only arise at the quantum level.

9.3.4 Quantization of renormalizable and strictly renormalizable theories

Let us decompose $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{R}^4}$, the space of translation-invariant local functionals on $\mathscr{E}$, into eigenspaces for the action of $\mathbb{R}_{>0}$. For $k \in \frac{1}{m}\mathbb{Z}$, we let $\mathscr{O}_{loc}^{(k)}(\mathscr{E})^{\mathbb{R}^4}$ be the subspace on which ρ_λ acts by λ^k. Let $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$ denote the direct sum of all the non-negative eigenspaces.

Let us suppose that we are interested in quantizing a classical theory, given by an interaction I, which is either strictly renormalizable or renormalizable. In the first case, I is in $\mathscr{O}_{loc}^{(0)}(\mathscr{E})^{\mathbb{R}^4}$, and in the second, it is in $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$.

By our initial assumptions, the Lie bracket on $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{R}^4}$ commutes with the action of $\mathbb{R}_{>0}$. Thus, if we have a strictly renormalizable classical theory, then $\mathscr{O}_{loc}^{(0)}(\mathscr{E})^{\mathbb{R}^4}$ is a cochain complex with differential $Q + \{I-,\}$. This is the cochain complex controlling first-order deformations of our classical theory as a strictly renormalizable theory. In physics terminology, this is the cochain complex of marginal deformations.

If we start with a classical theory which is simply renormalizable, then the space $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$ is a cochain complex under the differential $Q + \{I-,\}$. This is the cochain complex of renormalizable deformations.

Typically, the cochain complexes of marginal and renormalizable deformations are finite-dimensional. (This happens, for instance, for scalar field theories in dimensions greater than 2.)

Here is the quantization theorem for renormalizable and strictly renormalizable quantizations.

9.3.4.1 Theorem. *Fix a classical theory on $\mathbb{R}^n$ which is renormalizable with classical interaction I. Let $\mathcal{R}^{(n)}$ denote the set of renormalizable quantizations defined modulo $\hbar^{n+1}$. Then, given any element in $\mathcal{R}^{(n)}$, there is an obstruction to quantizing to the next order, which is an element*

$$O_{n+1} \in H^1\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}, Q + \{I, -\}\right).$$

If this obstruction vanishes, then the set of quantizations to the next order is a torsor for $H^0\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}\right)$.

This statement holds in the simplicial sense too: if $\mathcal{R}_{\triangle}^{(n)}$ denotes the simplicial set of renormalizable theories defined modulo $\hbar^{n+1}$ and quantizing a given classical theory, then there is a homotopy fibre diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{R}_{\triangle}^{(n+1)} \arrow[rr] \arrow[d] & & \mathcal{R}_{\triangle}^{(n)} \arrow[d] \\
0 \arrow[r] & \mathrm{DK}\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}[1], Q + \{I, -\}\right) &
\end{tikzcd}$$

On the bottom right DK *indicates the Dold-Kan functor from cochain complexes to simplicial sets.*

All of these statements hold for the (simplicial) sets of strictly renormalizable theories quantizing a given strictly renormalizable classical theory, except that we should replace $\mathscr{O}_{loc}^{(\geq 0)}$ by $\mathscr{O}_{loc}^{(0)}$ everywhere. Further, all these results hold in families iwth evident modifications.

Finally, if $\mathscr{G}\mathscr{F}$ denotes the simplicial set of translation-invariant gauge fixing conditions for our fixed classical theory (where we only consider gauge-fixing conditions which scale well with respect to ρ_λ as discussed earlier), then *the simplicial sets of (strictly) renormalizable theories with a fixed gauge fixing condition are fibres of a simplicial set fibred over $\mathcal{GF}$. As before, this means that the simplicial set of theories is independent up to homotopy of the choice of gauge fixing condition.*

This theorem is proved in Chapter 4 of Costello (2011b), and is the analog of the quantization theorem for theories without the renormalizability criterion.

Let us give some examples of how this theorem allows us to construct small-dimensional families of quantizations of theories where without the renormalizability criterion there would be an infinite dimensional space of quantizations.

Consider, as above, the massless ϕ^4 theory on $\mathbb{R}^4$, with interaction $\int \phi \mathrm{D} \phi + \phi^4$. At the classical level this theory is scale-invariant, and so strictly renormalizable. We have the following.

9.3.4.2 Lemma. *The space of strictly-renormalizable quantizations of the massless ϕ^4 theory in 4 dimensions which are also invariant under the $\mathbb{Z}/2$ action $\phi \mapsto -\phi$ is isomorphic to $\hbar\mathbb{R}[[\hbar]]$. That is, there is a single $\hbar$-dependent coupling constant.*

Proof We need to check that the obstruction group for this problem is zero, and the deformation group is one-dimensional. The obstruction group is zero for degree reasons, because for a theory without gauge symmetry the complex of local functionals is concentrated in degrees ≤ 0. To compute the deformation group, note that the space of local functionals which are scale invariant and invariant under $\phi \mapsto -\phi$ is two-dimensional, spanned by $\int \phi^4$ and $\int \phi \mathrm{D} \phi$. The quotient of this space by the image of the differential $Q + \{I, -\}$ is one dimensional, because we can eliminate one of the two possible terms by a change of coordinates in ϕ. □

Let us give another, and more difficult, example.

9.3.4.3 Theorem. *The space of renormalizable (or strictly renormalizable) quantizations of pure Yang-Mills theory on $\mathbb{R}^4$ with simple gauge Lie algebra $\mathfrak{g}$ is isomorphic to $\hbar\mathbb{R}[[\hbar]]$. That is, there is a single $\hbar$-dependent coupling constant.*

Proof The relevant cohomology groups were computed in Chapter 6 of Costello (2011b), where it was shown that the deformation group is one dimensional and that the obstruction group is $H^5(\mathfrak{g})$. The obstruction group is zero unless $\mathfrak{g} = \mathrm{sl}_n$ and $n \geq 3$. By considering the outer automorphisms of sl_n, it was argued in Costello (2011b) that the obstruction must always vanish. □

This theorem then tells us that we have an essentially canonical quantization of pure Yang-Mills theory on $\mathbb{R}^4$, and hence a corresonding factorization algebra.

The following is the main new result of this section.

9.3.4.4 Theorem. *The map from translation-invariant quantum theories on $\mathbb{R}^n$ to factorization algebras on $\mathbb{R}^n$ commutes with the local renormalization group flow.*

Proof Suppose we have a translation-invariant quantum theory on $\mathbb{R}^n$ with space of fields $\mathscr{E}$ and family of effective actions $\{I[\Phi]\}$. Recall that the RG flow on theories sends this theory to the theory defined by

$$I_\lambda[\Phi] = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}(\Phi)]).$$

We let $\Phi_\lambda = \lambda^k \rho_\lambda \Phi$. As we have seen in the proof of lemma 9.3.3.2, we have

$$\begin{aligned} P(\Phi_\lambda) &= \rho_\lambda(P(\Phi)) \\ \triangle_{\Phi_\lambda} &= \rho_\lambda(\triangle_\Phi). \end{aligned}$$

Suppose that $\{O[\Phi]\}$ is an observable for the theory $\{I[\Phi]\}$. First, we need to show that

$$O_\lambda[\Phi] = \rho_\lambda(O[\Phi_{-\lambda}])$$

is an observable for the theory $O_\lambda[\Phi]$. The fact that $O_\lambda[\Phi]$ satisfies the renormalization group flow equation is proved along the same lines as the proof that $I_\lambda[\Phi]$ satisfies the renormalization group flow equation in lemma 9.3.3.2.

If Obs^q_λ denotes the factorization algebra for the theory I_λ, then we have constructed a map

$$\begin{aligned}\mathrm{Obs}^q(\mathbb{R}^n) &\to \mathrm{Obs}^q_\lambda(\mathbb{R}^n)\\ \{O[\Phi]\} &\mapsto \{O_\lambda[\Phi]\}.\end{aligned}$$

The fact that $\triangle_{\Phi_\lambda} = \rho_\lambda(\triangle_\Phi)$ implies that this is a cochain map. Further, it is clear that this is a smooth map, and so a map of differentiable cochain complexes.

Next we need to check is the support condition. We need to show that if $\{O[\Phi]\}$ is in $\mathrm{Obs}^q(U)$, where $U \subset \mathbb{R}^n$ is open, then $\{O_\lambda[\Phi]\}$ is in $\mathrm{Obs}^q(\lambda^{-1}(U))$. Recall that the support condition states that, for all i, k, there is some parametrix Φ_0 and a compact set $K \subset U$ such that $O_{i,k}[\Phi]$ is supported in K for all $\Phi \leq \Phi_0$.

By making Φ_0 smaller if necessary, we can assume that $O_{i,k}[\Phi_\lambda]$ is supported on K for $\Phi \leq \Phi_0$. (If Φ is supported within ϵ of the diagonal, then Φ_λ is supported within $\lambda^{-1}\epsilon$.) Then, $\rho_\lambda O_{i,k}[\Phi_\lambda]$ will be supported on $\lambda^{-1}K$ for all $\Phi \leq \Phi_0$. This says that O_λ is supported on $\lambda^{-1}K$ as desired.

Thus, we have constructed an isomorphism

$$\mathrm{Obs}^q(U) \cong \mathrm{Obs}^q_\lambda(\lambda^{-1}(U)).$$

This isomorphism is compatible with inclusion maps and with the factorization product. Therefore, we have an isomorphism of factorization algebras

$$(\lambda^{-1})_* \mathrm{Obs}^q \cong \mathrm{Obs}^q_\lambda$$

where $(\lambda^{-1})_*$ indicates pushforward under the map given by multiplication by λ^{-1}. Since the action of the local renormalization group flow on factorization algebras on $\mathbb{R}^n$ sends $\mathcal{F}$ to $(\lambda^{-1})_*\mathcal{F}$, this proves the result. □

The advantage of the factorization algebra formulation of the local renormalization group flow is that it is very easy to define; it captures precisely the intuition that the renormalization group flow arises the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$. This theorem shows that the less-obvious definition of the renormalization group flow on theories, as defined in Costello (2011b), coincides with the very clear definition in the language of factorization algebras. The advantage of the definition presented in Costello (2011b) is that it is possible to compute with this definition, and that the relationship between this definition and how physicists define the β-function is more or less clear. For example, the one-loop β-function (one-loop contribution to the renormalization group flow) is calculated explicitly for the ϕ^4 theory in Costello (2011b).

9.4 Cotangent theories and volume forms

In this section we will examine the case of a cotangent theory, in which our definition of a quantization of a classical field theory acquires a particularly nice interpretation. Suppose that $\mathcal{L}$ is an elliptic L_∞ algebra on a manifold M describing an elliptic moduli problem, which we denote by $B\mathcal{L}$. As we explained in section 4.6, we can construct a classical field theory from $\mathcal{L}$, whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$. The main observation of this section is that a quantization of this classical field theory can be interpreted as a kind of "volume form" on the elliptic moduli problem $B\mathcal{L}$. This point of view was developed in Costello (2013a), and used in Costello (2011a) to provide a geometric interpretation of the Witten genus.

The relationship between quantization of field theories and volume forms was discussed already at the very beginning of this book, in Chapter I.2. There, we explained how to interpret (heuristically) the BV operator for a free field theory as the divergence operator for a volume form.

While this heuristic interpretation holds for many field theories, cotangent theories are a class of theories where this relationship becomes very clean. If we have a cotangent theory to an elliptic moduli problem $\mathcal{L}$ on a compact manifold, then the L_∞ algebra $\mathcal{L}(M)$ has finite-dimensional cohomology. Therefore, the formal moduli problem $B\mathcal{L}(M)$ is an honest finite-dimensional formal derived stack. We will find that a quantization of a cotangent theory leads to a volume form on $B\mathcal{L}(M)$ that is of a "local" nature.

Morally speaking, the partition function of a cotangent theory should be the volume of $B\mathcal{L}(M)$ with respect to this volume form. If, as we've been doing, we work in perturbation theory, then the integral giving this volume often does not converge. One has to replace $B\mathcal{L}(M)$ by a global derived moduli space of solutions to the equations of motion to have a chance at defining the volume. The volume form on a global moduli space is obtained by doing perturbation theory near every point and then gluing together the formal volume forms so obtained near each point.

This program has been successfully carried out in a number of examples, such as Costello (2011a); Gwilliam and Grady (2014); Li and Li (2016). For example, Costello (2011a) studied the cotangent theory to the space of holomorphic maps from an elliptic curve to a complex manifold, and it was shown that the partition function (defined in the way we sketched above) is the Witten elliptic genus.

9.4.1 A finite dimensional model

We first need to explain an algebraic interpretation of a volume form in finite dimensions. Let X be a manifold (or complex manifold or smooth algebraic variety; nothing we will say will depend on which geometric category we work in). Let $\mathscr{O}(X)$ denote the smooth functions on X, and let $\mathrm{Vect}(X)$ denote the vector fields on X.

If ω is a volume form on X, then it gives a divergence map

$$\mathrm{Div}_\omega : \mathrm{Vect}(X) \to \mathscr{O}(X)$$

defined via the Lie derivative:

$$\mathrm{Div}_\omega(V)\omega = \mathcal{L}_V \omega$$

for $V \in \mathrm{Vect}(X)$. Note that the divergence operator Div_ω satisfies the equations

$$\begin{aligned} \mathrm{Div}_\omega(fV) &= f\,\mathrm{Div}_\omega V + V(f). \\ \mathrm{Div}_\omega([V,W]) &= V\,\mathrm{Div}_\omega W - W\,\mathrm{Div}_\omega V. \end{aligned} \tag{$\dagger$}$$

The volume form ω is determined up to a constant by the divergence operator Div_ω.

Conversely, to give an operator $\mathrm{Div} : \mathrm{Vect}(X) \to \mathscr{O}(X)$ satisfying equations (†) is the same as to give a flat connection on the canonical bundle K_X of X, or, equivalently, to give a right D-module structure on the structure sheaf $\mathscr{O}(X)$.

9.4.1.1 Definition. *A* projective volume form *on a space X is an operator* $\mathrm{Div} : \mathrm{Vect}(X) \to \mathscr{O}(X)$ *satisfying equations (†).*

The advantage of this definition is that it makes sense in many contexts where more standard definitions of a volume form are hard to define. For example, if A is a quasi-free differential graded commutative algebra, then we can define a projective volume form on the dg scheme $\mathrm{Spec}\, A$ to be a cochain map $\mathrm{Der}(A) \to A$ satisfying equations (†). Similarly, if $\mathfrak{g}$ is a dg Lie or L_∞ algebra, then a projective volume form on the formal moduli problem $B\mathfrak{g}$ is a cochain map $C^*(\mathfrak{g}, \mathfrak{g}[1]) \to C^*(\mathfrak{g})$ satisfying equations (†).

9.4.2

There is a generalization of this notion that we will use where, instead of vector fields, we take any Lie algebroid.

9.4.2.1 Definition. *Let A be a commutative differential graded algebra over a base ring k. A* Lie algebroid *L over A is a dg A-module with the following extra data.*

(i) *A Lie bracket on L making it into a dg Lie algebra over k. This Lie bracket will be typically not A-linear.*
(ii) *A homomorphism of dg Lie algebras $\alpha : L \to \mathrm{Der}^*(A)$, called the anchor map.*
(iii) *These structures are related by the Leibniz rule*
$$[l_1, fl_2] = (\alpha(l_1)(f))\, l_2 + (-1)^{|l_1||f|} f[l_1, l_2]$$
for $f \in A$, $l_i \in L$.

In general, we should think of L as providing the derived version of a foliation. In ordinary as opposed to derived algebraic geometry, a foliation on a smooth affine scheme with algebra of functions A consists of a Lie algebroid L on A which is projective as an A-module and whose anchor map is fibrewise injective.

9.4.2.2 Definition. *If A is a commutative dg algebra and L is a Lie algebroid over A, then an L-projective volume form on A is a cochain map*

$$\mathrm{Div} : L \to A$$

satisfying

$$\begin{aligned} \mathrm{Div}(al) &= a \,\mathrm{Div}\, l + (-1)^{|l||a|}\alpha(l)a. \\ \mathrm{Div}([l_1, l_2]) &= l_1 \,\mathrm{Div}\, l_2 - (-1)^{|l_1||l_2|l_2} \,\mathrm{Div}\, l_1. \end{aligned}$$

Of course, if the anchor map is an isomorphism, then this structure is the same as a projective volume form on A. In the more general case, we should think of an L-projective volume form as giving a projective volume form on the leaves of the derived foliation.

9.4.3

Let us explain how this definition relates to the notion of quantization of P_0 algebras.

9.4.3.1 Definition. *Give the operad P_0 a $\mathbb{C}^\times$ action where the product has weight 0 and the Poisson bracket has weight 1. A graded P_0 algebra is a $\mathbb{C}^\times$-equivariant differential graded algebra over this dg operad.*

Note that, if X is a manifold, $\mathscr{O}(T^*[-1]X)$ has the structure of graded P_0 algebra, where the $\mathbb{C}^\times$ action on $\mathscr{O}(T^*[-1]X)$ is given by rescaling the cotangent fibers.

Similarly, if L is a Lie algebroid over a commutative dg algebra A, then $\mathrm{Sym}^*_A L[1]$ is a $\mathbb{C}^\times$-equivariant P_0 algebra. The P_0 bracket is defined by the bracket on L and the L-action on A; the $\mathbb{C}^\times$ action gives $\mathrm{Sym}^k L[1]$ weight $-k$.

9.4.3.2 Definition. *Give the operad BD over $\mathbb{C}[[\hbar]]$ a $\mathbb{C}^\times$ action, covering the $\mathbb{C}^\times$ action on $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, the product has weight 0, and the Poisson bracket has weight 1.*

Note that this $\mathbb{C}^\times$ action respects the differential on the operad BD, which is defined on generators by

$$\mathrm{d}(- * -) = \hbar\{-,-\}.$$

Note also that by describing the operad BD as a $\mathbb{C}^\times$-equivariant family of operads over $\mathbb{A}^1$, we have presented BD as a filtered operad whose associated graded operad is P_0.

9.4.3.3 Definition. *A* filtered BD algebra *is a BD algebra A with a $\mathbb{C}^\times$ action compatible with the $\mathbb{C}^\times$ action on the ground ring $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, and compatible with the $\mathbb{C}^\times$ action on the operad BD.*

9.4.3.4 Lemma. *If L is Lie algebroid over a dg commutative algebra A, then every L-projective volume form yields a filtered BD algebra structure on $\mathrm{Sym}_A^*(L[1])[[\hbar]]$, quantizing the graded P_0 algebra $\mathrm{Sym}_A^*(L[1])$.*

Proof If $\mathrm{Div} : L \to A$ is an L-projective volume form, then it extends uniquely to an order two differential operator $\triangle$ on $\mathrm{Sym}_A^*(L[1])$ which maps

$$\mathrm{Sym}_A^i(L[1]) \to \mathrm{Sym}_A^{i-1}(L[1]).$$

Then $\mathrm{Sym}_A^* L[1][[\hbar]]$, with differential $\mathrm{d} + \hbar\triangle$, gives the desired filtered BD algebra. □

9.4.4

Let $B\mathcal{L}$ be an elliptic moduli problem on a compact manifold M. The main result of this section is that there exists a special kind of quantization of the cotangent field theory for $B\mathcal{L}$ that gives a projective volume on this formal moduli problem $B\mathcal{L}$. Projective volume forms arising in this way have a special "locality" property, reflecting the locality appearing in our definition of a field theory.

Thus, let $\mathcal{L}$ be an elliptic L_∞ algebra on M. This gives rise to a classical field theory whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$, as described in section 4.6. Let us give the space $\mathscr{E}$ a $\mathbb{C}^\times$-action where $\mathcal{L}[1]$ has weight 0 and $\mathcal{L}^![-1]$ has weight 1. This induces a $\mathbb{C}^\times$ action on all associated spaces, such as $\mathscr{O}(\mathscr{E})$ and $\mathscr{O}_{loc}(\mathscr{E})$.

This $\mathbb{C}^\times$ action preserves the differential $Q + \{I, -\}$ on $\mathscr{O}(\mathscr{E})$, as well as the commutative product. Recall from section 5.2 that the subspace

$$\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E}) \subset \mathscr{O}(\mathscr{E})$$

of functionals with smooth first derivative has a Poisson bracket of cohomological degree 1, making it into a P_0 algebra. This Poisson bracket is of weight 1 with respect to the $\mathbb{C}^\times$ action on $\widetilde{\mathrm{Obs}}^{cl}(M)$, so $\widetilde{\mathrm{Obs}}^{cl}(M)$ is a graded P_0 algebra.

We are interested in quantizations of our field theory where the BD algebra $\mathrm{Obs}^q_\Phi(M)$ of (global) quantum observables (defined using a parametrix Φ) is a filtered BD algebra.

9.4.4.1 Definition. *A* cotangent quantization *of a cotangent theory is a quantization, given by effective interaction functionals $I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Φ, such that $I[\Phi]$ is of weight -1 under the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ of functionals.*

This $\mathbb{C}^\times$ action gives $\hbar$ weight -1. Thus, this condition means that if we expand

$$I[\Phi] = \sum \hbar^i I_i[\Phi],$$

then the functionals $I_i[\Phi]$ are of weight $i-1$.

Since the fields $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ decompose into spaces of weights 0 and 1 under the $\mathbb{C}^\times$ action, we see that $I_0[\Phi]$ is linear as a function of $\mathcal{L}^![-2]$, that $I_1[\Phi]$ is a function only of $\mathcal{L}[1]$, and that $I_i[\Phi] = 0$ for $i > 1$.

Remark: (i) The quantization $\{I[\Phi]\}$ is a cotangent quantization if and only if the differential $Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi$ preserves the $\mathbb{C}^\times$ action on the space $\mathscr{O}(\mathscr{E})[[\hbar]]$ of functionals. Thus, $\{I[\Phi]\}$ is a cotangent quantization if and only if the BD algebra $\mathrm{Obs}^q_\Phi(M)$ is a filtered BD algebra for each parametrix Φ.

(ii) The condition that $I_0[\Phi]$ is of weight -1 is automatic.

(iii) It is easy to see that the renormalization group flow

$$W\left(P(\Phi) - P(\Psi), -\right)$$

commutes with the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$.

$\Diamond$

9.4.5

Let us now explain the volume-form interpretation of cotangent quantization. Let $\mathcal{L}$ be an elliptic L_∞ algebra on M, and let $\mathscr{O}(B\mathcal{L}) = C^*(\mathcal{L})$ be the Chevalley-Eilenberg cochain complex of M. The cochain complexes $\mathscr{O}(B\mathcal{L}(U))$ for open subsets $U \subset M$ define a commutative factorization algebra on M.

As we have seen in section 3.1.3, we should interpret modules for an L_∞ algebra $\mathfrak{g}$ as sheaves on the formal moduli problem $B\mathfrak{g}$. The $\mathfrak{g}$-module $\mathfrak{g}[1]$ corresponds to the tangent bundle of $B\mathfrak{g}$, and so vector fields on $\mathfrak{g}$ correspond to the $\mathscr{O}(B\mathfrak{g})$-module $C^*(\mathfrak{g}, \mathfrak{g}[1])$.

Thus, we use the notation

$$\mathrm{Vect}(B\mathcal{L}) = C^*(\mathcal{L}, \mathcal{L}[1]);$$

this is a dg Lie algebra and acts on $C^*(\mathcal{L})$ by derivations (see Appendix B.2, for details).

For any open subset $U \subset M$, the $\mathcal{L}(U)$-module $\mathcal{L}(U)[1]$ has a submodule $\mathcal{L}_c(U)[1]$ given by compactly supported elements of $\mathcal{L}(U)[1]$. Thus, we have a sub-$\mathscr{O}(B\mathcal{L}(U))$-module

$$\mathrm{Vect}_c(B\mathcal{L}(U)) = C^*(\mathcal{L}(U), \mathcal{L}_c(U)[1]) \subset \mathrm{Vect}(B\mathcal{L}(U)).$$

This is in fact a sub-dg Lie algebra, and hence a Lie algebroid over the dg commutative algebra $\mathscr{O}(B\mathcal{L}(U))$. Thus, we should view the subspace $\mathcal{L}_c(U)[1] \subset \mathcal{L}_c(U)[1]$ as providing a foliation of the formal moduli problem $B\mathcal{L}(U)$, where two points of $B\mathcal{L}(U)$ are in the same leaf if they coincide outside a compact subset of U.

If $U \subset V$ are open subsets of M, there is a restriction map of L_∞ algebras $\mathcal{L}(V) \to \mathcal{L}(U)$. The natural extension map $\mathcal{L}_c(U)[1] \to \mathcal{L}_c(V)[1]$ is a map of $\mathcal{L}(V)$-modules. Thus, by taking cochains, we find a map

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \to \mathrm{Vect}_c(B\mathcal{L}(V)).$$

Geometrically, we should think of this map as follows. If we have an R-point α of $B\mathcal{L}(V)$ for some dg Artinian ring R, then any compactly-supported deformation of the restriction $\alpha\ |_U$ of α to U extends to a compactly supported deformation of α.

We want to say that a cotangent quantization of $\mathcal{L}$ leads to a "local" projective volume form on the formal moduli problem $B\mathcal{L}(M)$ if M is compact. If M is compact, then $\mathrm{Vect}_c(B\mathcal{L}(M))$ coincides with $\mathrm{Vect}(B\mathcal{L}(M))$. A local projective volume form on $B\mathcal{L}(M)$ should be something like a divergence operator

$$\mathrm{Div} : \mathrm{Vect}(B\mathcal{L}(M)) \to \mathscr{O}(B\mathcal{L}(M))$$

satisfying the equations (†), with the locality property that Div maps the subspace

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \subset \mathrm{Vect}(B\mathcal{L}(M))$$

to the subspace $\mathscr{O}(B\mathcal{L}(U)) \subset \mathscr{O}(B\mathcal{L}(M))$.

Note that a projective volume form for the Lie algebroid $\mathrm{Vect}_c(B\mathcal{L}(U))$ over $\mathscr{O}(B\mathcal{L}(U))$ is a projective volume form on the leaves of the foliation of $B\mathcal{L}(U)$ given by $\mathrm{Vect}_c(B\mathcal{L}(U))$. The leaf space for this foliation is described by the L_∞ algebra

$$\mathcal{L}_\infty(U) = \mathcal{L}(U)/\mathcal{L}_c(U) = \operatorname*{colim}_{K \subset U} \mathcal{L}(U \setminus K).$$

(Here the colimit is taken over all compact subsets of U.) Consider the one-point compactification U_∞ of U. Then the formal moduli problem $\mathcal{L}_\infty(U)$ describes the germs at ∞ on U_∞ of sections of the sheaf on U of formal moduli problems given by $\mathcal{L}$.

Thus, the structure we're looking for is a projective volume form on the fibers of the maps $B\mathcal{L}(U) \to B\mathcal{L}_\infty(U)$ for every open subset $U \subset M$, where the divergence operators describing these projective volume forms are all compatible in the sense described above.

What we actually find is something a little weaker. To state the result, recall (section 8.2) that we use the notation $\mathscr{P}$ for the contractible simplicial set of parametrices, and $\mathscr{C}\mathscr{P}$ for the cone on $\mathscr{P}$. The vertex of the cone $\mathscr{C}\mathscr{P}$ will denoted $\bar{0}$.

9.4.5.1 Theorem. *A cotangent quantization of the cotangent theory associated to the elliptic L_∞ algebra $\mathcal{L}$ leads to the following data.*

(i) *A commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$. The underlying graded algebra of this commutative dg algebra is $\mathscr{O}(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. The restriction of this commutative dg algebra to the vertex $\bar{0}$ of $\mathscr{C}\mathscr{P}$ is the commutative dg algebra $\mathscr{O}(B\mathcal{L})$.*

(ii) *A dg Lie algebroid* $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ *over* $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$*, whose underlying graded* $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$*-module is* $\mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$*. At the vertex* $\overline{0}$ *of* $\mathscr{C}\mathscr{P}$*, the dg Lie algebroid* $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ *coincides with the dg Lie algebroid* $\mathrm{Vect}_c(B\mathcal{L})$*.*

(iii) *We let* $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ *and* $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ *be the restrictions of* $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ *and* $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ *to* $\mathscr{P} \subset \mathscr{C}\mathscr{P}$*. Then we have a divergence operator*

$$\mathrm{Div}_{\mathscr{P}} : \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \mathscr{O}_{\mathscr{P}}(B\mathcal{L})$$

defining the structure of a $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ *projective volume form on* $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ *and* $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$*.*

Further, when restricted to the sub-simplicial set $\mathscr{P}_U \subset \mathscr{P}$ *of parametrices with support in a small neighborhood of the diagonal* $U \subset M \times M$*, all structures increase support by an arbitrarily small amount (more precisely, by an amount linear in* U*, in the sense explained in section 8.2).*

Proof This follows almost immediately from theorem 8.2.2.1. Indeed, because we have a cotangent theory, we have a filtered BD algebra

$$\mathrm{Obs}^q_{\mathscr{P}}(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P}), \widehat{Q}_{\mathscr{P}}, \{-,-\}_{\mathscr{P}}\right).$$

Let us consider the sub-BD algebra $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$, which, as a graded vector space, is $\mathscr{O}_{sm}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P})$ (as usual, $\mathscr{O}_{sm}(\mathscr{E})$ indicates the space of functionals with smooth first derivative).

Because we have a filtered BD algebra, there is a $\mathbb{C}^\times$-action on this complex $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$. We let

$$\mathscr{O}_{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^0$$

be the weight 0 subspace. This is a commutative differential graded algebra over $\Omega^*(\mathscr{P})$, whose underlying graded algebra is $\mathscr{O}(B\mathcal{L})$; further, it extends (using again the results of 8.2.2.1) to a commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$, which when restricted to the vertex is $\mathscr{O}(B\mathcal{L})$.

Next, consider the weight -1 subspace. As a graded vector space, this is

$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{P}) \oplus \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

We thus let

$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} / \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

The Poisson bracket on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$ is of weight 1, and it makes the space $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ into a sub Lie algebra.

We have a natural decomposition of graded vector spaces

$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \oplus \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

The dg Lie algebra structure on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ gives us

(i) The structure of a dg Lie algebra on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ (as the quotient of $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ by the differential Lie algebra ideal $\hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$).
(ii) An action of $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ by derivations; this defines the anchor map for the Lie algebroid structure on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.
(iii) A cochain map
$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$
This defines the divergence operator.

It is easy to verify from the construction of theorem 8.2.2.1 that all the desired properties hold. □

9.4.6

The general results about quantization of Costello (2011b) thus apply in this situation to show the following.

9.4.6.1 Theorem. *Consider the cotangent theory $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ to an elliptic moduli problem described by an elliptic L_∞ algebra $\mathcal{L}$ on a manifold M.*

The obstruction to constructing a cotangent quantization is an element in

$$H^1(\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^\times}) = H^1(\mathscr{O}_{loc}(B\mathcal{L})).$$

If this obstruction vanishes, then the simplicial set of cotangent quantizations is a torsor for the simplicial Abelian group arising from the cochain complex $\mathscr{O}_{loc}(B\mathcal{L})$ by the Dold-Kan correspondence.

As in section 3.5, we are using the notation $\mathscr{O}_{loc}(B\mathcal{L})$ to refer to a "local" Chevalley-Eilenberg cochain for the elliptic L_∞ algebra $\mathcal{L}$. If L is the vector bundle whose sections are $\mathcal{L}$, then as we explained in Costello (2011b), the jet bundle $J(L)$ is a D_M L_∞ algebra and

$$\mathscr{O}_{loc}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$

There is a de Rham differential (see section 4.3) mapping $\mathscr{O}_{loc}(B\mathcal{L})$ to the complex of local 1-forms,

$$\Omega^1_{loc}(B\mathcal{L}) = C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

The de Rham differential maps $\mathscr{O}_{loc}(B\mathcal{L})$ isomorphically to the subcomplex of $\Omega^1_{loc} B\mathcal{L})$ of closed local one-forms. Thus, the obstruction is a local closed 1-form on $B\mathcal{L}$ of cohomology degree 1: it is in

$$H^1(\Omega^1_{loc}(B\mathcal{L}).$$

Since the obstruction to quantizing the theory is the obstruction to finding a locally-defined volume form on $B\mathcal{L}$, we should view this obstruction as being the local first Chern class of $B\mathcal{L}$.

9.5 Correlation functions

So far in this chapter, we have proved the quantization theorem showing that from a field theory we can construct a factorization algebra. We like to think that this factorization algebra encodes most things one would want to with a quantum field theory in perturbation theory. To illustrate this, in this section, we will explain how to construct correlation functions form the factorization algebra, under certain additional hypothesis.

Suppose we have a field theory on a compact manifold M, with space of fields $\mathscr{E}$ and linearized differential Q on the space of fields. Furthermore, suppose that

$$H^*(\mathscr{E}(M), Q) = 0.$$

The complex $(\mathscr{E}(M), Q)$ is the tangent complex for the formal moduli space of solutions to the equation of motion to our field theory, at the base point around which we are doing perturbation theory. The statement that this tangent complex has no cohomology means that the trivial solution of the equation of motion has no deformations (up to whatever gauge symmetry we have). In other words, we are working with an isolated point in the moduli of solutions to the equations of motion.

As an example, consider a massive interacting scalar field theory on a compact manifold M, with action functional for example

$$\int_M \phi(\mathrm{D} + m^2)\phi + \phi^4$$

where $\phi \in C^\infty(M)$ and $m > 0$. Then, the complex $\mathscr{E}(M)$ of fields is the complex

$$C^\infty(M) \xrightarrow{\mathrm{D}+m^2} C^\infty(M).$$

Hodge theory tells us that this complex has no cohomology.

Let Obs^q denote the factorization algebra of quantum observables of a quantum field theory that satisfies this (classical) condition.

9.5.0.1 Lemma. *In this situation, there is a canonical isomorphism*

$$H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]].$$

Note that we usually work, for simplicity, with complex vector spaces; this result holds where everything is real too, in which case we find $\mathbb{R}[[\hbar]]$ on the right hand side.

Proof There is a spectral sequence

$$H^*(\mathrm{Obs}^{cl}(M))[[\hbar]] \twoheadrightarrow H^*(\mathrm{Obs}^q(M)).$$

Further, $\mathrm{Obs}^{cl}(M)$ has a complete decreasing filtration whose associated graded is the complex

$$\mathrm{Gr}\,\mathrm{Obs}^{cl}(M) = \prod_n \mathrm{Sym}^n(\mathscr{E}(M)^\vee)$$

with differential arising from the linear differential Q on $\mathscr{E}(M)$. The condition that $H^*(\mathscr{E}(M), Q) = 0$ implies that the cohomology of $\mathrm{Sym}^n(\mathscr{E}(M)^\vee)$ is also zero, so that $H^*(\mathrm{Obs}^{cl}(M)) = \mathbb{C}$. This shows that there is an isomorphism of $\mathbb{C}[[\hbar]]$-modules from $H^*(\mathrm{Obs}^q(M))$ to $\mathbb{C}[[\hbar]]$. To make this isomorphism canonical, we declare that the vacuum observable $|0\rangle \in H^0(\mathrm{Obs}^q(M))$ (that is, the unit in the factorization algebra) gets sent to $1 \in \mathbb{C}[[\hbar]]$. □

9.5.0.2 Definition. *As above, let* Obs^q *denote the factorization algebra of observables of a quantum field theory on* M *that satisfies* $H^*(\mathscr{E}(M), Q) = 0$.

Let $U_1, \dots, U_n \subset M$ *be disjoint open sets, and let* $O_i \in \mathrm{Obs}^q(U_i)$. *Define the* expectation value *(or correlation function) of the observables* O_i, *denoted by*

$$\langle O_1, \dots, O_n \rangle \in \mathbb{C}[[\hbar]],$$

to be the image of the product observable

$$O_1 * \cdots * O_n \in H^*(\mathrm{Obs}^q(M))$$

under the canonical isomorphism between $H^*(\mathrm{Obs}^q(M))$ *and* $\mathbb{C}[[\hbar]]$.

We have already encountered this definition when we discussed free theories (see section I.4.7). There we saw that this definition reproduced the usual physics definitions of correlation functions for free field theories.

10
Operator product expansions, with examples

We have constructed the factorization algebra of observables of an interacting quantum field theory, which encodes a huge amount of information about the theory. *A priori* it is not obvious what parts of this structure are fruitful to compute. In this section we will focus on a class of computations that have a clear meaning in mathematics and physics, and which are rather easy to implement in practice.

For readers already familiar with quantum field theory, we expect this chapter may be quite helpful in connecting established approaches to ours. In particular, we revisit here notions such as *local operators* and *operator product expansion* and identify them within the factorization algebra of quantum observables.

10.1 Point observables

Let Obs be a factorization algebra of observables on $\mathbb{R}^n$ for some translation-invariant quantum field theory. For every point $p \in \mathbb{R}^n$, we can consider *point observables* at p, which are the observables that live in the intersection of the observables $\mathrm{Obs}(V)$ as V ranges over neighbourhoods of p.

At the classical level, a point observable is a function on the fields of the theory that only depends on the value of the field and the value of its derivatives at p. At the quantum level, we define an observable $\mathcal{O}$

on V to be a family of functionals $\mathcal{O}[\Psi]$, one for every parametrix, such that the functionals $\mathcal{O}_{i,k}[\Psi]$ are supported on V for sufficiently small Ψ. A point observable is the same, except we strengthen the support condition to say that $\mathcal{O}_{i,k}[\Psi]$ is supported in an arbitrarily small neighborhood of p for sufficiently small Ψ.

Physicists often use the term "local operators" for point observables. (We use the term point observable to avoid any confusion with what we call local observables, which are integrals of point observables over the manifold.) Hence, point observables are the observables that are usually first studied in a quantum field theory.

We now formulate a precise definition.

10.1.0.1 Definition. *The* classical point observables $\mathrm{Obs}^{cl}(p)$ *are the limit of* $\mathrm{Obs}^{cl}(V)$ *as* V *runs over the open sets containing* p*. Likewise, the* quantum point observables $\mathrm{Obs}^{q}(p)$ *are the limit of* $\mathrm{Obs}^{q}(V)$ *as* V *runs over the open sets containing* p*.*

Remark: Here we mean the *strict* limit, in the sense that we take the limit as differentiable vector spaces in each cohomological degree. This definition is not particularly appealing mathematically, as the homotopy limit in $\mathrm{Ch}(\mathrm{DVS})$ is the only homotopically meaningful definition. As this chapter shows, however, this definition does capture and refine important notions in physics. In section 10.1.2 below, we discuss these issues and alternative approaches. $\Diamond$

Given the key role of local operators in conventional treatments of QFT, one can ask: why do we not try to formulate everything in terms of point observables, leading to a version of factorization algebras formulated in terms of points rather than open subsets? In such an approach, the factorization product would be replaced by the operator product expansion (OPE), which describes the factorization product of two point observables when they are very close to each other. We discuss OPE in detail below.

There are a number of reasons for our choice to emphasize observables living on open subsets rather than points. One reason is that it appears to be difficult to formulate an associativity axiom for point observables that is satisfied by an arbitrary interacting (non-conformal) quantum field theory. The associativity axiom would involve the expansion of the factorization product of three point observables when they are all close to each other. In an interacting QFT, it is not at all clear that this factorization product admits a well-behaved expansion.

Ultimately, we chose to work with our formulation based on open subsets because we found it to be a more flexible language, rich enough to capture both point observables and other classes of observables like Wilson lines.

10.1.1 Quantizing classical point observables

Given a classical point observable, we can ask if it lifts to a quantum one. Our main results tell us that a classical point observable at p must lift to a quantum observable defined in some neighborhood V of p. After all, the complex $\mathrm{Obs}^q(V)$ is flat over $\mathbb{C}[[\hbar]]$. There is no guarantee that the support is preserved, though, so the our main results do not imply automatically that classical point observables lift to quantum point observables. This statement does follow, however, from a result proven in Costello (2011b).

10.1.1.1 Proposition. *For any translation-invariant field theory on $\mathbb{R}^n$, the cochain complex $\mathrm{Obs}^q(p)$ of point observables is flat over $\mathbb{C}[[\hbar]]$, and it reduces modulo $\hbar$ to the complex of $\mathrm{Obs}^{cl}(p)$ of classical point observables.*

The proof below is a little technical, and it is not essential to understand it in order to follow the rest of this chapter. Some of the notation and key ideas are, however, useful, so we discuss them first.

First, notice that due to translation invariance, the point observables at p are all translates of point observables at the origin of $\mathbb{R}^n$.

10.1.1.2 Definition. *Given a point observable $\mathcal{O} \in \mathrm{Obs}^{cl}(0)$, let $\mathcal{O}(p)$ denote the translate of $\mathcal{O}$ to $p \in \mathbb{R}^n$.*

Note that because we can differentiate observables using the infinitesimal action of translation, $\mathrm{Obs}^{cl}(0)$ is a module for the algebra $\mathbb{R}[\partial_{x_1}, \ldots, \partial_{x_n}]$ of constant-coefficient differential operators. (If we use complex coefficients, it is a module for $\mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}]$.) A similar remark applies to the quantum point observables.

Second, we introduce a different description of classical point observables, useful in quantizing. Let $\mathscr{E}$ denote the space of fields, which is $C^\infty(\mathbb{R}^n) \otimes E$ for some graded vector space E, let $S \in \mathscr{O}_{loc}(\mathscr{E})$ denote the classical action functional.

Consider now action functionals

$$\widetilde{S} \in \mathscr{O}_{loc}(\mathscr{E} \oplus \Omega^d(\mathbb{R}^n))$$

that depend on background fields in $\Omega^n(\mathbb{R}^n)$. Restrict attention to such functionals that are at most linear in the field $\omega \in \Omega^d(\mathbb{R}^n)$ and that become the original action functional S when restricted to $\mathscr{E}$ (i.e., when $\omega = 0$). Furthermore, assume that $\widetilde{S}$ is translation invariant.

Any $\widetilde{S}$ is then of the form

$$\widetilde{S} = S + \int_{x \in \mathbb{R}^n} \omega\, \mathcal{O}(x),$$

where in the second term, we have a point observable $\mathcal{O} \in \mathrm{Obs}^{cl}(0)$ and we integrate its translates $\mathcal{O}(x)$ again the background field $\omega \in \Omega^d(\mathbb{R}^n)$. This construction establishes a correspondence between point observables and this class of translation-invariant local functionals, which is in fact an isomorphism. (To recover O from $\widetilde{S}$, take the functional derivative of $\widetilde{S}$ along the background field ω.) We thus know the following.

10.1.1.3 Lemma. *The point observables* $\mathrm{Obs}^{cl}(0)$ *are canonically identified with translation-invariant local functionals of both* $\mathscr{E}$ *and* $\Omega^n(\mathbb{R}^n)$ *that depend linearly on the background field in* $\Omega^n(\mathbb{R}^n)$.

This description will let us construct point operators at the quantum level. The methods developed in Costello (2011b) allow one to consider quantum field theories that depend on background fields, which we now use in our proof.

Proof of proposition The essential idea is that to produce a quantum point observable lifting a given classical one, we simply write a classical point observable as an action functional that depends on a background field, lift it to a quantum theory depending on the background field, and then extract the corresponding quantum point observable. We now make that idea precise.

Section 13 of chapter 2 of Costello (2011b) considers theories where the space of fields is decomposed as a direct sum into "propagating" fields (which will be quantized) and "background" fields (which remain classical). In our situation of interest, we have $\mathscr{E}$ as the propagating fields and $\Omega^d(\mathbb{R}^n)$ as the background fields. A quantization amounts to producing a collection of functionals $I_{b.g.}[\Psi]$, one for every parametrix Ψ, which are functions of the appropriate class of the fields in $\mathscr{E} \oplus \Omega^d(\mathbb{R}^n)$. Theorems 13.4.3 of chapter 2 and 3.1.2 of chapter 3 of Costello (2011b) imply that any classical theory with background fields lifts to a quantum theory with background fields. Here we do not ask that the quantum master equation hold; we only require that the other axioms of a field theory hold.

Based on our lemma above, we view a classical point observable $\mathcal{O}$ as encoded in an action functional $\widetilde{S}$. We thus focus on quantizations of the form

$$I_{b.g.}[\Psi] = I[\Psi] + I'[\Psi],$$

where $I[\Psi]$ is the quantization of the field theory with which we started (and hence does not depend on the field $\omega \in \Omega^d(\mathbb{R}^n)$) and where $I'[\Psi]$ is linear in the background field $\omega \in \Omega^d(\mathbb{R}^n)$.

Such action functionals are required to have smooth first derivative. This condition means that the functional

$$\mathcal{O}[\Psi] = \frac{\partial}{\partial \delta_0} I'[\Psi]$$

is well-defined, where we differentiate $I'[\Psi]$ along the background field given by the δ-function at the origin in $\mathbb{R}^n$. (*A priori* we can only differentiate against smooth ω, so this extension is a condition.)

The renormalization group flow satisfied by $I'[\Psi]$ implies that $\mathcal{O}[\Psi]$ is a quantum observable on some neighbourhood of 0. The locality condition satisfied by $I'[\Psi]$ implies that $\mathcal{O}[\Psi]$ becomes supported on an arbitrarily small neighborhood of 0, as $\Psi \to 0$. In other words, the collection $\{\mathcal{O}[\Psi]\}$ is a point observable. □

10.1.2 An extended remark on defining point observables

Our book's central theme is that perturbative field theory is encoded cleanly and effectively in factorization algebras, and so we want to show how important ideas from physics translate into the language of factorization algebras. At times, however, there is a tension between these two domains, especially when relating concrete field-theoretic computations with the rigors of higher abstract nonsense. (Such tensions are familiar in the history of math: think of the adoption and internalization of sheaf theory into algebraic geometry or of homological algebra into topology.) In this chapter, this tension appears in the definition of point observables.

On the one hand, there are the usual examples from physics, built from the delta function, its derivatives, and polynomials built from such linear functionals on the fields. It is a standard fact that the distributions — continuous linear functionals on smooth functions — that are supported at a point p are spanned by δ_p and its derivatives $\partial^\mu \delta_p$. In more sophisticated language, if $C^\infty(V)'$ denotes the continuous linear dual space to $C^\infty(V)$ for some open set $V \subset \mathbb{R}^n$, then

$$C^\infty(p)' = \lim_{\text{opens } V \ni p} C^\infty(V)' \cong \mathbb{C}\{\delta_p, \partial_1 \delta_p, \ldots, \partial^\mu \delta_p, \ldots\}_{\mu \in \mathbb{N}^n}.$$

These are the linear local operators on a scalar field theory. One takes polynomials in these operators to produce the nonlinear local operators.

Our definition of $\mathrm{Obs}^{cl}(p)$ recapitulates this approach but using the graded space of fields, as we work with BV formalism for field theories.

On the other hand, from the perspective of factorization algebras, the observables Obs, whether classical or quantum, take values in the ∞-category $\mathrm{Ch}(\mathrm{DVS})$. Hence, when we ask for the observables supported at a point p, it seems natural to take $\lim_{V \ni p} \mathrm{Obs}(V)$ in that ∞-category. To distinguish it from the strict limit in the 1-category of cochain complexes, the term *homotopy limit* is often used. For a sequence of cochain complexes, the limit and homotopy limit are typically different; they only coincide under strong hypotheses. A classic manifestation is $\lim^1$, first popularized by Milnor; the Mittag-Leffler condition guarantees a class of examples where the limit and homotopy limit coincide.

Hence we should *not* expect that the point observables from this factorization perspective agree with the point observables from the field theory perspective. It would be interesting to compute the homotopy limits of the observables for field theories, such as the ones examined in this chapter. Undoubtedly there are interesting extensions and generalizations of notions from physics in the realm of factorization algebras.

Before returning to the general situation, let us point out a concrete example of the issue just raised: how the limit vs. homotopy limit distinction plays out for free topological field theories.

Consider the difference between Ω^*_c and $\overline{\Omega}^*_c$ on $\mathbb{R}^n$. On any disk $D(0,r)$, the compactly supported smooth de Rham forms must have support with radius less than r, so the strict limit $\lim_{r\to 0}\Omega^*_c(D(0,r))$ is 0. There are no *smooth* forms with support at the origin. On the other hand, there are compactly supported *distributional* forms with support at the origin, built from the delta function and its derivatives, as well as the k-form versions thereof. Hence the strictlimit $\lim_{r\to 0}\overline{\Omega}^*_c(D(0,r))$ is nonzero.

If we want our answer for the local operators to be meaningful from the perspective of homological algebra, we must make arguments up to quasi-isomorphism. The inclusion $\Omega^*_c \hookrightarrow \overline{\Omega}^*_c$ is a quasi-isomorphism, so the "correct" answer should be the same for both complexes. The notion of *homotopy limit* captures this requirement in a precise way.

In this case, the situation can be seen concretely. If we evaluate on any disk $D(0,r)$, the compactly supported smooth de Rham forms are quasi-isomorphic to $\mathbb{R}[-n]$ by the Poincaré lemma. In other words, the complex does not care about the size of the disk, which makes sense in that de Rham complexes measure topological information. The homotopy limit as $r\to 0$ is thus simply $\mathbb{R}[-n]$, or any complex quasi-isomorphic to it. A direct computation shows that there is a quasi-isomorphism

$$\lim_{r\to 0}\overline{\Omega}^*_c(D(0,r)) \to \mathbb{R}[-n]$$

from the strict limit to $\mathbb{R}[-n]$, as required.

This discussion scales up quickly to the observables of a free topological theory. Recall that we have two models for classical observables, using the distributional or smooth forms. The above argument scales up to show that the strict limit of the distributional model *does* compute the homotopy limit, but the strict limit of the smeared observables is zero. One can piggyback off these observations to discuss the quantum observables and also interacting topological theories: the strict limit of the distributional observables does encode the homotopy limit.

Already, even for holomorphic field theories, the situation becomes notably more complicated. We suspect that nontrivial features of elliptic complexes are necessary for analogs of the topological result.

We want to conclude by recasting the field-theoretic definition in a more positive light. Lemma 10.1.1.3 tells us that point observables admit an alternative definition that evades the use of strict limits: they are a class of translation-invariant local functionals, namely linear couplings between the field theory of interest and a background scalar field. In the language of symmetries developed in the next Part, the background scalar fields can be seen as defining a Lie algebra that acts on the field theory of interest, with each point observable encoding the action of one element of the Lie algebra.

10.2 The operator product expansion

Now consider two point observables $\mathcal{O}_1$ and $\mathcal{O}_2$. Place $\mathcal{O}_1$ at the origin 0 and $\mathcal{O}_2$ at $x \in \mathbb{R}^n \setminus \{0\}$. Pick a radius r bigger than the length $\|x\|$ of x. Choose a radius $\epsilon > 0$ small enough so that $\mathcal{O}_1$ lives on a small disc $D(0,\epsilon)$ around the origin, $\mathcal{O}_2$ lives on a small disc $D(x,\epsilon)$, and there is an inclusion

$$D(0,\epsilon) \sqcup D(x,\epsilon) \hookrightarrow D(0,r)$$

of disjoint discs. Hence there is the factorization product

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(x) \in \mathrm{Obs}(D(0,r)),$$

which is independent of ϵ. (Our focus will ultimately be on the quantum observables but our current discussion applies to classical observables too, so we use Obs for the moment and avoid a superscript.)

As x varies within $D(0,r)$ — but avoids the origin — this product varies smoothly as an element of $\mathrm{Obs}(D(0,r))$. Thus the product is an element

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(x) \in C^\infty(D(0,r) \setminus \{0\}, \mathrm{Obs}(D(0,r))).$$

Clearly, if $\|x\| < r'$ for some $r' < r$, then this family of observables lies in the subcomplex $\mathrm{Obs}(D(0,r'))$ of $\mathrm{Obs}(D(0,r))$.

We are interested in computing $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ modulo those observable-valued functions of x that are non-singular at $x = 0$. This quantity will be called the *operator product expansion* (OPE) or, more precisely, the singular part of the operator product expansion.

Remark: For classical observables, there is a canonical smooth extension of products to the origin because classical point observables form a commutative algebra. Hence the OPE of classical observables is defined and there is no singular part. From here on, we discuss quantum observables. $\Diamond$

The precise way to define "up to non-singular functions of x" is a bit technical. In this book, the space of observables on any disc D is a differentiable vector space. This setting makes it easy to talk about smooth families of observables depending on $x \in D(0,r) \setminus \{0\}$ that extend to a smooth family of observables on the whole disc $D(0,r)$. Such observables are certainly non-singular at $x = 0$, but we would like to consider a larger class of non-singular observables. For instance, we want to view quantities like the norm $\|x\|$ or $\|x\| \log \|x\|$ as being non-singular at $x = 0$. These functions are continuous but not differentiable at the origin.

To make our approach work, we need to define what it means for a family of observables to depend *continuously* on $x \in D(0,r)$. Recall that for a theory on $\mathbb{R}^n$, a quantum observable is a collection $\mathcal{O}[\Phi]$ of functionals on the space of fields on $\mathbb{R}^n$, satisfying some additional properties. It admits an expansion

$$\mathcal{O}[\Phi] = \sum \hbar^i \mathcal{O}_{i,k}[\Phi]$$

where each $\mathcal{O}_{i,k}[\Phi]$ is a polynomial of degree k on the space of fields $\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E$, with E a finite-dimensional graded vector space. We will view $\mathcal{O}_{i,k}[\Phi]$ as an element

$$\mathcal{O}_{i,k}[\Phi] \in \mathcal{D}_c(\mathbb{R}^{nk}) \otimes (E^*)^{\otimes k}$$

where $\mathcal{D}_c$ refers to the space of compactly supported distributions. We will now treat $\mathcal{D}_c(\mathbb{R}^{nk})$ as a topological vector space, rather than a differentiable vector space.

Suppose we have a family $\mathcal{O}(x)$ of observables parametrized by a topological space X. We say that this family is *continuous* if for all indices i and k and for all parametrices Φ, the map

$$\begin{array}{ccc} X & \to & \mathcal{D}_c(\mathbb{R}^{nk}) \otimes (E^*)^{\otimes k} \\ x & \mapsto & \mathcal{O}_{i,k}(x)[\Phi] \end{array}$$

is continuous.

We now formulate a precise definition of the singular part of the OPE. As we will work repeatedly with a punctured disk, we introduce a convenient notation.

10.2.0.1 Definition. *If D denotes the open disk $D(x,r)$ centered at $x \in \mathbb{R}^n$ and of radius r, let $\mathring{D}$ denote the punctured open disk $D \setminus \{x\}$ where the center is removed. Likewise $\mathring{\mathbb{R}}^n$ denotes $\mathbb{R}^n \setminus \{0\}$.*

Fix a translation-invariant field theory on $\mathbb{R}^n$ and consider the product of two point observables as a function of position.

10.2.0.2 Definition. *Let*

$$\alpha, \beta \in C^\infty(\mathring{D}, \mathrm{Obs}^q(D))$$

denote two smooth families of quantum observables on a ball D, depending smoothly on $x \in \mathring{D}$. If their difference $\alpha - \beta$ extends to a continuous map from $D \to \mathrm{Obs}^q(D)$, we say they differ by non-singular functions *of $x \in D$ and write*

$$\alpha \simeq \beta$$

to denote that α and β have the same singular part.

This notion allows us define the *operator product expansion*.

10.2.0.3 Definition. *Let $\mathcal{O}, \mathcal{O}' \in \mathrm{Obs}^q(0)$ be point observables in a translation-invariant field theory on $\mathbb{R}^n$, and let $\mathcal{O}(x)$, $\mathcal{O}'(x)$ denote their translates to $x \in \mathbb{R}^n$. An* operator product expansion *(or OPE, for short) is an expression of the form*

$$\sum_{i=1}^{\infty} \hbar^i \sum_r F_{ir}(x) \mathcal{O}_{ir}(0)$$

where $\mathcal{O}_{ir}(0) \in \mathrm{Obs}(0)$ *and* $F_{ir}(x) \in C^\infty(\mathring{\mathbb{R}}^n)$ *such that*

(i) *at each order in* $\hbar$, *the sum over* r *is finite, and*
(ii) *the product satisfies*

$$\mathcal{O}(0) \cdot \mathcal{O}(x) \simeq \sum_{i=1}^{\infty} \hbar^i \sum_r F_{ir}(x) \mathcal{O}_{ir}(0)$$

as observable-valued functions in $C^\infty(\mathring{\mathbb{R}}^n, \mathrm{Obs}^q(\mathbb{R}^n))$.

Our usage of the term OPE means that we take the expansion only up non-singular functions of x.

Note that the existence of an operator product expansion is a *property* of a factorization algebra, and not extra data. The factorization algebra encodes the factorization product, which is an observable-valued function of x, while the existence of the OPE means that this product admits a well-behaved asymptotic expansion.

There is a simple criterion that guarantees the existence of OPE for theories.

Claim. *Fix a free theory on* $\mathbb{R}^n$, *invariant under the action of* $\mathbb{R}_{>0}$ *by scaling on* $\mathbb{R}^n$, *such that that the* $\mathbb{R}_{>0}$ *action on the space of point observables is diagonalizable with discrete spectrum (e.g., with integer eigenvalues).*

For any field theory obtained by adding a translation-invariant interaction term to this free theory, each pair of point observables has a unique operator product expansion.

This kind of claim appears throughout the physics literature, often somewhat implicitly. Scale-invariant free theories are quite common — and so the claim assures the existence of operator product expansion in many situations. (Under the stronger condition of being a conformal field theory, such OPE play a crucial role.) Note that the eigenvalues here are the scaling dimensions of point observables (*aka* local operators). When the fields are built from sections of tensor or spinor bundles, these scaling dimensions satisfy the condition above.

We will not give a detailed proof of this claim in this book, as it is a bit technical and also somewhat off-topic. We will prove it explicitly, however, to leading order in $\hbar$.

Before jumping into detailed computations, let us see heuristically why we should expect this claim to be true. Choose a smooth function $f(x)$ that is zero when $\|x\| < \epsilon$ and one when $\|x\| > 2\epsilon$. Then for any two point observables $\mathcal{O}_1, \mathcal{O}_2$, we have

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(x) f(x) \simeq 0.$$

Thus the singular part in $\mathcal{O}_1(0)\mathcal{O}_2(x)$ only depends on the product for $\|x\|$ arbitrarily small. This singular part can be taken to arise in

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(x) \in C^\infty(\mathring{D}_\epsilon, \mathrm{Obs}^q(D_\epsilon)),$$

where D_ϵ is the disc of radius ϵ around 0 and ϵ is arbitrarily small.

This observation suggests why we should expect the singular part of $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ to be a point observable: it can be realized as an observable on an arbitrarily small disc around a point.

The content of the claim is then entirely analytical. It is simply the statement that $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ admits a reasonable asymptotic expansion of some form. The singular parts of this expansion will automatically be given in terms of local functionals.

10.3 The OPE to first order in $\hbar$

In this section we will prove the existence of the OPE at order $\hbar$ in any translation-invariant theory on $\mathbb{R}^n$. Along the way, we will derive an explicit formula for the OPE in terms of Feynman diagrams. This OPE gives a kind of Poisson bracket on the algebra of classical observables, which we summarize before delving into the construction of the OPE. (The reader may prefer to read in the other order.)

10.3.1 A Poisson structure modulo $\hbar^2$

Below we verify the first order OPE exists and that singular part of $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ modulo $\hbar^2$ is encoded by a family of point observables.

As suggestive notation, let

$$\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE}$$

denote the x-dependent observable given by the singular part of $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ modulo $\hbar^2$. We use a bracket here because this operation behaves like a Poisson bracket in following precise sense, and we call it the *semi-classical OPE*.

10.3.1.1 Lemma. *For any classical point observables $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ in a translation-invariant theory on $\mathbb{R}^n$, there is an equivalence*

$$\begin{aligned}&\{\mathcal{O}_1(0)\mathcal{O}_2(0), \mathcal{O}_3(x)\}_{OPE}\\ &\qquad \simeq \mathcal{O}_1(0)\{\mathcal{O}_2(0), \mathcal{O}_3(x)\}_{OPE} \pm \mathcal{O}_2(0)\{\mathcal{O}_1(0), \mathcal{O}_3(x)\}_{OPE}\end{aligned}$$

where juxtaposition indicates the product of classical observables. Similarly, the bracket is a derivation in the second factor.

In words, $\{-,-\}_{OPE}$ behaves like an x-dependent Poisson bracket on $\mathrm{Obs}^{cl}(0)$. This bracket is a broad generalization, to arbitrary $\mathbb{R}^n$, of notions that have appeared before: compare with vertex Poisson algebras that arise in the setting of chiral CFT on $\mathbb{R}^2$ (see, for example, chapter 16 of Frenkel and Ben-Zvi (2004)). The proof of this lemma and the proposition below come after we show existence of the OPE in the next subsection.

For convenience of reference, we now summarize the structure we have unveiled on the point observables due to quantization to first order in $\hbar$. Recall that $C^\omega(U)$ denotes the real-analytic functions on an open subset $U \subset \mathbb{R}^n$, and note that the equivalence relation $\simeq$ of Definition 10.2.0.2 makes sense with the analytic functions $C^\omega(\mathring{\mathbb{R}}^n)$.

10.3.1.2 Proposition. *For a translation-invariant theory on $\mathbb{R}^n$, the classical point observables $\mathrm{Obs}^{cl}(0)$ form a differential graded commutative algebra.*

Moreover, $\mathrm{Obs}^{cl}(0)$ has a canonically defined bi-derivation

$$\{-,-\}_{OPE} : \mathrm{Obs}^{cl}(0) \otimes \mathrm{Obs}^{cl}(0) \to \mathrm{Obs}^{cl}(0) \otimes \left(C^\omega(\mathring{\mathbb{R}}^n)/\simeq\right),$$

where the dependence on x is real-analytic. This bracket is compatible with the differential d on classical observables:

$$\mathrm{d}\{\mathcal{O}_1(0), \mathcal{O}_2(x)\} = \{\mathrm{d}\mathcal{O}_1(0), \mathcal{O}_2(x)\} \pm \{\mathcal{O}_1(0), \mathrm{d}\mathcal{O}_2(x)\}.$$

Translation along $\mathbb{R}^n$ acts in a highly controlled way on this bracket, and we discuss these properties in detail in Lemma 10.3.3.1. These drastically simplify the computation of the bracket $\{-,-\}_{OPE}$: it is entirely determined by its value on those linear point functionals of the fields that do not involve any derivatives (cf. with descendants in CFT).

10.3.2 The explicit construction of the OPE modulo $\hbar^2$

The first thing we show is that the OPE at first order in $\hbar$ only depends on classical observables, and not their lift to quantum observables defined modulo $\hbar^2$.

10.3.2.1 Lemma. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be any two classical point observables in a translation-invariant theory on $\mathbb{R}^n$. Let $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$ be any lifts of these to quantum observables defined modulo $\hbar^2$. Then the OPE between $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$ at order $\hbar$ is independent of the choice of quantum lifts of the classical observables $\mathcal{O}_1$ and $\mathcal{O}_2$.*

Proof Recall that OPE is defined straightforwardly for classical point observables, with no singular part. If we change $\widetilde{\mathcal{O}}_1$ by $\hbar\mathcal{O}'$, then the OPE between $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$ changes by the OPE between $\hbar\mathcal{O}'$ and $\widetilde{\mathcal{O}}_2$. But $\hbar\mathcal{O}'(0)\cdot\widetilde{\mathcal{O}}_2(x)$ is non-singular at order $\hbar$, as it is the same as the classical OPE between $\mathcal{O}'$ and $\widetilde{\mathcal{O}}_2$. □

This result tells us that the order $\hbar$ term in the OPE is canonically associated to classical point observables. Now, we need to find a formula for the order $\hbar$ term in the OPE. We will derive it from a formula for the factorization product of quantum point observables.

For any two quantum point observables, the product $\mathcal{O}_1(0)\cdot\mathcal{O}_2(x)$ is a quantum observable. Unpacking that definition, we have a family $(\mathcal{O}_1(0)\cdot\mathcal{O}_2(x))\,[\Phi]$ over the space of parametrices. According to our formula for this product, it is defined by the limit

$$(\mathcal{O}_1(0)\cdot\mathcal{O}_2(x))\,[\Phi] = \lim_{\Psi\to 0} W_\Psi^\Phi(I[\Psi], \widetilde{\mathcal{O}}_1(0)[\Psi]\widetilde{\mathcal{O}}_2(x)[\Psi]) \quad (10.3.2.1)$$

where $\widetilde{\mathcal{O}}_i[\Psi]$ are arbitrary lifts of our classical observables to quantum observables with parametrix Ψ.

Using the diagrammatic expression for W_Ψ^Φ, we see that the order $\hbar$ term in equation (10.3.2.1), before we take the $\Psi \to 0$ limit, is given by a sum of diagrams of the following types:

(i) Disconnected diagrams with two connected components. Each connected component must contain a special vertex labelled by $\mathcal{O}_1(0)[\Psi]$ or $\mathcal{O}_2(x)[\Psi]$.

(ii) Connected trees with two special vertices, labelled by $\mathcal{O}_1(0)[\Psi]$ and $\mathcal{O}_1(x)[\Psi]$. All other vertices are labelled by the interaction terms $I[\Psi]$. The edges are labelled by $P(\Phi) - P(\Psi)$. There are external edges labelled by the field ϕ of the theory; the amplitude of the diagram is a functional of this field.

To clarify this description, note that when one initially constructs a diagram, one treats $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ as a single observable sitting at a single vertex, but it is naturally to decompose that vertex into the $\mathcal{O}_1(0)$ and $\mathcal{O}_2(x)$ factors, which become distinct vertices. For example, a connected tree with two special vertices will have precisely one loop if one glues the special vertices together, which explains why such a diagram has order $\hbar$.

Disconnected diagrams can not have any singularities as $x \to 0$. They therefore do not contribute to the singular part of the OPE, so we only need to consider the connected diagrams.

For a connected diagram to have order exactly $\hbar$, all of its vertices must be labelled by the coefficient of $\hbar^0$ in whatever functional is at the vertex. That is, the two special vertices are labelled by $\mathcal{O}_i[\Psi]$ modulo $\hbar$, and the internal vertices are labelled by $I[\Psi]$ modulo $\hbar$.

Modulo $\hbar$, the limit as $\Psi \to 0$ of $\mathcal{O}_i[\Psi]$ is simply the original classical observable $\mathcal{O}_i$. Similarly, the limit as $\Psi \to 0$ of $I[\Psi]$ is the interaction term in the original classical Lagrangian, which we call I. Therefore, once we take the $\Psi \to 0$ limit, we are left with a sum over connected trees with two special vertices, where $\mathcal{O}_1(0)$ and $\mathcal{O}_2(0)$ label the special vertices, the classical interaction term I labels the other vertices, and the propagator $P(\Phi)$ labels internal edges.

There is one further simplification before we reach the final diagrammatic expression. The result of our calculation will be an observable

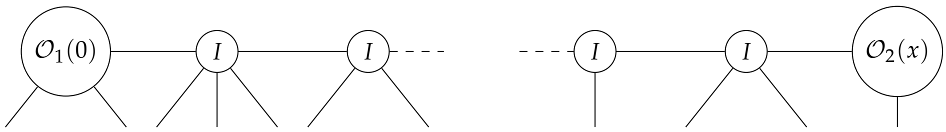

Figure 10.1 A typical bridge diagram appearing in a theory which has interaction terms of order 3, 4, and 5 in the field. Each internal vertex can carry any of the interaction terms, leading in this case to a sum over all diagrams where the internal vertices are of valency 3, 4 or 5. The vertices at the beginning and end have a valency reflecting the order of the observables $\mathcal{O}_i$ as polynomials in the fundamental fields.

that is defined modulo $\hbar^2$ and is zero modulo $\hbar$. It is therefore $\hbar$ times a classical observable, and therefore has a $\Phi \to 0$ limit. We will calculate this $\Phi \to 0$ limit, which is a classical observable defined as a functional of the fields.

We will say that a diagram that appears in the sum-over-trees is a *bridge* if every time we cut an edge, the vertices labelled by $\mathcal{O}_1(0)$ and $\mathcal{O}_2(x)$ end up on different components. Bridges are all given by a sequence of vertices, starting with that labelled by $\mathcal{O}_1(0)$ and ending with the one labelled by $\mathcal{O}_2(x)$, so that consecutive vertices in the sequence are connected by a propagator. Any connected diagram that appears in the sum-over-trees is obtained by grafting some trees onto a bridge.

It turns out that only bridges contribute to the $\Phi \to 0$ limit of the product $\{\mathcal{O}(0) \cdot \mathcal{O}(x)\}[\Phi]$. A non-bridge diagram contributes zero in the $\Phi \to 0$ limit.

10.3.2.2 Proposition. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be any two classical point observables in a translation-invariant theory on $\mathbb{R}^n$. Modulo $\hbar^2$ and modulo terms that are non-singular functions of x, the product $\lim_{\Phi\to 0} (\mathcal{O}_1(0) \cdot \mathcal{O}_2(x))\,[\Phi]$ has a diagrammatic expression*

$$\lim_{\Phi\to 0} (\mathcal{O}_1(0) \cdot \mathcal{O}_2(x))\,[\Phi] \simeq \hbar \sum_{\textit{bridges } \Gamma} W_\Gamma$$

where the sum is over bridges with two special vertices labelled by the functionals $\mathcal{O}_1(0)$ and $\mathcal{O}_2(x)$. All other vertices of a bridge are labelled by the classical interaction I, and the internal edges are labelled by the propagator $P(\Phi)$ for any parametrix Φ.

Up to terms that are non-singular as $\|x\| \to 0$, this expression is independent of the parametrix Φ.

The result of this diagrammatic calculation is a functional of the fields, which we view as a classical observable multiplied by $\hbar$.

Proof As we have seen, before we take the $\Phi \to 0$ limit, we find an arbitrary connected tree with special vertices labelled by $\mathcal{O}_1(0)$, $\mathcal{O}_2(x)$. Any such tree is a bridge, with other trees grafted on to it. That is, if we let W_0^Φ denote the tree-level RG flow on observables from scale 0 to Φ, we have at order $\hbar$,

$$(\mathcal{O}_1(0) \cdot \mathcal{O}_2(x))\,[\Phi] = \hbar W_0^\Phi \left(\sum_{\text{bridges } \Gamma} W_\Gamma \right)$$

(the point is that W_0^Φ is given by grafting trees with propagator $P(\Phi)$).

In the $\Phi \to 0$ limit, W_0^Φ drops out.

Next, let's show that the contribution of bridges is independent of $P(\Phi)$. Note that if we change the parametrix Φ, the propagator $P(\Phi)$ changes by a smooth kernel (i.e., with no singularities). An infinitesimal change of Φ will change the contribution of the bridges so that precisely one edge is labelled by a smooth kernel, and all the other edges are labelled by the propagator $P(\Phi)$. Because the diagram is a bridge, if we cut the edge that is labelled by the smooth kernel, then the vertices labelled by $\mathcal{O}(0)$ and $\mathcal{O}(x)$ lie on different connected components.

The presence of the smooth kernel on an edge that separates $\mathcal{O}(0)$ from $\mathcal{O}(x)$ guarantees that the amplitude of the diagram is non-singular as $x \to 0$. Since we are only interested in the product $\mathcal{O}(0) \cdot \mathcal{O}(x)$ modulo non-singular terms, we find that the contribution of bridges is independent of the parametrix. □

This result has a number of corollaries.

10.3.2.3 Corollary. *For any translation-invariant theory on $\mathbb{R}^n$, the operator product expansion in the sense of Definition 10.2.0.3 exists to order $\hbar$: for any classical point observables $\mathcal{O}_1, \mathcal{O}_2$, the product $\mathcal{O}_1(0)\mathcal{O}_2(x)$ is well-defined modulo $\hbar^2$. Furthermore, all the functions of $x \in \mathring{\mathbb{R}}^n$ appearing in this expansion are real analytic.*

Proof This claim follows from an explicit analysis of the sum-over-bridges formula above. Since the question is entirely analytical, we can assume without loss of generality that our space of fields consists of one or several scalar fields. We denote the input field — placed at all external lines in the sum-over-bridges formula — by ϕ. We examine each bridge diagram Γ separately. Let the value $W_{\Gamma,x}$ be the *amplitude* of the diagram Γ with $\mathcal{O}_2$ inserted at x.

Let m denote the number of internal vertices of Γ, so that this amplitude $W_{\Gamma,x}$ is written as an integral over $\mathbb{R}^{nm}$:

$$W_{\Gamma,x}(\Phi,\phi) = \int_{\mathbb{R}^{nm}} F_{\Gamma,x}(\Phi,\phi). \tag{10.3.2.2}$$

Note that the integrand depends not only an input field ϕ but also on a choice of parametrix, so we fix throughout a parametrix Φ and suppress it from the notation below.

We use $x_1,\dots,x_m$ to denote the points in $\mathbb{R}^n$ associated to each vertex, and we set $x_0 = 0$ and $x_{m+1} = x$. The external lines attached to each vertex contribute a polynomial built from ϕ and finitely many of its derivatives. (Derivatives may appear because the interaction terms in the Lagrangian may involve derivatives.) On the edge connecting vertex i with vertex $i+1$, we place the propagator $P(\Phi)(x_i - x_{i+1})$. We include the case that $i = 0$ and $i+1 = m+1$.

As a function the variables x_i and x_{i+1} attached to the vertices, the propagator $P(\Phi)$ vanishes when

$$\|x_i - x_{i+1}\| \gg 0,$$

by definition. Therefore for some constant C, the integrand vanishes F_Γ whenever $\|x_i\| > C$ for some point x_i. We thus take our domain of integration to be the region where $\|x_i\| < C$ for all i.

If we take the insertion point $x = x_{m+1}$ to be some finite distance C' from the origin (i.e., $\|x\| > C'$), then the integral $W_{\Gamma,x}(\phi)$ converges absolutely, for each fixed ϕ. The result is therefore a smooth function of x (for fixed ϕ) as long as $x \neq 0$. The integral may diverge if $x \to 0$ and thus the amplitude $W_{\Gamma,x}$ diverges, leading to singular terms in the OPE.

The amplitude $W_{\Gamma,x}$ is also a continuous function on the space of fields. Hence we can assume without loss of generality that the input field ϕ is a polynomial, as polynomials are dense in all fields by Weierstrass approximation.

Assume then that ϕ is homogeneous of degree k as a function on $\mathbb{R}^n$. If $k \gg 0$, the value of the integral $W_{\Gamma,x}(\phi)$ is a function of x that is continuous at $x = 0$, by the following argument.

On the domain of integration, there is some natural number l such that the propagator is bounded above by $\|x_i - x_{i+1}\|^{-l}$, because the propagator is constructed using a parametrix, which manifestly has this kind of property from standard analytic results about the Laplacian for the Euclidean metric on $\mathbb{R}^n$. Thus, we find that the absolute value of the original integral is bounded above by an integral where each propagator has been replaced by $\|x_i - x_{i+1}\|^{-l}$ and each monomial involving the rth factor $x_{i,r}$ of a point $x_i = (x_{i,1}, \ldots, x_{i,n})$ has been replaced by its absolute value. This new integrand is homogeneous under rescaling all the x_i and x. In fact, we can arrange that it is homogeneous of positive weight by taking $k \gg 0$. After integrating it over the x_i, we get a function of x that is homogenous of positive weight under rescaling of x. This function will therefore be bounded above in absolute value by its supremum on the unit sphere times $\|x\|^m$ for some m. In other words, for some $K \gg 0$ and for any field ϕ homogeneous of degree $k > K$,

$$|W_{\Gamma,x}(\phi)| \leq \|x\|^m .$$

Therefore the amplitude $W_{\Gamma,x}(\phi)$ is continuous at $x = 0$ and does not contribute to the OPE.

For fields ϕ that are homogeneous with degree less than some K (such as a constant field), the amplitude $W_{\Gamma,x}(\phi)$ may diverge as $x \to 0$. For an arbitrary field ϕ, we can use its Taylor approximation to deduce that the singular part of the amplitude $W_{\Gamma,x}(\phi)$ will only depend on the value of ϕ and finitely many of its derivatives at 0 (up to order K). Hence it is a point observable.

The coefficient of the point observables appearing in the OPE will be encoded in the sum of the amplitudes of bridges with the background field ϕ being a polynomial, as above. These coefficients can be interpreted as a composition of operators on the space of fields. Each edge gives an operator that is a convolution with the propagator $P(\Phi)$. Each vertex gives an operator that is some differential operator with polynomial coefficients, since we assume ϕ is polynomial. Any convolution of this form will always lead to a function of x that is real-analytic up to non-singular terms in x. $\square$

10.3.3 The Poisson bracket, redux

Recall that we use $\{-,-\}_{OPE}$ as suggestive notation for the order $\hbar$ term in the OPE:

$$\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} \simeq \lim_{\hbar \to 0} \hbar^{-1} \mathcal{O}_1(0) \cdot \mathcal{O}_2(x).$$

The first step in justifying the Poisson flavor of this binary operator is to prove Lemma 10.3.1.1 that it is a derivation in each entry.

Proof of Lemma 10.3.1.1 In the sum-over-bridges formula for $\{\mathcal{O}_1(0)\mathcal{O}_2(0), \mathcal{O}_3(x)\}_{OPE}$, an edge of the bridge connects to the observable $\mathcal{O}_2(0)$ or to $\mathcal{O}_1(0)$, but not to both. Therefore the sum-over-bridges reduces to two sums, one of which is $\mathcal{O}_1(0)$ times the sum-over-bridges giving $\{\mathcal{O}_2(0), \mathcal{O}_3(x)\}_{OPE}$, and the other is $\mathcal{O}_2(0)$ times the sum-over-bridges giving $\{\mathcal{O}_1(0), \mathcal{O}_3(x)\}_{OPE}$. $\square$

Similarly, the OPE $\{-,-\}_{OPE}$ satisfies the Leibniz rule for the differential on classical observables:

$$\mathrm{d}\{\mathcal{O}_1(0), \mathcal{O}_2(x)\} = \{\mathrm{d}\mathcal{O}_1(0), \mathcal{O}_2(x)\} \pm \{\mathcal{O}_1(0), \mathrm{d}\mathcal{O}_2(x)\}.$$

This equation is an immediate consequence of the fact that the differential on quantum observables is a derivation for the factorization product. In sum, we have shown Proposition 10.3.1.2.

The one-loop OPE also satisfies some compatibility conditions with the action of differentiation on observables. To state them, we need to introduce notation for two different notions of derivative (or infinitesimal translation) because there are two places where we can differentiate. Note that in the definition of the OPE, we deal with spaces of the form $C^\infty(U', \mathrm{Obs}(U))$ where U and U' are open subsets of $\mathbb{R}^n$. Here the observables themselves can be differentiated: for $i = 1, \ldots, n$ we let D_i denote the action of differentiation with respect to x_i on observables. Likewise, functions on U', with values in *any* differentiable vector space, can be differentiated: we denote this differentiation by $\frac{\partial}{\partial x_i}$. Thus, elements of $C^\infty(U', \mathrm{Obs}(U))$ can be differentiated in two separate ways.

Something special happens for point observables: on an open subset $U \subset \mathbb{R}^n$ containing x,

$$D_i \mathcal{O}(x) = \frac{\partial}{\partial x_i} \mathcal{O}(x) \in \mathrm{Obs}^q(U)$$

for any point observable $\mathcal{O}$. This equality follows from the definition of D_i and of $\mathcal{O}(x)$, because the observable $\mathcal{O}(x)$ is obtained by translating the observable $\mathcal{O}(0)$ to x and D_i is defined by the infinitesimal action of translation.

Recall that the operators D_i are derivations for the factorization product. Hence for any classical point observables $\mathcal{O}_1$ and $\mathcal{O}_2$, we have

$$D_i\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} = \{D_i\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} + \{\mathcal{O}_1(0), D_i\mathcal{O}_2(x)\}_{OPE},$$

and so

$$\begin{aligned}\frac{\partial}{\partial x_i}\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} &= \{\mathcal{O}_1(0), \frac{\partial}{\partial x_i}\mathcal{O}_2(x)\}_{OPE} \\ &= \{\mathcal{O}_1(0), D_i\mathcal{O}_2(x)\}_{OPE}.\end{aligned}$$

In sum, we have the following.

10.3.3.1 Lemma. *This bi-derivation $\{-,-\}_{OPE}$ satisfies the following properties:*

(i) *It is compatible with differentiation of classical observables:*

$$D_i\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} = \{D_i\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} + \{\mathcal{O}_1(0), D_i\mathcal{O}_2(x)\}_{OPE}$$
$$\frac{\partial}{\partial x_i}\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} = \{\mathcal{O}_1(0), D_i\mathcal{O}_2(x)\}_{OPE}.$$

(ii) *It has the symmetry property:*

$$\{\mathcal{O}_1(0), \mathcal{O}_2(x)\}_{OPE} \simeq \pm e^{\sum x_i D_i}\{\mathcal{O}_2(0), \mathcal{O}_1(-x)\}_{OPE}.$$

Proof It remains to show the symmetry property, which follows by noting that

$$\mathcal{O}_1(0)\mathcal{O}_2(x) = e^{\sum x_i D_i}\left(\mathcal{O}_1(-x)\mathcal{O}_2(0)\right),$$

since $e^{\sum x_i D_i}$ is the operator of translation by the vector $x = (x_1, \ldots, x_n)$.

Note that because the symmetry identity only holds modulo singular functions of x, only finitely many terms in the expansion of the exponential $e^{\sum x_i D_i}$ will appear in the equation. □

10.4 The OPE in the ϕ^4 theory

Let us now explore the computation of the semi-classical OPE in an explicit example. Consider here ϕ^4 theory on $\mathbb{R}^4$, whose action functional is

$$S(\phi) = -\int \tfrac{1}{2}\phi\triangle\phi + \int \tfrac{1}{4!}\phi^4.$$

Let $\mathcal{O} \in \mathrm{Obs}(0)$ be the point observable

$$\mathcal{O}(\phi) = \phi(0),$$

which evaluates the field ϕ at 0. In other words, the translate $\mathcal{O}(x)$ is the delta-function δ_x.

We will now calculate $\{\mathcal{O}(0), \mathcal{O}(x)\}_{OPE}$. As explained above, this notation indicates the order $\hbar$ term in the OPE, viewed as a classical observable.

10.4.0.1 Lemma. *The semi-classical OPE satisfies*

$$\{\mathcal{O}(0), \mathcal{O}(x)\}_{OPE} \simeq \frac{1}{4\pi^2}\left(\|x\|^{-2} - (\log\|x\|)\phi(0)^2\right).$$

Remark: There are many other computations of the bracket $\{-,-\}_{OPE}$ for scalar field theories, and the interested reader is encouraged to perform some of them. For instance, it is interesting to analyze the ϕ^4 theory in dimensions other than 4. As the dimension increases, more and more diagrams can contribute to the OPE. In dimension 6, for instance, bridges with three internal vertices can contribute, leading to expressions like

$$\{\phi(0), \phi(x)\}_{OPE} = C\phi(0)^6 \log\|x\| + \text{ terms of lower order in } \phi$$

for some constant C. (Here, we are abusing notation in a way commonly done by physicists: we are writing $\phi(0)$ for the delta function observable on fields that sends ϕ to its value $\phi(0)$.) The lower order terms are the contributions from diagrams with two or fewer internal vertices.

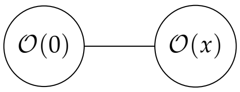

Figure 10.2 The first diagram contributing to the operator product expansion

These expressions can be quite complicated and may involve operators built from derivatives of ϕ. It is instructive to compute some of these terms. In dimensions lower than 4, fewer and fewer diagrams can contribute, until one finds that in dimension 2 the order $\hbar$ OPE coincides with that for the free theory. (Verifying this claim is also a useful exercise.) $\Diamond$

Proof We will prove this claim by applying the sum-over-bridges formula (10.3.2.2) for the order $\hbar$ OPE. It turns out that only two diagrams can contribute.

The first diagram we need to compute has only two vertices, and it is illustrated in Figure 10.2. This diagram has a single edge, labelled by the propagator $P(\Phi)$ associated to an arbitrary parametrix. This propagator satisfies

$$P(\Phi) \simeq \frac{1}{4\pi^2} \frac{1}{\|x' - x''\|^2}$$

up to smooth functions on $\mathbb{R}^4 \times \mathbb{R}^4$. One end of the propagator is evaluated at zero, and the other at x. Therefore we find that this diagram contributes the factor

$$\hbar \frac{1}{4\pi^2 \|x\|^2}$$

to the OPE of $\mathcal{O}(0)$ and $\mathcal{O}(x)$.

Next, let us analyze the contribution of the next bridge, illustrated in Figure 10.3. We will take our parametrix Φ to be of the form

$$\frac{F(\|x - x'\|)}{\|x - x'\|^2},$$

where $F(r)$ is a non-increasing function of r that is 0 for $r \gg 0$ and is $\frac{1}{4\pi^2}$ in a neighbourhood of $r = 0$. (For the scalar theory we are considering, the parametrix and the propagator associated to the parametrix are the same thing. For a gauge theory, they are different.)

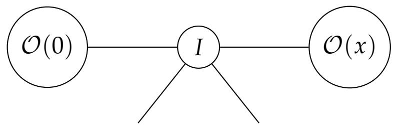

Figure 10.3 The second diagram contributing to the OPE calculation in the ϕ^4 theory.

The diagram in Figure 10.3 contributes the functional of ϕ given by

$$\frac{\hbar}{2}\int_{x'\in\mathbb{R}^2}\phi(x')^2\frac{F(\|x'\|)}{\|x'\|^2}\frac{F(\|x-x'\|)}{\|x-x'\|^2}\mathrm{d}^4x' \tag{10.4.0.1}$$

where x' labels the interior vertex. More generally, the bridge with $2+n$ vertices contributes the functional

$$\frac{\hbar}{2^n}\int_{\mathbb{R}^{4n}}\phi(x_1)^2\cdots\phi(x_n)^2\frac{F(x_1)}{\|x_1\|^2}\frac{F(x_2-x_1)}{\|x_2-x_1\|^2}\cdots\frac{F(x-x_n)}{\|x-x_n\|^2}\mathrm{d}^4x_1\cdots\mathrm{d}^4x_n \tag{10.4.0.2}$$

where $x_1,\ldots,x_n\in\mathbb{R}^4$ label the interior vertices. This functional of ϕ is quite complicated. But if we recall that we are only interested in the result up to non-singular functions of x, we find the answer simplifies.

Let's calculate the integral when $n=1$. Write $\phi(x)$ as the sum $\phi(0)+\phi_0(x)$ where $\phi_0(0)=0$. The integral is then

$$\begin{aligned}&\frac{1}{2}\int_{x'\in\mathbb{R}^4}\phi(0)^2\frac{F(\|x'\|)}{\|x'\|^2}\frac{F(\|x-x'\|)}{\|x-x'\|^2}\mathrm{d}^4x'\\&\quad+\frac{1}{2}\int_{x'\in\mathbb{R}^4}\left(2\phi(0)\phi_0(x')+\phi_0(x')^2\right)\frac{F(\|x'\|)}{\|x'\|^2}\frac{F(\|x-x'\|)}{\|x-x'\|^2}\mathrm{d}^4x'.\end{aligned} \tag{10.4.0.3}$$

We will first show that the integral on the second line is non-singular as $x\to 0$. Let C be a constant such that $F(\|x\|)=0$ if $\|x\|\geq C$. Since $\phi_0(0)=0$, we can find some constant D such that

$$|\phi_0(x)|\leq D\,\|x\|\ \text{ if }\ \|x\|\leq C.$$

This inequality means that the integrand on the second line of (10.4.0.3) is bounded above in absolute value by

$$D'\,\|x'\|\,\frac{F(\|x'\|)}{\|x'\|^2}\frac{F(\|x-x'\|)}{\|x-x'\|^2}\mathrm{d}^4x'$$

for some constant D' that depends on $\phi(0)$. This integral is absolutely convergent uniformly in x. The result of this integral is therefore a continuous function of x, which does not contribute to the singular part of the OPE.

We conclude that the term in the singular part of the OPE associated to the diagram of Figure 10.3 is given by

$$\phi(0)^2 \frac{1}{2} \int_{x'} \frac{F(\|x'\|)F(\|x - x'\|)}{\|x'\|^2 \, \|x - x'\|^2} \mathrm{d}^4 x'. \tag{10.4.0.4}$$

This integral can be evaluated explicitly up to terms that are non-singular as $x \to 0$. The result is

$$-\phi(0)^2 \frac{1}{4\pi^2} \log \|x\| \,,$$

which is the second term in our putative OPE.

So far, we have found that our operator product expansion is

$$\mathcal{O}(0) \cdot \mathcal{O}(x) \simeq \frac{\hbar}{4\pi^2} \left(\frac{1}{\|x\|^2} - \log \|x\| \right) + \cdots .$$

To complete the calculation, we need to verify that the bridge diagrams with four and more vertices can not contribute. If we consider the integral in (10.4.0.2), we note that we can bound $\phi(x_i)$ above by a constant on the domain of integration (which is bounded because the functions $F(\|x_i - x_{i+1}\|)$ are zero if $\|x_i - x_{i+1}\|$ are large). The integrand is therefore bounded above in absolute value by

$$\frac{F(x_1)}{\|x_1\|^2} \frac{F(x_2 - x_1)}{\|x_2 - x_1\|^2} \cdots \frac{F(x - x_n)}{\|x - x_n\|^2}.$$

The integral of this integrand over the parameters $x_1, \ldots, x_n$ in $\mathbb{R}^4$ is absolutely convergent, uniformly in x. The result of the integral is therefore a function of x with no singularities at $x = 0$, so that these diagrams do not contribute to the OPE. □

10.4.1 A heat kernel approach

Let us remark on another way to perform this calculation, which can be applied to many other field theories. The calculation we are performing is independent of the parametrix. Instead of regularizing the

Green's function by writing it as $F(x)\|x\|^{-2}$, we can instead write it as an integral of the heat kernel K_t from $t = 0$ to some L. This choice of parametrix is the one used extensively in Costello (2011b).

If we use this parametrix, the integral (10.4.0.4) becomes

$$\frac{1}{2}\int_{x'\in\mathbb{R}^4}\int_{t,s=0}^{L} K_t(0,x')K_s(x',x)\mathrm{d}t\mathrm{d}s\mathrm{d}^4x'.$$

The integral over x' gives a convolution of the heat kernel K_t with the heat kernel K_s. The result is K_{t+s}. Therefore we need to calculate

$$\frac{1}{2}\int_{t,s=0}^{L} K_{t+s}(0,x)\mathrm{d}t\mathrm{d}s.$$

It is convenient to change the domain of integration in t and s. Instead of asking that t, s live in the square $[0, L]\times[0, L]$, we can integrate over the triangle where $t, s \geq 0$ and $t + s \leq L$. The difference between the two integrals will be an integral over a domain where $t + s \geq L$. As a function of x, the heat kernel $K_{t+s}(0, x)$ is smooth at $x = 0$ as long as we bound $t + s$ below. Therefore, up to a smooth function of x, we will get the same answer if we integrate over the domain $\{t + s \leq L\}$.

It is straightforward to verify

$$\frac{1}{2}\int_{t+s\leq L} K_{t+s}(0,x)\,\mathrm{d}t\,\mathrm{d}s = \frac{1}{2}\int_{u=0}^{L} uK_u(0,x)\mathrm{d}u.$$

As

$$K_u(x) = \frac{1}{(4\pi u)^2}e^{-\|x\|^2/4u},$$

we find

$$\begin{aligned}\frac{1}{2}\int_0^L uK_u(x)\mathrm{d}u &= \frac{1}{8\pi^2}\int_{\|x\|^2/4L}^{\infty} t^{-1}e^{-t}\mathrm{d}t\\ &= -\frac{1}{8\pi^2}\,\mathrm{Ei}(-\|x\|^2/4L)\end{aligned}$$

where Ei is a special function called the exponential integral, which has an expansion

$$\mathrm{Ei}(t) = \gamma + \log|t| + \{\text{terms continuous at } t = 0\}.$$

(Here γ is the Euler-Mascheroni constant.) We conclude that the result of the integral (10.4.0.4) is $-\frac{1}{4\pi^2}\log\|x\|$, up to non-singular terms in x.

10.5 The operator product for holomorphic theories

It is often interesting to study translation-invariant theories on $\mathbb{R}^n$ for which the action of certain translations is homotopically trivial. For instance, in a topological theory, all translations act in a homotopically trivial way. Something similar happens for a holomorphic theory on $\mathbb{R}^{2n}$. In that case, half of the translations — those given by $\partial_{\bar{z}_i}$ in a set of complex coordinates — act homotopically trivial.

We will focus on holomorphically translation-invariant theories on $\mathbb{C}^n$ when $n \geq 2$. In that case, at first sight, one would expect that the OPE is trivial at the level of cohomology, by the following argument.

We use $\mathring{\mathbb{C}}^n$ to denote $\mathbb{C}^n \setminus \{0\}$, as in the notation $\mathring{D}$ from earlier.

By definition the operators $\partial_{\bar{z}_i}$ act homotopically trivially on the theory, and so the OPE map

$$
\begin{aligned}
\{-,-\}_{OPE} : H^*(\mathrm{Obs}^{cl}(0)) \otimes H^*(\mathrm{Obs}^{cl}(0)) \\
\to \left(C^\omega(\mathring{\mathbb{C}}^n) / \simeq \right) \otimes H^*(\mathrm{Obs}^{cl}(0))
\end{aligned}
$$

actually takes values in *holomorphic* functions on $\mathring{\mathbb{C}}^n$ modulo those functions that extend across the origin. But Hartogs' theorem tells us that all holomorphic functions on $\mathring{\mathbb{C}}^n$ extend to holomorphic functions on $\mathbb{C}^n$, so that the singular part of the OPE appears to be trivial.

We will find, however, that there is a non-trivial secondary operation of cohomological degree $1-n$ that is built from the OPE and from the homotopies that make the action of $\partial_{\bar{z}_i}$ homotopically trivial. This secondary operation is the analog, in the world of holomorphic theories, of the degree $1-n$ Poisson bracket that exists on the cohomology of an E_n algebra, and which appears as the bracket $\{-,-\}_{OPE}$ on the classical point observables of a topological field theory.

To explain where this secondary structure comes from, we will need to enhance our discussion of OPE to be compatible with Dolbeault complexes, and so we will begin by recalling the role these complexes play in holomorphic theories. Our main result in this section then appears as Proposition 10.5.3.1, which characterizes the secondary operation.

Finally, we discuss a simple holomorphic field theory on $\mathbb{C}^2$ that arises as the holomorphic twist of the free chiral multiplet.

10.5.1 Recollections on holomorphic factorization algebras

Holomorphically translation-invariant factorization algebras were studied in Chapter I.5. There, we saw that a holomorphically translation-invariant factorization algebra on $\mathbb{C}^n$ gives rise to a coalgebra over a colored cooperad PDiscs_n of polydiscs in $\mathbb{C}^n$, whose definition we now review. (Recall that a polydisc is an n-fold product of 1-dimensional discs. We use polydiscs because they are Stein manifolds, so that their Dolbeault cohomology is concentrated in degree 0.)

If $z \in \mathbb{C}^n$, let

$$PD_r(z) = \{w \in \mathbb{C}^n \mid |w_i - z_i| < r \text{ for } 1 \leq i \leq n\}$$

denote the polydisc of radius r around z. Let

$$\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s) \subset (\mathbb{C}^n)^k$$

denote any configuration of points $z_1, \dots, z_k \in \mathbb{C}^n$ with the property that the closures of the polydiscs $PD_{r_i}(z_i)$ are disjoint and contained in the polydisc $PD_s(0)$.

Given a holomorphically translation-invariant theory on $\mathbb{C}^n$ with factorization algebra Obs^q of quantum observables, the factorization product extends to a cochain map

$$\begin{aligned} &\mathrm{Obs}^q(PD_{r_1}) \times \cdots \times \mathrm{Obs}^q(PD_{r_k}) \\ &\qquad\qquad \to \Omega^{0,*}(\mathrm{PDiscs}_n(r_1, \dots, r_k \mid s), \mathrm{Obs}^q(PD_s)), \end{aligned}$$

where $\mathrm{Obs}^q(PD_r)$ denotes the value of Obs^q on the polydisc of radius r with any center.

Let us focus on the map when $k = 2$, the first polydisc PD_{r_1} is centered at the origin, and the second is centered at some point z. Taking the limit as the radii $r_i \to 0$, we find that there is a cochain map

$$\mathrm{Obs}^q(0) \otimes \mathrm{Obs}^q(0) \to \Omega^{0,*}(\mathring{D}_s, \mathrm{Obs}^q(PD_s))$$

that sends pairs of point observables to an observable on the punctured polydisc. This map is nearly an OPE, so we might hope that, working up to elements of the Dolbeault complex that extend across the origin, we will find an OPE operation roughly of the form

$$\mathrm{Obs}^q(0) \otimes \mathrm{Obs}^q(0) \to \left(\Omega^{0,*}(\mathring{D}_s)/\simeq\right) \otimes \mathrm{Obs}^q(0).$$

There is a technical problem here, however. The relation $\simeq$ addresses whether a function f can be extended continuously across the origin. But even if f is continuous at the origin, its derivative need not be continuous; in particular, $\overline{\partial} f$ may not be continuous at the origin. Hence we need to be more careful when we try to quotient $\Omega^{0,*}(\mathring{D}_s)$ by the subspace of Dolbeault forms that extend continuously across the origin, since the subspace may not be a subcomplex.

10.5.2 The equivalence relation $\simeq$ for the Dolbeault complex

To solve this problem, we note that there was an arbitrary choice involved in what we mean by "non-singular." We decided that the non-singular functions which we wanted to discard in the OPE were those that extend across the origin as continuous functions. One can instead make a weaker equivalence relation, where the non-singular functions are those which extend across the origin as k-times differentiable functions for some $k > 0$, i.e., C^k functions. With this weaker equivalence relation the OPE expansion will contain more terms. For instance, with the C^1 equivalence relation, the function $r \log r$ will appear in the OPE, as it is continuous but not C^1 at $r = 0$.

Let

$$f \simeq^{(k)} g$$

indicate that two functions $f, g \in C^\omega(U \setminus 0)$ differ by a function in $C^k(U)$, where $U \subset \mathbb{R}^n$ is an open subset containing the origin. In our construction of the OPE, there was nothing special about our choice of the C^0 equivalence relation: we could have used the C^k equivalence relation instead. Therefore, for any translation-invariant classical theory on $\mathbb{R}^n$, we have a C^k semi-classical OPE

$$\{-,-\}^{(k)}_{OPE} : \mathrm{Obs}^{cl}(0) \otimes \mathrm{Obs}^{cl}(0) \to \left(C^\omega(\mathbb{R}^n \setminus 0)/\simeq^{(k)}\right)$$

satisfying all the properties listed in Proposition 10.3.1.2. This OPE is given by the same sum-over-bridges formula derived in Proposition 10.3.2.2, except that the contribution of each bridge is now taken up to the addition of a C^k function instead of a C^0 function.

We can use these C^k equivalence relations to produce a quotient of the Dolbeault complex of $\mathring{\mathbb{C}}^n$, which is still a complex.

10.5.2.1 Definition. *Let*

$$\Omega^{0,*}(\mathring{\mathbb{C}}^n)/\simeq^{(n-*)}$$

denote the quotient of the Dolbeault complex $\Omega^{0,*}(\mathring{\mathbb{C}}^n)$ where in each degree i, we apply the equivalence relation $\simeq^{(n-i)}$ to $\Omega^{0,i}(\mathring{\mathbb{C}}^n)$.

In this definition, we are quotienting by a sub-complex because if we have an Dolbeault form

$$\alpha \in \Omega^{0,i}(\mathring{\mathbb{C}}^n)$$

that extends across the origin as a C^{n-i} Dolbeault form, then $\overline{\partial}\alpha$ extends across the origin as a C^{n-i-1} Dolbeault form.

To understand the structure of the OPE using this refined notion of equivalence, we need to understand the cohomology groups of this complex, which are described by the next lemma.

10.5.2.2 Lemma. *The cohomology groups*

$$H^k\left(\Omega^{0,*}(\mathring{\mathbb{C}}^n)/\simeq^{(n-*)}\right). \tag{10.5.2.1}$$

vanish unless $k = n-1$.

Moreover, there is a pairing

$$\begin{array}{ccc} \mathbb{C}[z_1,\dots,z_n] \otimes H^{n-1}\left(\Omega^{0,*}(\mathring{\mathbb{C}}^n)/\simeq^{(n-*)}\right) & \to & \mathbb{C} \\ F(z_i)\otimes\alpha & \mapsto & \int_{S^{2n-1}} F(z_i)\wedge\alpha\, \mathrm{d}z_1\dots\mathrm{d}z_n \end{array}$$

by integrating over the unit $2n-1$-sphere S^{2n-1} the wedge product of a form α in $\Omega^{0,n-1}(\mathring{\mathbb{C}}^n)$ with the $(n,0)$-form $F(z_i)\mathrm{d}z_1\dots\mathrm{d}z_n$. The induced map

$$H^{n-1}\left(\Omega^{0,*}(\mathring{\mathbb{C}}^n)/\simeq^{(n-*)}\right) \hookrightarrow (\mathbb{C}[z_1,\dots,z_n])^\vee$$

is an embedding.

Proof Throughout the proof, let $\Omega^{0,i}_{(n-i)}(\mathbb{C}^n)$ denote the Dolbeault forms the are smooth away from the origin and of class C^{n-i} at the origin. Let $\Omega^{0,*}_{(n-*)}(\mathbb{C}^n)$ denote the complex built from these forms, with the Dolbeault differential $\overline{\partial}$.

There is a short exact sequence of cochain complexes

$$0 \to \Omega^{0,*}_{(n-*)}(\mathbb{C}^n) \to \Omega^{0,*}(\mathring{\mathbb{C}}^n) \to \left(\Omega^{0,*}(\mathbb{C}^n \setminus 0)/ \simeq^{(n-*)}\right) \to 0.$$

The cohomology of the middle term $\Omega^{0,*}(\mathring{\mathbb{C}}^n)$ can be computed by an easy Čech cohomology calculation associated to the cover of $\mathring{\mathbb{C}}^n$ by the n open sets obtained by removing a coordinate hyperplane from $\mathbb{C}^n$.

One finds that it is the space $\mathrm{Hol}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$ in degree 0 (recall Hartogs' theorem), and it is zero in degrees not equal to $n-1$. In degree $n-1$, the natural integration pairing

$$\int_{S^{2n-1}} : \Omega^{0,n-1}(\mathring{\mathbb{C}}^n) \otimes \mathbb{C}[z_1, \dots, z_n] \mathrm{d}z_1 \dots \mathrm{d}z_n \to \mathbb{C}$$

gives an embedding

$$H^{n-1}_{\overline{\partial}}(\mathring{\mathbb{C}}^n) \hookrightarrow \mathbb{C}[z_1, \dots, z_n]^\vee.$$

To complete the proof of the lemma, we need to verify that the cohomology of $\Omega^{0,*}_{(n-*)}(\mathbb{C}^n)$ consists of holomorphic functions in degree 0. In other words, we need to check that the inclusion map

$$\Omega^{0,*}_{(n-*)}(\mathbb{C}^n) \hookrightarrow \Omega^{0,*}(\mathbb{C}^n)$$

induces an isomorphism on cohomology. The fact that cohomology in degree 0 of $\Omega^{0,*}_{(n-*)}(\mathbb{C}^n)$ consists of holomorphic functions follows from elliptic regularity. To check that the higher cohomology vanishes, one uses the usual proof of the $\overline{\partial}$-Poincaré lemma applies, which applies with the degree of regularity we are using.

□

10.5.3 The main result on holomorphic OPE

Now we can finally describe the structure of the semi-classical OPE of a holomorphic theory.

10.5.3.1 Proposition. *For a holomorphically translation-invariant classical field theory on $\mathbb{C}^n$, the semi-classical OPE*

$$\{-,-\}^{(n)}_{OPE} : \mathrm{Obs}^{cl}(0) \otimes \mathrm{Obs}^{cl}(0) \to \left(C^\infty(\mathring{\mathbb{C}}^n)/ \simeq^{(n)}\right) \otimes \mathrm{Obs}^{cl}(0)$$

extends to a cochain map

$$\{-,-\}^{\overline{\partial}}_{OPE} : \mathrm{Obs}^{cl}(0) \otimes \mathrm{Obs}^{cl}(0) \to \left(\Omega^{0,*}(\mathring{\mathbb{C}}^n)/ \simeq^{(n-*)}\right) \otimes \mathrm{Obs}^{cl}(0).$$

When $n = 1$ and our theory is equivariant for rotation, recall that the observables determine a vertex algebra (see Theorem I.5.3.3). In this case, this semi-classical OPE determines a vertex Poisson algebra.

Proof We will follow the argument in section I.5.2. Since we start with a holomorphically translation-invariant theory, for $i = 1, \dots n$, we have commuting operators η_i acting on $\mathrm{Obs}^{cl}(0)$ that are of cohomological degree -1 and that satisfy

$$\left[\mathrm{d}_{\mathrm{Obs}^{cl}(0)}, \eta_i\right] = D_{\overline{z}_i},$$

where $D_{\overline{z}_i}$ indicates the action of differentiation with respect to $\overline{z}_i$ on $\mathrm{Obs}^{cl}(0)$.

Now consider the expression

$$\left\{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\right\}_{OPE} \in \left(\Omega^{0,*}(\mathring{\mathbb{C}}^n)/ \simeq^{(n-*)}\right) \otimes \mathrm{Obs}^{cl}(0).$$

which is the same formula used in Chapter I.5. We need to check that this expression defines a cochain map. To do it, we will calculate the failure of the differential $\mathrm{d}_{\mathrm{Obs}^{cl}(0)}$ to satisfy the Leibniz rule for this operation. For concision's sake, let d be shorthand for $\mathrm{d}_{\mathrm{Obs}^{cl}(0)}$. Then we compute

$$\begin{aligned}
\mathrm{d}\{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE} =& \{\mathrm{d}O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE} \pm \{O, \mathrm{d}e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE} \\
=& \{\mathrm{d}O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE} \pm \{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} \mathrm{d}O'\}_{OPE} \\
& \mp \{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} \mathrm{d}\overline{z}_j D_{\overline{z}_j} O'\}_{OPE}.
\end{aligned}$$

Using the fact that applying $D_{\overline{z}_i}$ to $O'(x)$ is the same as differentiating with respect to the position x of the operator O', we find that

$$\begin{aligned}
\mathrm{d}\{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE} =& \{\mathrm{d}O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE} \pm \{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} \mathrm{d}O'\}_{OPE} \\
& \mp \sum \mathrm{d}\overline{z}_j \partial_{\overline{z}_j} \{O, e^{-\sum \eta_i \mathrm{d}\overline{z}_i} O'\}_{OPE}
\end{aligned}$$

so that

$$\{\mathrm{d}O, e^{-\sum \eta_i \mathrm{d}\bar{z}_i} O'\}_{OPE} \pm \{O, e^{-\sum \eta_i \mathrm{d}\bar{z}_i} \mathrm{d}O'\}_{OPE} = \mathrm{d}\{O, e^{-\sum \eta_i \mathrm{d}\bar{z}_i} O'\}_{OPE} \pm \sum \mathrm{d}\bar{z}_j \partial_{\bar{z}_j} \{O, e^{-\sum \eta_i \mathrm{d}\bar{z}_i} O'\}_{OPE},$$

as desired. □

At the level of cohomology, the semi-classical OPE for holomorphically translation-invariant theories takes a particularly nice form. It is a map

$$H^*(\mathrm{Obs}^{cl}(0)) \otimes H^*(\mathrm{Obs}^{cl}(0)) \to H^*\left(\Omega^{0,*}(\mathring{\mathbb{C}}^n)/\simeq^{(n-*)}\right) \otimes H^*(\mathrm{Obs}^{cl}(0)),$$

and we know some of the cohomology groups by Lemma 10.5.2.2. In particular, we know that the Dolbeault cohomology classes appearing in the OPE can be detected by integrating them over the $2n-1$-sphere against polynomials in the coordinates z_i. Hence the content of the holomorphic OPE is encoded in a sequence of bracket operations

$$\{-,-\}^{k_1 \dots k_n}_{OPE} : H^*(\mathrm{Obs}^{cl}(0)) \otimes H^*(\mathrm{Obs}^{cl}(0)) \to H^*(\mathrm{Obs}^{cl}(0))$$

defined by the formula

$$\{O, O'\}^{k_1 \dots k_n}_{OPE} = \sum_{i=1}^{n} \int_{S^{2n-1}} \{O, \eta_1 \cdots \widehat{\eta_i} \cdots \eta_n O'\}_{OPE}\, z_1^{k_1} \cdots z_n^{k_n} \mathrm{d}\bar{z}_1 \cdots \widehat{\mathrm{d}\bar{z}_i} \cdots \mathrm{d}\bar{z}_n \mathrm{d}^n z.$$

This expression is given in terms of the ordinary OPE between the operator O and the auxiliary operators $\eta_1 \dots \widehat{\eta_i} \dots \eta_n O'$ built from O'. This collection of brackets is a systematic generalization to higher dimensions of the Laurent (or mode) expansion of the OPE for vertex algebras.

10.5.4 A remark on topological field theories

These results can be generalized to topological field theories in a straightforward way, and also to theories where some directions are topological and others are holomorphic. Suppose we have a translation-invariant theory on $\mathbb{R}^n \times \mathbb{C}^m$ where all the translations on $\mathbb{R}^n$ act homotopically trivial and where the anti-holomorphic translations $\partial_{\bar{z}_i}$ act homotopically trivial. We denote the homotopies trivializing these actions by η_a and η'_i where a runs from 1 to n and i from 1 to m, respectively.

For any $U \subset \mathbb{R}^n \times \mathbb{C}^m$, consider the mixed de Rham/Dolbeault complex that we denote $\mathcal{A}^*(U)$. This complex is the quotient of the full de Rham complex $\Omega^*(U)$ by the differential ideal generated by $\mathrm{d}z_i$.

With the same proof as we just gave, one can show that the expression

$$\left\{ O, e^{-\sum \eta_a \mathrm{d}x_a + \eta_i' \mathrm{d}\bar{z}_i} O' \right\}_{OPE}$$

defines a cochain map

$$\mathrm{Obs}^{cl}(0) \otimes \mathrm{Obs}^{cl}(0) \to \left(\mathcal{A}^*((\mathbb{R}^n \times \mathbb{C}^m) \setminus \{0\}) / \simeq^{(n+m-*)} \right) \otimes \mathrm{Obs}^{cl}(0).$$

For a topological field theory on $\mathbb{R}^n$, the cohomology of $\Omega^*(\mathbb{R}^n \setminus 0)/ \simeq^{(-*)}$ consists of $\mathbb{C}$ in degree $n-1$. Thus, at the level of cohomology, the semi-classical OPE gives $H^*(\mathrm{Obs}^{cl}(0))$ a bi-derivation of degree $1-n$, exactly as we would expect from the semi-classical limit of an E_n algebra.

10.5.5 The holomorphic twist of the free $N = 1$ chiral multiplet on $\mathbb{R}^4$

Many holomorphic theories arise as twists of supersymmetric theories. In these examples, the auxiliary operators $\eta_1 \cdots \widehat{\eta_i} \cdots \eta_n O'$ are called *supersymmetric descendents* of O', because the operators η_i come from certain supersymmetries of the physical theory. We will now quickly demonstrate an example of how these techniques yield efficient tools for computing OPE in an example.

Consider the field theory in complex dimension 2 whose fields are $\alpha \in \Omega^{0,*}(\mathbb{C}^2)$ and $\beta \in \Omega^{2,*}(\mathbb{C}^2)[1]$, equipped with the odd symplectic pairing given by wedge and integration, so α and β pair as $\int \alpha\beta$. (This theory is cotangent-type.) Here, as usual, we work in the BV formalism. In the language of physics, the action functional is

$$S(\alpha, \beta) = \int_{\mathbb{C}^2} \beta \, \bar{\partial} \alpha,$$

and this theory is the holomorphic twist of the free $N = 1$ chiral multiplet.

We will refer to the graded components of Dolbeault degree i as α^i or β^i. At the level of cohomology, the commutative algebra of classical point observables is the freely generated by the observables

$$A_{n,m}(\alpha,\beta) = \partial_{z_1}^n \partial_{z_2}^m \alpha^0(0) \text{ and } B_{n,m}(\alpha,\beta) = \partial_{z_1}^n \partial_{z_2}^m \beta^0(0).$$

These are of cohomological degree 0 and 1 respectively.

The operators η_i, which are the homotopies trivializing the action of $\partial_{\overline{z}_i}$, act on the fields by contracting with $\partial_{\overline{z}_i}$. Applied to the observables $A_{n,m}$, $B_{n,m}$, we get

$$\eta_i A_{n,m}(\alpha,\beta) = \partial_{z_1}^n \partial_{z_2}^m \iota_{\partial_{\overline{z}_i}} \alpha^1(0) \text{ and } \eta_i B_{n,m}(\alpha,\beta) = \partial_{z_1}^n \partial_{z_2}^m \iota_{\partial_{\overline{z}_i}} \beta^1(0).$$

Thus, $\eta_1 A_{n,m}$ picks up the $\mathrm{d}\overline{z}_1$ component of $\alpha^1 \in \Omega^{0,1}(\mathbb{C}^2)$.

The propagator of the theory is the Green's function for the Dolbeault operator, which is

$$\mathrm{d}z_1 \mathrm{d}z_2 \frac{\overline{z}_1 \mathrm{d}\overline{z}_2 - \overline{z}_2 \mathrm{d}\overline{z}_1}{(|z_1|^2 + |z_2|^2)^2}$$

up to factors of π. We have the non-trivial OPEs

$$\{A_{0,0}, \eta_1 B_{0,0}(z_1,z_2)\}_{OPE} = \frac{-\overline{z}_2}{(|z_1|^2 + |z_2|^2)^2}$$

and

$$\{A_{0,0}, \eta_2 B_{0,0}(z_1,z_2)\}_{OPE} = \frac{\overline{z}_1}{(|z_1|^2 + |z_2|^2)^2}.$$

The holomorphic OPE

$$\{-,-\}_{OPE}^{n,m} : H^*(\mathrm{Obs}^{cl}(0)) \otimes H^*(\mathrm{Obs}^{cl}(0)) \to H^*(\mathrm{Obs}^{cl}(0)).$$

satisfies

$$\begin{aligned} \{A_{0,0}, B_{0,0}\}_{OPE}^{n,m} &= \int_{S^3} z_1^n z_2^m \frac{\overline{z}_1 \mathrm{d}\overline{z}_2 - \overline{z}_2 \mathrm{d}\overline{z}_1}{(|z_1|^2 + |z_2|^2)^2} \mathrm{d}z_1 \mathrm{d}z_2 \\ &= C \delta_{n=0} \delta_{m=0} \end{aligned}$$

for some non-zero constant C. Thus, the holomorphic OPE between these two operators is non-zero.

The holomorphic OPE between the operators $A_{n,m}$ and $B_{r,s}$ are determined from those between $A_{0,0}$ and $B_{0,0}$ by the compatibility conditions between the OPE and differentiation described in proposition 10.3.1.2. For instance, we have

$$\{A_{r,s}, B_{a,b}\}^{n,m}_{OPE} = \sum \int_{S^3} \mathrm{d}\bar{z}_i \{A_{r,s}(0,0), \eta_i B_{a,b}(z_1,z_2)\}_{OPE} z_1^n z_2^m \mathrm{d}z_1 \mathrm{d}z_2.$$

Since $B_{a,b} = D_{z_1} B_{a-1,b}$ and since

$$\{O(0,0), D_{z_1} O'(z_1,z_2)\}_{OPE} = \partial_{z_1} \{O(0,0), O'(z_1,z_2)\}_{OPE},$$

we find

$$\begin{aligned}\sum_i \int_{S^3} &\{A_{r,s}(0,0), \eta_i B_{a,b}(z_1,z_2)\}_{OPE}\, z_1^n z_2^m \mathrm{d}\bar{z}_i \mathrm{d}z_1 \mathrm{d}z_2 \\ &= \sum_i \int_{S^3} \partial_{z_1} \{A_{r,s}(0,0), \eta_i B_{a-1,b}(z_1,z_2)\}_{OPE}\, z_1^n z_2^m \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}\bar{z}_i.\end{aligned}$$

Integrating by parts, we find that

$$\{A_{r,s}, B_{a,b}\}^{n,m}_{OPE} = -n \{A_{r,s}, B_{a-1,b}\}^{n-1,m}_{OPE}.$$

Similarly, the properties derived in proposition 10.3.1.2 tell us that

$$\begin{aligned}D_{z_1} \{A_{r,s}(0,0), \eta_i B_{a,b}(z_1,z_2)\}_{OPE} &= \{D_{z_1} A_{r,s}(0,0), \eta_i B_{a,b}(z_1,z_2)\}_{OPE} \\ &\quad + \{A_{r,s}(0,0), D_{z_1} \eta_i B_{a,b}(z_1,z_2)\}_{OPE}.\end{aligned}$$

We deduce the identity

$$D_{z_1} \{A_{r,s}, B_{a,b}\}^{n,m} = \{A_{r+1,s}, B_{a,b}\}^{n,m}_{OPE} + \{A_{r,s}, B_{a+1,b}\}^{n,m}_{OPE}.$$

Since $\{A_{r,s}, B_{a,b}\}^{n,m}_{OPE}$ is a multiple of the identity operator in this free theory, the left hand side of this equation is zero. These identities reduce all the holomorphic OPEs to the computation of $\{A_{0,0}, B_{0,0}\}^{n,m}_{OPE}$.

10.6 Quantum groups and higher-dimensional gauge theories

Computations of the bracket $\{-,-\}_{OPE}$ can be done for many field theories. In this section we will analyze an example related to quantum groups that play an important role in recent work on geometric representation theory, as we discuss below. This example is more involved than any other we consider in this book, and it is included to illustrate how the techniques apply in contemporary research.

10.6.1 Higher-dimensional Chern-Simons theories and quantum groups

According to the philosophy of Costello (2013b, 2017); Costello et al. (2019), quantum groups are associated to Chern-Simons-type field theories in various dimensions. Ordinary Chern-Simons theory is a topological theory defined in three dimensions, but it has cousins where one or more of the real directions has been complexified. If only one real direction is complexified (e.g., $\mathbb{R}^3$ is replaced by $\mathbb{R}^2 \times \mathbb{C}$), we find the four-dimensional Chern-Simons theory studied in those references. In fact, depending on whether the complexified direction is $\mathbb{C}$, $\mathbb{C}^\times$, or an elliptic curve, the corresponding quantum groups are the Yangian, the quantum loop group, and the elliptic quantum group, respectively. These are one-variable quantum groups: they are quantizations of the Lie algebra of maps from a Riemann surface to a Lie algebra.

If two real directions of ordinary Chern-Simons theory are complexified (e.g., $\mathbb{R}^3$ is replaced by $\mathbb{R} \times \mathbb{C}^2$), one finds a 5-dimensional Chern-Simons theory studied in Costello (n.d., 2017). The corresponding quantum groups are two-variable quantum groups: they are quantizations of the Lie algebra of polynomial maps from an affine algebraic surface to a finite-dimensional Lie algebra. (To make the construction work at the quantum level, the affine algebraic surface needs to be symplectic. See Costello (n.d.) for details.)

In the literature, two-variable quantum groups have been constructed where the affine algebraic surface is $\mathbb{C}^2$, $\mathbb{C}^\times \times \mathbb{C}$, or $\mathbb{C}^\times \times \mathbb{C}^\times$. The corresponding quantum groups are called the deformed double current algebra Guay (2007); Guay and Yang (2017), the affine Yangian Maulik and Okounkov (2019); Schiffmann and Vasserot (2013), and the quantum toroidal algebra Hernandez (2009). These quantum groups play an important role in recent work in algebraic geometry Maulik and Okounkov (2019); Schiffmann and Vasserot (2013), representation theory Guay (2007); Guay and Yang (2017), and string theory Gaiotto and Oh (2019). So far, a rigorous proof that these two-variable quantum groups are those corresponding to 5-dimensional Chern-Simons theory is only available in the case that the surface is $\mathbb{C}^2$, but it is expected to hold in general.

10.6.2 Five-dimensional Chern-Simons theory on $\mathbb{R} \times \mathbb{C}^2$

We will take the affine algebraic surface to be $\mathbb{C}^2$. The fields and the action functional of the theory are defined in terms of a differential graded associative algebra

$$\begin{aligned}\mathcal{A}(\mathbb{R} \times \mathbb{C}^2) &= \Omega^*(\mathbb{R} \times \mathbb{C}^2)/\langle \mathrm{d}z_1, \mathrm{d}z_2 \rangle \\ &= \Omega^*(\mathbb{R}) \widehat{\otimes} \Omega^{0,*}(\mathbb{C}^2).\end{aligned}$$

On the first line, we have described the algebra as the quotient of forms on $\mathbb{R} \times \mathbb{C}^2$ by the differential ideal generated by $\mathrm{d}z_1$ and $\mathrm{d}z_2$. Because it is a differential ideal, the de Rham operator on $\Omega^*(\mathbb{R} \times \mathbb{C}^2)$ descends to a differential on $\mathcal{A}$, which we will call $\mathrm{d}_{\mathcal{A}}$. Denoting the coordinate on $\mathbb{R}$ by t, we write explicitly

$$\mathrm{d}_{\mathcal{A}} = \mathrm{d}t\, \partial_t + \mathrm{d}\bar{z}_1\, \partial_{\bar{z}_1} + \mathrm{d}\bar{z}_2\, \partial_{\bar{z}_2}.$$

That is, $\mathrm{d}_{\mathcal{A}}$ is the sum of the de Rham operator on $\mathbb{R}$ with the Dolbeault operator on $\mathbb{C}^2$.

The fundamental field of the theory is an element

$$\alpha \in \mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N[1].$$

The action functional is the Chern-Simons functional

$$\int_{\mathbb{R}\times\mathbb{C}^2} \mathrm{d}z_1, \mathrm{d}z_2\, CS(\alpha) = \int_{\mathbb{R}\times\mathbb{C}^2} \mathrm{d}z_1\, \mathrm{d}z_2\, \left(\tfrac{1}{2}\operatorname{Tr}(\alpha\, \mathrm{d}_{\mathcal{A}}\alpha) + \tfrac{1}{3}\operatorname{Tr}(\alpha^3) \right).$$

We are writing, as usual, the action functional in the BV formalism.

This theory is associated to the local dg Lie algebra $\mathcal{A} \otimes \mathfrak{gl}_N$ on $\mathbb{R} \times \mathbb{C}^2$. The corresponding formal moduli problem is the moduli of GL_N-bundles on $\mathbb{R} \times \mathbb{C}^2$ with a holomorphic structure on $\mathbb{C}^2$, a connection along $\mathbb{R}$, and the compatibility that the parallel transport along $\mathbb{R}$ gives an isomorphism of holomorphic bundles on $\mathbb{C}^2$. The dg Lie algebra has a natural invariant pairing

$$\langle \omega \otimes M, \omega' \otimes M' \rangle = \operatorname{Tr}(MM') \int \mathrm{d}z_1, \mathrm{d}z_2 \wedge \omega \wedge \omega'$$

given by tracing over the matrix factor and integrating the forms wedged with a holomorphic volume form on $\mathbb{C}^2$.

The first hint of the two-variable quantum groups in this field theory can be seen by taking its cohomology:

$$H^*(\mathcal{A}(\mathbb{C}^2) \otimes \mathfrak{gl}_N) = \mathrm{Hol}(\mathbb{C}^2) \otimes \mathfrak{gl}_N,$$

where $\mathrm{Hol}(\mathbb{C}^2)$ denotes the space of holomorphic functions on $\mathbb{C}^2$. In words, the cohomology is concentrated in degree 0, by the $\bar{\partial}$-Poincaré lemma, and it is the Lie algebra of holomorphic maps from $\mathbb{C}^2$ to $\mathfrak{gl}_N$.

Remark: In Costello (2017) it was important that we study a certain deformation of this theory, where the dg commutative algebra $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2)$ deforms into a non-commutative dg algebra. At the level of cohomology, this deformation amounts to replacing the algebra $\mathrm{Hol}(\mathbb{C}^2)$ of holomorphic functions on $\mathbb{C}^2$ (i.e., the cohomology of $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2)$) by a deformation quantization using the Poisson tensor $\partial_{z_1} \wedge \partial_{z_2}$. The Moyal product is a classic formula for this deformation quantization, and it extends to deform the algebra structure on $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2)$ as well. If c denotes the deformation parameter, the deformed product $\alpha *_c \beta$ has the form

$$\alpha\beta + \frac{c}{2}\epsilon_{ij}\frac{\partial}{\partial z_i}\alpha\frac{\partial}{\partial z_j}\beta + \frac{c^2}{2^2 \cdot 2!}\epsilon_{i_1 j_1}\epsilon_{i_2 j_2}\left(\frac{\partial}{\partial z_{i_1}}\frac{\partial}{\partial z_{i_2}}\alpha\right)\left(\frac{\partial}{\partial z_{j_1}}\frac{\partial}{\partial z_{j_2}}\beta\right) + \cdots$$

where ϵ_{ij} is the alternating tensor and we have used the summation convention.

It was essential to use the Moyal product in Costello (2017) because there are potential obstructions to quantizing the undeformed 5-dimensional Chern-Simons theory to all orders in $\hbar$. These potential obstructions are given by degree 1 cohomology classes in the complex of local functionals. Certain symmetries of the theory guarantee, however, that these obstructions can only appear at order $\hbar^2$ and higher. Since we are doing a semi-classical analysis, i.e., working to order $\hbar$ only, they will not appear in our story and we can use the undeformed version of 5-dimensional Chern-Simons theory. $\Diamond$

10.6.3 The main result

The goal of this section is to show a precise relationship with quantum groups using the OPE technology we have developed in this chapter.

To state the main result, we need the concept of Koszul duality of dg associative algebras, a rather advanced topic for which we recommend Polishchuk and Positselski (2005); Positselski (1993); Fløystad (2006); Keller (2006) as starting points into the vast literature. In section 11.2.1 we discuss Koszul duality briefly, with an emphasis on the dg algebras arising from a Lie algebra, which is the primary situation of interest here.

In brief, the *Koszul dual* $A^!$ of a differential graded algebra A with an augmentation $\varepsilon : A \to \mathbb{C}$ is

$$A^! = \mathrm{RHom}_A(\mathbb{C}, \mathbb{C}),$$

where we view $\mathbb{C}$ as an A-module via the augmentation. (We emphasize that the dual depends on the augmentation.) The product in $A^!$ is by composition of homomorphisms. This dual algebra is well-defined up to quasi-isomorphism, and one can provide explicit dg models, when needed.

For example, the polynomial algebra $\mathbb{C}[x]$ of functions on the affine line can be augmented by evaluating functions at the origin $x = 0$, and the Koszul dual algebra is $\mathbb{C}[\epsilon]$, where ϵ has degree one. (This example generalizes to duality between symmetric and exterior algebras.) As a more subtle example — and crucial for us here — the enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, with the canonical augmentation that annihilates the ideal generated by $\mathfrak{g}$, has Koszul dual given by the Chevalley-Eilenberg cochains $C^*(\mathfrak{g})$. The subtlety here is that we must treat $C^*(\mathfrak{g})$ as a *filtered* dg algebra, with the filtration

$$F^i C^*(\mathfrak{g}) = C^{\geq i}(\mathfrak{g}).$$

This setting is treated in detail in Costello (2013b). We use this fact in the main result.

10.6.3.1 Proposition. *The point observables of the theory associated to $\mathcal{A} \otimes \mathfrak{gl}_N$ on $\mathbb{R} \times \mathbb{C}^2$ satisfy:*

(i) *At the classical level, the point operators form a dg commutative algebra quasi-isomorphic to $C^*(\mathfrak{gl}_N[[z_1, z_2]])$. The Koszul dual of this dg algebra is $U(\mathfrak{gl}_N[[z_1, z_2]])$.*

(ii) *At the quantum level, modulo $\hbar^2$, the point operators form a dg algebra quasi-isomorphic to a non-trivial deformation of $C^*(\mathfrak{gl}_N[[z_1, z_2]])$, where the deformation is determined by an explicit Poisson bracket. The Koszul* *dual of this algebra of point operators is an explicit first-order deformation of* $U(\mathfrak{gl}_N[[z_1, z_2]])$ *into a "quantum universal enveloping algebra." The form of this deformation is studied in Guay (2007); Costello (2017).*

There are a few technical subtleties to bear in mind about our use of Koszul duality. First, the factorization algebra associated to any quantum or classical field theory comes equipped with a filtration, and it is this filtration that we use. It corresponds, under the quasi-isomorphism mentioned in the proposition, to the filtration

$$F^i C^*(\mathfrak{gl}_N[[z_1, z_2]]) = C^{\geq i}(\mathfrak{gl}_N[[z_1, z_2]])$$

on $C^*(\mathfrak{gl}_N[[z_1, z_2]])$. The second point to bear in mind is that the observables have a unique augmentation (up to homotopy). This happens at either the quantum or classical level, because H^0 of the algebra of observables is simply $\mathbb{C}[[\hbar]]$, and negative cohomology vanishes.

The remainder of this chapter provides proof of the proposition, but we break it up over several subsections to develop the relevant ideas.

10.6.4 The algebra of classical point operators of 5-dimensional Chern-Simons theory

We are interested in point observables. Hence we need to consider a nested sequence of open sets whose intersection is the origin, and we need to determine the limit of the classical observables over this sequence.

We thus take an open subset in $\mathbb{R} \times \mathbb{C}^2$ of the form $I \times D$, where I is an interval in $\mathbb{R}$ and D is a polydisc in $\mathbb{C}^2$. The cohomology of the fields $\mathcal{A}(I \times D)$ is $\mathrm{Hol}(D)$, the algebra of holomorphic functions on D, and therefore the algebra of classical observables in $I \times D$ is quasi-isomorphic to $C^*(\mathrm{Hol}(D) \otimes \mathfrak{gl}_N)$.

There is a canonical map on the fields

$$\mathrm{Hol}(D) \otimes \mathfrak{gl}_N \to \mathfrak{gl}_N[[z_1, z_2]]$$

given by taking Taylor series at the origin in D. Observe that if a functional on $\mathrm{Hol}(D) \otimes \mathfrak{gl}_N$ only depends on the value of the fields and their derivatives at the origin, then the functional factors through this Taylor expansion map. We conclude that the algebra of point observables satisfies

$$\mathrm{Obs}^{cl}(0) \simeq C^*(\mathfrak{gl}_N[[z_1, z_2]]).$$

It is important to note that we are taking here the *continuous* Lie algebra cochains of the topological Lie algebra $\mathfrak{gl}_N[[z_1, z_2]]$, so we now turn to unpacking what that means.

Recall that the continuous linear dual of $\mathbb{C}[[z_1, z_2]]$ is $\mathbb{C}[\partial_{z_1}, \partial_{z_2}]$, as follows. Given a constant-coefficient differential operator $D \in \mathbb{C}[\partial_{z_1}, \partial_{z_2}]$, there is a continuous linear functional L_D on $\mathbb{C}[[z_1, z_2]]$ sending $f(z_1, z_2)$ to $Df(0)$, the value of Df at the origin. That is, there is a map $D \mapsto D\delta_0$ sending such a differential operator D to a distribution with support at the origin. This map is, in fact, an isomorphism. Hence, we see that the classical point observables satisfy

$$C^*(\mathfrak{gl}_N[[z_1, z_2]]) = \mathrm{Sym}^*(\mathfrak{gl}_N^\vee \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}][-1])$$

as a graded vector space. We have shown the first item of Proposition 10.6.3.1.

10.6.5 The Poisson bracket and the OPE

In the connection between quantum groups and field theories, the quantum group arises as the Koszul dual of the algebra of quantum point observables. In our five-dimensional theory, the Koszul dual of the classical point observables $C^*(\mathfrak{gl}_N[[z_1, z_2]])$ is $U(\mathfrak{gl}_N[[z_1, z_2]])$. The quantum group associated to the quantized theory will be a deformation of this enveloping algebra as an associative algebra.

To identify this deformation, our strategy is to view the real direction of our space $\mathbb{C}^2 \times \mathbb{R}$ as a kind of time coordinate and to treat our theory as a mechanical system in this direction. By construction, the factorization algebra for this theory is locally constant along $\mathbb{R}$ and holomorphic along $\mathbb{C}^2$, both at the classical and quantum levels. Thanks to the local constancy along $\mathbb{R}$, the product of observables in this direction should determine an associative algebra. In particular, the quantization of the theory should deform this product.

More formally, let Obs denote the factorization algebra associated to the theory we are studying (either at the quantum or classical level). Let $\pi : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}$ be the projection map. The factorization algebra we want is almost $\pi_* \operatorname{Obs}$. Due to analytical issues, however, it is more convenient to consider the subalgebra $\pi_*^{\mathrm{res}} \operatorname{Obs}$ given by those observables that, order by order in $\hbar$, are finite sums of eigenvectors for the $U(1) \times U(1)$ action that rotates each coordinate of $\mathbb{C}^2$.

One can check that $\pi_*^{\mathrm{res}} \operatorname{Obs}$ is also a locally constant factorization algebra on $\mathbb{R}$. Therefore, it defines a homotopy associative algebra. Further, for any interval $I \subset \mathbb{R}$ containing the origin, the inclusion

$$\operatorname{Obs}(0) \hookrightarrow \pi_*^{\mathrm{res}} \operatorname{Obs}(I)$$

is a quasi-isomorphism. In this way, we equip the point observables $\operatorname{Obs}(0)$ with a homotopy associative algebra structure, using the factorization product of $\pi_*^{\mathrm{res}} \operatorname{Obs}$ on $\mathbb{R}$.

We now compute this product explicitly using the techniques developed in this chapter. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two point observables in $\operatorname{Obs}(0)$. Consider the factorization product

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(t) \in (\pi_*^{\mathrm{res}} \operatorname{Obs})(\mathbb{R}) \simeq \operatorname{Obs}(0)$$

for $t \in \mathbb{R} \setminus 0$. This punctured line has two components, and in each component, the product does not change under small variations of t (up to cohomologically exact terms) because our factorization algebra is locally constant on $\mathbb{R}$. When $t > 0$, it is the product in our homotopy associative algebra structure on $\operatorname{Obs}(0)$. When $t < 0$, it is the product in the opposite algebra. Therefore, the commutator of the product satisfies

$$[\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1(0)\mathcal{O}_2(t) - \mathcal{O}_1(0)\mathcal{O}_2(-t)$$

for any $t > 0$.

This expression allows us to relate the operator product expansion in the t-direction to the commutator. The factorization product $\mathcal{O}_1(0)\mathcal{O}_2(t)$ is locally constant as a function of t, up to exact terms, while the operator product expansion picks up the singular parts of this product. But in this one-dimensional situation, singular parts becomes quite simple: the space of singular functions on $\mathbb{R} \setminus \{0\}$ that are locally constant is spanned by the step function $\operatorname{Sign}(t)$, which is 1 on $\mathbb{R}_{>0}$ and -1 on

$\mathbb{R}_{<0}$. We thus conclude that the operator product expansion is

$$\mathcal{O}_1(0)\mathcal{O}_2(t) \simeq [\mathcal{O}_1, \mathcal{O}_2]\,\mathrm{Sign}(t),$$

where as before $\simeq$ indicates equality up to non-singular functions of t.

To first order in $\hbar$, we then have two descriptions of the same situation. On the other hand, by the general theory of deformation quantization, the commutator bracket to first order in $\hbar$ is given by a Poisson bracket, which makes the classical point observables into a Poisson algebra. On the one hand, by our work in this chapter, we know the OPE modulo $\hbar$ determines a bracket $\{\mathcal{O}_1(0), \mathcal{O}_2(t)\}_{OPE}$. Hence we can conclude that

$$\{\mathcal{O}_1(0), \mathcal{O}_2(t)\}_{OPE} = \{\mathcal{O}_1, \mathcal{O}_2\}\,\mathrm{Sign}(t), \tag{10.6.5.1}$$

which tells us that the OPE for the quantized theory determines the Poisson bracket that contols the deformation quantization we want to identify (i.e., the quantum group).

10.6.6 Computing the Poisson bracket

Our next goal is to compute the Poisson bracket $\{\mathcal{O}_1, \mathcal{O}_2\}$ for two elements in

$$C^*(\mathfrak{gl}_N[[z_1, z_2]]) \simeq \mathrm{Obs}^{cl}(0),$$

which we will read off the OPE bracket $\{-,-\}_{OPE}$ by equation (10.6.5.1). It suffices to compute the bracket of linear observables, because the OPE bracket is a derivation in each factor and so we can compute it for polynomial observables by iterations of the Leibniz rule.

We have seen that the linear dual of $\mathfrak{gl}_N[[z_1, z_2]]$ is $\mathfrak{gl}_N[\partial_{z_1}, \partial_{z_2}]$. Concretely, a linear operator has the form

$$\mathcal{O}(\alpha) = \mathrm{Tr}(M(D\alpha)(0))$$

for some $N \times N$-matrix M and some constant-coefficient differential operator $D \in \mathbb{C}[\partial_{z_1}, \partial_{z_2}]$. Here $\alpha \in \mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N[1]$ is the fundamental field, and by $(D\alpha)(0)$ we mean evaluation at the origin of the function $D\alpha$ in $\mathcal{A}^0(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N[1]$.

For convenience we introduce some useful notation to describe such

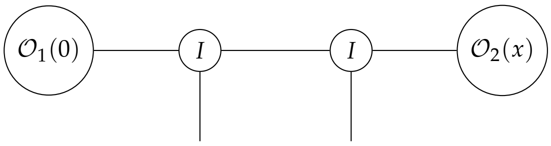

Figure 10.4 The unique bridge with two internal vertices that contributes to our OPE calculation.

operators. Let $\{E_{ij}\}$ denote the standard linear basis for $\mathfrak{gl}_N$ given by the elementary matrices. (That is, E_{ij} is zero in every entry except the (i,j)th, where it is one.) Then, thanks to our description of linear operators above, we see that $\{E_{ij}\,\partial_{z_1}^l \partial_{z_2}^k\}$ forms a linear basis for the linear observables $\mathfrak{gl}_N[\partial_{z_1}, \partial_{z_2}]$, since these span the matrix-valued, constant-coefficient differential operators.

In this section we will not compute the OPE between arbitrary linear observables, just a minimal collection that determines what we need here. In fact, we will only compute the OPE between such linear operators where the differential operator D is the identity but where M is an arbitrary matrix. If we replace $\mathfrak{gl}_N$ by a simple Lie algebra such as $\mathfrak{sl}_N$, all OPEs are determined from this restricted class by certain algebraic consistency conditions.

Before delving into the OPE computation, we record the remarkably simple answer for the most basic such observables.

10.6.6.1 Lemma. *The Poisson bracket on the classical point observables satisfies*

$$\{E_{ij}, E_{kl}\} = (E_{il}\partial_{z_1})(E_{kj}\partial_{z_2}) - (E_{il}\partial_{z_2})(E_{kj}\partial_{z_1}), \tag{10.6.6.1}$$

where on the left hand side E_{ij} means the operator $E_{ij}\,\partial_{z_1}^0 \partial_{z_2}^0$, which involves no interesting differential operator.

This lemma follows from a general computation, which we now undertake. According to Proposition 10.3.2.2, these OPEs are given by a sum-over-bridges formula. In the theory at hand, for each $n \geq 0$ there is exactly one bridge with $2+n$ vertices that contributes to the calculation. The bridge with $n = 2$ is illustrated in figure 10.4; this bridge plays the most important role in our computation.

We could calculate the amplitude of each diagram contributing to the

OPE by performing an integral, as we did in the scalar field case. In this example, however, we will introduce a more efficient method. To compute the amplitude of a diagram such as that in figure 10.4, we compose operators acting on the space of fields $\mathcal{A} \otimes \mathfrak{gl}_N$, instead of performing integrals. (Note that the natural pairing on this space of fields lets us turn an integral kernel into an operator and vice versa.) This operator approach gives the same answer as the integrals, because the integral expressions describe convolutions that compute the composition of operators.

For instance, instead of viewing the propagator $P(\Phi)$ as an integral kernel, we view it as the linear operator

$$\mathrm{d}^*_{\mathcal{A}} \triangle_\Phi^{-1}$$

on $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$. Here we have chosen the gauge-fixing operator

$$\begin{aligned} \mathrm{d}^*_{\mathcal{A}} &= 2\overline{\partial}^*_{\mathbb{C}^2} + \mathrm{d}^*_{\mathbb{R}} \\ &= \iota_{\partial_t}\partial_t + 4\iota_{\partial_{\overline{z}_1}}\partial_{z_1} + 4\iota_{\partial_{\overline{z}_2}}\partial_{z_2} \end{aligned}$$

The operator $\triangle_\Phi^{-1}$ is given by convolution with the regularized Green's kernel defined by the parametrix Φ. (We give an explicit formula for this kernel below, when it is needed.) By construction, $\triangle_\Phi^{-1}$ is a two-sided inverse — up to smoothing operators that depend on Φ — to the Laplacian

$$\begin{aligned} \triangle &= [\mathrm{d}_{\mathcal{A}}, \mathrm{d}^*_{\mathcal{A}}] \\ &= \partial_t^2 + 4\partial_{z_1}\partial_{\overline{z}_1} + 4\partial_{z_2}\partial_{\overline{z}_2} \\ &= \partial_t^2 + \sum_i (\partial_{x_i}^2 + \partial_{y_i}^2), \end{aligned}$$

where on the final line we expand $z_j = x_j + \mathrm{i}y_j$ and hence obtain the standard Euclidean Laplacian. (Recall that a smoothing operator is an operator that sends an input distribution to a smooth output. Smoothing operators are given by convolution with smooth kernels.)

Thus, each edge in the diagram gives the operator $\mathrm{d}^*_{\mathcal{A}} \triangle_\Phi^{-1}$. Likewise, at each interior vertex, its external leg takes as input a background field α in $\mathcal{A} \otimes \mathfrak{gl}_N[1]$. Each interior vertex then gives the operator $\mathrm{Ad}\,\alpha$ of bracketing with α, which is linear on the Lie algebra $\mathcal{A} \otimes \mathfrak{gl}_N[1]$. It remains to describe the effect of the initial and final vertices.

We read the diagram from left to right. The initial vertex — the left-most vertex in the diagram in figure 10.4 — is associated to an operator that is a linear functional on $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$. Via the pairing on $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$, this linear functional is an element of the distributional completion $\overline{\mathcal{A}}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$ of this space. We choose

$$\mathcal{O}_1(\alpha) = \mathrm{Tr}(M_1\, \alpha(0))$$

for some $M_1 \in \mathfrak{gl}_N$. The corresponding element of $\overline{\mathcal{A}}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$ is $\delta_0 M_1$. (We have abused notation slightly by viewing the δ-function at the origin as an element of $\overline{\mathcal{A}}^3(\mathbb{R} \times \mathbb{C}^2)$ by removing $\mathrm{d}z_1 \mathrm{d}z_2$ from the usual δ-function, viewed as a distributional 5-form on $\mathbb{R} \times \mathbb{C}^2$.)

At the final vertex — the rightmost vertex in figure 10.4 — we place our second operator

$$(\mathcal{O}_2(x))\,(\alpha) = \mathrm{Tr}(M_2\, \alpha(x)),$$

which we also view as a linear functional on $\mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$ and supported at some point $x \in \mathbb{C}^2 \times \mathbb{R}$. We are only interested in the case where $x = (0, 0, t) \in \mathbb{C}^2 \times \mathbb{R}$, so we view the operator as a function of t from here on.

Composing these operators, we find that the amplitude for the diagram with n internal vertices is

$$\mathrm{Tr}\left[M_2\, \mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1} \left((\mathrm{Ad}\, \alpha)\mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1}\right)^n M_1 \delta_0\right](t).$$

Here $\alpha \in \mathcal{A}(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N[1]$ is the field placed at the external legs, and the trace is taken in the fundamental representation of $\mathfrak{gl}_N$. We read this composition of operators from the right to left: we take $M_1\delta_0$, apply the operator $(\mathrm{Ad}\, \alpha)\mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1}$ n times, apply the operator $M_2 \mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1}$, trace over the $\mathfrak{gl}_N$ indices, and finally evaluate at t.

At the level of cohomology, each operator arising in the diagram only depends on the component of α in $\mathcal{A}^0$. We will therefore assume that α has cohomological degree -1 and so lies in $\mathcal{A}^0(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$. This restriction greatly simplifies the problem, so that the only nontrivial diagram has precisely 3 edges, and so 2 internal vertices. In other words figure 10.4 is the only one that matters.

Justifying this claim that only this digram matters is the following argument by cohomological degrees. Note that we must end up with something in $\mathcal{A}^0(\mathbb{R} \times \mathbb{C}^2)$, because otherwise the evaluation at t will be zero. Vertices do not change the degree because it is bracketing with a degree 0 element $\alpha \in \mathcal{A}^0(\mathbb{R} \times \mathbb{C}^2) \otimes \mathfrak{gl}_N$. Every edge of the diagram involves one copy of the operator $\mathrm{d}^*_{\mathcal{A}}$, which changes degree by 1. Finally, the δ-function δ_0 is in $\overline{\mathcal{A}}^3(\mathbb{R} \times \mathbb{C}^2)$. Thus we need exactly 3 edges, as asserted.

We have thus found that the only non-trivial contribution to the one-loop OPE comes from the expression

$$\left\{ \mathrm{Tr} \left(M_2 \mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1} (\mathrm{Ad}\, \alpha) \mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1} (\mathrm{Ad}\, \alpha) \mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1} M_1 \delta_0 \right) \right\} (t). \quad (10.6.6.2)$$

This quantity is a continuous function of α. Therefore we can assume without loss of generality that α is a polynomial, by Weierstrass approximation. At the level of cohomology, α only needs to be a polynomial in z_1, z_2, not in $\bar{z}_1, \bar{z}_2$. Thus we can write α as a linear combination

$$\alpha = \sum f_i N_i$$

for some matrices N_i and polynomial functions f_i of z_1 and z_2. We can now explicitly describe the OPE.

10.6.6.2 Lemma. *Consider the operators*

$$\mathcal{O}_i(\alpha) = \mathrm{Tr}(M_i \alpha(0))$$

associated to $N \times N$-matrices M_i. When the field α takes the form $f^i N_i$ for f_i functions on $\mathbb{R} \times \mathbb{C}^2$ and N_i matrices, the OPE of these operators is the operator

$$\begin{aligned} &\{\mathcal{O}_1(0), \mathcal{O}_2(t)\}_{OPE}(f^i N_i) \\ &= (\delta_{t \geq 0} - \delta_{t \leq 0}) \frac{1}{8\pi} \epsilon_{kl} (\partial_{z_k} f^i)(0) (\partial_{z_l} f^j)(0)\, \mathrm{Tr}(M_1 (\mathrm{Ad}\, N_i)(\mathrm{Ad}\, N_j) M_2), \end{aligned}$$

where ϵ_{kl} is the alternating symbol, and we sum over repeated indices..

Lemma 10.6.6.1 is a quick consequence of this result. We note that we absorbed the factors of π into the constant $\hbar$ in that expression.

Proof Note that $\mathrm{d}^*_{\mathcal{A}}$ commutes with $\triangle_{\Phi}^{-1}$ and that $(\mathrm{d}^*_{\mathcal{A}})^2 = 0$. This property means that we can rewrite the quantity in the expression 10.6.6.2 as

$$\mathrm{Tr}(M_2\, \mathrm{Ad}(N_i)\, \mathrm{Ad}(N_j) M_1) \left\{ \mathrm{d}^*_{\mathcal{A}} \triangle_{\Phi}^{-1} [\mathrm{d}^*_{\mathcal{A}}, f_i] \triangle_{\Phi}^{-1} [\mathrm{d}^*_{\mathcal{A}}, f_j] \triangle_{\Phi}^{-1} \delta_0 \right\} (t).$$

We have factored the OPE into a product of a Lie-theoretic factor $\mathrm{Tr}(M_2 \,\mathrm{Ad}(N_i)\,\mathrm{Ad}(N_j) M_1)$ and an analytic factor that depends on the f_i. Since the f_i are holomorphic, the operator $[\mathrm{d}^*_{\mathcal{A}}, f_i]$ becomes

$$[\mathrm{d}^*_{\mathcal{A}}, f_i] = 4(\partial_{z_1} f_i)\iota_{\partial_{\bar{z}_1}} + 4(\partial_{z_2} f_i), \iota_{\partial_{\bar{z}_2}}$$

where the operators $\iota_{\partial_{\bar{z}_l}}$ denote contraction with the vector field $\partial_{\bar{z}_l}$. These commute with $\triangle_\Phi^{-1}$ and so the analytic factor of the OPE simplifies to

$$16\,\epsilon_{kl} \left\{ \partial_t \triangle_\Phi^{-1}(\partial_{z_k} f_i) \triangle_\Phi^{-1}(\partial_{z_l} f_j) \triangle_\Phi^{-1} \iota_{\partial_{\bar{z}_k}} \iota_{\partial_{\bar{z}_l}} \iota_{\partial_t} \delta_0^{\mathcal{A}} \right\}(t) \qquad (10.6.6.3)$$

where ϵ_{kl} is the alternating symbol.

In this expression, $\delta_0^{\mathcal{A}}$ is the δ-function at the origin, viewed as a distributional element of the algebra $\mathcal{A}$ in degree 3. Once we apply the contraction $\iota_{\partial_{\bar{z}_k}} \iota_{\partial_{\bar{z}_l}} \iota_{\partial_t}$ in formula (10.6.6.3), we replace $\delta_0^{\mathcal{A}}$ by the δ-function without any dependence on differential forms, which we denote δ_0.

Thus, by stripping off the Lie algebra factor, we have reduced the problem to one entirely about functions on $\mathbb{R} \times \mathbb{C}^2$. We now need to compute

$$\partial_t \triangle_\Phi^{-1}(\partial_{z_1} f) \triangle_\Phi^{-1}(\partial_{z_2} g) \triangle_\Phi^{-1} \delta_0 \qquad (10.6.6.4)$$

for two arbitrary functions f and g. To do this calculation we need to describe the propagator explicitly.

The propagator for our gauge fixing and parametrix Φ is an element

$$P(\Phi) \in \mathfrak{gl}_N \otimes \mathfrak{gl}_N \otimes \overline{\mathcal{A}}(\mathbb{R} \times \mathbb{C}^2) \widehat{\otimes} \overline{\mathcal{A}}(\mathbb{R} \times \mathbb{C}^2).$$

It is a tensor product of the quadratic Casimir $c_{\mathfrak{gl}_N}$ in $\mathfrak{gl}_N \otimes \mathfrak{gl}_N$ with an integral kernel in $\overline{\mathcal{A}}(\mathbb{R} \times \mathbb{C}^2) \widehat{\otimes} \overline{\mathcal{A}}(\mathbb{R} \times \mathbb{C}^2)$. The Green's kernel for the scalar Laplacian on $\mathbb{R} \times \mathbb{C}^2$ is

$$\frac{1}{\left((t-t')^2 + \sum |z_i - z_i'|^2\right)^{3/2}},$$

up to factors of 2 and π. Our parametrix is this Green's function multiplied by $F(t-t', |z_i - z_i'|)$, where F is a function on $\mathbb{R}^3$ that is 1 near the origin and 0 outside a ball of finite radius. Then

$$P(\Phi) = c_{\mathfrak{gl}_N} \mathrm{d}^*_{\mathcal{A}} \left\{ \frac{F(t-t', |z_i - z_i'|)}{\left((t-t')^2 + \sum |z_i - z_i'|^2\right)^{3/2}} \mathrm{d}(t-t')\mathrm{d}(\overline{z}_1 - \overline{z}_1')\mathrm{d}(\overline{z}_2 - \overline{z}_2') \right\}$$

is the propagator for our theory.

We are interested in the function (10.6.6.4) up to non-singular terms. We will show by dimensional analysis that this quantity is non-singular unless $\partial_{z_1} f$ and $\partial_{z_2} g$ are constant.

Under the action of uniform scaling of $R \times \mathbb{C}^2$, a holomorphic monomial $z_1^m z_2^n$ has weight $m+n$. Thus, the delta-function δ_0 has weight -5, and each derivative, such as ∂_t, has weight -1. The operator $\triangle^{-1}$ increases weight by 2. The operator $\triangle_\Phi^{-1}$ does as well, up to non-singular terms. That is, if ρ denotes the vector field corresponding to rescaling $\mathbb{R} \times \mathbb{C}^2$, then

$$[\rho, \triangle_\Phi^{-1}]D = 2\triangle_\Phi^{-1} D + \text{ a smooth operator}$$

for any distribution D. Thus, if we assume that the functions f, g are homogeneous polynomials of weights k, l, then the function in (10.6.6.4) has weight $-5+6+k+l-3$, up to non-singular terms, with the 6 from the three copies of $\triangle_\Phi^{-1}$ and the -3 from the three derivatives.

This function in (10.6.6.4) will be non-singular whenever $k+l > 2$. Note that if $k = 0$ or $l = 0$, the function is zero because f, g are differentiated in equation (10.6.6.4). In particular, the only way to get a nontrivial singular OPE is if $f = z_1$ and $g = z_2$. This choice reduces equation (10.6.6.4) to computing

$$\partial_t \triangle_\Phi^{-1} \triangle_\Phi^{-1} \delta_0.$$

To compute the square of $\triangle_\Phi^{-1}$ applied to δ_0, we use the technique from section 10.4.1, where we used the heat kernel to determine the OPE for the scalar field.

If our parametrix is given by the heat kernel, then we have

$$\int_{s=0}^{L} s K_s \simeq \triangle_\Phi^{-1} \triangle_\Phi^{-1} \delta_0,$$

exactly as we found in the case of the scalar field. Letting $r^2 = t^2 +$

$|z_1|^2 + |z_2|^2$, we have

$$K_s = \frac{1}{(4\pi s)^{3/2}} e^{-r^2/4s}.$$

Since we are interested in evaluating the result at $z_i = 0$, we find that we need to compute the t-derivative of

$$\int_{s=0}^{L} \left(\frac{1}{(4\pi)^{3/2} s^{1/2}} e^{-t^2/4s} \right) \mathrm{d}t.$$

This integral can be computed to be

$$\frac{1}{8\pi^{3/2}} |t| \left(\pi^{1/2} \operatorname{erf}(t) - 1 \right) + \text{ smooth functions of } t$$

where the error function $\operatorname{erf}(t)$ is defined by

$$\operatorname{erf}(t) = 2\pi^{-1/2} \int_0^t e^{-u^2} \mathrm{d}u.$$

As $|t| \operatorname{erf}(t)$ has a continuous first derivative, it will not contribute to our OPE. Hence

$$\partial_t \triangle_\Phi^{-1} \triangle_\Phi^{-1} \delta_0 \simeq (\delta_{t \geq 0} - \delta_{t \leq 0}) \frac{1}{8\pi},$$

which yields the analytic factor of the OPE. □

10.6.7 The first-order deformation of the Koszul dual algebra

We have now identified the Poisson bracket on $C^*(\mathfrak{gl}_N[[z_1, z_2]]) \simeq \mathrm{Obs}^{cl}(0)$ arising from the first-order OPE. As we discussed earlier, this bracket identifies the deformation quantization of these observables, and under Koszul duality it will also identify the deformation quantization of the algebra $U(\mathfrak{gl}_N[[z_1, z_2]])$, which is the Koszul dual to the algebra of classical point operators of our 5-dimensional theory.

Let us explain how to calculate this first-order deformation explicitly. The generators $E_{ij} z_1^k z_2^l$ of the algebra $U(\mathfrak{gl}_N[[z_1, z_2]])$ are linear dual to the generators $E_{ji} \partial_{z_1}^k \partial_{z_2}^l$ of the algebra $C^*(\mathfrak{gl}_N[[z_1, z_2]])$, up to a factor. By turning off the differential on $C^*(\mathfrak{gl}_N[[z_1, z_2]])$ and replacing $U(\mathfrak{gl}_N[[z_1, z_2]])$ by the symmetric algebra $\mathrm{Sym}^* \mathfrak{gl}_N[[z_1, z_2]]$, the Koszul duality becomes the very simplest kind of Koszul duality: that between an exterior algebra on a vector space V and the symmetric algebra $Sym(V^*)$ on its dual.

Let us recall how deformations of these symmetric and exterior algebras transform under Koszul duality. If V is a vector space, then a quadratic-coefficient Poisson bracket (i.e., a bi-vector field) on the algebra $\mathrm{Sym}(V^*)$ is an element of $\wedge^2 V \otimes \mathrm{Sym}^2 V^*$. (We are not assuming that this Poisson bracket satisfies the Jacobi identity.) This Poisson bracket gives a first-order deformation of $\mathrm{Sym}(V^*)$, and so a first-order deformation of its Koszul dual $\wedge V$. This first-order deformation is again given by a quadratic-coefficient Poisson bracket on $\wedge V$, now viewed as an element of $\mathrm{Sym}^2 V^* \otimes \wedge^2 V$.

The map on quadratic Poisson brackets induced by Koszul duality is simply the isomorphism between $\mathrm{Sym}^2 V^* \otimes \wedge^2 V$ and $\wedge^2 V \otimes \mathrm{Sym}^2 V^*$. All that happens is that the interpretation of the two tensor factors are reversed. Let us see how this identification works in terms of a basis on V. Let v_i be a basis V and v^i the dual basis of V^*. Suppose we have an element π^{ij}_{kl} in $\mathrm{Sym}^2 V^* \otimes \wedge^2 V$. It determines a Poisson bracket on $\mathrm{Sym}(V^*)$ by the formula

$$\{v^i, v^j\} = \pi^{ij}_{kl} v^k v^l,$$

and it gives a Poisson bracket on $\wedge V$ by the formula

$$\{v_i, v_j\} = \pi^{kl}_{ij} v_k v_l,$$

in which the role of upper and lower indices has been reversed.

This analysis allows us to understand the first-order deformation of the associative algebra structure on $U(\mathfrak{gl}_N[[z_1, z_2]])$ because we have an explicit formula in terms of a basis. By reading formula 10.6.6.1 backwards, we find

$$[E_{ij} z_1, E_{kl} z_2] = \hbar E_{il} E_{kj} \text{ if } i \neq l, j \neq k. \tag{10.6.7.1}$$

This expression is part of a first-order deformation of $U(\mathfrak{gl}_N[[z_1, z_2]])$ into an associative algebra that is not a universal enveloping algebra.

Thus we have shown the second part of Proposition 10.6.3.1.

Remark: In fact, one can show that this first-order deformation is *uniquely* determined by equation (10.6.7.1), together with the fact that it is GL_N-invariant and that the elements in $\mathfrak{gl}_1[[z_1, z_2]]$ remain central. The idea is that firstly, any deformation of the commutator of $E_{ij}z_1$ with $E_{jk}z_2$ can be absorbed into a redefinition of $E_{ik}z_1z_2$, and so is not relevant. Secondly, the Lie algebra $\mathfrak{sl}_N[[z_1, z_2]]$ is generated by elements that are constant or linear along $\mathbb{C}^2$, i.e., by elements of $\mathfrak{sl}_N$ and by elements $M z_1$, $N z_2$ for $M, N \in \mathfrak{sl}_N$. $\Diamond$

PART THREE

A FACTORIZATION ENHANCEMENT OF NOETHER'S THEOREM

11
Introduction to Noether's theorems

Symmetries play a key role in field theory, just as actions of Lie groups and algebras play a key role in mathematics. In field theory, the central result is Noether's theorem, which says, loosely speaking, that any symmetry has an associated conserved quantity. For instance, in a one-dimensional theory, invariance under time translation corresponds to conservation of energy, which is described by the Hamiltonian function of the theory.

Our goal in this part of the book is to provide a refinement of Noether's theorem in the language of factorization algebras; there will be both classical and quantum versions.

The basic approach is local over a manifold X. Given a field theory on X, the symmetries will be encoded as a local Lie algebra $\mathcal{L}$ (or local L_∞ algebra) on X. From the local Lie algebra $\mathcal{L}$, we can produce its enveloping factorization algebra $U\mathcal{L}$; associated to a central extension of $\mathcal{L}$, there is the twisted enveloping factorization algebra $U_\alpha\mathcal{L}$. (The enveloping factorization algebra of a local Lie algebra is discussed in detail in Sect I.3.6.) Recall that $\mathcal{L}$ is a locally constant sheaf of Lie algebras on the real line $\mathbb{R}$, the enveloping factorization algebra encodes the universal enveloping algebra of the Lie algebra given by the stalk of $\mathcal{L}$. Hence one should view the enveloping factorization algebra as a natural generalization of the notion of universal enveloping algebra.

In the BV formalism, the quantum observables of a field theory form a factorization algebra Obs^q on X. Our formulation of Noether's theorem says that if a field theory on X has a local Lie algebra $\mathcal{L}$ of symmetries, then there is a map of factorization algebras

$$U_\alpha \mathcal{L} \to \mathrm{Obs}^q$$

from the twisted enveloping factorization algebra of $\mathcal{L}$ to quantum observables, where α is a central extension determined by the compatibility of quantization with the symmetry.

At the classical level, the result has a similar flavor, but we replace $U_\alpha \mathcal{L}$ by its associated graded with respect to a natural filtration. The associated graded $\mathrm{Gr}\, U_\alpha \mathcal{L}$ has the structure of a P_0 algebra in factorization algebra, and it can be interpreted as the universal P_0 factorization algebra containing $\mathcal{L}$, which we denote by $U^{P_0}\mathcal{L}$. Our formulation of classical Noether's theorem asserts that if $\mathcal{L}$ acts on a classical field theory, we have a map of P_0 factorization algebras

$$U_\alpha^{P_0}(\mathcal{L}) \to \mathrm{Obs}^{cl},$$

where, again, a central extension might be required.

What these theorems tell us is that every symmetry — encoded by a section of $\mathcal{L}$ — is represented by some observable of the field theory. We identify this observable with the corresponding conserved current.

This formulation of Noether's theorem encompasses many constructions familiar in physics. For instance, in a chiral conformal field theory on a Riemann surface Σ, the holomorphic vector fields provide a symmetry and so we can take

$$\mathcal{L} = \Omega^{0,*}(\Sigma, T_\Sigma^{1,0}).$$

This result then says that we obtain a factorization algebra map

$$\rho : Vir_c \to \mathrm{Obs}^q$$

where Vir_c is the Virasoro factorization algebra obtained by taking the enveloping factorization algebra with the central extension with level c. As shown in Williams (2017), this factorization algebra recovers the usual Virasoro vertex algebra, and hence the Noether theorem ensures that we obtain the usual inner action of the Virasoro vertex algebra on the vertex algebra arising from this chiral conformal field theory.

In this chapter, we will overview the key algebraic ideas that underpin these results and then give a precise statement of the theorems. Subsequent chapters will give proofs of these factorization Noether theorems, while also examining concrete examples and explaining how more standard versions, such as Ward identities or the usual Noether theorem, follow from our results.

11.1 Symmetries in the classical BV formalism

In the setting of Hamiltonian mechanics and symplectic geometry, the discussion of symmetries is very clean. A compelling feature of the BV formalism is that it rephrases the Lagrangian approach in a Hamiltonian style, as we will now see.

11.1.1 The ordinary symplectic version

Let's begin by recalling the approach to symmetries in Hamiltonian mechanics. Let (X, ω) be a symplectic manifold. Then the natural symmetries are the symplectic vector fields $\mathrm{SympVect}(X)$, which is the sub Lie algebra of smooth vector fields that preserve the symplectic form. Explicitly, we have

$$\mathrm{SympVect}(X) = \{Z \in \mathrm{Vect}(X) \mid L_Z\omega = 0\}.$$

Note that the Lie algebra of symplectomorphisms is precisely this Lie algebra.

For $\mathfrak{g}$ an ordinary Lie algebra, a Lie algebra map $\rho : \mathfrak{g} \to \mathrm{SympVect}(X)$ encodes the idea that $\mathfrak{g}$ is a symmetry of the mechanical system. But such a map does *not* realize the symmetry as an observable of the system. To do that, we need to produce a lift

$$\begin{tikzcd} & \mathscr{O}(X) \arrow[d, "Ham"] \\ \mathfrak{g} \arrow[ur, dashed, "\widetilde{\rho}"] \arrow[r] & \mathrm{SympVect}(X) \end{tikzcd}$$

where the vertical map sends a function f to its Hamiltonian vector field $\{f, -\}$.

The kernel of the map Ham is just the constant functions, and the cokernel is isomorphic to $H^1(X)$, since the isomorphism $\omega^\sharp : \mathrm{Vect}(X) \cong \Omega^1(X)$ identifies $\mathrm{SympVect}(X)$ with closed 1-forms and Hamiltonian vector fields with exact 1-forms. Hence, there exists a lift $\widetilde{\rho}$ only if the composite $\mathfrak{g} \to \mathrm{coker}(Ham) \cong H^1(X)$ is trivial. If a lift exists, it is unique up to a scalar. Even if the lift is obstructed, we can centrally extend $\mathfrak{g}$ to get a sequence of maps

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d, "="] & \widetilde{\mathfrak{g}} \arrow[r] \arrow[d, "\widetilde{\rho}"] & \mathfrak{g} \arrow[d, "\rho"] \\ \mathbb{C} \arrow[r] & \mathscr{O}(X) \arrow[r] & \mathrm{SympVect}(X) \end{tikzcd}$$

In short, symmetries can be realized as observables, up to a central extension.

We can rephrase this situation fully in the setting of Poisson algebras, which will help us articulate generalizations. Note that $\mathrm{Sym}(\mathfrak{g})$ is a Poisson algebra by extending the Lie bracket. For instance, given $x, y, z \in \mathfrak{g}$, we have

$$\{x, yz\} = [x, y]z + [x, z]y$$

where concatenation like yz means the commutative product. Indeed, $\mathrm{Sym}(\mathfrak{g})$ is the *enveloping Poisson algebra* of $\mathfrak{g}$, as Sym is the left adjoint to the forgetful functor from Poisson algebras to Lie algebras. Thus, a Lie algebra map $\mathfrak{g} \to \mathscr{O}(X)$ determines a map of Poisson algebras $\mathrm{Sym}(\mathfrak{g}) \to \mathscr{O}(X)$.

Remark: We have not specified here a Hamiltonian function H, and so we have only discussed kinematics and not dynamics. Given H, the natural symmetries are symplectic vector fields that commute with the Hamiltonian vector field $\{H, -\}$. Thus we should look to give a map of Lie algebras $\widetilde{\rho} : \mathfrak{g} \to \mathscr{O}(X)$ such that $\{H, \widetilde{\rho}(x)\} = 0$ for every $x \in \mathfrak{g}$. In the BV setting, we replace H with the action functional S and asks for a cochain map. $\lozenge$

11.1.2 The shifted symplectic version

A parallel story applies in the classical BV setting. Here X is a -1-symplectic space and so the commutative dg algebra $\mathscr{O}(X)$ has a 1-shifted Poisson bracket. There is also a dg Lie algebra $\mathrm{Vect}(X)$ of derivations of

$\mathscr{O}(X)$, and there is another dg Lie algebra $\mathrm{SympVect}(X)$ of derivations that preserve the shifted symplectic structure. (One should take care to give the correct derived definitions — as we will do later — but here we assume this has been done and focus on the analogy with ordinary symplectic geometry.) There is a dg Lie algebra map

$$\begin{array}{rccc} Ham: & \mathscr{O}(X)[-1] & \to & \mathrm{SympVect}(X) \\ & f & \mapsto & \{f,-\} \end{array}$$

that produces the Hamiltonian vector field of a function.

We can now mimic the story of symmetries in ordinary symplectic geometry. For $\mathfrak{g}$ a dg Lie algebra, a dg Lie algebra map $\rho : \mathfrak{g} \to \mathrm{SympVect}(X)$ encodes the idea that $\mathfrak{g}$ is a symmetry of the classical BV theory. But such a map does *not* realize the symmetry as an observable. To do that, we need to produce a lift

$$\begin{tikzcd} & \mathscr{O}(X)[-1] \arrow[d, "Ham"] \\ \mathfrak{g} \arrow[r] \arrow[ur, dashed, "\widetilde{\rho}"] & \mathrm{SympVect}(X) \end{tikzcd}$$

where the vertical map sends a function f to its Hamiltonian vector field $\{f,-\}$.

Again, the map *Ham* has kernel $\mathbb{C}[-1]$, with the shift arising from the shift of $\mathscr{O}(X)$. (Note that the kernel is also the derived kernel here.) Finding a lift $\widetilde{\rho}$ is the same as splitting an exact sequence of the form

$$0 \to \mathbb{C}[-1] \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of homotopy Lie algebras. Exact sequences like this are classified by elements in the first Lie algebra cohomology group $H^1(\mathfrak{g})$. If this cohomology class vanishes, then the set of ways of splitting the extension of homotopy Lie algebras is a torsor for $H^0(\mathfrak{g})$.

We remark that for any dg Lie algebra $\mathfrak{g}$, the commutative dg algebra $\mathrm{Sym}(\mathfrak{g}[1])$ obtains a natural shifted Poisson bracket by extending the shifted Lie bracket. In fact, the functor $\mathrm{Sym}(-[1])$ is left adjoint to the "forgetful" functor from 1-shifted Poisson algebras to dg Lie algebras (where one forgets the commutative product and shifts). Thus a dg Lie algebra map $\mathfrak{g} \to \mathscr{O}(X)[-1]$ determines a map $\mathrm{Sym}(\mathfrak{g}[1]) \to \mathscr{O}(X)$ of 1-shifted Poisson algebras.

Thus, the map $\mathfrak{g} \to C^\infty(X)$ is entirely analogous to the map that appears in our formulation of classical Noether's theorem. Indeed, we defined a field theory to be a sheaf of formal moduli problems with a -1-shifted symplectic form. The P_0 Poisson bracket on the observables of a classical field theory is analogous to the Poisson bracket on observables in classical mechanics. Our formulation of Noether's theorem can be rephrased as saying that, after passing to a central extension, an action of a sheaf of Lie algebras by symplectic symmetries on a sheaf of formal moduli problems is Hamiltonian.

This similarity is more than just an analogy. After some non-trivial work, one can show that our formulation of Noether's theorem, when applied to classical mechanics, yields the statement discussed above about actions of a Lie algebra on a symplectic manifold. The key result one needs in order to translate is a result of Safronov (2018) and Nick Rozenblyum.

Observables of classical mechanics form a locally constant P_0 factorization algebra on the real line, and so encode an E_1 algebra in P_0 algebras, by a theorem of Lurie discussed in Section I.6.4. Safronov and Rozenblyum show that E_1 algebras in P_0 algebras are the same as P_1, that is ordinary Poisson, algebras. This identification allows us to translate the shifted Poisson bracket on the factorization algebra on $\mathbb{R}$ of observable of classical mechanics into the ordinary unshifted Poisson bracket that is more familiar in classical mechanics, and to translate our formulation of Noether's theorem into the statement about Lie algebra actions on symplectic manifolds discussed above.

11.1.3 A field theory version

We have just seen that symmetries in shifted symplectic geometry work much as they do in ordinary symplectic geometry, but the extensions and obstructions are shifted. This bit of algebra must now be integrated into the context of field theory. In particular, we want to enforce *locality*, so that the Lie algebra acts locally on the spacetime manifold.

Defining symmetries

Let us unravel this situation in more explicit terms. Suppose we have a classical BV theory on a manifold M, whose fields are $\mathscr{E}$ and whose action functional is S. Then $\mathrm{Obs}^{cl} = (\mathscr{O}(\mathscr{E}), \{S, -\})$ would seem to play the role of $\mathscr{O}(X)$ above. Likewise, there is a dg Lie algebra Vect of derivations of Obs^{cl}, and a subalgebra SympVect of derivations preserving the shifted symplectic pairing.

To enforce locality, however, we work with certain subcomplexes consisting functions or derivations that are local in nature, i.e., constructed out of polydifferential operators. We will start by discussing the local version of the dg Lie algebra of symplectic vector fields. This is the dg Lie algebra $\mathscr{O}_{loc}[-1]$ of (shifted by one) local functionals, defined in Chapter 3.5.1. It is important to bear in mind that local functionals are taken up to the addition of an additive constant. This is why they should be thought of as the analog in our context of symplectic vector fields. Reintroducing the additive constant will give rise to the exact sequence relating Hamiltonian and symplectic vector fields.

Note that the dg Lie algebra $\mathscr{O}_{loc}[-1]$ controls the symmetries and deformations of a classical field theory, just as the dg Lie algebra of symplectic vector fields controls the symmetries and deformations of a shifted symplectic manifold. Note also that the BV bracket $\{-,-\}$ is well-defined on $\mathscr{O}_{loc}$, so we avoid an analytical issue about the shifted Poisson structure on $\mathscr{O}(\mathscr{E})$.

11.1.3.1 Definition. *A dg Lie algebra* $\mathfrak{g}$ acts by local symmetries *on the classical BV theory if there is a map of dg Lie algebras* $\rho : \mathfrak{g} \to \mathscr{O}_{loc}[-1]$*.*

Here is a simple example. Suppose we have a free scalar field theory on a manifold M, with n fields $\phi_i \in C^\infty(M)$. In the BV formalism, we also introduce n anti-fields $\psi_i \in \Omega^d(M)$, where d is the dimension of M. The odd symplectic pairing is $\int \phi_i \psi_i$, and the action functional is $\int \mathrm{d}\phi_i \wedge *\mathrm{d}\phi_i$. In both formulae we use the summation convention.

This theory is acted on by the Lie algebra $\mathfrak{so}(n)$. To describe the action in the sense of 11.1.3.1 we need to construct, for every anti-symmetric matrix $B \in \mathfrak{so}(n)$, a local functional of cohomological degree 1. We require the BV bracket of these local functionals to reproduce the commutator in $\mathfrak{so}(n)$. The desired local functionals are simply

$$S_B = \sum B_{ij}\psi_i\phi_j.$$

One can compute easily that $\{S_B, S_C\} = S_{[B,C]}$ for matrices $B, C \in \mathfrak{so}(n)$.

From global to local symmetries

So far we have kept $\mathfrak{g}$ as just a dg Lie algebra, but it is natural to consider instead a *local* Lie algebra on M. This step constitutes a generalization, since we can always replace a dg Lie algebra $\mathfrak{g}$ by the local Lie algebra $\Omega^*_M \otimes \mathfrak{g}$. This replacement does nothing on small opens, in the sense that the Poincaré lemma ensures

$$\mathfrak{g} \xrightarrow{\simeq} \Omega^*(D) \otimes \mathfrak{g}$$

for any disk D, but it allows us to use differential-geometric or field-theoretic techniques, which makes computations easier, and has interesting global consequences on manifolds with interesting topology.

Moreover, once we are working in this totally local setting, it is possible to talk about maps of Lie algebras that are local on M and hence to discuss maps of factorization algebras. We pursue this line of thought throughout our discussion of the factorization Noether theorems.

11.2 Koszul duality and symmetries via the classical master equation

Throughout this book we work in a perturbative setting, which gives access to several useful algebraic tools. A running theme of this book has been the utility of phrasing constructions both in terms of dg Lie and commutative algebras, and we lean heavily into this theme in developing the Noether theorems. The key issue here is to have several ways to describe or encode a map of Lie algebras.

11.2.1 Koszul duality

Given that we use homotopical algebra in the BV formalism, it is natural to use a more flexible, homotopical version of map between Lie algebras. Recall that a homotopy-coherent map of dg Lie algebras or L_∞ algebras

$$\rho : \mathfrak{g} \rightsquigarrow \mathfrak{h}$$

is in fact a map of coaugmented dg cocommutative coalgebras

$$\rho : C_*\mathfrak{g} \to C_*\mathfrak{h},$$

which can also be described by the dual map of augmented dg commutative algebras

$$\rho^\vee : C^*\mathfrak{h} \to C^*\mathfrak{g}.$$

Such a map encodes the idea of a map that does not respect the bracket or Jacobi relation on the nose, but only up to exact terms. See Appendix A.1 for further discussion.

Remark: This notion of a homotopy-coherent map encompasses, of course, the usual, stricter notion, and it hence provides added flexibility and examples. Thus, in practice we modify Definition 11.1.3.1 by replacing a strict map of dg Lie algebras everywhere by a homotopy-coherent map. This generalization becomes particularly useful in the quantum setting. $\Diamond$

There is yet another way to describe such a map $\rho : \mathfrak{g} \rightsquigarrow \mathfrak{h}$. Because $C^*\mathfrak{h}$ is semifree — i.e., the underlying graded-commutative algebra is a symmetric algebra — the map $\rho^\vee$ is determined by where it sends the generators

$$\mathfrak{h}^\vee[-1] = \mathrm{Sym}^1(\mathfrak{h}^\vee[-1]) \subset \widehat{\mathrm{Sym}}(\mathfrak{h}^\vee[-1]) = C^\sharp\mathfrak{h}.$$

The generators must land in the augmentation ideal $C^*_{red}(\mathfrak{g}) \subset C^*\mathfrak{g}$ to ensure that the map of algebras respects the augmentations. Hence $\rho^\vee$ is encoded by a degree zero element

$$A_\rho \in \mathrm{Hom}(\mathfrak{h}^\vee[-1], C^*_{red}(\mathfrak{g})) \cong \mathfrak{h}[1] \otimes C^*_{red}(\mathfrak{g}),$$

which is conventionally viewed as a degree one element of $\mathfrak{h} \otimes C^*_{red}(\mathfrak{g})$. The condition that $\rho^\vee$ is a map of dg commutative algebras is encoded by the Maurer-Cartan equation

$$\mathrm{d}A_\rho + \frac{1}{2}[A_\rho, A_\rho] = 0,$$

where the differential d and bracket $[-,-]$ mean those on the dg Lie algebra $\mathfrak{h} \otimes C^*_{red}(\mathfrak{g})$. (It is a straightforward but illuminating exercise to verify that this equation does indeed correspond to giving a map of dg algebras.)

Remark: The identification of (homotopy-coherent) maps of Lie algebras with maps of augmented commutative algebras is a standard and central example of Koszul duality. Loosely speaking, this duality means that one can often translate between Lie algebras and commutative algebras, although one obtains a perfect translation, such as an equivalence of categories, only under restrictive hypotheses. We use this translation mechanism throughout the book but do not rely on more subtle aspects of Koszul duality. There should be important applications of this notion to field theory and factorization algebras. $\Diamond$

11.2.2 Rephrasing classical BV symmetries

In our primary context, the -1-symplectic space X we work with is a formal stack, and so it is encoded by a dg Lie algebra or L_∞ algebra $\mathcal{M}$ with an invariant pairing. We assume here that $\mathcal{M}$ is finite-dimensional: essentially we are studying quantum field theory on a point. Later we will generalize this analysis to the case when $\mathcal{M}$ is a local L_∞ algebra on some space.

The 1-shifted Poisson algebra of function on $\mathcal{M}$ is

$$\mathscr{O}(X) = C^*\mathcal{M},$$

and we will freely refer to X as $B\mathcal{M}$. The symplectic vector fields are all Hamiltonian here, and so we have

$$\mathrm{SympVect}(X) = C^*_{red}(\mathcal{M})[-1],$$

where we have identified the reduced Chevalley-Eilenberg cochains with the quotient of $\mathscr{O}(X)$ by the constants.

Giving a homotopy-coherent map of Lie algebras

$$\rho : \mathfrak{g} \rightsquigarrow \mathrm{SympVect}(X)$$

is thus equivalent to solving the Maurer-Cartan equation in the dg Lie algebra

$$C^*_{red}(\mathcal{M})[-1] \otimes C^*_{red}(\mathfrak{g}).$$

In other words, we look for a degree one element $A \in C^*_{red}(\mathcal{M})[-1] \otimes C^*_{red}(\mathfrak{g})$ such that

$$\mathrm{d}A + \frac{1}{2}\{A, A\} = 0, \qquad (\dagger)$$

where $\{-,-\}$ denotes the shifted Poisson (or BV) bracket on $C^*_{red}(\mathcal{M})$ and d denotes the differential determined by the differentials on $C^*_{red}(\mathcal{M})$ and $C^*_{red}(\mathfrak{g})$ under tensor product.

This equation for A is, in essence, a version of the classical master equation. To be precise, consider the 1-shifted Poisson algebra

$$C^*(\mathcal{M}) \otimes C^*_{red}(\mathfrak{g}),$$

where the bracket is simply the Poisson bracket on $C^*\mathcal{M}$ extended to be $C^*_{red}(\mathfrak{g})$-linear. Note that by construction, the subcomplex $C^*_{red}(\mathfrak{g}) \hookrightarrow C^*\mathcal{M} \otimes C^*_{red}(\mathfrak{g})$ is Poisson-central and hence determines an ideal in the sense of Lie algebras. In consequence, the quotient

$$(C^*\mathcal{M}[-1] \otimes C^*_{red}(\mathfrak{g}))/C^*_{red}(\mathfrak{g})[-1]$$

inherits a dg Lie algebra structure. It is straightforward to check that the inclusion $C^*_{red}\mathcal{M} \hookrightarrow C^*\mathcal{M}$ determines an isomorphism

$$C^*_{red}(\mathcal{M})[-1] \otimes C^*_{red}(\mathfrak{g}) \xrightarrow{\cong} (C^*\mathcal{M}[-1] \otimes C^*_{red}(\mathfrak{g}))/C^*_{red}(\mathfrak{g})[-1]$$

of dg Lie algebras.

Under this identification, our Maurer-Cartan equation ($\dagger$) amounts to asking for a degree zero element B in the Poisson algebra that satisfies the classical master equation *modulo constants in the base ring*: that is, we want $\mathrm{d}B + \frac{1}{2}\{B, B\}$ to live in the constants $C^*_{red}(\mathfrak{g}) \subset C^*\mathcal{M} \otimes C^*_{red}(\mathfrak{g})$, i.e.,

$$\mathrm{d}B + \frac{1}{2}\{B, B\} \in C^*_{red}(\mathfrak{g}).$$

Equivalently, we want $\mathrm{d}B + \frac{1}{2}\{B, B\}$ to vanish in the quotient space $(C^*\mathcal{M} \otimes C^*_{red}(\mathfrak{g}))/C^*_{red}(\mathfrak{g})$. The desired element A is simply the image of such a solution B in $C^*_{red}(\mathcal{M}) \otimes C^*_{red}(\mathfrak{g})$.

Given the central importance of this dg Lie algebra in classifying symmetries, we introduce some notation.

11.2.2.1 Definition. *Let $B\mathcal{M}$ be a -1-shifted symplectic formal derived stack, where $\mathcal{M}$ is an L_∞ algebra. For $\mathfrak{g}$ an L_∞ algebra, let*

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) = \left(C^*\mathcal{M}[-1] \otimes C^*_{red}(\mathfrak{g})\right) / C^*_{red}(\mathfrak{g})[-1]$$

denote the dg Lie algebra whose Maurer-Cartan solutions encode homotopy-coherent actions of $\mathfrak{g}$ on $B\mathcal{M}$ as symplectic symmetries.

Equivariant classical observables

This description of a $\mathfrak{g}$-action on a formal shifted symplectic manifold is the one most convenient for the BV formalism. One of its compelling features is that it reduces the question of exhibiting a map to a familiar challenge: solving a master equation. We can thus rephrase the challenge of exhibiting $\mathfrak{g}$ as a symmetries of a classical BV theory as follows.

Suppose $B\mathcal{M}$ is a -1-symplectic formal stack, described by an L_∞ algebra $\mathcal{M}$ with -3-shifted invariant pairing. That means there is a degree zero element S in the graded-commutative algebra

$$\mathscr{O}_{red}(\mathcal{M}[1]) = \widehat{\mathrm{Sym}}^{>0}(\mathcal{M}^\vee[-1])$$

such that $\{S, S\} = 0$. Note that we are solving here the classical master equation modulo constants, which here is $\mathbb{C}$ since that is the base ring.

To exhibit $\mathfrak{g}$ as symmetries of this classical BV theory is to find a $\mathfrak{g}$-equivariant solution to the classical master equation. Explicitly, we mean that working over the base ring $C^*(\mathfrak{g}) = \mathscr{O}(B\mathfrak{g})$, we find a degree zero element S^{eq} in the graded-commutative algebra

$$\mathscr{O}_{red}(\mathcal{M}[1]) \otimes \mathscr{O}(B\mathfrak{g})$$

such that

- $\mathrm{d}_\mathfrak{g} S^{eq} + \frac{1}{2}\{S^{eq}, S^{eq}\} = 0$ where $\mathrm{d}_\mathfrak{g}$ is the Chevalley-Eilenberg differential on $C^*(\mathfrak{g}) = \mathscr{O}(B\mathfrak{g})$.
- S^{eq} agrees with the original solution S when reduced along the augmentation $C^*\mathfrak{g} \to \mathbb{C}$ (geometrically, this correspond to specializing to the base point of $B\mathfrak{g}$).

In homotopy theory, to make something G-equivariant is to describe it in families over the classifying space BG. Hence, we view S^{eq} as encoding a $\mathfrak{g}$-equivariant classical BV theory, since working over the base ring $C^*\mathfrak{g}$ amounts to providing a family of BV theories over $B\mathfrak{g}$.

The differential $\mathrm{d}_\mathfrak{g} + \{S^{eq}, -\}$ makes $\mathscr{O}(\mathcal{M}[1]) \otimes \mathscr{O}(\mathfrak{g}[1])$ into a semifree commutative dga (which is also augmented). Therefore it corresponds to an L_∞ structure on the graded vector space $\mathcal{M} \oplus \mathfrak{g}$. By assumption, the natural map $\mathscr{O}(\mathcal{M}[1]) \otimes \mathscr{O}(\mathfrak{g}[1]) \to \mathscr{O}(\mathcal{M}[1])$ is a map of differential graded algebras, where $\mathscr{O}(\mathcal{M}[1])$ is identified with $C^*(\mathcal{M})$. It follows that the L_∞ structure on $\mathcal{M} \oplus \mathfrak{g}$ is such that there is a short exact sequence of L_∞ algebras

$$0 \to \mathcal{M} \to \mathcal{M} \oplus \mathfrak{g} \to \mathfrak{g}.$$

Therefore the L_∞ structure on $\mathcal{M} \oplus \mathfrak{g}$ is a semi-direct product L_∞ algebra, which we can write $\mathfrak{g} \ltimes \mathcal{M}$.

We call the total complex $C^*(\mathfrak{g} \ltimes \mathcal{M}) = \mathscr{O}(B(\mathfrak{g} \ltimes \mathcal{M})[1])$ the total complex $\mathscr{O}(\mathfrak{g} \ltimes \mathcal{M})$ the *equivariant classical observables*.

Inner actions and the classical master equation

The notion of an *inner* action has a natural articulation along these lines as well. Since $\mathbb{C}$ is Poisson-central, the constant functions form a Lie ideal inside $C^*(\mathcal{M})[-1] \otimes C^*(\mathfrak{g})$, so that the quotient $C^*(\mathcal{M})[-1] \otimes C^*(\mathfrak{g})/\mathbb{C}$ is a dg Lie algebra. There is a canonical isomorphism $C^*_{red}(\mathfrak{g} \oplus \mathcal{M}) \to C^*(\mathcal{M}) \otimes C^*(\mathfrak{g})/\mathbb{C}$, along which we transfer the (shifted) Lie bracket.

To ask for an inner action is to ask for a solution S^{in} to the classical master equation in

$$\mathscr{O}_{red}(\mathcal{M}[1] \oplus \mathfrak{g}[1]) = \widehat{\mathrm{Sym}}^{>0}(\mathcal{M}^\vee[-1] \oplus \mathfrak{g}^\vee[-1])$$

such that $\{S^{in}, S^{in}\} = 0$ modulo constants, here $\mathbb{C}$. This equation means that S^{in} encodes a homotopy-coherent map of Lie algebras $\mathfrak{g} \rightsquigarrow C^*(\mathcal{M})[-1]$, rather than to $C^*_{red}(\mathcal{M})[1]$. Given its importance, we introduce some convenient notation.

11.2.2.2 Definition. *Let $B\mathcal{M}$ be a -1-shifted symplectic formal derived stack, where $\mathcal{M}$ is an L_∞ algebra with a -3 shifted invariant pairing. For $\mathfrak{g}$ an L_∞*

algebra, let

$$\mathrm{InnerAct}(\mathfrak{g}, \mathcal{M}) = (C^*(\mathcal{M})[-1] \otimes C^*(\mathfrak{g}))\,/\mathbb{C}$$

denote the dg Lie algebra whose Maurer-Cartan solutions encode homotopy-coherent actions of $\mathfrak{g}$ *on* $B\mathcal{M}$ *as Hamiltonian symmetries.*

Note that by construction, there is a short exact sequence of dg Lie algebras

$$C^*_{red}(\mathfrak{g}) \to \mathrm{InnerAct}(\mathfrak{g}, \mathcal{M}) \to \mathrm{Act}(\mathfrak{g}, \mathcal{M}),$$

where $C^*_{red}(\mathfrak{g})$ is central in $\mathrm{InnerAct}(\mathfrak{g}, \mathcal{M})$. Thus, if we are given an action S^{eq}, specified by a solution to the Maurer-Cartan equation in $\mathrm{Act}(\mathfrak{g}, \mathcal{M})$, then the obstruction to lifting it to an inner action S^{in} lives in $C^*_{red}(\mathfrak{g})$. Indeed, by hypothesis, we know that S^{eq} satisfies the classical master equation modulo $C^*_{red}(\mathfrak{g})$ and so the failure to satisfy the classical master equation modulo $\mathbb{C}$ must live in $C^*_{red}(\mathfrak{g})$. In other words, the obstruction $\{S^{eq}, S^{eq}\}$ is a degree one cocycle in $C^*_{red}(\mathfrak{g})$.

Geometrically, reduced functions $\mathscr{O}_{red}(B\mathfrak{g}) = C^*_{red}(\mathfrak{g})$ can be identified with closed 1-forms $\Omega^1_{cl}(B\mathfrak{g})$, so this anomaly in $H^1(B\mathfrak{g}, \Omega^1_{cl})$ can be viewed as a being the first Chern class of a line bundle.

11.2.3 Noether's theorem as a map of P_0 algebras

We have seen that if we have a $\mathfrak{g}$-action on a -1-shifted formal sympectic stack $X = B\mathcal{M}$, then there is a shifted central extension $\widehat{\mathfrak{g}}$ and a map of homotopy Lie algebras $\widehat{\mathfrak{g}} \to \mathscr{O}(X)$. In this section we will see how to extend this to a map from a P_0 algebra built from $\widehat{\mathfrak{g}}$. This formulation will generalize well to the quantum setting, and to the factorization algebra setting, which is our ultimate goal.

There is a forgetful functor from P_0 algebras to Lie algebras. This functor has a left adjoint, which is the universal P_0 algebra receiving a map from a given Lie algebra.

11.2.3.1 Definition. *The* enveloping P_0 algebra *of a dg Lie algebra* $\mathfrak{g}$ *is*

$$U^{P_0}(\mathfrak{g}) = \mathrm{Sym}^*(\mathfrak{g}[1])$$

the free graded commutative algebra with differential given by the derivation

$$\mathrm{d}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y).$$

It has a shifted Poisson structure characterized by the fact that when restricted to the generators $\mathfrak{g}[1]$*, it is a shift of the Lie bracket on* $\mathfrak{g}$*.*

For formal reasons, there is a version of this construction which takes a homotopy Lie algebra to a homotopy P_0 algebra, but we will not dwell on this generalization.

More generally, suppose that we have a class in $\alpha \in H^1(C^*_{red}(\mathfrak{g}))$, determining a central extension

$$0 \to \mathbb{C}[-1] \cdot c \to \widehat{\mathfrak{g}} \to \mathfrak{g}.$$

Then the central element $c \in \widehat{\mathfrak{g}}[1]$ lifts to a Poisson central element c in $U^{P_0}(\widehat{\mathfrak{g}}) = \mathrm{Sym}^*(\widehat{\mathfrak{g}}[1])$. We can form the quotient

$$U^{P_0}_{\alpha}(\mathfrak{g}) = U^{P_0}(\widehat{\mathfrak{g}}) / \ (c = 1)$$

determined by setting the central element c to 1. This quotient inherits a P_0 structure, because c is Poisson central.

This quotient $U^{P_0}_{\alpha}(\mathfrak{g})$ is called the *twisted enveloping P_0 algebra* associated to $\mathfrak{g}$ and the class $\alpha \in H^1(C^*_{red}(\mathfrak{g}))$.

In this language, Noether's theorem in the classical BV formalism has the following formulation.

11.2.3.2 Lemma. *Suppose we have a* $\mathfrak{g}$*-action on a* -1*-shifted formal symplectic stack* $X = B\mathcal{M}$*, and let* $\alpha \in H^1(C^*(\mathfrak{g}))$ *denote the obstruction to making this an inner action.*

There is then a canonical map of P_0 algebras

$$U^{P_0}_{\alpha}(\mathfrak{g}) \to \mathscr{O}(X) \tag{11.2.3.1}$$

which encodes the action.

Proof This result is easily proved. By definition, we have a map of homotopy Lie algebras from $\mathfrak{g} \to \mathscr{O}(X)_{red}[-1]$. This map extends to a homomorphism from $\widehat{\mathfrak{g}} \to \mathscr{O}(X)[-1]$, where $\widehat{\mathfrak{g}}$ is the central extension pulled back from the central extension

$$0 \to \mathbb{C}[-1] \to \mathscr{O}(X)[-1] \to \mathscr{O}(X)_{red}[-1].$$

By the universal property of an enveloping P_0 algebra, we find a homomorphism from $U^{P_0}(\widehat{\mathfrak{g}}) \to \mathscr{O}(X)$ of P_0 algebras, which sends the central element in $\widehat{\mathfrak{g}}[1]$ to $1 \in \mathscr{O}(X)$. Hence this homomorphism descends to a map from the twisted enveloping P_0 algebra $U_\alpha^{P_0}(\mathfrak{g})$. □

11.3 Symmetries in the quantum BV formalism

As in the classical setting, we begin by discussing the algebraic story before bringing in field theory.

11.3.1 The algebraic essentials

Let X be a -1-symplectic space so that $\mathscr{O}(X)$ is a 1-shifted Poisson algebra. Let $\mathscr{O}(X)^q$ denote some BD quantization, so that the underlying graded algebra is $\mathscr{O}(X)[[\hbar]]$ but the differential has the form $\mathrm{d}_{\mathscr{O}} + \hbar\, \mathrm{d}^q$. There are two distinct questions that interest us here:

- Given an action of $\mathfrak{g}$ on X, does it admit a lift to an action on the quantization $\mathscr{O}(X)^q$?
- Given an action of $\mathfrak{g}$ on the quantization $\mathscr{O}(X)^q$, does it admit a lift to an inner action?

The tools used to answer each question are similar to those used in the classical setting. In particular, we will use the notion of a $\mathfrak{g}$-equivariant solution to the quantum master equation to encode and analyze symmetries of a quantum BV theory.

The equivariant quantum master equation

Let us now analyze symmetries in terms of equivariant solutions to the quantum master equation. As in the classical setting, we will restrict attention to the setting of formal stacks and use Koszul duality.

We now let $X = B\mathcal{M}$ so that $\mathscr{O}(X) = C^*\mathcal{M}$. We suppose that we have fixed a BD quantization $(C^*\mathcal{M})^q$ of $C^*\mathcal{M}$ by deforming the differential $\{S, -\}$ to $\{S, -\} + \hbar\, \mathrm{d}^q$. Then, $\mathscr{O}(X)^q$ is a dg Lie algebra, and the constant functions $\mathbb{C}[[\hbar]]$ are a Lie ideal. Given a Lie algebra (or homotopy Lie algebra) $\mathfrak{g}$, we say that a $\mathfrak{g}$-action at the quantum level is a map of homotopy Lie algebras

$$\rho^q : \mathfrak{g} \to \mathscr{O}(X)^q/\mathbb{C}[[\hbar]].$$

This is just the obvious quantum version of the notion of classical $\mathfrak{g}$-action.

As in our analysis of the classical version, such a $\mathfrak{g}$-action is the same as a solution of the Maurer-Cartan equation in the dg Lie algebra $C^*_{reg}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathscr{O}(X)^q_{red}$ (where we use the notation $\mathscr{O}(X)^q_{red}$ for the quotient of $\mathscr{O}(X)^q$ by $\mathbb{C}[[\hbar]]$). Again, following our classical analysis, such a solution to the Maurer-Cartan equation is the same as an element

$$S^{eq} \in C^*(\mathfrak{g}) \otimes \mathscr{O}(X)^q_{red}$$

with the following properties:

- S^{eq} satisfies the quantum master equation

  $$\mathrm{d}_{\mathfrak{g}} S^{eq} + \hbar \triangle S^{eq} + \tfrac{1}{2}\{S^{eq}, S^{eq}\} = 0.$$

  Here $\mathrm{d}_{\mathfrak{g}}$ indicates the Chevalley-Eilenberg differential on $C^*(\mathfrak{g})$, and $\triangle$ is the BV Laplacian.
- When reduced modulo the maximal ideal in $C^*(\mathfrak{g})$, S^{eq} becomes the original solution $S \in \mathscr{O}(X)^q_{red}$ to the quantum master equation.

As in the classical setting, we can also give a definition of an inner action of a Lie algebra on a BV theory. Given a homotopy Lie algebra $\mathfrak{g}$, and a BD algebra $\mathscr{O}(X)_q$ quantizing a formal -1-shifted symplectic stack $X = B\mathcal{M}$, then an inner action of $\mathfrak{g}$ is by definition a map of homotopy Lie algebras

$$\mathfrak{g} \to \mathscr{O}(X)_q.$$

This is the same as a solution to the Maurer-Cartan equation in $C^*(\mathfrak{g}) \otimes_{\mathbb{C}} \mathscr{O}(X)_q$. It is also equivalent to given a functional

$$S^{inner} \in \left(C^*(\mathfrak{g}) \otimes_{\mathbb{C}} \mathscr{O}(X)_q\right)/\mathbb{C}[[\hbar]]$$

with the following properties:

- S^{inner} satisfies the quantum master equation
$$\mathrm{d}_{\mathfrak{g}} S^{inner} + \hbar \triangle S^{inner} + \tfrac{1}{2}\{S^{inner}, S^{inner}\} = 0.$$
- S^{inner} It reduces modulo the maximal ideal of $C^*(\mathfrak{g})$ to the original solution $S \in \mathscr{O}(X)^q$ to the quantum master equation that defines or BV theory.

Just as in the classical case, there are dg Lie algebras whose Maurer-Cartan solutions describe actions and inner actions.

11.3.1.1 Definition. *Let $X = B\mathcal{M}$ be a -1-shifted formal dg stack, where $\mathcal{M}$ is a homotopy Lie algebra with an invariant pairing of degree -3. Suppose we have chosen a quantization of this BV theory, given by a solution to the quantum master equation $S \in \mathscr{O}(\mathcal{M}[1])^q$ reducing modulo $\hbar$ to the solution to the classical master equation encoded by the Lie algebra structure on $\mathcal{M}$.*

Define a dg Lie algebra
$$\mathrm{Act}^q(\mathfrak{g}, \mathcal{M}) = \left(\mathscr{O}^q(\mathcal{M}[1])[-1] \otimes C^*_{red}(\mathfrak{g})\right) / C^*_{red}(\mathfrak{g})[[\hbar]][-1]$$
with differential $\mathrm{d}_{\mathfrak{g}} + \{S, -\} + \hbar\triangle$. Then a solution to the Maurer-Cartan equation on $\mathrm{Act}^q(\mathfrak{g}, \mathcal{M})$ is the same as a $\mathfrak{g}$-action on the BV theory.

Similarly, define
$$\mathrm{InnerAct}^q(\mathfrak{g}, \mathcal{M})\ \left(\mathscr{O}^q(\mathcal{M}[1])[-1] \otimes C^*_{red}(\mathfrak{g})\right) / \mathbb{C}[[\hbar]][-1]$$
A solution to the Maurer-Cartan equation in this is an inner action of $\mathfrak{g}$ on the BV theory given by the quantization of $\mathcal{M}$.

Note that there is a short exact sequence
$$0 \to C^*_{red}(\mathfrak{g})[[\hbar]][-1] \to \mathrm{InnerAct}^q(\mathfrak{g}, \mathcal{M}) \to \mathrm{Act}^q(\mathfrak{g}, \mathcal{M}) \to 0.$$
This tells us that the obstruction to lifting an inner action to an action is an element of $H^1(C^*_{red}(\mathfrak{g}))[[\hbar]]$. As in the classical case, this implies that an action of $\mathfrak{g}$ will lift to an inner action once we replace $\mathfrak{g}$ by a shifted central extension determined by the obstruction class in $H^1(C^*_{red}(\mathfrak{g}))[[\hbar]]$.

We can also ask whether an action of $\mathfrak{g}$ at the classical level lifts to an action at the quantum level, or similarly for an inner action. $\mathrm{Act}^q(\mathfrak{g}, \mathcal{M})$

is a dg Lie algebra in the category of free $\mathbb{C}[[\hbar]]$-modules, which reduces modulo $\hbar$ to $\mathrm{Act}(\mathfrak{g}, \mathcal{M})$. The problem of lifting a Maurer-Cartan solution in $\mathrm{Act}^q(\mathfrak{g}, \mathcal{M})$ defined modulo $\hbar^n$ to one defined modulo $\hbar^{n+1}$ is given by a cohomology class in $H^2(\mathrm{Act}(\mathfrak{g}, \mathcal{M}))$. We can thus understand, by obstruction theory, whether an action at the classical level lifts to one at the quantum level. A simlar remark holds for lifting inner actions to the quantum level.

A quantized Noether's map

Now let us analyze the quantum version of lemma 11.2.3.2. This lemma told us that a $\mathfrak{g}$-action $X = B\mathcal{M}$ on a classical field theory led to a map from a twisted enveloping P_0 algebra of $\mathfrak{g}$ to the P_0 algebra $\mathscr{O}(X)$. At the quantum level, we need to replace the enveloping P_0 algebra of $\mathfrak{g}$ by the enveloping BD algebra, which we now introduce.

We claim that the Chevalley-Eilenberg chains $C_*\mathfrak{g}$ is a natural BD quantization of the enveloping P_0 algebra $\mathrm{Sym}(\mathfrak{g}[1])$, since the Chevalley-Eilenberg differential encodes the Lie bracket by

$$\mathrm{d}_{CE}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y) + [x, y].$$

More accurately, to keep track of the $\hbar$-dependency in BD algebras, we introduce a kind of Rees construction.

11.3.1.2 Definition. *The* enveloping BD algebra *of a dg Lie algebra* $\mathfrak{g}$ *is* $U^{BD}(\mathfrak{g})$, *the graded-cocommutative coalgebra in* $\mathbb{C}[[\hbar]]$*-modules*

$$\mathrm{Sym}(\mathfrak{g}[1])[[\hbar]] \cong \mathrm{Sym}_{\mathbb{C}[[\hbar]]}(\mathfrak{g}[[\hbar]][1])$$

with differential given by the coderivation

$$\mathrm{d}(xy) = \mathrm{d}_{\mathfrak{g}}(x)y \pm x\,\mathrm{d}_{\mathfrak{g}}(y) + \hbar[x, y].$$

We occasionally use the notation $U^{BD}(\mathfrak{g})$, *and hope that it will not cause confusion.*

It is straightforward to verify that $U^{BD}(\mathfrak{g})$ is a BD algebra. Moreover, as one can check by direct computation, this functor U^{BD} determines the left adjoint to the forgetful functor from BD algebras to Lie algebras. (One can clearly use L_∞ algebras as well.) This definition also makes sense when $\mathfrak{g}$ is a dg Lie algebra in the category of free modules over

$\mathbb{C}[[\hbar]]$. In this more general context, the symmetric algebra $\mathrm{Sym}^* \mathfrak{g}[1]$ is taken over $\mathbb{C}[[\hbar]]$ and the formula for the BD structure is unchanged.

More generally, suppose we have a $\hbar$-linear central extension

$$0 \to \mathbb{C}[[\hbar]][-1] \to \widehat{\mathfrak{g}}_\hbar \to \mathfrak{g}[[\hbar]] \to 0$$

determined by a cohomology class $\alpha \in H^1(C^*_{red}(\mathfrak{g}))[[\hbar]]$. Then we can make the following definition.

11.3.1.3 Definition. *The* α-twisted enveloping BD algebra $U^{BD}_\alpha(\mathfrak{g})$ *is the quotient of* $U^{BD}(\widehat{\mathfrak{g}})$ *by the relation setting the central element equal to* 1.

We now see that the map of quantum symmetries ρ^q determines canonically a map of *BD* algebras

$$\rho^q : U^{BD}(\mathfrak{g}) \to \mathscr{O}(X)^q.$$

11.3.1.4 Lemma. *Suppose we have a* $\mathfrak{g}$*-action on a BV theory quantizing a formal* -1 *shifted symplectic dg stack X. Let* $\alpha \in H^1(C^*_{red}(\mathfrak{g}))[[\hbar]]$ *is the obstruction to making this an inner action. There is thus a map of BD algebras*

$$\rho^q : U^{BD}(\mathfrak{g}) \to \mathscr{O}(X)^q.$$

Proof This claim is a simple formal consequence of the universal property of the enveloping BD algebra, as in lemma 11.2.3.2. □

This map manifestly dequantizes to the map $\rho : U^{P_0}_\alpha(\mathfrak{g}) \to \mathscr{O}(X)$ of P_0 algebras, since $U^{BD}_\alpha(\mathfrak{g})$ is given by deforming the differential of $\mathrm{Sym}(\mathfrak{g}[1])$ in an $\hbar$-dependent fashion.

This articulation of quantum symmetries has a nice pay-off: it says that quantizing a symmetry amounts to lifting a map of P_0 algebras to a map of *BD* algebras.

It also provides a natural home for elements that behave like symmetries but are not elements of a Lie algebra, namely the "polynomials" that appear in $U^{BD}_\alpha(\mathfrak{g})$. Under the map ρ^q, these nonlinear elements go to nonlinear observables. In this sense, $U^{BD}_\alpha(\mathfrak{g})$ plays a role in the BV formalism analogous to the role that the enveloping algebra $U\mathfrak{g}$ plays in traditional algebra. For instance, the Casimir element is quite useful in representation theory but it is quadratic in $U\mathfrak{g}$ and hence is not itself in $\mathfrak{g}$. At the level of factorization algebras, this formulation of the theorem will be important: the factorization version of enveloping BD algebras will include objects like the Virasoro and Kac-Moody vertex algebras.

11.3.2 Porting the story to field theory

So far, we have discussed everything over a space-time manifold that is a point. Our goal, of course, is to analyze Noether's theorem for honest field theories over a non-trivial space-time manifold.

The homological algebra we have developed in this chapter will apply, but with a few additional analytical subtleties, as usual. In the factorization-theoretic version of this story, it will be important to replace a Lie algebra $\mathfrak{g}$ of symmetries with a local Lie algebra $\mathcal{L}$. The enveloping *BD* algebra $U^{BD}\mathcal{L}$ is then a factorization algebra, and as we will see, current algebras from physics, such as the Kac-Moody or Virasoro chiral algebras, appear as examples of this construction. If $\mathcal{L}$ acts by quantum symmetries on a quantum BV theory, we will thus obtain a map of factorization algebras

$$U^{BD}\mathcal{L} \to \mathrm{Obs}^q$$

that realizes currents as quantum observables. On global sections, one recovers Ward-Takahashi identities, but the local story implies, for instance, statements about algebras and vertex algebras, for one- and two-dimensional field theories respectively.

12

Noether's theorem in classical field theory

A central result in field theory is Noether's theorem, which states that there is a one-to-one correspondence between continuous symmetries of a field theory and conserved currents. In this chapter we will develop a version of Noether's theorem for classical field theories in the language of factorization algebras. In the following chapter, we will develop the analogous theorem for quantum field theories.

12.1 An overview of the main theorem

In the approach to classical field theory developed in this book, the statement runs as follows.

Suppose we have a classical field theory on a manifold X, and let $\widetilde{\mathrm{Obs}}^{cl}$ denote the P_0 factorization algebra of observables of the classical theory and let Obs^{cl} denote the quasi-isomorphic factorization algebra containing it, as developed in Chapter 5. The P_0 structure on $\widetilde{\mathrm{Obs}}^{cl}$ makes $\widetilde{\mathrm{Obs}}^{cl}[-1]$ into a precosheaf of dg Lie algebras, by forgetting the commutative multiplication and shifting. Moreover, this dg Lie algebra $\widetilde{\mathrm{Obs}}^{cl}(U)[-1]$ acts on $\mathrm{Obs}^{cl}(U)$ by the shifted Poisson bracket.

Now suppose that $\mathcal{L}$ is a local L_∞ algebra on X that acts on our classical field theory. This action also naturally induces an action on the global observables $\mathrm{Obs}^{cl}(X)$. (We will define soon precisely what we mean by an action.) Let $\mathcal{L}_c$ denote the precosheaf of L_∞ algebras on X given by compactly supported section of $\mathcal{L}$. We wish to refine the action of $\mathcal{L}$ on $\mathrm{Obs}^{cl}(X)$ to a local action involving $\mathcal{L}_c$ and Obs^{cl}, which furthermore realizes a symmetry as an observable.

Our formulation of Noether's theorem involves shifted central extensions of this precosheaf $\mathcal{L}_c$. Such central extensions were discussed in Section I.3.6; we are interested in -1-shifted central extensions that fit into short exact sequences

$$0 \to \underline{\mathbb{C}}[-1] \to \widehat{\mathcal{L}}_c \to \mathcal{L}_c \to 0,$$

where $\underline{\mathbb{C}}$ is the constant precosheaf.

The theorem is then the following.

Theorem. *If a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory with observables Obs^{cl}, then there is a -1-shifted central extension $\widehat{\mathcal{L}}_c$ of the precosheaf $\mathcal{L}_c$ of L_∞ algebras on X, and a map of precosheaves of L_∞ algebras*

$$\widehat{\mathcal{L}}_c \rightsquigarrow \widetilde{\mathrm{Obs}}^{cl}[-1]$$

that sends the central element of $\widehat{\mathcal{L}}_c$ to the observable 1 in $\widetilde{\mathrm{Obs}}^{cl}(U)[-1]$ for every open subset U.

This map is, by construction, compatible with the action of the cosheaf $\mathcal{L}_c$ on Obs^{cl} induced by the action $\mathcal{L}_c$ on the field theory. Let us explain the form of this compatibility.

When the dg Lie algebra $\widetilde{\mathrm{Obs}}^{cl}(U)[-1]$ acts on $\mathrm{Obs}^{cl}(U)$ by the Poisson bracket, the constant observable 1 acts by zero. The L_∞ map we just discussed therefore gives an action of $\widehat{\mathcal{L}}_c(U)$ on $\mathrm{Obs}^{cl}(U)$, which descends to an action of $\mathcal{L}_c(U)$ because the central element acts by zero.

Theorem. *In the situation of the preceding theorem, the action of $\mathcal{L}_c(U)$ coming from the L_∞-map $\widehat{\mathcal{L}}_c \rightsquigarrow \widetilde{\mathrm{Obs}}^{cl}$ and the action coming from the action of $\mathcal{L}$ on the classical field theory coincide up to a homotopy.*

12.2 Symmetries of a classical field theory

We will start by examining what it means for a homotopy Lie algebra to act on a field theory. We are particularly interested in what it means for a *local L_∞ algebra* to act on a classical field theory. Recall from Section 3.1.3 that a local L_∞ algebra $\mathcal{L}$ is a sheaf of L_∞ algebras given by sections of a graded vector bundle L and whose L_∞-structure maps are polydifferential operators.

We know from Chapter 4 that a perturbative classical field theory is described by an elliptic moduli problem on X with a degree -1 symplectic form. Equivalently, it is described by a local L_∞ algebra $\mathcal{M}$ on X equipped with an invariant pairing of degree -3, as discussed in Section 4.2. Therefore, an action of $\mathcal{L}$ on $\mathcal{M}$ should be an L_∞ action of $\mathcal{L}$ on $\mathcal{M}$. Thus, the first thing we need to understand is what it means for one L_∞ algebra to act on the other.

12.2.1 Actions of L_∞ algebras

We develop the notion starting from more concrete situations. If $\mathfrak{g}, \mathfrak{h}$ are ordinary Lie algebras, then it is straightforward to say what it means for $\mathfrak{g}$ to act on $\mathfrak{h}$: there is a map of Lie algebras from $\mathfrak{g}$ to $\mathrm{Der}(\mathfrak{h})$. One can then define the associated semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$. This semi-direct product lives in a short exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \to 0.$$

Conversely, one can recover the action of $\mathfrak{g}$ on $\mathfrak{h}$ (up to equivalence) from the data of such a short exact sequence of Lie algebras.

We take this construction as a model for the action of one L_∞ algebra $\mathfrak{g}$ on another L_∞ algebra $\mathfrak{h}$.

12.2.1.1 Definition. *An* action of an L_∞ algebra $\mathfrak{g}$ on an L_∞ algebra $\mathfrak{h}$ *is an L_∞-algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$, which we denote $\mathfrak{g} \ltimes \mathfrak{h}$, with the property that the maps in the exact sequence of vector spaces*

$$0 \to \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \to 0$$

are strict maps of L_∞ algebras.

Remark: We note:

(i) The set of actions of $\mathfrak{g}$ on $\mathfrak{h}$ enriches to a simplicial set, whose n-simplices are families of actions over the dg algebra $\Omega^*(\triangle^n)$.
(ii) There are other possible notions of action of $\mathfrak{g}$ on $\mathfrak{h}$ that might seem more natural to the reader. For instance, an abstract notion is to say that an action of $\mathfrak{g}$ on $\mathfrak{h}$ is an L_∞ algebra $\widetilde{\mathfrak{h}}$ with a map $\phi : \widetilde{\mathfrak{h}} \rightsquigarrow \mathfrak{g}$ and an isomorphism of L_∞ algebras between the homotopy fibre $\phi^{-1}(0)$ and $\mathfrak{h}$. One can show that this fancier definition is equivalent to the concrete one just proposed, in the sense that the two ∞-groupoids of possible actions are equivalent.

If $\mathfrak{h}$ is finite dimensional, the dg Lie algebra of derivations of $C^*(\mathfrak{h})$ can be identified with the module $C^*(\mathfrak{h}, \mathfrak{h}[1])$, equipped with a natural Lie bracket. It is then straightforward to verify the following.

12.2.1.2 Lemma. *An action of $\mathfrak{g}$ on $\mathfrak{h}$ is equivalent to an L_∞-algebra map $\mathfrak{g} \rightsquigarrow C^*(\mathfrak{h}, \mathfrak{h}[1])$.*

Remark: We view $C^*(\mathfrak{h}, \mathfrak{h}[1])$ as the dg Lie algebra of vector fields on the formal moduli problem $B\mathfrak{h}$. This lemma shows that an action of $\mathfrak{g}$ on $\mathfrak{h}$ is the same as an action of $\mathfrak{g}$ on the formal moduli problem $B\mathfrak{h}$ which *may not* preserve the base point of $B\mathfrak{h}$. $\Diamond$

12.2.2 Actions of local L_∞ algebras

Now let us return to the setting of local L_∞ algebras and define what it means for one local L_∞ algebra to act on another.

12.2.2.1 Definition. *Let $\mathcal{L}$ and $\mathcal{M}$ be local L_∞ algebras on X. Then* an action of $\mathcal{L}$ on $\mathcal{M}$ *is an local L_∞ structure on $\mathcal{L} \oplus \mathcal{M}$, which we denote $\mathcal{L} \ltimes \mathcal{M}$, such that the exact sequence of maps of sheaves*

$$0 \to \mathcal{M} \to \mathcal{L} \ltimes \mathcal{M} \to \mathcal{L} \to 0$$

consists of maps of L_∞ algebras.

More explicitly, this definition says that $\mathcal{M}$, with its original L_∞ structure, is both a sub-L_∞ algebra of $\mathcal{M} \ltimes \mathcal{L}$, but also an L_∞ ideal: if at least one element of $\mathcal{M}$ is the input to an operation, the output lands in $\mathcal{M}$.

A classical field theory involves a local L_∞ algebra with a pairing, so we need to articulate a compatibility between the pairing and the action to define a symmetry of a theory.

12.2.2.2 Definition. *Suppose that $\mathcal{M}$ has an invariant pairing $\langle -, - \rangle$. An action of $\mathcal{L}$ on $\mathcal{M}$* preserves the pairing *if for any compactly supported sections $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathcal{L}$ and $\{\beta_1, \ldots, \beta_s\}$ of $\mathcal{M}$, the expression*

$$\langle l_{r+s}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s), \beta_{s+1} \rangle$$

is graded totally anti-symmetric under permutation of the β_i.

Consider then a classical field theory defined by a local L_∞ algebra $\mathcal{M}$ with an invariant pairing of degree -3.

12.2.2.3 Definition. *Let $\mathcal{M}$ be a local L_∞ algebra with an invariant pairing of degree -3, so that it encodes a classical field theory by Definition 4.2.0.4. If $\mathcal{L}$ acts on $\mathcal{M}$ and preserves its pairing, we call it an* action of a local L_∞ algebra $\mathcal{L}$ on the classical field theory.

We will often refer to a classical field theory with an action of a local L_∞ algebra as an *equivariant classical field theory*.

The definition we just gave is a little abstract. We can make it more concrete, by relating it to action functionals, along the lines of Section 11.2.2. (Some readers might prefer to examine Example 12.3.1 before reading the general discussion.)

Recall that the space of fields of the classical field theory associated to $\mathcal{M}$ is $\mathcal{M}[1]$ and that the L_∞ structure on $\mathcal{M}$ is entirely encoded in the action functional

$$S \in \mathscr{O}_{loc}(\mathcal{M}[1]),$$

which satisfies the classical master equation $\{S, S\} = 0$. Indeed, $\mathscr{O}_{loc}(\mathcal{M}[1])[-1]$ has a Lie structure via the BV bracket $\{-, -\}$, and the L_∞ structure is given by a Maurer-Cartan element in that graded Lie algebra. (Recall that the notation $\mathscr{O}_{loc}$ always indicates local functionals modulo constants. *A priori* the bracket is defined on all local functionals, but the constants are central for the BV bracket, so we can equip the quotient with a Lie bracket.)

An action of a local L_∞ algebra $\mathcal{L}$ on $\mathcal{M}$ can also be encoded in a certain local functional, which depends on $\mathcal{L}$. Before providing this approach, we need to describe the precise space of functionals that arise in formulating it.

If X denotes the space-time manifold on which $\mathcal{L}$ and $\mathcal{M}$ are sheaves, then $\mathcal{L}(X)$ is an L_∞ algebra. Thus, we can form the Chevalley-Eilenberg cochain complex

$$C^*(\mathcal{L}(X)) = (\mathscr{O}(\mathcal{L}(X)[1]), \mathrm{d}_{\mathcal{L}})$$

as well as its reduced version $C^*_{red}(\mathcal{L}(X))$, which is the maximal ideal of that dg commutative algebra. In section 3.5.2 we also defined the local version of this, which is a subcomplex

$$C^*_{red,loc}(\mathcal{L}) \subset C^*_{red}(\mathcal{L}(X))$$

of local cochains.

The completed tensor product

$$C^*(\mathcal{L}(X)) \widehat{\otimes} C^*_{red,loc}(\mathcal{M})[-1]$$

is a dg Lie algebra, as it is a tensor product of a dg commutative algebra and a dg Lie algebra. There is a subcomplex

$$C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M})[-1]$$

which one can check explicitly is a sub dg Lie algebra. The Lie bracket comes from the invariant pairing on $\mathcal{M}$. The complex $C^*_{red,loc}(\mathcal{L})[-1]$ is a subcomplex which is in the center of this Lie bracket.

By restricting local cochains to the sub Lie algebra $\mathcal{M} \subset \mathcal{L} \oplus \mathcal{M}$, we get a map of dg Lie algebras

$$C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M})[-1] \to C^*_{red,loc}(\mathcal{M})[-1]. \tag{12.2.2.1}$$

12.2.2.4 Definition. *We let* $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ *denote the kernel of this map of dg Lie algebras. We let* $\mathrm{Act}(\mathcal{L}, \mathcal{M})$ *be the quotient of* $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ *by the central subcomplex* $C^*_{red,loc}(\mathcal{L})$*.*

Note that as a cochain complex, we can identify

$$\mathrm{Act}(\mathcal{L}, \mathcal{M}) = C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / (C^*_{red,loc}(\mathcal{L}) \oplus C^*_{red,loc}(\mathcal{M}).$$

This identification does not make sense as an isomorphism of dg Lie algebras, because $C^*_{red,loc}(\mathcal{M})$ is not an ideal in $C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M})$. This is because the cochain map 12.2.2.1 splits as a cochain map, but not as a map of dg Lie algebras.

This notation allows us to offer an alternative description of the action of a local L_∞ algebra on a classical field theory.

12.2.2.5 Lemma. *Let $\mathcal{M}$ be a local L_∞ algebra with an invariant pairing of degree -3, so that it encodes a classical field theory. To give an action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory is the same as to give a Maurer-Cartan element $S^{\mathcal{L}}$ in* $\mathrm{Act}(\mathcal{L}, \mathcal{M})$*, i.e., satisfying*

$$(\mathrm{d}_{\mathcal{L}} + \mathrm{d}_{\mathcal{M}})S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0.$$

Such a Maurer-Cartan element is a solution to a kind of $\mathcal{L}$-equivariant classical master equation modulo local functionals depending just on $\mathcal{L}$ or $\mathcal{M}$.

This lemma suggests that we view a classical field theory with an action of $\mathcal{L}$ as a family of classical field theories over the sheaf of formal moduli problems $B\mathcal{L}$. Further justification for this idea will be offered in Proposition 12.3.0.2.

Proof Given such an $S^{\mathcal{L}}$, then

$$\mathrm{d}_{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} + \{S^{\mathcal{L}}, -\}$$

defines a differential on $\mathscr{O}(\mathcal{L}(X)[1] \oplus \mathcal{M}(X)[1])$. The classical master equation implies that this differential is of square zero, so that it defines an L_∞ structure on $\mathcal{L}(X) \oplus \mathcal{M}(X)$. Moreover, the locality condition on $S^{\mathcal{L}}$ guarantees that it produces a local L_∞ algebra structure. Direct examination shows that this L_∞ structure respects the exact sequence

$$0 \to \mathcal{M} \to \mathcal{L} \oplus \mathcal{M} \to \mathcal{L} \to 0$$

and the invariant pairing on $\mathcal{M}$. □

The Maurer-Cartan equation satisfied by $S^{\mathcal{L}}$ can be rephrased as follows. Let $S \in \mathscr{O}_{loc}(\mathcal{M}[1])$ be the solution to the classical master equation for our classical theory. Let

$$S^{tot} = S^{\mathcal{L}} + S \in C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{L}).$$

Then the Maurer-Cartan equation for $S^{\mathcal{L}}$ is equivalent to the classical master equation

$$d_{\mathcal{L}} S^{tot} + \tfrac{1}{2}\{S^{tot}, S^{tot}\} = 0. \qquad (12.2.2.2)$$

Here, as before, $d_{\mathcal{L}}$ is the Chevalley-Eilenberg differential associated to the Lie bracket on $\mathcal{L}$.

We can then interpret S^{tot} as defining a field theory with fields $\mathcal{M}[1]$ and "background fields" $\mathcal{L}[1]$. For example, if $\mathcal{L} = \Omega_X^{\leq 1} \otimes \mathfrak{g}$, for some Lie algebra $\mathfrak{g}$, then we are specifying an action functional S^{tot} which depends on the fields in $\mathcal{M}[1]$ and on a background connection $A \in \Omega^1(X) \otimes \mathfrak{g}$. The dependence of S^{tot} on $\Omega^0(X) \otimes \mathfrak{g}[1]$ tells us how gauge transformations act on the fields on $\mathcal{M}[1]$, and the classical master equation 12.2.2.2 tells us that the system is gauge invariant, where we include the action of gauge transformations on both the fields in $\mathcal{M}[1]$ and on the connection in $\Omega^1(X) \otimes \mathfrak{g}$.

In this special case, giving an action of $\mathcal{L} = \Omega_X^{\leq 1} \otimes \mathfrak{g}$ on a field theory specified by $\mathcal{M}$ is equivalent to giving a gauge-invariant action depending on a background gauge field. This is a familiar manifestation of symmetry in physics.

At this stage it is worthwhile introducing a very simple example of an action of a local L_∞ algebra on a theory, to which we can apply our constructions.

Example: Consider a chiral free fermion on a Riemann surface Σ. In the BV formalism, the fields are

$$\Psi \in \Pi\Omega^{1/2,*}(\Sigma)$$

where Π denotes parity shift. (We work here with $\mathbb{Z}/2 \times \mathbb{Z}$ graded vector spaces, where the $\mathbb{Z}/2$ counts fermion number and $\mathbb{Z}$ is the cohomological grading. Both gradings contribute to signs.)

We let ψ denote the component of Ψ in $\Omega^{1/2,0}$ and $\psi^{a.f.}$ denote the component in $\Omega^{1/2,1}$. The field ψ is the usual fermion, and $\psi^{a.f.}$ is its anti-field. The odd symplectic pairing is $\frac{1}{2}\int \Psi \wedge \Psi = \int \psi\psi^{a.f.}$, and the Lagrangian is

$$S = \frac{1}{2}\int \Psi \overline{\partial} \Psi.$$

The corresponding L_∞ algebra is

$$\mathcal{M} = \Pi\Omega^{1/2,*}(\Sigma)[-1].$$

The only non-trivial L_∞ operations on $\mathcal{M}$ are $l_1 = \overline{\partial}$.

We can take n copies of $\mathcal{M}$,

$$\mathcal{M}^{\oplus n} = \pi\Omega^{1/2,*}(\Sigma, \mathbb{C}^n)[-1].$$

This has an action of the local Lie algebra

$$\mathcal{L} = \Omega^{0,*}(\Sigma, \mathfrak{so}(n))$$

which combines the $\mathfrak{so}(n)$ action on $\mathbb{C}^n$ with the wedge product of Dolbeault forms. The corresponding semi-direct product L_∞ algebra is the dg Lie algebra

$$\mathcal{L} \ltimes \mathcal{M}^{\oplus n} = \Omega^{0,*}(\Sigma, \mathfrak{so}(n)) \ltimes \pi\Omega^{1/2,*}(\Sigma, \mathbb{C}^n)[-1].$$

The corresponding equivariant action functional, for $\alpha \in \mathcal{L}[1]$ and $\Psi \in \mathcal{M}^{\oplus n}[1]$, is

$$S^{\mathcal{L}}(\alpha, \Psi) = \tfrac{1}{2} \int \Psi \alpha \Psi. \tag{12.2.2.3}$$

Adding to this the original Lagrangian we find

$$S^{tot} = S^{\mathcal{L}} + S = \tfrac{1}{2} \int \Psi(\overline{\partial} + \alpha)\Psi.$$

If we restrict to $\alpha^1 \in \Omega^{0,1}(\Sigma, \mathfrak{so}(n))$. ,we find the action functional for a free fermion in the presence of a background gauge field. Restricting to $\alpha^0 \in \Omega^{0,0}(\Sigma, \mathfrak{so}(n))[1]$, we find the Lagrangian

$$\int \psi \alpha^0 \psi^{a.f.}.$$

Since $\psi^{a.f.}$ is the anti-field and α^0 is the ghost, this term describes the action of $\Omega^{0,0}(\Sigma, \mathfrak{so}(n))$ on the fields of the free fermion system. $\Diamond$

12.2.3 Inner actions

In definition 12.2.2.4, we defined a dg Lie algebra $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ so that there is a short exact sequence of dg Lie algebras

$$0 \to \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) \to C^*_{red,loc}(\mathcal{M}) \to 0.$$

12.2.3.1 Definition. *An* inner action *of $\mathcal{L}$ on $\mathcal{M}$ is a Maurer-Cartan element*

$$S^{\mathcal{L}} \in \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}).$$

Viewed as an $\mathcal{L}$-dependent local functional on the classical field theory, $S^{\mathcal{L}}$ is of cohomological degree 0 *and satisfies the master equation*

$$\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0.$$

(Recall we work with the shift $C^*_{loc}(\mathcal{M})[-1]$ to obtain an un*shifted Lie algebra.)*

In the example of a chiral free fermion, the equivariant action in (12.2.2.3) is easily seen to be an inner action. Indeed, the classical master equation holds without having to drop any terms which depend on just $\mathcal{L}[1]$. This is because $\S^{\mathcal{L}}$ is quadratic in $\mathcal{M}[1]$, so that the Poisson bracket $\{S^{\mathcal{L}}, S^{\mathcal{L}}\}$ and the differential $\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}}$ are both quadratic in $\mathcal{M}[1]$.

This observation generalizes.

12.2.3.2 Lemma. *Given an action of $\mathcal{L}$ on a field theory $\mathcal{M}$, there is an obstruction class in $H^1(C^*_{red,loc}(\mathcal{L}))$ such that the action extends to an inner action if and only if this class vanishes.*

Proof There is a short exact sequence of dg Lie algebras

$$0 \to C^*_{red,loc}(\mathcal{L}) \to \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \to \mathrm{Act}(\mathcal{L}, \mathcal{M}) \to 0,$$

and $\mathscr{O}_{loc}(\mathcal{L}[1])$ is central. The result then follows from general facts about Maurer-Cartan simplicial sets.

The obstruction is calculated explicitly as follows. Suppose we have an action functional

$$S^{\mathcal{L}} \in \mathrm{Act}(\mathcal{L}, \mathcal{M}).$$

There is a natural inclusion $C^*_{red}(\mathcal{M}(X)) \hookrightarrow C^*(\mathcal{M}(X))$ that allows us to view $S^{\mathcal{L}}$ as a functional

$$\widetilde{S}^{\mathcal{L}} \in \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \subset C^*_{red}(\mathcal{L}(X)) \otimes C^*(\mathcal{M}(X))$$

using the . The obstruction is simply the failure of $\widetilde{S}^{\mathcal{L}}$ to satisfy the Maurer-Cartan equation in $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$. ◻

Let us now briefly remark on some refinements of this lemma, which give more control over the obstruction class.

Define a subcomplex $C^{\geq 2}_{red,loc}(\mathcal{L})$ of $C^*_{red,loc}(\mathcal{L})$ consisting of those local cochains which involve two or more sections of $\mathcal{L}$. These correspond to local functionals on $B\mathcal{L}$ which vanish to second order at the base point.

12.2.3.3 Lemma. *If a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory $\mathcal{M}$, then the obstruction to extending $\mathcal{L}$ to an inner action lifts naturally from $C^*_{red,loc}(\mathcal{L})$ to the subcomplex $C^{\geq 2}_{red,loc}(\mathcal{L})$.*

Proof Suppose that the action of $\mathcal{L}$ on $\mathcal{M}$ is encoded by an action functional $S^{\mathcal{L}}$, as before. The obstruction is

$$\left(\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\}\right)\Big|_{\mathscr{O}_{loc}(\mathcal{L}[1])} \in \mathscr{O}_{loc}(\mathcal{L}[1]).$$

Here, $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{M}}$ are the Chevalley-Eilenberg differentials for the two L_∞ algebras.

We need to verify that no terms in this expression can be linear in $\mathcal{L}$. Recall that the functional $S^{\mathcal{L}}$ has no linear terms. Further, the differentials $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{M}}$ respect the filtration by polynomial degree, so that they cannot produce a functional with a linear term from a functional that does not have a linear term. $\square$

Remark: There is a somewhat more general situation where this lemma is false. It can be convenient to work with families of classical field theories over some dg ring R with a nilpotent ideal I, where one allows the L_∞ algebra $\mathcal{M}$ describing the field theory to be curved, as long as the curving vanishes modulo I. This situation is encountered in the study of σ-models: see Costello (2011a); Grady and Gwilliam (2015); Li and Li (2016). When $\mathcal{M}$ is curved, the differential $\mathrm{d}_{\mathcal{M}}$ does not preserve the filtration by polynomial degree, so that this argument fails. $\Diamond$

Let us briefly discuss a special case where the obstruction vanishes.

12.2.3.4 Lemma. *Suppose that the action of $\mathcal{L}$ on $\mathcal{M}$ is encoded by an L_∞ structure on $\mathcal{L} \ltimes \mathcal{M}$ such that the components of the L_∞ structure maps l_n which map $\mathcal{L}^{\otimes n}$ to $\mathcal{M}$ all vanish. (When viewed as an action of $\mathcal{L}$ on the sheaf of formal moduli problems $B\mathcal{M}$, this conditions means that $\mathcal{L}$ preserves the base point of $B\mathcal{M}$.) Then the action of $\mathcal{L}$ extends canonically to an inner action.*

Proof We need to verify that the obstruction

$$\left(\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\}\right)\Big|_{\mathscr{O}_{loc}(\mathcal{L}[1])} \in \mathscr{O}_{loc}(\mathcal{L}[1]).$$

is identically zero. Our assumptions on $S^{\mathcal{L}}$ mean that it is at least quadratic as a function on $\mathcal{M}[1]$. Hence the obstruction is also at least quadratic as a function of $\mathcal{M}[1]$, so that it is zero when restricted to being a function of just $\mathcal{L}[1]$. □

12.2.4 Sheafifying actions of ordinary Lie algebras

We have explained what we mean by an action of a local L_∞ algebra on a classical field theory, and what it means to give an inner action. We should also relate this notion to the more familiar and concrete notion of an ordinary Lie algebra acting a field theory.

In particular, our formulation of Noether's theorem will be phrased in terms of the action of a local L_∞ algebra on a field theory. In practice, however, we are often presented first with the action of an ordinary, finite-dimensional L_∞-algebra on a theory, and we would like to apply Noether's theorem to this situation. Thus, we need to be able to reformulate this kind of ordinary action as an action of a local L_∞ algebra.

Hence, let $\mathfrak{g}$ be an L_∞ algebra, which we assume to be finite-dimensional for simplicity. (Note that we are not working with a *sheaf* of L_∞ algebras here.) Let $C^*(\mathfrak{g})$ be its Chevalley-Eilenberg cochains, viewed as a pronilpotent commutative dg algebra. Finally, consider a classical field theory, represented as an elliptic L_∞ algebra $\mathcal{M}$ with an invariant pairing. Then we define the notion of a $\mathfrak{g}$-action on $\mathcal{M}$ as follows.

12.2.4.1 Definition. *An* action of $\mathfrak{g}$ on $\mathcal{M}$ *is any of the following equivalent data:*

(i) *An L_∞ structure on $\mathfrak{g} \oplus \mathcal{M}(X)$ such that the exact sequence*

$$\mathcal{M}(X) \to \mathfrak{g} \ltimes \mathcal{M}(X) \to \mathfrak{g}$$

is a sequence of maps of L_∞-algebras, the structure maps

$$\mathfrak{g}^{\otimes n} \otimes \mathcal{M}(X)^{\otimes m} \to \mathcal{M}(X)$$

are poly-differential operators in the $\mathcal{M}$-variables, and the action preserves the pairing.

(ii) *An L_∞-morphism $\mathfrak{g} \rightsquigarrow \mathscr{O}_{loc}(B\mathcal{M})[-1]$. (The shift ensures that $\mathscr{O}_{loc}(B\mathcal{M})[-1]$ is an unshifted dg Lie algebra.)*

(iii) *An degree one element $S^{\mathfrak{g}}$ in the dg Lie algebra*

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) = C^*_{red}(\mathfrak{g}) \otimes \mathscr{O}_{loc}(B\mathcal{M})[-1]$$

that satisfies the Maurer-Cartan equation

$$\mathrm{d}_{\mathfrak{g}} S^{\mathfrak{g}} + \mathrm{d}_{\mathcal{M}} S^{\mathfrak{g}} + \tfrac{1}{2}\{S^{\mathfrak{g}}, S^{\mathfrak{g}}\} = 0.$$

(That is, $S^{\mathfrak{g}}$ is a solution to a $\mathfrak{g}$-equivariant version of the classical master equation.)

It is straightforward to verify that these three notions are identical. The third version of the definition can be viewed as saying that a $\mathfrak{g}$-action on a classical field theory is a family of classical field theories over the dg ring $C^*(\mathfrak{g})$ that reduces to the original classical field theory modulo the maximal ideal $C^{>0}(\mathfrak{g})$. It will be this last version of the definition that generalizes to the quantum level.

The following lemma shows how to relate this notion with the local version already introduced.

12.2.4.2 Lemma. *Let $\mathfrak{g}$ be an L_∞-algebra. There is a canonical homotopy equivalence between the simplicial sets describing*

(i) *Actions of $\mathfrak{g}$ on a fixed classical field theory on a manifold X.*

(ii) *Actions of the local L_∞ algebra $\Omega^*_X \otimes \mathfrak{g}$ on the same classical field theory.*

The appearance of the sheaf $\Omega^*_X \otimes \mathfrak{g}$ should not be surprising: it is a fine resolution of the constant sheaf of L_∞ algebras with value $\mathfrak{g}$. This replacement of $\mathfrak{g}$ by $\Omega^*_X \otimes \mathfrak{g}$ is, as always, a convenient way to approach locally constant constructions via tools with a differential-geometric flavor. Indeed, it codifies familiar manipulations from physics, where a global symmetry is made local.

We remark that the lemma can be further generalized to show that for any locally constant sheaf of L_∞ algebras $\mathfrak{g}$, an action of $\mathfrak{g}$ on a theory is the same thing as an action of the de Rham resolution of $\mathfrak{g}$.

Proof We have already explained how each space of actions arises from the simplicial set of Maurer-Cartan elements for a dg Lie algebra. Hence, it will suffice to give a quasi-isomorphism of dg Lie algebras

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) \to \mathrm{Act}(\mathfrak{g}, \mathcal{M}),$$

since one then obtains a weak equivalence between the simplicial sets of Maurer-Cartan elements. (We remark that we work slightly outside the most familiar deformation-theory set-up, as we are taking the Maurer-Cartan simplicial sets associated to just a dg Lie algebra rather than to a dg Lie algebra tensored with a nilpotent commutative algebra. This is legitimate, however, as the dg Lie algebras we are considering are both pro-nilpotent).

Let $\mathcal{M}$ denote a local L_∞ algebra encoding a classical field theory, and consider the trivial action of the local L_∞ algebra $\Omega^*_X \otimes \mathfrak{g}$ on $\mathcal{M}$.

Since constant functions include into the de Rham complex, we obtain an inclusion of L_∞ algebras

$$\mathfrak{g} \hookrightarrow \Omega^*(X) \otimes \mathfrak{g}$$

and hence a map of dg commutative algebras

$$C^*_{red}(\Omega^*(X) \otimes \mathfrak{g}) \to C^*_{red}(\mathfrak{g}).$$

Tensoring with $\mathscr{O}_{loc}(\mathcal{M}[1])$ yields a map of dg Lie algebras

$$\begin{tikzcd} C^*_{red,loc}(\Omega^*(X) \otimes \mathfrak{g}) \otimes \mathscr{O}_{loc}(\mathcal{M}[1]) \arrow[r] \arrow[d, equal] & C^*_{red}(\mathfrak{g}) \otimes \mathscr{O}_{loc}(\mathcal{M}[1]) \arrow[d, equal] \\ \mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) & \mathrm{Act}(\mathfrak{g}, \mathcal{M}), \end{tikzcd}$$

as desired. It remains to show that this map is a quasi-isomorphism.

To show this, we use a D_X-module description of the left hand side. Let $J(\mathcal{M})$ refer to the D_X-modules of jets of sections of $\mathcal{M}$, and let $J(\Omega^*_X)$ refer to jets of the de Rham complex.

Recall that the local functionals for $\mathcal{M}$ are Lagrangian densities up to total derivatives. One can phrase this statement in terms of tensor products of D_X-modules:

$$\mathscr{O}_{loc}(\mathcal{M}[1]) = \mathrm{Dens}_X \otimes_{D_X} C^*_{red}(J(\mathcal{M})),$$

as shown in Lemma 6.6.1 of Chapter 5 of Costello (2011b). (Here $C^*_{red}(J(\mathcal{M}))$ is a description of Lagrangians.) The cochain complex underlying the dg Lie algebra $\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M})$ thus has the following interpretation in the language of D_X-modules:

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) = \mathrm{Dens}_X \otimes_{D_X} \left(C^*_{red}(J(\Omega^*_X) \otimes_{\mathbb{C}} \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \right).$$

On the other hand, the cochain complex $\mathrm{Act}(\mathfrak{g}, \mathcal{M})$ has the D_X-module interpretation

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) = \mathrm{Dens}_X \otimes_{D_X} \left(C^*_{red}(C^\infty_X \otimes \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \right).$$

Thankfully, there is a natural map of D_X-modules

$$C^\infty_X \to J(\Omega^*_X)$$

given by sending a smooth function to its jet, and this map is a quasi-isomorphism by the Poincaré lemma. In consequence, the natural map

$$C^*_{red}(J(\Omega^*_X) \otimes_{\mathbb{C}} \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \to C^*_{red}(C^\infty_X \otimes \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M}))$$

is a quasi-isomorphism of D_X-modules. Moreover, both sides of this map are flat as left D_X-modules, as $C^*_{red}(J(\mathcal{M}))$ is a flat D_X-module. It follows that this map is still a quasi-isomorphism after tensoring over D_X with Dens_X. □

This lemma has a modification when the L_∞ algebra $\mathcal{M}$ is concentrated in degrees 1 and 2. This happens when $\mathcal{M}$ is the L_∞ algebra associated to a field theory with no gauge symmetry.

12.2.4.3 Lemma. *In this situation, the action of $\Omega^*_X \otimes \mathfrak{g}$ on $\mathcal{M}$ factors through the truncation $\Omega^{\leq 1}_X \otimes \mathfrak{g}$.*

Proof This is for simple grading reasons. Specifying an action of $\Omega^*_X \otimes \mathfrak{g}$ is the same as specifying a Maurer-Cartan element in $\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M})$, i.e., a component of

$$\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right)[-1].$$

This Maurer-Cartan element is a local functional of an element in $\Omega^*_X \otimes \mathfrak{g}[1]$ and $\mathcal{M}[1]$. It is also of cohomological degree 0. Therefore it can not depend on fields in $\Omega^{\geq 2}_X \otimes \mathfrak{g}[1]$, and must descend to an element of $\mathrm{Act}(\Omega^{\leq 1}_X \otimes \mathfrak{g}, \mathcal{M})$. □

The reason to mention this lemma is to connect with a standard statement in physics, which is that if we have a field theory with an action of $\mathfrak{g}$ then we can couple to a background $\mathfrak{g}$ connection in a gauge invariant way. Such a coupling is precisely the same as an action of $\Omega^{\leq 1}_M \otimes \mathfrak{g}$. The dependence of the equivariant action functional on $\Omega^1_M \otimes \mathfrak{g}$ gives us the terms in the action which depend on the background connection, and the dependence on $\Omega^0_M \otimes \mathfrak{g}$ tells us how gauge transformations act. The Maurer-Cartan equation implies gauge invariance.

Notice that this lemma requires some hypothesis: we assume that our space of fields $\mathcal{M}[1]$ lives in cohomological degrees 0 and 1. This happens when the field theory associated to $\mathcal{M}$ is defined by an action functional without gauge symmetry. The lemma is false without this hypothesis, so that the standard physics statement is also false without this hypothesis.

Example: Let us see how this works in our running example of the chiral free fermion on a Riemann surface Σ. Recall that the fields are $\Psi \in \Pi\Omega^{1/2,*}(\Sigma, \mathbb{C}^n)$ with Lagrangian $\delta_{ij} \int \Psi_i \overline{\partial} \Psi_j$. This has an obvious action of the constant sheaf of Lie algebras $\mathfrak{so}(n)$. The action is given by the functional $\int \Psi_i M_{ij} \Psi_j$, where $M \in \mathfrak{so}(n)$.

We have seen that this action extends in a unique way, up to homotopy, to an action of the de Rham resolution of this constant sheaf. Because of this uniqueness result, we simply need to write down *some* action of the de Rham resolution $\Omega^*_\Sigma \otimes \mathfrak{so}(n)$.

Earlier, we described an action of $\Omega^{0,*}_\Sigma \otimes \mathfrak{so}(n)$. The functional describing this action is $\int \Psi A \Psi$, where $A \in \Omega^{0,*}_\Sigma \otimes \mathfrak{so}(n)$. There is a homomorphism

$$\Omega^*_\Sigma \otimes \mathfrak{so}(n) \to \Omega^{0,*}_\Sigma \otimes \mathfrak{so}(n)$$

which gives rise to an action of $\Omega^*_\Sigma \otimes \mathfrak{so}(n)$. $\Diamond$

12.3 The factorization algebra of equivariant classical observables

When an L_∞ algebra $\mathcal{L}$ acts on another L_∞ algebra $\mathcal{M}$, the dg commutative algebra $C^*(L \ltimes \mathcal{M})$ describes the functions on the formal moduli space $B\mathcal{M}$ that are $\mathcal{L}$-invariant up to homotopy. In the setting of field theory, we are thinking about the observables that are invariant with respect to the symmetry encoded by $\mathcal{L}$. This line of thought suggests the following definition, which builds upon the commutative factorization algebra $C^*(\mathcal{L})$ that assigns to an open subset U, the dg commutative algebra $C^*(\mathcal{L}(U))$.

12.3.0.1 Definition. *On a manifold X, let $\mathcal{L}$ be a local L_∞ algebra acting on a classical field theory encoded by a local L_∞ algebra $\mathcal{M}$ with invariant pairing. The* equivariant classical observables *are*

$$\begin{aligned}\mathrm{Obs}^{cl}_{\mathcal{L}} &= C^*(\mathcal{L} \ltimes \mathcal{M}) \\ &= C^*(\mathcal{L}, \mathrm{Obs}^{cl}).\end{aligned}$$

Here $\mathrm{Obs}^{cl}(X)$ denotes the classical observables for the theory associated to $\mathcal{M}$; its structure as an $\mathcal{L}$-module manifestly depends on the action of $\mathcal{L}$ on $\mathcal{M}$.

It is clear that this notion lifts to the level of prefactorization algebras, since $\mathcal{L}$ and $\mathcal{M}$ are both local in nature. In fact, we showed in Section I.6.6 that $C^*(\mathcal{L} \ltimes \mathcal{M})$ satisfies the local-to-global axiom for a factorization algebra. By construction it is a $C^*(\mathcal{L})$-module in factorization algebras.

We now turn to the more refined question of the P_0 structure.

12.3.0.2 Proposition. *Let $\mathcal{M}$ encode a classical field theory with an action of $\mathcal{L}$. The sub-factorization algebra*

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathcal{L}} = C^*(\mathcal{L}, \widetilde{\mathrm{Obs}}^{cl})$$

of $\mathrm{Obs}^{cl}_{\mathcal{L}}$ is a P_0 factorization algebra, and the bracket is linear with respect to $C^*(\mathcal{L})$-module structure. It is, moreover, quasi-isomorphic to $\mathrm{Obs}^{cl}_{\mathcal{L}}$ as factorization algebras.

Proof We extend the results of Section 5.4 in a straightforward way. By definition, $\widetilde{\mathrm{Obs}}^{cl}_{\mathcal{L}}$ consists of those functionals that have smooth first derivative but only in the $\mathcal{M}$-directions. We equip these with a P_0 structure by extending the P_0 structure on $\widetilde{\mathrm{Obs}}^{cl}$ in a $C^*(\mathcal{L})$-linear fashion to $\widetilde{\mathrm{Obs}}^{cl}_{\mathcal{L}}$. Hence, those functionals that lie in $C^*(\mathcal{L}(U))$ are central for this Poisson bracket. By construction we thus have a P_0-factorization algebra over the factorization algebra $C^*(\mathcal{L})$. □

Example: Let's see return to our example of a chiral free fermion. As we have seen in Chapter I.5, when dealing with a chiral theory in dimension 2, it is often useful to consider those observables on a disc which are in some eigenspace for the S^1 action rotating the disc. Typically, these eigenspaces are finite dimensional (at the level of cohomology). The direct sum of these eigenspaces has the structure of a vertex algebra. An operator which is in the k eigenspaces for this S^1 action is said to have spin k. For theories like a free fermion, where fields are spinors, k may be a half-integer.

If we just consider the chiral free fermion, the cohomology of the space of $\mathrm{Obs}^{cl}(D)$ is a commutative algebra. The subalgebra consisting of those elements that are finite direct sums of S^1-eigenvectors is freely generated by the fermionic point observables (or operators)

$$\Psi \mapsto \partial_z^k \Psi_i(0) \tag{12.3.0.1}$$

of spin $k + \frac{1}{2}$. (These are simply holomorphic derivatives of the delta function at the center of the disk.) As such it is the exterior algebra on the vector space $\mathbb{C}^n[\partial_z]$. Abusing notation slightly, we will refer to these operations as $\partial_z^k \Psi_i$.

Let us now consider the equivariant observables for the action of $\Omega^{0,*}_\Sigma \otimes \mathfrak{so}(n)$, where again we consider operators which are finite sums of S^1 eigenvectors. The algebra of equivariant observables is a dg commutative algebra freely generated by the operators (12.3.0.1), and by new operators

$$\partial_z^k \mathsf{c}_{ij} : \alpha \mapsto \partial_z^k \alpha_{ij}(0)$$

for $\alpha \in \Omega^{0,*}(D, \mathfrak{so}(n))[1]$. In physics the operator c_{ij} is referred to as the c-ghost operators, and the other operators $\partial_z^k \mathsf{c}_{ij}$ are its derivatives using the natural action of $\mathbb{C}[\partial_z]$ on operators on a disc.

The operators $\partial_z^k \mathsf{c}_{ij}$ are in cohomological degree 1, and so generate a free exterior algebra on $\mathfrak{so}(n) \otimes \mathbb{C}[\partial_z]$.

The commutative algebra generated by the operators $\partial_z^k \Psi_i$ and $\partial_z^k \mathsf{c}_{ij}$ acquires a differential, which comes from the Chevalley-Eilenberg differential for the action of $\Omega^{0,*}_\Sigma \otimes \mathfrak{so}(n)$ on $\Pi\Omega^{0,*}_\Sigma \otimes \mathbb{C}^n$. Explicitly, the differential is characterized by the feature that

$$\begin{aligned} \mathrm{d}\Psi_i &= \mathsf{c}_{ij}\Psi_j \\ \mathrm{d}\mathsf{c}_{ij} &= \mathsf{c}_{ij}\mathsf{c}_{jk}. \end{aligned}$$

and by the requirement d commutes with ∂_z. After taking $\overline{\partial}$-cohomology, the space of classical operators we are considering is $C^*(\mathfrak{so}_n[[z]], \wedge^*(\mathbb{C}^n[\partial_z]))$. ♢

12.3.1 Simple examples of local L_∞ algebras acting on a classical field theory

There are many examples one can construct of local L_∞ algebras acting on classical field theories.

Example: Here is an example that demonstrates how our notion of action goes beyond what is usually considered a symmetry of a field theory in our physics. In derived mathematics, symmetries and deformations can be thought of on a unified footing: a deformation is an action of a graded Lie algebra concentrated in degree 1. This suggests that, whenever we have a deformation of a Lagrangian, we should find an action of a sheaf of Lie algebras with a component in degree 1.

In fact, we saw this in the previous example, where the deformation of the Lagrangian of a free fermion by a term $\int \psi A \psi$, where $A \in \Omega^{0,1}(\Sigma, \mathfrak{so}(n))$, was interpreted as part of an action of the dg Lie algebra $\Omega^{0,*}(\Sigma, \mathfrak{so}(n))$. The phenomenon is very general, however.

Consider the free scalar field theory on an oriented Riemannian d-dimensional manifold (X, g_0). In the BV formalism, the fields are $\phi \in C^\infty(X)$ and $\psi \in \Omega^d_X[-1]$. The odd symplectic pairing is $\int \phi\psi$, and the action functional is $\frac{1}{2}\int \mathrm{d}\phi \wedge *\mathrm{d}\phi$.

We can deform this theory by adding on the term

$$S^{source} = \int f\phi$$

where $f \in \Omega^d(X)$ is a background field. This is a source term.

We claim that this deformation can be viewed as an action of the Abelian Lie algebra $\Omega^d(X)[-1]$. In fact, this is entirely tautological: to define an action, the functional S^{source} would need to satisfy the classical master equation

$$\{S^{source} + S, S^{source} + S\} = 0$$

(where S is the original action functional). This is true by virtue of the fact that neither S nor S^{source} involve the anti-field ψ.

We can similarly consider the case of ϕ^4 theory, and define anaction of $\mathcal{L} = \Omega^d_X[-1]$ by setting

$$S^{\mathcal{L}}(f, \phi) = \frac{1}{2} \int \phi \triangle_{g_0} \phi + \int f \phi \, \mathrm{d}Vol_{g_0} + \frac{1}{4!} \int \phi^4 \mathrm{d}Vol_{g_0},$$

Again, this action functional is the usual way to encode a source.

We thus see that our generalized concept of symmetry (and our generalized Noether's theorem) is very broad, and encodes such familiar concepts as a source. $\Diamond$

Here is another simple example, which shows the importance of L_∞ actions.

Example: Consider the theory of n free scalar fields in dimension 2. The corresponding dg Lie algebra is the Abelian dg Lie algebra

$$\mathcal{M} = \Omega^{0,0}_\Sigma \otimes \mathbb{C}^n[-1] \xrightarrow{\partial\bar{\partial}} \Omega^{1,1}_\Sigma \otimes \mathbb{C}^n[-2].$$

We will let $\Phi \in \mathcal{M}[1]$ denote a field of the theory, which has two components ϕ in degree 0 and ψ in degree 1. The odd symplectic pairing is $\int \phi\psi$, and the Lagrangian is $\frac{1}{2} \int \phi \partial\bar{\partial}\phi$.

This theory has an action of $\mathfrak{so}(n)$, which must extend to an action of $\Omega^*_\Sigma \otimes \mathfrak{so}(n)$. Further, as the fields of the theory are concentrated in degrees 0 and 1, this action must come from an action of $\mathcal{L} = \Omega^{\leq 1}_\Sigma \otimes \mathfrak{so}(n)$. To specify such an action, we must specify a functional $S^{\mathcal{L}}$ which depends on an element of $\mathcal{L}[1]$ and a field in $\mathcal{M}[1]$ and which satisfies the relevant master equation.

The functional is the following. If $A \in \Omega^1_\Sigma \otimes \mathfrak{so}(n)$, and $\mathsf{c} \in \Omega^0_\Sigma \otimes$

$\mathfrak{so}(n)[1]$, we can give a functional $S^{\mathcal{L}}$ by the formula

$$S^{\mathcal{L}}(A, \mathsf{c}, \phi, \psi) = \int \phi_i \mathsf{c}_{ij} \psi_j + \tfrac{1}{2} \int \overline{\partial} \phi_i A^{1,0}_{ij} \phi_j + \tfrac{1}{2} \int A^{0,1}_{ij} \phi_j \partial \phi_i + \tfrac{1}{2} \int A^{0,1}_{ij} \phi_j A^{1,0}_{ik} \phi_k.$$

The first term in this expression – the one involving c – tells us how gauge symmetries act. Because c_{ij} is coupled to $\phi_i \psi_j$, which is the Hamiltonian function on the odd symplectic manifold of fields for the usual action of a generator of $\mathfrak{so}(n)$, we see that gauge transformations act in the usual way.

The remaining terms tell us how the fields couple to a background connection. They are obtained by simply taking the original action, which is $\frac{1}{2} \int \overline{\partial} \phi \partial \phi$, and replacing each derivative by a covariant derivative. The resulting action is gauge invariant, where we include gauge transformations of the fields ϕ and the connection A. This implies that $S^{\mathcal{L}}$ satisfies the Maurer-Cartan equation defining an action.

We note that the functional $S^{\mathcal{L}}$ contains a term quadratic in A. This implies that there is a non-trivial L_∞ action of $\mathcal{L}$ on the fields, which contains an l_3 term. Explicitly, the map

$$l_3 : \wedge^2 \mathcal{L} \otimes \mathcal{M} \to \mathcal{M}$$

is given by the map

$$\begin{aligned} \wedge^2 \Omega^1(\Sigma, \mathfrak{so}(n)) \otimes \Omega^0(\Sigma, \mathbb{C}^n) &\to \Omega^2(\Sigma, \mathbb{C}^n) \\ A_1 \otimes A_2 \otimes \phi_0 &\mapsto (A_1^{1,0} \wedge A_2^{0,1} - A_2^{1,0} \wedge A_1^{0,1}) \phi. \end{aligned}$$

It is clear that the term in $S^{\mathcal{L}}$ which is quadratic in A is necessary for gauge invariance: the functional

$$\int A^{0,1}_{ij} \phi_j \partial \phi_i + \int \overline{\partial} \phi_i A^{1,0}_{ij} \phi_j$$

is not invariant under simultaneuous gauge transformations of the connection A and of the scalar field ϕ_i. This implies that it is necessary to include the l_3 term to find an action of $\Omega^{\leq 1}_{\Sigma} \otimes \mathfrak{so}(n)$ on the field theory defined by $\mathcal{M}$. $\Diamond$

A large class of examples arises through the following result, which is the analogue of the fact that an action on a manifold Y lifts canonically to a symplectic action on its cotangent bundle T^*Y.

12.3.1.1 Lemma. *If $\mathcal{L}$ acts on an elliptic L_∞ algebra $\mathcal{M}$, then $\mathcal{L}$ acts on the cotangent theory for $\mathcal{M}$.*

Proof This claim is immediate by naturality, but we also write down explicitly the semi-direct product L_∞ algebra describing the action. Note that $\mathcal{L} \ltimes \mathcal{M}$ acts linearly on the shifted coadjoint module

$$(\mathcal{L} \ltimes \mathcal{M})^! [-3] = \mathcal{L}^! [-3] \oplus \mathcal{M}^! [-3].$$

Further, $\mathcal{L}^! [-3]$ is a submodule for this action, so that we can form the quotient, which is naturally equivalent to $\mathcal{M}^! [-3]$. Hence we consider the associated extension

$$(\mathcal{L} \ltimes \mathcal{M}) \ltimes \mathcal{M}^! [-3],$$

which is the desired semi-direct product. □

Remark: Note that this construction is simply the -1-shifted relative cotangent bundle to the map $B(\mathcal{L} \ltimes \mathcal{M}) \to B\mathcal{L}$. ◊

12.4 The classical Noether's theorem

As we showed in lemma 3.5.3.2, there is a bijection between classes in $H^1(C^*_{red,loc}(\mathcal{L}))$ and local central extensions of $\mathcal{L}_c$ shifted by -1.

12.4.0.1 Theorem. *Let $\mathcal{M}$ encode a classical field theory, and let $\widetilde{\mathrm{Obs}}^{cl}$ be the classical observables of the field theory $\mathcal{M}$, equipped with its P_0 structure. Let a local L_∞ algebra $\mathcal{L}$ act on $\mathcal{M}$, and let $\widehat{\mathcal{L}}_c$ be the central extension corresponding to the obstruction class $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ for lifting $\mathcal{L}$ to an inner action.*

There is then a map of precosheaves of L_∞-algebras

$$\widehat{\mathcal{L}}_c \rightsquigarrow \widetilde{\mathrm{Obs}}^{cl}[-1]$$

that sends the central element c to the unit $1 \in \widetilde{\mathrm{Obs}}^{cl}[-1]$. (Note that, after the shift, the unit 1 is in cohomological degree 1, as is the central element c.)

Remark: The linear term in the L_∞-morphism is a map of precosheaves of cochain complexes from $\widehat{\mathcal{L}}_c$ to $\mathrm{Obs}^{cl}[-1]$. The fact that we have such a map of precosheaves implies that we have a map of commutative dg factorization algebras

$$\widehat{\mathrm{Sym}}^*(\widehat{\mathcal{L}}_c[1]) \to \widetilde{\mathrm{Obs}}^{cl},$$

which, as above, sends the central element to 1. This formulation is the one that will quantize: we will find a map from a certain Chevalley-Eilenberg chain complex of $\widehat{\mathcal{L}}_c[1]$ to quantum observables. ◊

Remark: Lemma 12.2.3.3 implies that the central extension $\widehat{\mathcal{L}}_c$ is split canonically as a presheaf of cochain complexes:

$$\widehat{\mathcal{L}}_c(U) = \mathbb{C}[-1] \oplus \mathcal{L}_c(U).$$

Thus, we have a map of precosheaves of cochain complexes $\mathcal{L}_c \to \mathrm{Obs}^{cl}$. The same argument will show that that this cochain map extends to a continuous map from the distributional completion $\overline{\mathcal{L}}_c(U)$ to Obs^{cl}. ◊

Before giving the proof, we rehearse the argument in the finite-dimensional setting, in the case when the central extension splits. Let $\mathfrak{g}, \mathfrak{h}$ be L_∞ algebras, and suppose that $\mathfrak{h}$ is equipped with an invariant pairing of degree -3, so that $C^*(\mathfrak{h})$ is a P_0 algebra. Recall that an L_∞ map $\mathfrak{g} \rightsquigarrow C^*(\mathfrak{h})[-1]$ is encoded by an element

$$G \in C^*_{red}(\mathfrak{g}) \otimes C^*(\mathfrak{h})$$

of cohomological degree 0 satisfying the Maurer-Cartan equation

$$\mathrm{d}_{\mathfrak{g}} G + \mathrm{d}_{\mathfrak{h}} G + \tfrac{1}{2}\{G, G\} = 0,$$

where $\mathrm{d}_{\mathfrak{g}}, \mathrm{d}_{\mathfrak{h}}$ are the Chevalley-Eilenberg differentials for $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and $\{-,-\}$ denotes the Poisson bracket coming from the P_0 structure on $C^*(\mathfrak{h})$.

Let us now consider the case of a central extension. Suppose that we have an element

$$G \in C^*_{red}(\mathfrak{g}) \otimes C^*(\mathfrak{h})$$

of degree 0 and an obstruction element $\alpha \in C^*_{red}(\mathfrak{g})$ of degree 1 such that

$$\mathrm{d}_{\mathfrak{g}} G + \mathrm{d}_{\mathfrak{h}} G + \tfrac{1}{2}\{G, G\} = \alpha \otimes 1.$$

Let $\widehat{\mathfrak{g}}$ be the -1-shifted central extension determined by α, so that there is a short exact sequence

$$0 \to \mathbb{C}[-1] \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

The data of G and α is the same as a map of L_∞ algebras $\widehat{\mathfrak{g}} \to C^*(\mathfrak{h})[-1]$ that sends the central element of $\widehat{\mathfrak{g}}$ to $1 \in C^*(\mathfrak{h})$.

To see this, choose a splitting $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C} \cdot c$ where the central element c is of degree 1. Let $c^\vee$ be the linear functional on $\widehat{\mathfrak{g}}$ that is zero on $\mathfrak{g}$ and that sends c to 1. The image of α under the natural map $C^*(\mathfrak{g}) \to C^*(\widehat{\mathfrak{g}})$ is made exact by $c^\vee$, viewed as a zero-cochain in $C^*(\widehat{\mathfrak{g}})$. It follows that

$$G + c^\vee \otimes 1 \in C^*_{red}(\widehat{\mathfrak{g}}) \otimes C^*(\mathfrak{h})$$

satisfies the Maurer-Cartan equation, and therefore defines (as above) an L_∞-map $\widehat{\mathfrak{g}} \to C^*(\mathfrak{h})[-1]$. This L_∞-map sends c to 1 because G only depends on c by the term $c^\vee \otimes 1$.

Now let us turn to the proof of theorem 12.4.0.1.

Proof Let us apply the remarks we have made about the finite-dimensional case to the setting of factorization algebras.

Suppose we have an action of a local L_∞-algebra $\mathcal{L}$ on a classical field theory $\mathcal{M}$. Let

$$\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1]) = C^*_{red,loc}(\mathcal{L})$$

be a 1-cocycle representing the obstruction to lifting to an inner action on $\mathcal{M}$. Let

$$\widehat{\mathcal{L}}_c = \mathcal{L}_c \oplus \underline{\mathbb{C}}[-1]$$

be the corresponding central extension. By the definition of α, we have a functional

$$S^{\mathcal{L}} \in \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) = C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{M})$$

satisfying the Maurer-Cartan equation

$$\mathrm{d}S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = \alpha.$$

We now want to understand the precosheaf aspects of this data.

Let us first remind the reader of some properties of local functionals. For any vector bundle E on our manifold M with space of sections $\mathscr{E}(U)$, a local functional in $\mathscr{O}_{loc}(\mathscr{E})$ does *not* define a function on the space $\mathscr{E}(M)$ of global sections (unless M is compact). A local functional does, however, determine a function on the space $\mathscr{E}_c(M)$ of compactly supported global sections. More generally, if $e \in \mathscr{E}_c(M)$ and $F \in \mathscr{O}_{loc}(\mathscr{E})$, we can differentiate F with respect to e. *A priori* $\partial_e F$ is a functional on $\mathscr{E}_c(M)$, but it turns out that it extends to a functional on $\mathscr{E}(M)$.

More generally, if $e \in \mathscr{E}_c(U)$, then $\partial_e F$ gives rise to an element of $\mathscr{O}(\mathscr{E}(U))$.

One can restate this by saying that if we have a local functional which is homogeneous of degree n, and we place elements of $\mathscr{E}_c(U)$ on at least one of its inputs, then we can place elements of $\mathscr{E}(U)$ on the remaining inputs.

Applying these observations to the situation at hand, we find that for every open subset $U \subset M$, we have an injective cochain map

$$\Phi : C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{M}) \to C^*_{red}(\mathcal{L}_c(U)) \widehat{\otimes} \widetilde{C}^*(\mathcal{M}(U)),$$

where $\widehat{\otimes}$ refers to the completed tensor product and $\widetilde{C}^*(\mathcal{M}(U))$ refers to the subcomplex of $C^*(\mathcal{M}(U))$ consisting of functionals with smooth first derivative.

The cochain map Φ is in fact a map of dg Lie algebras, where the Lie bracket arises as usual from the pairing on $\mathcal{M}$. Thus, for every U, we have an element

$$S^{\mathcal{L}}(U) \in C^*_{red}(\mathcal{L}_c(U)) \widehat{\otimes} \widetilde{C}^*(\mathcal{M}(U))$$

satisfying the Maurer-Cartan equation

$$\mathrm{d}S^{\mathcal{L}}(U) + \tfrac{1}{2}\{S^{\mathcal{L}}(U), S^{\mathcal{L}}(U)\} = \alpha(U).$$

It follows, as in the finite-dimensional case discussed above, that $S^{\mathcal{L}}(U)$ gives rise to a map of L_∞ algebras

$$\widehat{\mathcal{L}}_c(U) \rightsquigarrow \widetilde{C}^*(\mathcal{M}(U))[-1] = \widetilde{\mathrm{Obs}}^{cl}(U)[-1]$$

sending the central element c in $\widehat{\mathcal{L}}_c(U)$ to the unit $1 \in \widetilde{\mathrm{Obs}}^{cl}(U)$. The fact that $S^{\mathcal{L}}$ is local implies immediately that it is a map of precosheaves.

□

12.4.1 Noether's theorem as a map of P_0 factorization algebras

A classical field theory is encoded by its P_0 factorization algebra Obs^{cl}. It turns out that the classical Noether's theorem can be reformulated as saying that an action of $\mathcal{L}$ on a classical theory leads to a homomorphism of P_0 factorization algebras from one built from $\mathcal{L}$ to classical observables. This version of the statement has the advantage that it leads to a clean statement at the quantum level.

We have already seen the non-factorization version of this statement in section 11.2.3. There, we saw that we must replace the (finite-dimensional) L_∞ algebra $\mathfrak{g}$ of symmetries by its enveloping P_0 algebra $U\mathfrak{g}$, which is the universal P_0 algebra containing a sub-Lie algebra $\mathfrak{g}$.

We will implement the same construction in the world of factorization algebras, under the simplifying assumption that our local Lie algebra $\mathcal{L}$ is simply a dg Lie algebra, and not an L_∞ algebra; and that the map of pre-cosheaves of homotopy Lie algebras $\widehat{\mathcal{L}}_c \to \mathrm{Obs}^{cl}$ of theorem 12.4.0.1 is a strict map of dg Lie algebras, and not an L_∞ map.

12.4.1.1 Definition. *Let $\mathcal{L}$ be a local Lie algebra on a manifold M, let $\alpha \in H^1(C^*_{loc,red}(\mathcal{L}))$ be a local cohomology class, and let $\widehat{\mathcal{L}}_c$ be the corresponding central extension with central element in degree 1.*

Define a P_0 factorization algebra $\mathbb{U}^{P_0}_\alpha(\mathcal{L})$ as follows. We first define $\mathbb{U}^{P_0}(\widehat{\mathcal{L}},$ by saying that for every open subset $U \subset M$, we have an isomorphism of dg commutative algebras

$$\mathbb{U}^{P_0}(\mathcal{L})(U) = \mathrm{Sym}^*(\widehat{\mathcal{L}}_c(U)[1]).$$

The right hand side of this equation has a P_0 structure coming from the Lie bracket on $\widehat{\mathcal{L}}_c(U)$. This is a P_0 factorization algebra in the category of modules for $\mathbb{C}[c]$, which acts on each $\mathbb{U}^{P_0}_\alpha(\mathcal{L})(U)$ by multiplication with the central element $c \in \widehat{\mathcal{L}}_c(U)[1]$.

We then define

$$\mathbb{U}^{P_0}_\alpha(\mathcal{L})(U) = \mathbb{U}^{P_0}(\mathcal{L})(U) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}$$

to be obtained by specializing the central element c to 1.

12.4.1.2 Theorem. *Suppose that a local Lie algebra $\mathcal{L}$ acts on a classical* *field theory with P_0 factorizaton algebra of observables* Obs^{cl}*, and that* $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ *is the obstruction to making this an inner action.*

Then, there is a homormorphism of P_0 factorization algebras

$$\mathbb{U}^{P_0}_\alpha(\mathcal{L}) \to \mathrm{Obs}^{cl}.$$

Proof This claim follows almost immediately from theorem 12.4.0.1. This theorem shows that we have a map of precosheaves of dg Lie algebras from $\widehat{\mathcal{L}}_c \to \mathrm{Obs}^{cl}[-1]$ which sends the central element c to 1. Shifting by 1, we get a map of shifted Lie algebra $\widehat{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}$. Because Obs^{cl} is a commutative algebra, this extends canonically to a map of prefactorization algebras

$$\mathbb{U}^{P_0}(\widehat{\mathcal{L}}) = \mathrm{Sym}^*(\widehat{\mathcal{L}}_c[1]) \to \mathrm{Obs}^{cl}.$$

Since the central element in $\widehat{\mathcal{L}}_c[1]$ gets sent to $1 \in \mathrm{Obs}^{cl}$, this descends to a map of P_0 factorization algebras $\mathbb{U}^{P_0}_\alpha(\mathcal{L}) \to \mathrm{Obs}^{cl}$. □

12.5 Conserved currents

The standard formulations of Noether's theorem associate to each symmetry a *conserved current*, which is a $d-1$-form valued in Lagrangians if the field theory lives on an oriented manifold X of dimension d. (Compare with a local functional, which is a top form valued in Lagrangians.) Such a current determines an observable by picking a closed submanifold of codimension 1: for each field, evaluate the current on the field and integrate over submanifold to compute the associated *charge* of the field. In practice, people often work on spacetime manifolds that are products $N \times \mathbb{R}$ and compute the charge along $N \times \{t\}$ at some fixed time t. The charge is conserved because it is invariant under time translation.

This section is devoted to explaining why the version of classical Noether's theorem that we just presented leads to this more traditional statement. Similar remarks will hold for the quantum version of Noether's theorem.

In our formalism, we generalize the usual notion of conserved current with the following definition, which will be valid at the quantum level as well.

12.5.0.1 Definition. *A* conserved current *in a field theory is a map of precosheave*

$$J : \overline{\Omega}_c^*[1] \to \mathrm{Obs}^{cl}$$

to the factorization algebra of classical observables.

Such a J can be viewed as a form valued in classical observables. For each open subset U, J determines a closed, degree 0 element

$$J(U) \in \Omega^*(U)[d-1] \widehat{\otimes} \mathrm{Obs}^{cl}(U),$$

and this element is compatible with inclusions of open subsets in the obvious way. In particular, consider the component

$$J(U)^{d-1,0} \in \Omega^{d-1}(U) \widehat{\otimes} \mathrm{Obs}^{cl}(U)^0,$$

where the superscript in $\mathrm{Obs}^{cl}(U)^0$ indicates cohomological degree 0. This term $J(U)^{d-1,0}$ is an $d-1$-form valued in observables, which is precisely what is traditionally called a current.

Let us now explain why our definition means that this current is conserved (up to homotopy). A little notation is needed to explain this point. If $N \subset X$ is a closed subset, let

$$\mathrm{Obs}^{cl}(N) = \lim_{N \subset U} \mathrm{Obs}^{cl}(U)$$

be the limit of observables on open neighbourhoods U of N. Thus, an element of $\mathrm{Obs}^{cl}(N)$ is an observable defined on every open neighbourhood of N, in a way compatible with inclusions of open sets. When $p \in X$ is a point, the complex $\mathrm{Obs}^{cl}(p)$ is the algebra of point operators we defined in Chapter 10 in the discussion of the OPE.

Remark: For this discussion, we could use either the homotopy limit or the ordinary limit. We choose to use the ordinary limit because because we chose to use the ordinary limit in the discussion of point observables (functional analytic complications make the homotopy limit somewhat difficult to work with). $\Diamond$

Now let J be a conserved current, in the sense of the definition above. For every compact codimension 1 oriented submanifold $N \subset X$, the delta-distribution on N is an element

$$[N] \in \overline{\Omega}^1(U)$$

defined for every open neighbourhood U of N. Applying J, we get an element

$$J[N] \in \mathrm{Obs}^{cl}(U)$$

for every neighbourhood U of N. This element is compatible with inclusions $U \hookrightarrow U'$, and so defines an element of $\mathrm{Obs}^{cl}(N)$. This quantity is to be identified with what is usually called the *charge* of a conserved current, which is the integral of the current over a codimension 1 submanifold.

The fact that a current is conserved should say that the corresponding charge only depends on the homology class of the codimension 1 submanifold on which we integrate it. Let us see why this is true in our approach.

Consider what happens when we have a cobordism between two such submanifolds: let $M \subset X$ be a codimension 0 submanifold with boundary $\partial M = N \amalg N'$. Then the fact that J is cochain map tells us that

$$\mathrm{d}J[M] = J[N] - J[N'] \in \mathrm{Obs}^{cl}(M),$$

and so the cohomology class $[J[N]]$ of $J[N]$ does not change if N is changed by a cobordism. In particular, if our space-time manifold is a product

$$X = N \times \mathbb{R},$$

then the cohomology class $[J[N_t]]$ associated to the submanifold $N \times \{t\}$ is independent of "time" t. This condition is precisely the traditional formulation of a conserved current.

With these ideas in place, we can now explain why our version of Noether's theorem produces a conserved current from a symmetry.

12.5.0.2 Lemma. *Suppose we have a classical field theory on a manifold X that has an infinitesimal symmetry. To this data, our formulation of Noether's theorem produces a conserved current.*

Proof A theory with an infinitesimal symmetry is acted on by the abelian

Lie algebra $\mathbb{R}$ (or $\mathbb{C}$). Lemma 12.2.4.2 shows us that such an action is equivalent to the action of the Abelian local dg Lie algebra Ω^*_X. Lemma 12.2.3.3 implies that the central extension $\widehat{\mathcal{L}}_c$ is split as a cochain complex. We thus get a map

$$\Omega^*_{X,c}[1] \to \mathrm{Obs}^{cl}.$$

The remark following theorem 12.4.0.1 tells us that this map extends to a continuous cochain map

$$\overline{\Omega}^*_{X,c}[1] \to \mathrm{Obs}^{cl},$$

which is our definition of a conserved current. □

12.6 Examples of classical Noether's theorem

We now outline some examples of this construction. Chapter 14 contains a more extensive treatment of some other examples, with a discussion of their quantizations as well.

12.6.1 Examples where the obstruction is trivial

For all of the examples described here, the central extension of the local L_∞ algebra of symmetries is trivial.

Example: Let us revisit the example of a source, which is the first example considered in section 12.3.1. There, we saw that the free scalar field theory with Lagrangian $\int_X \phi \, \mathrm{D} \, \phi$ has an action of the Abelian local Lie algebra $\Omega^d_X[-1]$. The equivariant Lagrangian describing the action contains an interaction term $\int f\phi$, where f is a section of Ω^d_X and plays the role of a background field.

The Noether map is then the map

$$\begin{aligned} \Omega^d_c(U) &\to \mathrm{Obs}^{cl}(U) \\ f &\mapsto \int f\phi. \end{aligned}$$

This map then describes the observables obtained by integrating the field ϕ against an arbitrary function.

We remark that the most fundamental point observable (or local operator) appears by extending this map to the distributional forms $\overline{\Omega}^d(X)$: for every point $p \in X$, the delta-function δ_p maps to a point observable in $\mathrm{Obs}^{cl}(p)$. This observable varies smoothly with p.

Consider the case $X = \mathbb{R}^n$. Then any point observable supported at the origin — built from a polynomial of the value of ϕ and its derivatives at the origin — defines an action of $\overline{\Omega}^d[-1]$. At the quantum level, it is sometimes useful to define an operator supported at a point to be an action of $\overline{\Omega}^d[-1]$. $\Diamond$

Example: Let us revisit another running example of a system of n chiral free fermions on a Riemann surface Σ, acted on by the local Lie algebra $\Omega^{0,*}_\Sigma \otimes \mathfrak{so}_n$. The fields of the fermionic theory are $\Psi_i \in \Omega^{1/2,*}(\Sigma)$, and the map produced by theorem 12.4.0.1 is the map

$$\begin{array}{ccc} \Omega^{0,*}_c(U) \otimes \mathfrak{so}(n)[1] & \mapsto & \mathrm{Obs}^{cl}(U) \\ A_{ij} & \mapsto & \int \Psi_i A_{ij} \Psi_j. \end{array}$$

As in the remark following theorem 12.4.0.1, this map extends to a map from the distributional completion $\overline{\Omega}^{0,*}_c(U) \otimes \mathfrak{so}(n)[1]$.

Take U to be a disc D, and let us consider elements of $\overline{\Omega}^{0,*}_c(D)$ which are finite sums of eigenvectors under the S^1 action rotating D. A basis for the cohomology of this spaces is provided by the delta-function δ_p (where $p \in D$ is the center of the disc) and its z-derivatives. The Noether map is

$$\partial_z^n \delta_p M_{ij} \mapsto \partial_z^n (\Psi_i(p) M_{ij} \Psi_j(p)) \in \mathrm{Obs}^{cl}(D).$$

These are precisely the Kac-Moody currents that, at the quantum level, describe the action of the Kac-Moody vertex algebra on the vertex algebra of n free fermions. $\Diamond$

12.6.2 Geometric examples of classical field theories with an action of a local L_∞ algebra

Many field theories are defined on a class of manifolds, not just on a fixed manifold. For example, the free scalar field theory makes sense on any Riemannian manifold; Yang-Mills theory makes sense (classically) on any manifold equipped withe a conformal class of metric; and chiral conformal field theories, such as the $\beta\gamma$ system or the free chiral fermion, make sense on any Riemann surface. This kind of situation can be formalized by saying that a theory is acted on by a local L_∞ algebra associated to that type of manifold. The observables built from this action by the Noether map define the stress-energy tensor for the appropriate class of geometry.

Below we will examine explicitly theories associated with Riemannian, conformal, or complex geometry. But the method generalizes to any geometric structure on a manifold that can be described by a combination of differential equations and symmetries.

Riemannian geometry

We start by introducing the local dg Lie algebra $\mathcal{L}^{Riem}$ on a Riemannian manifold X that controls deformations of X as a Riemannian manifold. This local dg Lie algebra acts on field theories that are defined on Riemannian manifolds; we will exhibit this action explicitly in the case of scalar field theories. The current associated to the action of this Lie algebra is the stress-energy tensor.

Let (X, g_0) be a Riemannian manifold of dimension d. Consider the local dg Lie algebra $\mathcal{L}^{Riem}(X)$ consisting of the sheaf Vect of smooth vector fields in degree 0 and the sheaf $\Gamma(X, \mathrm{Sym}^2 TX)$ in degree 1. The differential d sends a vector field V to $\mathrm{d}V = \mathscr{L}_V g_0$, the Lie derivative of the metric g_0 along V. The bracket is given by the Lie bracket of vector fields in degree 0 and by the Lie derivative acting on symmetric 2-tensors for the action of a degree 0 element on a degree 1 element.

The degree 0 component thus encodes infinitesimal diffeomorphisms of X, and the zeroth cohomology is the vector space of Killing fields for g_0. The degree 1 component encodes infinitesimal deformations of the metric g_0, and the first cohomology identifies first-order deformations modulo infinitesimal diffeomorphism. Hence, $\mathcal{L}^{Riem}(X)$ is the dg Lie algebra describing the formal neighbourhood of (X, g_0) in the moduli space of Riemannian manifolds.

Example: For simplicity, let us now suppose X is oriented. Consider the free scalar field theory on X, defined by the Abelian elliptic dg Lie algebra

$$\mathcal{M}(X) = C^\infty(X)[-1] \xrightarrow{\triangle_{g_0}} \Omega^d(X)[-2]$$

where the superscript indicates cohomological degree, and

$$\triangle_{g_0} = \mathrm{d} * \mathrm{d}$$

is the Laplacian for the metric g_0, modified to land in top forms. This dg Lie algebra describes the formal moduli space of harmonic functions on (X, g_0). The pairing on $\mathcal{M}$ is the integration pairing.

To describe the action of $\mathcal{L}^{Riem}(X)$ on $\mathcal{M}(X)$, we specify an action functional $S^{\mathcal{L}}$ that couples the fields in $\mathcal{L}^{Riem}(X)$ to those in $\mathcal{M}(X)$. Let $\phi, \psi \in \mathcal{M}(X)[1]$ denote fields of cohomological degree 0 and 1, respectively, and let $V \in \mathrm{Vect}(X)$ and $\alpha \in \Gamma(X, \mathrm{Sym}^2 TX)$ denote elements of $\mathcal{L}(X)$. We define

$$S^{\mathcal{L}}(\phi, \psi, V, \alpha) = \int \phi(\triangle_{g_0+\alpha} - \triangle_{g_0})\phi + \int (V\phi)\psi.$$

On the right hand side we interpret $\triangle_{g_0+\alpha}$ as a formal power series in the field α; in other words, we expand the dependence of that Laplacian on α order by order, as in usual perturbation theory. The fact that $S^{\mathcal{L}}$ satisfies the master equation follows from the fact that the Laplacian $\triangle_{g_0+\alpha}$ is covariant under infinitesimal diffeomorphisms:

$$\triangle_{g_0+\alpha} + \epsilon[V, \triangle_{g_0+\alpha}] = \triangle_{g_0+\alpha+\epsilon\mathcal{L}_V g_0+\epsilon\mathscr{L}_V\alpha}.$$

One can rewrite this assertion in the language of L_∞ algebras by Taylor expanding $\triangle_{g_0+\alpha}$ in powers of α. The resulting L_∞-algebra $\mathcal{L}^{Riem}(X) \ltimes \mathcal{M}(X)$ describes the formal moduli space of Riemannian manifolds together with a harmonic function ϕ.

Note that the classical master equation holds on the nose, and not just modulo functionals that depend only on the fields in $\mathcal{L}$. This means that we have an inner action of $\mathcal{L}^{Riem}(X)$ on the theory, and not just an action. The resulting L_∞ map from $\mathcal{L}_c^{Riem}(X)$ to classical observables encodes the *stress-energy tensor*: the dependence of the theory on the background metric. $\Diamond$

Example: Let us modify the previous example by considering a scalar field theory with a polynomial interaction, so that the action functional has the form

$$\int \phi \triangle_{g_0} \phi + \sum_{n\geq 2} \frac{\lambda_n}{n!} \phi^n \, \mathrm{d}Vol_{g_0}.$$

Hence we are deforming the Abelian dg Lie algebra from the preceding example to obtain a non-Abelian L_∞ algebra $\mathcal{M}_\lambda$ with brackets l_n defined by

$$l_n : C^\infty(X)^{\otimes n} \to \Omega^d(X)$$

where

$$l_n(\phi_1, \ldots, \phi_n) = \lambda_n \phi_1 \cdots \phi_n \mathrm{d}Vol_{g_0}.$$

The action of $\mathcal{L}^{Riem}$ on $\mathcal{M}$ is defined, as in the preceding example, by specifying an action functional $S^{\mathcal{L}}$ that couples the two types of fields:

$$S^{\mathcal{L}}(\phi, \psi, V, \alpha) + S(\phi, \psi) = \int \phi \triangle_{g_0+\alpha} \phi + \sum_{n\geq 2} \lambda_n \tfrac{1}{n!} \phi^n \mathrm{d}Vol_{g_0+\alpha} + \int (V\phi)\psi.$$

The associated L_∞-algebra $\mathcal{L}^{Riem}(X) \ltimes \mathcal{M}_\lambda(X)$ describes the formal moduli space of Riemannian manifolds together with a function ϕ satisfying a nonlinear PDE depending on the interaction. $\Diamond$

Conformal geometry

We next discuss classical conformal field theories.

As above, let (X, g_0) be a Riemannian manifold. Define a local dg Lie algebra $\mathcal{L}^{conf}$ on X by setting the degree 0 component to be $\mathrm{Vect}(X) \oplus C^\infty(X)$ and the degree 1 component to be $\Gamma(X, \mathrm{Sym}^2 TX)$. In other words, we have added to $\mathcal{L}^{Riem}$ a copy of $C^\infty(X)$ that encodes Weyl rescalings of the metric. The differential on $\mathcal{L}^{conf}(X)$ is

$$\mathrm{d}(V, f) = \mathscr{L}_V g_0 + f g_0$$

where $V \in \mathrm{Vect}(X)$ and $f \in C^\infty(X)$. The Lie bracket is defined by saying that $\mathrm{Vect}(X)$ acts on everything by Lie derivative, and that

$$[f, \alpha] = f\alpha$$

for $f \in C^\infty(X)$ and $\alpha \in \Gamma(X, \mathrm{Sym}^2 TX)$. The Weyl rescalings form an Abelian sub-Lie algebra.

We observe that $H^0(\mathcal{L}^{conf}(X))$ is the Lie algebra of conformal symmetries of X, since $\mathrm{d}(V, f) = 0$ implies that the infinitesimal deformation $\mathscr{L}_V g_0$ is just given by a Weyl rescaling $-f g_0$. Likewise, we note that $H^1(\mathcal{L}^{conf}(X))$ is the space of first-order conformal deformations of X. Thus $\mathcal{L}^{conf}$ describes the formal neighborhood in the moduli space of conformal metrics of the conformal class of (X, g_0). The inclusion $\mathcal{L}^{Riem} \to \mathcal{L}^{conf}$ of dg Lie algebras encodes the formal version of the quotient map from the moduli of Riemannian manifolds to the moduli of conformal manifolds. The local dg Lie algebra $\mathcal{L}^{conf}$ will act on any classical conformal field theory; indeed, this assertion is almost tautological, inasmuch as a conformal field theory means a family of field theories over the moduli of conformal manifolds.

Example: Consider the case of the free scalar field theory in dimension 2. Let $\mathcal{M}$ be the elliptic Abelian dg Lie algebra

$$C^\infty(X) \xrightarrow{\triangle_{g_0}} \Omega^2(X),$$

as described in a previous example (as before the complex is in degrees $1, 2$ when we view it as an L_∞ algebra).

It is a straightforward and standard computation to verify that in dimension 2, the action functional is invariant under Weyl rescaling. (In more detail, a Weyl rescaling of the metric $g \mapsto g' = e^f g$ just rescales the Laplacian on functions by the factor e^{-f}; this scaling factor on the Laplacian is canceled by the scaling factor in the volume form $\mathrm{d}Vol_g$.) Hence, the action of $\mathcal{L}^{Riem}$ on $\mathcal{M}$ extends to an action of $\mathcal{L}^{conf}$ by having $C^\infty(X)$ act trivially.

This action does not extend to the scalar field with a polynomial interaction, because no polynomial interaction is conformally invariant, as the volume form $\mathrm{d}Vol_{g_0}$ changes with Weyl scaling. However, if we have n scalar fields, so that $\mathcal{M}$ is

$$C^\infty(X) \otimes \mathbb{R}^n \xrightarrow{\triangle_{g_0}} \Omega^2(X) \otimes \mathbb{R}^n$$

we have a non-trivial deformation by choosing a metric on $\mathbb{R}^n$. If we vary the flat metric δ_{ij} on $\mathbb{R}^n$ by $\delta_{ij} + h_{ij}$, then the Lagrangian giving the deformation is

$$\int_{\mathbb{R}^2} h_{ij}(\phi) \mathrm{d}\phi_i *_{g_0} \mathrm{d}\phi_j$$

where the Hodge star operator is defined with respect to g_0. This theory is manifestly conformally invariant, although only at the classical level. ◊

There are many other, but more complicated, examples of this nature. If X is a conformal 4-manifold, then Yang-Mills theory on X is conformally invariant at the classical level. The same goes for self-dual Yang-Mills theory. One can explicitly write an action of $\mathcal{L}^{conf}$ on the elliptic L_∞-algebra on X describing either self-dual or full Yang-Mills theory.

Complex geometry

We now turn to the case of holomorphic field theories. (An extensive treatment can be found in Williams (2020).)

Let X be a complex manifold. Consider the local dg Lie algebra

$$\mathcal{L}^{hol}(X) = \Omega^{0,*}(X, T^{1,0}X),$$

equipped with the Dolbeault differential $\overline{\partial}$ and the Lie bracket of vector fields. The zeroth cohomology of this sheaf is the sheaf of holomorphic vector fields on X, and $\mathcal{L}^{hol}$ is a resolution with a differential-geometric flavor. A holomorphic classical field theory will be a theory with an action of $\mathcal{L}^{hol}(X)$.

Remark: A stronger notion of holomorphicity might require the field theory to be acted on by the group of holomorphic diffeomorphisms of X, such that the derivative of this action extends to an action of the local dg Lie algebra $\mathcal{L}^{hol}$. ◊

Let us now give some examples of field theories acted on by $\mathcal{L}^{hol}$.

Example: Let X be a complex manifold of complex dimension d, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra for a complex Lie group G. Given a holomorphic principal G-bundle $P \to X$, consider the adjoint bundle $\mathrm{ad}\, P$, which is a holomorphic vector bundle. The associated Dolbeault complex $\Omega^{0,*}(X, \mathrm{ad}\, P)$ is the dg Lie algebra that describes the formal neighborhood of P in the moduli of principal G-bundles on X. Form the cotangent theory associated to this formal moduli space, which is a classical field theory encoded by

$$\mathcal{M} = \Omega^{0,*}(X, \mathrm{ad}\, P) \oplus \Omega^{d,*}(X, (\mathrm{ad}\, P)^\vee)[d-3].$$

This theory is a holomorphic version of BF theory. For simplicity, we will focus on the case of the trivial bundle so that we are working with Dolbeault forms valued in $\mathfrak{g}$ and $\mathfrak{g}^\vee$, respectively.

As discussed in Costello (2013a), this example is of interest to physics. For instance, when $d = 2$ it describes a holomorphic twist of $N = 1$ supersymmetric gauge theory. In addition, one can express holomorphic twists of supersymmetric σ-models in these terms via the formalism of L_∞ spaces developed in Costello (2011a, 2013a); Grady and Gwilliam (2015); Gwilliam et al. (2020).

The dg Lie algebra $\mathcal{L}^{hol}(X)$ acts by Lie derivative on the Dolbeault complex $\Omega^{k,*}(X)$ for holomorphic k-forms. An explicit description is given by the Cartan homotopy formula

$$\mathcal{L}_V \omega = [\iota_V, \partial]\omega,$$

where the contraction

$$\begin{array}{ccc} \Omega^{0,*}(X, TX) \times \Omega^{k,*}(X) & \to & \Omega^{k-1,*}(X) \\ (V, \omega) & \mapsto & \iota_V \omega \end{array}$$

is the $\Omega^{0,*}(X)$-linear extension of the contraction with smooth vector fields. In this way, $\mathcal{L}$ acts on $\mathcal{M}$. This action preserves the invariant pairing.

It is straightforward to express this action in terms of an $\mathcal{L}$-dependent action functional. If $\alpha \in \Omega^{0,*}(X, \mathfrak{g})[1]$, $\beta \in \Omega^{d,*}(X, \mathfrak{g}^\vee)[d-2]$ and $V \in \Omega^{0,*}(X, TX)[1]$, we define

$$S^{\mathcal{L}}(\alpha, \beta, V) = \int \left\langle \beta, (\overline{\partial} + \mathcal{L}_V)\alpha \right\rangle + \tfrac{1}{2} \left\langle \beta, [\alpha, \alpha] \right\rangle .$$

Note that the fields α, β, V can be of mixed degree.

Consider the situation where $V \in \Omega^{0,*}(X, TX)$ is a cocycle of cohomological degree 1. Then V determines a deformation of complex structure of X, and the $\overline{\partial}$-operator for this deformed complex structure is $\overline{\partial} + \mathcal{L}_V$. The action functional $S^{\mathcal{L}}$ therefore describes the variation of the original action functional S as we vary the complex structure on X. Other terms in $S^{\mathcal{L}}$ encode the fact that the functional S is invariant under holomorphic symmetries of X. $\Diamond$

We will return to this example throughout our discussion of Noether's theorem. For instance, in dimension $d = 1$ and with $\mathfrak{g}$ Abelian, we will see in Section 14.2 that the quantized construction leads to a version of the Segal-Sugawara construction: a map from the Virasoro vertex algebra to the vertex algebra associated to a free $\beta\gamma$ system.

Now let us consider a larger set of symmetries that appears in field theories defined on a complex manifold X together with a holomorphic principal G-bundle $P \to X$. When X is a Riemann surface, field theories of this form play an important role in the mathematics of chiral conformal field theory.

The relevant local dg Lie algebra $\mathcal{L}$ is

$$\mathcal{L}(X) = \Omega^{0,*}(X, TX) \ltimes \Omega^{0,*}(X, \mathrm{ad}\, P)$$

so that $\mathcal{L}(X)$ is the semi-direct product of the Dolbeault resolution of holomorphic vector fields with the Dolbeault complex with coefficients in the adjoint bundle ad P. Thus, $\mathcal{L}(X)$ is the dg Lie algebra controlling deformations of X as a complex manifold equipped with a holomorphic G-bundle. It is also known as the Atiyah algebroid At_P of P.

Example: Let V be a finite-dimensional representation of G. Consider the Abelian elliptic moduli problem $\Omega^{0,*}(X, P \times^G V)[-1]$ describing holomorphic sections of the associated bundle. Form the cotangent theory, encoded by

$$\mathcal{M}(X) = \Omega^{0,*}(X, V)[-1] \oplus \Omega^{0,*}(X, V^\vee)[d-2].$$

This classical theory is manifestly acted on by the local L_∞ algebra $\mathcal{L}$ we described above, since deformations of the principal bundle P deform the associated bundle as well.

When V is a trivial representation, one can view this theory as a holomorphic σ-model into V. (For nontrivial representations, it is a kind of twisted σ-model.) Hence, as a generalization, we could replace the vector space V by a complex manifold M with a G-action and consider the cotangent theory to the moduli of holomorphic maps to M. $\Diamond$

Example: Consider the $\beta\gamma$ system on $\mathbb{C}$, as developed in Section I.5.4. The dg Lie algebra $\mathcal{M}$ describing this theory is

$$\mathcal{M}(\mathbb{C}) = \Omega^{0,*}(\mathbb{C}, V)[-1] \oplus \Omega^{1,*}(\mathbb{C}, V^*)[-1],$$

where the field γ denotes a section of the first summand and the field β

denotes a section of the second. Consider the dg Lie algebra

$$\mathcal{L} = \Omega^{0,*}(\mathbb{C}, T^{1,0}\mathbb{C}),$$

the Dolbeault resolution of holomorphic vector fields on $\mathbb{C}$. This Lie algebra $\mathcal{L}$ acts on $\mathcal{M}$ by Lie derivative. The action functional encoding this action is

$$S^{\mathcal{L}}(\beta, \gamma, V) = \int (\mathscr{L}_V \beta)\gamma,$$

where $\beta \in \Omega^{0,*}(\mathbb{C}, V)$, $\gamma \in \Omega^{1,*}(\mathbb{C}, V^*)$ and $V \in \Omega^{0,*}(\mathbb{C}, T\mathbb{C})$.

In this case there is no central extension. Therefore, we have a map

$$\Phi : \mathcal{L}_c[1] \to \mathrm{Obs}^{cl}$$

of precosheaves of cochain complexes. At the cochain level, this map is easy to describe: it simply sends a compactly supported vector field $V \in \Omega_c^{0,*}(U, TU)[1]$ to the observable

$$\Phi(V)(\beta, \gamma) = \int_U (\mathscr{L}_V \beta)\, \gamma.$$

We are interested in what this construction does at the level of cohomology.

Let us work on an open annulus $A \subset \mathbb{C}$. We have seen in Section I.5.4 that the cohomology of $\mathrm{Obs}^{cl}(A)$ can be expressed in terms of the dual of the space of holomorphic functions on A:

$$H^0(\mathrm{Obs}^{cl}(A)) = \widehat{\mathrm{Sym}}^* \left(\mathrm{Hol}(A)^\vee \otimes V^\vee \oplus \Omega^1_{hol}(A)^\vee \otimes V\right),$$

where $\mathrm{Hol}(A)$ denotes holomorphic functions on A, $\Omega^1_{hol}(A)$ denotes holomorphic 1-forms, and we are taking the continuous linear duals of these spaces. Further, we use, as always, the completed tensor product when defining the symmetric algebra. The other cohomology groups of $\mathrm{Obs}^{cl}(A)$ vanish.

In a similar way, we can identify

$$H^*(\Omega_c^{0,*}(A, T^{1,0}A)) = H^*(\Omega^{0,*}(A, K_A^{\otimes 2}))^\vee[-1],$$

by Serre duality.

The residue pairing gives a dense embedding

$$\mathbb{C}[t, t^{-1}]\mathrm{d}t \subset \mathrm{Hol}(A)^\vee.$$

There is a concrete map at the cochain level

$$R : \mathbb{C}[t, t^{-1}][-1] \to \Omega_c^{0,*}(A)$$

that realizes this residue embedding, defined as follows. Choose a smooth function f on the annulus that takes value 1 in a neighborhood of the outer boundary and value 0 in a neighborhood of the inner boundary. Then, $\overline{\partial} f$ has compact support. The map R sends a polynomial $P(t)$ to $\overline{\partial}(fP)$. One can check that this construction is compatible with the residue pairing: if $Q(t)\mathrm{d}t$ is a holomorphic one-form on the annulus, then

$$\oint P(t)Q(t)\mathrm{d}t = \int_A \overline{\partial}(f(t,\bar{t})P(t))Q(t)\mathrm{d}t,$$

using Stokes' theorem.

In fact, the residue pairing tells us that a dense subspace of $H^1(\mathcal{L}_c(A))$ is

$$\mathbb{C}[t, t^{-1}]\partial_t \subset H^1(\Omega_c^{0,*}(A, TA)).$$

We therefore want to describe a map

$$\Phi : \mathbb{C}[t, t^{-1}]\partial_t \to \widehat{\mathrm{Sym}}^* \left(\mathrm{Hol}(A)^\vee \otimes V^\vee \oplus \Omega^1_{hol}(A)^\vee \otimes V \right).$$

In other words, given an element $P(t)\partial_t \in \mathbb{C}[t, t^{-1}]\mathrm{d}t$, we need to describe a functional $\Phi(P(t)\partial_t)$ on the space of pairs

$$(\beta, \gamma) \in \mathrm{Hol}(A) \otimes V \oplus \Omega^1_{hol}(A) \otimes V^\vee.$$

From what we have explained so far, it is easy to calculate that this functional is

$$\Phi(P(t)\partial_t)(\beta, \gamma) = \oint \left(P(t)\partial_t \beta(t)\right) \gamma(t).$$

The reader familiar with the theory of chiral conformal field theory and vertex algebras will see that this is the classical limit of a standard formula for the Virasoro current.

For the quantum analogue of this situation, see Section 14.2. $\lozenge$

12.7 Noether's theorem and the operator product expansion

Consider a translation-invariant field theory on $\mathbb{R}^n$. Let $\mathrm{Obs}^{cl}(0)$ denote the classical point observables, as considered in Chapter 10. In that chapter, we saw how the cochain complex $\mathrm{Obs}^{cl}(0)$ has an additional algebraic structure coming from the operator product expansion. It encodes the singularities appearing in the factorization product to leading order in $\hbar$.

The space of classical point observables $\mathrm{Obs}^{cl}(0)$ is a dg commutative algebra. The operator product expansion (OPE) gives a linear map

$$\{-,-\}_{OPE} : \mathrm{Obs}^{cl}(0) \otimes \mathrm{Obs}^{cl}(0) \to (C^\omega(\mathbb{R}^n \setminus 0) / \simeq) \otimes \mathrm{Obs}^{cl}(0).$$

On the right hand side, $C^\omega(\mathbb{R}^n \setminus 0) / \simeq$ is the space of real-analytic functions on $\mathbb{R}^n \setminus 0$ modulo those that extend across the origin as continuous functions.

The linear map $\{-,-\}_{OPE}$ is a cochain map, and it is a derivation in each factor for the commutative algebra structure on classical observables. Further properties of $\{-,-\}_{OPE}$ are listed in Proposition 10.3.1.2.

Now suppose that our classical theory on $\mathbb{R}^n$ is acted on by a translation-invariant local Lie algebra $\mathcal{L}$ on $\mathbb{R}^n$. We have seen that this action gives rise to a map of P_0 factorization algebras from the enveloping P_0 factorization algebra $\mathbb{U}_\alpha^{P_0}(\mathcal{L})$ of $\mathcal{L}$ to the P_0 factorization algebra Obs^{cl} of classical observables.

One can ask if there is a similar statement one can make about the OPE? Is it possible to define the analog of the bracket $\{-,-\}_{OPE}$ on the point observables in the factorization algebra $\mathbb{U}_\alpha^{P_0}(\mathcal{L})$? If so, does the map of 12.4.1.2 respect the map $\{-,-\}_{OPE}$?

In this section we will answer these questions in the affirmative, under an additional hypotheses on $\mathcal{L}$: we will assume that $\mathcal{L}$ is *elliptic*, in the sense that the differential on $\mathcal{L}$ makes $\mathcal{L}$ into an elliptic complex. We will also make the simplifying assumption that $\mathcal{L}$ is not an L_∞ algebra, but an ordinary dg Lie algebra. (This assumption is not strictly necessary, but it makes arguments a little easier. The L_∞ versions can be readily supplied at the cost of extra notation.)

12.7.1 The OPE bracket for local Lie algebras

The first thing we need to do is to define the analog for $\mathcal{L}$ of the space of point observables. To do it we first note, by Lemma 12.2.3.3, that the central extension $\widehat{\mathcal{L}}$ is split when viewed as an extension of cochain complexes. The cochain complex that will play the role of point observables will not depend on the central extension, although the OPE bracket will depend on the central extension.

Suppose that the local Lie algebra $\mathcal{L}$ arises as sections of a graded vector bundle L on $\mathbb{R}^n$. Since everything is translation-invariant, L is a trivial vector bundle. For any open $U \subset \mathbb{R}^n$, let $\overline{\mathcal{L}}(U)$ refer to the distributional sections of L, and let $\overline{\mathcal{L}}(0)$ denote the space of distributional sections that are supported at the origin in $\mathbb{R}^n$. It is the costalk at the origin of the cosheaf $\overline{\mathcal{L}}$.

The analog of point observables will be $\mathrm{Sym}^* \overline{\mathcal{L}}(0)$. Our next task is to define an OPE map

$$\{-,-\}_{OPE} : \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]) \otimes \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]) \to (C^\omega(\mathbb{R}^n \setminus 0)/\simeq) \otimes \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]).$$

Here, we are following the terminology introduced in Chapter 10: $C^\omega(\mathbb{R}^n \setminus 0)$ is the space of real-analytic functions on $\mathbb{R}^n$, and the equivalence relation $\simeq$ identifies two real analytic functions if they differ by a function that extends across 0 as a continuous function.

It is in the definition of $\{-,-\}_{OPE}$ that we will use the assumption that $\mathcal{L}(\mathbb{R}^n)$ is an elliptic complex. In fact, we will make a slightly stronger assumption on $\mathcal{L}(\mathbb{R}^n)$, that puts us in a familiar setting, even though it is not strictly necessary. We assume that there is a gauge-fixing operator $\mathrm{d}_{\mathcal{L}}^{GF}$ such that the commutator $[\mathrm{d}_{\mathcal{L}}, d_{\mathcal{L}}^{Q^{GF}}]$ is a generalized Laplacian. This assumption puts us in the situation we have studied extensively in field theory.

If we refer to the commutator $[\mathrm{d}_{\mathcal{L}}, d_{\mathcal{L}}^{Q^{GF}}]$ as $\triangle_{\mathcal{L}}$, then we can define a parametrix

$$\Phi : \overline{\mathcal{L}}_c(\mathbb{R}^n) \to \overline{\mathcal{L}}_c(\mathbb{R}^n)$$

that is an inverse to $\triangle_{\mathcal{L}}$ up to a smoothing operator. We finally define the parametrix for the differential $\mathrm{d}_{\mathcal{L}}$ to be

$$P(\Phi) = \mathrm{d}_{\mathcal{L}}^{GF}\Phi : \overline{\mathcal{L}}_c(\mathbb{R}^n) \to \overline{\mathcal{L}}_c(\mathbb{R}^n),$$

which satisfies

$$[\mathrm{d}_{\mathcal{L}}, P(\Phi)] = \mathrm{Id} + \text{a smooth operator}.$$

We will use this parametrix to define the OPE bracket on $\mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1])$. Since this bracket will be a derivation in each factor, it suffices to define it on the generators $\overline{\mathcal{L}}(0)[1]$ of this dg commutative algebra.

The OPE bracket will be defined in terms of the Lie bracket on the central extension $\widehat{\mathcal{L}}$ of $\mathcal{L}$. Since the central extension splits when viewed as an extension of cochain complexes, we can view this Lie bracket as a map

$$[-,-] : \mathcal{L}_c(\mathbb{R}^n) \otimes \mathcal{L}_c(\mathbb{R}^n) \to \mathcal{L}_c(\mathbb{R}^n) \oplus \mathbb{C} \cdot c[-1].$$

If we have inputs $J \in \overline{\mathcal{L}}(0)$ and $J' \in \overline{\mathcal{L}}(\mathbb{R}^n)$, where J' is smooth near 0,then

$$[J, J'] \in \overline{\mathcal{L}}(0) \oplus \mathbb{C} \cdot c[-1]$$

is well-defined because of the local nature of the central extension and of the Lie bracket on $\mathcal{L}$.

12.7.1.1 Definition. *The OPE bracket*

$$\{-,-\}_{OPE} : \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]) \otimes \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]) \\ \to (C^{\omega}(\mathbb{R}^n \setminus 0)/\simeq) \, \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1])$$

is defined on the generators in $\overline{\mathcal{L}}(0)[1]$ *by*

$$\{J(0), J'(x)\}_{OPE} \quad = \quad -(-1)^{|J|}[J(0), P(\Phi)J'(x)] \quad \in \quad \overline{\mathcal{L}}(0)[1] \oplus \mathbb{C}.$$

On the right hand side, we apply the parametrix $P(\Phi)$ to $J'(x)$ to get an element of $\overline{\mathcal{L}}(\mathbb{R}^n)$ with singularities only at x. By bracketing with $J(0)$ using the Lie bracket on the central extension $\widehat{\mathcal{L}}$, we find an element of $\overline{\mathcal{L}}(0)[1] \oplus \mathbb{C}$.

12.7.1.2 Proposition. *The operation*

$$\{-,-\}_{OPE} : \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]) \otimes \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]) \to \\ (C^{\omega}(\mathbb{R}^n \setminus 0)/\simeq) \, \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1])$$

satisfies all the properties listed in Proposition 10.3.1.2.

Proof Let us first check that $\{-,-\}_{OPE}$ is a cochain map on the generators $\overline{\mathcal{L}}(0)$. We have

$$\mathrm{d}_{\mathcal{L}}[J(0), P(\Phi)J'(x)] = [\mathrm{d}_{\mathcal{L}}J(0), P(\Phi)J'(x)] + (-1)^{|J|}[J(0), \mathrm{d}_{\mathcal{L}}P(\phi)J'(x)].$$

Now,

$$\mathrm{d}_{\mathcal{L}}(P(\Phi)J'(x)) = -P(\Phi)\mathrm{d}_{\mathcal{L}}J'(x) + J'(x) + \rho(x)$$

where the remainder term $\rho(x)$ is a smooth function of x, i.e., with no singularities at 0. (This term arises from the fact that $[\mathrm{d}_{\mathcal{L}}, P(\Phi)]$ is the identity plus a smoothing operator.)

Dropping the terms non-singular in x, we find

$$\begin{aligned}\mathrm{d}_{\mathcal{L}}[J(0), P(\Phi)J'(x)] = [\mathrm{d}_{\mathcal{L}}J(0), P(\Phi)J'(x)] - (-1)^{|J|}[J(0), P(\Phi)\mathrm{d}_{\mathcal{L}}J'(x)]\\ + (-1)^{|J|}[J(0), J'(x)].\end{aligned}$$

Since $J(0)$, $J'(x)$ have disjoint support, the last term vanishes. Incorporating the signs in the definition of $\{-,-\}_{OPE}$, we see that it is a cochain map.

The bracket $\{-,-\}_{OPE}$ is extended to a a bilinear operator on $\mathrm{Sym}^*(\overline{L}(0))$ by asking that it is a derivation in each factor. The remaining things to check are the compatibilities between $\{-,-\}_{OPE}$ and the action of differentiation on $\overline{\mathcal{L}}(0)$. These, however, are not difficult and can be verified along the same lines as the proof of the corresponding statement in Chapter 10. □

The OPE bracket on classical point observables captures the singularities in the factorization product between quantum point observables, to leading order in $\hbar$. The OPE bracket we build from a local Lie algebra has a similar interpretation, but where the algebra of quantum observables is replaced by the twisted enveloping factorization algebra $\mathbb{U}_\alpha\mathcal{L}$, as defined in section I.3.6.

By definition,

$$(\mathbb{U}_\alpha\mathcal{L})\,(U) = C_*(\widehat{\mathcal{L}}_c(U)) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}$$

is obtained from the Lie algebra chains of $\widehat{\mathcal{L}}_c(U)$ by setting the central element to 1. As such, there is an isomorphism of graded vector spaces

$$(\mathbb{U}_\alpha\mathcal{L})\,(U) = \mathrm{Sym}^*(\mathcal{L}_c(U)[1]).$$

The complex $\mathbb{U}_\alpha(\mathcal{L})(U)$ has a filtration where

$$F^k \mathbb{U}_\alpha(\mathcal{L})(U) = \mathrm{Sym}^{\leq k}(\mathcal{L}_c(U)[1]).$$

In particular, the subspace given by F^1, which is $\mathcal{L}_c(U)[1] \oplus \mathbb{C}$, is a subcomplex. By using Lemma 12.2.3.3, which tells us that the extension provided by α is trivial as an extension of cochain complexes, we see that in fact $\mathcal{L}_c(U)[1]$ is naturally a subcomplex of $\mathbb{U}_\alpha(\mathcal{L})(U)$. By varying the open subset U, we find that there is a map of precosheaves of cochain complexes

$$\mathcal{L}_c[1] \to \mathbb{U}_\alpha(\mathcal{L}).$$

Passing to cohomology, we find that there is a natural map

$$H^*(\mathcal{L}_c(U)[1]) \to H^*(\mathbb{U}_\alpha(\mathcal{L})(U)).$$

Since we assume that the differential on $\mathcal{L}$ is elliptic, $\mathcal{L}_c(U)[1]$ and $\overline{\mathcal{L}}_c(U)[1]$ are quasi-isomorphic by the Atiyah-Bott lemma. Hence, for any point $p \in U$, we have a map of graded vector spaces

$$\rho : H^*(\overline{\mathcal{L}}_c(p)[1]) \to H^*(\mathbb{U}_\alpha(\mathcal{L})(U)).$$

Here, as before, by $\overline{\mathcal{L}}_c(p)$ we mean those elements of $\overline{\mathcal{L}}_c(U)$ which are supported at p.

We assumed that our local Lie algebra $\mathcal{L}$ on $\mathbb{R}^n$ is translation-invariant. Thus any class $J(0) \in H^*(\overline{\mathcal{L}}_c(0)[1])$ can be translated to $J(x) \in H^*(\overline{\mathcal{L}}_c(x))$ for any $x \in \mathbb{R}^n$. Now fix two elements $J, J' \in H^*(\overline{\mathcal{L}}_c(0)[1])$. Taking ϵ small, we view

$$\rho(J(0)) \in H^*(\mathbb{U}_\alpha(\mathcal{L})(D(0,\epsilon))),$$
$$\rho(J'(x)) \in H^*(\mathbb{U}_\alpha(\mathcal{L})(D(x,\epsilon)))$$

The factorization product of these elements is then an element

$$J(0) \cdot J'(x) \in H^*\left(C^\infty(\mathbb{R}^n \setminus 0, \mathbb{U}_\alpha(\mathcal{L})(\mathbb{R}^n))\right).$$

(We can take $x \in \mathbb{R}^n \setminus 0$, as opposed to $\mathbb{R}^n \setminus D(0, 2\epsilon)$, by using that fact that this construction works for arbitrarily small ϵ and that the factorization product is independent of the choice of ϵ as long as $\|x\| > 2\epsilon$.)

12.7.1.3 Lemma. *The OPE bracket*

$$\{J(0), J'(x)\}_{OPE} \in (C^\omega(\mathbb{R}^n \setminus 0)/\simeq) \otimes (H^*(\overline{\mathcal{L}}_c(0)[1]) \oplus \mathbb{C})$$

as defined in Definition 12.7.1.1 satisfies

$$\{J(0), J'(x)\}_{OPE} \simeq \rho(J(0)) \cdot \rho(J'(x)).$$

That is, $\{J(0), J'(x)\}_{OPE}$ and $\rho(J(0)) \cdot \rho(J'(x))$ differ only by functions of x that take valued in $\mathbb{U}_\alpha(\mathcal{L})(\mathbb{R}^n)$ and that are continuous at $x = 0$.

In other words, the OPE bracket $\{J(0), J'(x)\}_{OPE}$ captures the singular part of the factorization product between the elements $\rho(J(0))$ and $\rho(J'(x))$ in the cohomology of $\mathbb{U}_\alpha(\mathcal{L})$. This lemma is the analog of Proposition 10.3.2.2, which tells us that the OPE bracket on classical observables can understood either by an explicit formula, or as the singular part in the factorization product of quantum observables.

Proof To prove this claim, we take cochain representatives $\widetilde{J}(0)$ and $\widetilde{J}'(x)$, which are elements of $\mathcal{L}_c(D(0,\epsilon))[1]$ and $\mathcal{L}_c(D(x,\epsilon))[1]$ respectively. We choose, as in the definition of the OPE bracket, an operator $\mathrm{d}^*_{\mathcal{L}}$ on $\mathcal{L}$ such that the commutator $\triangle = [\mathrm{d}_{\mathcal{L}}, \mathrm{d}^*_{\mathcal{L}}]$ is a generalized Laplacian operator. We take a parametrix Φ for $\triangle$, which we assume to be of the form

$$\begin{aligned} \Phi : \mathcal{L}_c(\mathbb{R}^n) &\to \mathcal{L}_c(\mathbb{R}^n) \\ \Phi(\alpha) &= f(x)\triangle^{-1}\alpha \end{aligned}$$

where $\triangle^{-1}$ indicates the standard Green's operator, and $f(x) \in C^\infty_c(\mathbb{R}^n)$ is some function which is 1 in a ball $x = 0$ of radius R, and of compact support. The result will, as usual, be independent of the choice of parametrix.

We let $P(\Phi)$ be the composition $d^*_{\mathcal{L}}\Phi$. For any closed element $\alpha \in \mathcal{L}_c(\mathbb{R}^n)[1]$,

$$\mathrm{d}_{\mathcal{L}}(P(\Phi)\alpha) = \alpha + S(\alpha).$$

If α is supported in a ball of radius ϵ around 0, then $S(\alpha)$ is supported in the region where $R < |x| < R + \epsilon$.

Let us calculate the differential on $\mathbb{U}_\alpha(\mathcal{L})$ applied to the factorization product of $\widetilde{J}(0)$ with $P(\Phi)\widetilde{J}'(x)$. We find

$$\mathrm{d}_{\mathbb{U}_\alpha(\mathcal{L})}\left(P(\Phi)\widetilde{J}'(x) \cdot J(0)\right) = [P(\Phi)\widetilde{J}'(x), \widetilde{J}(0)] + \widetilde{J}'(x) \cdot \widetilde{J}(0) + S\widetilde{J}'(x) \cdot \widetilde{J}(0).$$

Note that $S\widetilde{J}'(x)$ has support disjoint from 0 for small x, so that $S\widetilde{J}'(x) \cdot$

$\widetilde{J}(0)$ extends smoothly across $x = 0$. We find that, modulo things which extend smoothly across 0 and after passing to cohomology,

$$\widetilde{J}'(x) \cdot \widetilde{J}(0) \simeq -[P(\Phi)\widetilde{J}'(x), \widetilde{J}(0)].$$

Switching the order of $\widetilde{J}$ and $\widetilde{J}'$ on both sides, we pick up a sign of $(-1)^{|\widetilde{J}(0)|}$, which appears in our definition of $\{-,-\}_{OPE}$. □

We now give some examples.

Example: Let $\mathcal{L}$ be the Abelian local Lie algebra

$$\mathcal{L}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^n(R^n)[-1]$$

that consists of functions on $\mathbb{R}^n$ in degrees 0, top forms in degree 1, and with differential the Laplacian. We let ϕ be an element of $\mathcal{L}$ in degree 0, and ψ an element in degree 1. Define a central extension

$$\widehat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C} \cdot c[-1]$$

with Lie bracket

$$[\phi, \psi] = c \int_{\mathbb{R}^n} \phi\psi$$

We can thus identify $\mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1])$ with the point observables in the free scalar field. The OPE bracket $\{-,-\}_{OPE}$ defined using the Lie bracket on $\widehat{\mathcal{L}}$ coincides with that for the point observables in a free scalar field. ♢

Example: More generally, suppose that we have a translation-invariant free field theory on $\mathbb{R}^n$ associated to an Abelian local Lie algebra $\mathcal{M}$, with a pairing of degree -3. We let $\mathcal{L} = \mathcal{M}[1]$. The invariant pairing on $\mathcal{M}$ gives rise to a 1-shifted central extension $\widehat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C} \cdot c[-1]$, where the bracket of two elements $\alpha, \beta \in \mathcal{L}_c$ is $c \langle \alpha, \beta \rangle$, $\langle -,- \rangle$ indicating the invariant pairing on $\mathcal{M}$.

Let Obs^{cl} be the corresponding commutative factorization algebra, and let $\mathrm{Obs}^{cl}(0)$ denote the dg commutative algebra of point observables. Thus,

$$\mathrm{Obs}^{cl}(0) = \mathrm{Sym}^*(\overline{\mathcal{M}}^!(0)[-1]) = \mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1]),$$

where we have used the invariant pairing to identify $\mathcal{M}^![-1]$ with $\mathcal{L}[1]$. This OPE bracket $\{-,-\}_{OPE}$ on $\mathrm{Obs}^{cl}(0)$ coincides with that on $\mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1])$. ♢

Example: Consider $\mathcal{L} = \Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$, with the trivial central extension. The cohomology of $\overline{\mathcal{L}}(0)[1]$ is given by $\mathfrak{g}$, with representatives given by $X\delta_{x=0}$ where $X \in \mathfrak{g}$.

The Green's function for the Laplacian on $\mathbb{R}$ is $\frac{1}{2}|x|$. The parametrix for the de Rham operator is the composition of d^* with the operator of convolution with the Green's function. Explicitly, it is the linear map

$$\begin{array}{rlcl} P: & \overline{\Omega}^1_c(\mathbb{R}) & \to & \Omega^0(\mathbb{R}) \\ & f(x)\mathrm{d}x & \mapsto & \partial_{x'} \int_x f(x)\mathrm{d}x\, \frac{1}{2}\,|x - x'| \\ & & & = \frac{1}{2}\int_x f(x)\mathrm{d}x\, (\delta_{x<x'} - \delta_{x>x'})\,. \end{array}$$

If $X, Y \in \mathfrak{g}$, the OPE bracket $\{X\delta_{x=0}, Y\delta_{x=\epsilon}\}$ is given by $-[X\delta_{x=0}, P(Y\delta_{x=\epsilon})]$. Since

$$P\delta_{x=\epsilon} = -\tfrac{1}{2}\delta_{x>\epsilon} + \tfrac{1}{2}\delta_{x<\epsilon},$$

we find

$$\{X\delta_{x=0}, Y\delta_{x=\epsilon}\}_{OPE} = \tfrac{1}{2}[X, Y]\,(\delta_{\epsilon<0} - \delta_{\epsilon>0})\,.$$

We should interpret this example as follows. If we take the product of X times Y with Y to the right of X, minus the product taken in the other order, the answer is $[X, Y]$.

This example is the semi-classical version of the computation in Section I.3.4, where we saw that the enveloping factorization algebra of the sheaf of Lie dg Lie algebras $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ is given by the universal enveloping algebra of $\mathfrak{g}$. $\Diamond$

Example: This example is the holomorphic version of the previous example, and it is the semi-classical limit of the construction of the Kac-Moody algebra as an enveloping factorization algebra presented in Section I.5.5. In this example, we take $\mathcal{L}$ to be the local Lie algebra $\Omega^{0,*}_{\mathbb{C}} \otimes \mathfrak{g}$ on $\mathbb{C}$, with trivial central extension. Then the cohomology of $\overline{\mathcal{L}}(0)[1]$ is provided by $\mathfrak{g} \otimes \mathbb{C}[\partial_z]$. Representatives for these cohomology classes are provided by $X\,\partial_z^n \delta_{z=0}$.

The OPE bracket is determined by that between $X\delta_{z=0}$ and $Y\delta_{z=u}$. Since the parametrix P (morally $\overline{\partial}^{-1}$ applied to $\delta_{z=0}$ is $\frac{1}{2\pi i}\frac{1}{z-u}$, we find

$$\{X\delta_{z=0}, Y\delta_{z=u}\}_{OPE} = \frac{-1}{2\pi i}\frac{1}{u}.$$

This bracket is the semi-classical limit in the OPE of the Kac-Moody vertex algebra, at level zero. (Note that we are recovering the vertex Poisson structure.) $\Diamond$

12.7.2 Classical Noether's theorem and the OPE

In this subsection we will show that our version of classical Noether's theorem is compatible with the OPE brackets on both the local Lie algebra and on the classical point observables.

Suppose that a translation-invariant elliptic local Lie algebra $\mathcal{L}$ acts on a translation-invariant classical field theory. Consider the Noether map from theorem 12.4.0.1:

$$\mathcal{L}_c \to \mathrm{Obs}^{cl}.$$

By the remark following theorem 12.4.0.1, this map extends to a map from the distributional completion $\overline{\mathcal{L}}_c$. It therefore gives a cochain map

$$\mu : \overline{\mathcal{L}}_c(0) \to \mathrm{Obs}^{cl}(0)$$

between the costalks of these cosheaves.

Because $\mathrm{Obs}^{cl}(0)$ is a commutative dg algebra, the map μ determines a map of dg commutative algebras

$$\mu : \mathrm{Sym}^* \overline{\mathcal{L}}_c(0) \to \mathrm{Obs}^{cl}(0),$$

which in turn gives rise to a map

$$[\mu] : \mathrm{Sym}^* H^*(\overline{\mathcal{L}}_c(0)) \to H^* \mathrm{Obs}^{cl}(0)$$

of graded commutative algebras.

12.7.2.1 Proposition. *Suppose a translation-invariant elliptic local Lie algebra $\mathcal{L}$ acts on a translation-invariant classical field theory. Suppose further that both the translation-invariant classical field theory and the action of $\mathcal{L}$ can be defined at the quantum level, modulo $\hbar^2$.*

Then the map $[\mu]$ intertwines the OPE brackets on $\mathrm{Sym}^* H^*((\overline{\mathcal{L}}_c(0))$ and on $H^*(\mathrm{Obs}^{cl}(0))$. That is, if $J, J' \in \mathrm{Sym}^* H^*(\overline{\mathcal{L}}_c(0))$, then

$$[\mu]\left(\{J(0), J'(x)\}_{OPE}\right) = \{[\mu](J)(0), [\mu](J')(0)\}_{OPE}$$

as elements in $(C^\omega(\mathbb{R}^n \setminus 0)/\simeq) \otimes H^*(\mathrm{Obs}^{cl}(0))$.

Remark: The assumption that the field theory and the $\mathcal{L}$ action can be defined modulo $\hbar^2$ is not strictly necessary; it does facilitate the proof. $\Diamond$

Proof To prove this claim, it suffices to take J, J' to be elements of

$H^*(\overline{\mathcal{L}}_c(0))$. As we will see in Chapter 13, the classical Noether map extends to a quantum Noether map of the form

$$\mu_q : \mathrm{Rees}\, \mathbb{U}_\alpha(\mathcal{L}) \to \mathrm{Obs}^q$$

where $\mathrm{Rees}\, \mathbb{U}_\alpha(\mathcal{L})$ is a Rees construction applied to the enveloping factorization algebra $\mathbb{U}_\alpha(\mathcal{L})$. The quantization parameter $\hbar$ is the Rees parameter. (See Chapter 13 for more details on the Rees construction.)

Lemma 12.7.1.3 tells us that using the map

$$\rho : H^*(\overline{\mathcal{L}}_c(0)) \to H^*(\mathbb{U}_\alpha(\mathcal{L}_c)(D(0,\epsilon))),$$

we can equate the OPE bracket of two elements $J, J' \in H^*(\overline{\mathcal{L}}_c(0))$ with the singularities in the factorization product of $\rho(J)(0)$, $\rho(J')(x)$. If we perform the same calculation in the Rees algebra $\mathrm{Rees}\, \mathbb{U}_\alpha(\mathcal{L})$ and include the Rees parameter $\hbar$, we will find that

$$\rho(J)(0) \cdot \rho(J')(x) \simeq \hbar\{J(0), J'(x)\}_{OPE},$$

where we view $\rho(J)$ and $\rho(J')$ as elements of $H^*(\mathrm{Rees}\, \mathbb{U}_\alpha(\mathcal{L})(D(0,\epsilon)),$ with ϵ being sufficiently small.

The map μ_q of 12.7.2 respects the factorization product, and therefore so does the map $[\mu_q]$ obtained from μ_q by passing to cohomology. This feature tells us that

$$[\mu_q]\rho(J)(0) \cdot [\mu_q]\rho(J)(x) = [\mu_q](\rho(J(0)) \cdot \rho(J(x))).$$

Further, $[\mu_q]\rho(J)$ is a lift of $[\mu](J)$ to a quantum observable defined modulo $\hbar^2$. Thus, we conclude that

$$[\mu]\,\{J(0), J(x)\}_{OPE} = \{[\mu]J(0), [\mu]J(x)\}_{OPE}$$

as desired. □

Let us finish with an example of the OPE formulation of classical Noether's theorem.

Example: Let us revisit the example where $\mathcal{L}$ be the Abelian local Lie algebra

$$\mathcal{L}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^n(R^n)[-1]$$

with ϕ in degree 0 a function on $\mathbb{R}^n$ in degrees 0 and ψ a top form in degree 1. Define a central extension

$$\widehat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C} \cdot c[-1]$$

with Lie bracket

$$[\phi, \psi] = c \int_{\mathbb{R}^n} \phi\psi$$

Let $\mathcal{M} = \mathcal{L}[-1]$. Then $\mathcal{M}$ is the Abelian local Lie algebra with invariant pairing of degree -3, which describes the free scalar field theory on $\mathbb{R}^n$.

There is an action of $\mathcal{L}$ on $\mathcal{M}$, given by the canonical translation action of a vector space on itself (here, shifted). The semi-direct product $\mathcal{L} \rtimes \mathcal{M}$ describing this action is

$$\mathcal{L} \rtimes \mathcal{M} = (\mathcal{L} \oplus \mathcal{M}, \mathrm{d}),$$

where the differential is the identity map $\mathcal{L} \to \mathcal{M} = \mathcal{L}[-1]$.

We can thus identify $\mathrm{Sym}^*(\overline{\mathcal{L}}(0)[1])$ with the point observables in the free scalar field. The OPE bracket $\{-,-\}_{OPE}$ defined using the Lie bracket on $\widehat{\mathcal{L}}$ coincides with that for the point observables in a free scalar field. ◊

13

Noether's theorem in quantum field theory

In this chapter we explore how to quantize the map that assigns a current to a symmetry, using the BV approach. Our main result is a quantum version of Noether's theorem in the language of factorization algebras, which recovers our classical Noether's theorem when $\hbar \to 0$. As a demonstration of this method, we discuss how the local index theorem arises, following Rabinovich (2020).

13.1 The quantum Noether's theorem

So far, we have explained the classical version of Noether's theorem, which states that given an action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory, we have a central extension $\widetilde{\mathcal{L}}_c$ of the precosheaf $\mathcal{L}_c$ of L_∞-algebras, and a map of precosheaves of L_∞ algebras

$$\widetilde{\mathcal{L}}_c \to \mathrm{Obs}^{cl}[-1].$$

Our quantum Noether's theorem provides a version of this at the quantum level. Before we explain this theorem, we need to introduce some algebraic ideas about enveloping algebras of homotopy Lie algebras.

13.1.1 Reformulating the classical case

Recall the notion of the enveloping P_0 algebra $U^{P_0}(\mathfrak{g})$ of a dg Lie algebra, from Definition 11.2.3.1. Explicitly, this enveloping P_0 algebra is

$$U^{P_0}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}[1]),$$

with the obvious commutative product and with the 1-shifted Poisson bracket that is the unique biderivation that agrees with the 1-shifted Lie bracket on $\mathfrak{g}[1]$. This construction provides a left adjoint to the forgetful functor from P_0 algebras to dg Lie algebras.

Furthermore, if we have a shifted central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathbb{C}[-1]$, determined by a class $\alpha \in H^1(\mathfrak{g})$, we define the *twisted* enveloping P_0 algebra

$$U_\alpha^{P_0}(\mathfrak{g}) = U^{P_0}(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}$$

obtained from $U^{P_0}(\widetilde{\mathfrak{g}})$ by specializing the central parameter to 1.

For formal reasons, a version of these construction holds in the world of L_∞ algebras. If a dg commutative algebra is equipped with a 1-shifted L_∞ structure such that all higher brackets are multi-derivations for the product structure, then it defines a *homotopy P_0 algebra*. (The point is that the operad describing such gadgets is naturally quasi-isomorphic to the operad P_0.)

In particular, if $\mathfrak{g}$ is an L_∞ algebra, one can directly construct such a homotopy P_0 algebra: take the commutative algebra is $\mathrm{Sym}(\mathfrak{g}[1])$ and give it the unique shifted L_∞ structure where $\mathfrak{g}[1]$ is a sub-L_∞ algebra and all higher brackets are derivations in each variable. This L_∞ structure makes $\mathrm{Sym}(\mathfrak{g}[1])$ into a homotopy P_0 algebra, and one can show that it is the homotopy enveloping P_0 algebra of $\mathfrak{g}$. We will denote it by $U^{P_0}(\mathfrak{g})$ as well, and suppress explicit mention of the fact that this is a homotopy P_0 algebra. (For deeper discussion of these ideas and their quantum analogues, see Braun and Lazarev (2013); Bashkirov and Voronov (2017); Gwilliam and Haugseng (2018).)

It is straightforward to formulate a version of these notions at the level of factorization algebras, along the lines of Section I.3.6.

13.1.1.1 Definition. *If $\mathcal{L}$ is a local L_∞ algebra on a manifold X, its* enveloping P_0 factorization algebra $\mathbb{U}^{P_0}\mathcal{L}$ *assigns to each open subset $U \subset X$, the homotopy P_0 algebra $U^{P_0}(\mathcal{L}_c(U))$.*

Similarly, given a cocycle $\alpha \in H^1_{loc}(\mathcal{L}(X))$, the twisted *enveloping P_0 factorization algebra is given by*

$$\mathbb{U}^{P_0}_\alpha\mathcal{L}(U) = U^{P_0}_\alpha(\widetilde{\mathcal{L}}_c(U))$$

for every open $U \subset X$. Here $\widetilde{\mathcal{L}}_c$ denotes the central extension of the precosheaf $\mathcal{L}_c$ of L_∞ algebras associated to α.

Note that $\mathbb{U}^{P_0}\mathcal{L}$ is a homotopy P_0 algebra object in factorization algebras.

We now rephrase the classical version of Noether's theorem.

13.1.1.2 Theorem. *Suppose that a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory, whose factorization algebra of classical observables is* Obs^{cl}. *If the obstruction to lifting this action to an inner action is a local cocycle α in $H^1_{loc}(\mathcal{L}(X))$, then there is a map*

$$\mathbb{U}^{P_0}_\alpha(\mathcal{L}_c) \to \mathrm{Obs}^{cl}$$

of homotopy P_0 factorization algebras.

The universal property of $U^{P_0}_\alpha(\mathcal{L}_c)$ means that this theorem is a formal consequence of the version of Noether's theorem that we have already proved. At the level of commutative factorization algebras, this map is obtained just by taking the cochain map $\widetilde{\mathcal{L}}_c(U)[1] \to \mathrm{Obs}^{cl}(U)$ and extending it in the unique way to a map of dg commutative algebras

$$\mathrm{Sym}(\widetilde{\mathcal{L}}_c(U)[1]) \to \mathrm{Obs}^{cl}(U),$$

before specializing by setting the central parameter to 1. There are higher homotopies making it into a map of homotopy P_0 algebras, but we will not write them down explicitly. (They come from the higher homotopies making the map $\widetilde{\mathcal{L}}_c(U) \to \mathrm{Obs}^{cl}(U)$ into a map of L_∞ algebras.)

This formulation of classical Noether's theorem is clearly ripe for quantization. We must simply replace classical observables by quantum observables, and the enveloping P_0 algebra by the enveloping BD algebra.

13.1.2 The quantum version

Recall the notion of the enveloping *BD* algebra $U^{BD}(\mathfrak{g})$ of a dg Lie algebra, from Definition 11.3.1.2. Concretely,

$$U^{BD}(\mathfrak{g}) = C_*(\mathfrak{g})[[\hbar]] = \mathrm{Sym}^*(\mathfrak{g}[-1])[[\hbar]]$$

with differential $\mathrm{d}_{\mathfrak{g}} + \hbar \mathrm{d}_{CE}$, where $\mathrm{d}_{\mathfrak{g}}$ is the internal differential on $\mathfrak{g}$ and d_{CE} is the Chevalley-Eilenberg differential. The commutative product and Lie bracket are the $\hbar$-linear extensions of those on the enveloping P_0 algebra we discussed above. It is straightforward to modify this discussion for L_∞ algebras and obtain a homotopy *BD* algebra, in analogy with our discussion of the enveloping P_0 algebra.

Likewise, given a shifted central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathbb{C}[-1]$, determined by a class $\alpha \in H^1(\mathfrak{g})$, the *twisted* enveloping *BD* algebra

$$U^{BD}_\alpha(\mathfrak{g}) = U^{BD}(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}$$

obtained from $U^{BD}(\widetilde{\mathfrak{g}})$ by specializing the central parameter to 1.

There is a natural version of these notions at the level of factorization algebras, along the lines of Section I.3.6.

13.1.2.1 Definition. *If $\mathcal{L}$ is a local L_∞ algebra on a manifold X, its* enveloping *BD* factorization algebra $\mathbb{U}^{BD}\mathcal{L}$ *assigns to each open subset $U \subset X$, the homotopy BD algebra $U^{BD}(\mathcal{L}_c(U))$.*

Similarly, given a cocycle $\alpha \in H^1_{loc}(\mathcal{L}(X))$, the twisted *enveloping BD factorization algebra is given by*

$$\mathbb{U}^{BD}_\alpha \mathcal{L}(U) = U^{BD}_\alpha(\widetilde{\mathcal{L}}_c(U))$$

for every open $U \subset X$. Here $\widetilde{\mathcal{L}}_c$ denotes the central extension of the precosheaf $\mathcal{L}_c$ of L_∞ algebras associated to α.

Now we can state the quantum version of Noether's theorem, modulo explaining precisely what it means for a Lie algebra to act on a quantum field theory, which we do in Section 13.2.

13.1.2.2 Theorem. *Suppose a local L_∞ algebra $\mathcal{L}$ acts on a quantum field theory, whose factorization algebra of quantum observables is* Obs^q*. There is* *then an $\hbar$-dependent local cocycle α in $H^1_{loc}(\mathcal{L})[[\hbar]]$ and a map*

$$\mathbb{U}^{BD}_{\alpha}(\mathcal{L}_c) \to \mathrm{Obs}^q$$

of factorization algebras.

The map will arise by deforming the classical map in a way compatible with the quantization of the field theory.

13.1.3 Interpretation via Noether currents

In Section 12.5, we discussed the relationship between this formulation of Noether's theorem and the traditional point of view, via currents. Let us explain some aspects of this story that are slightly different in the quantum and classical settings.

Suppose that an ordinary Lie algebra $\mathfrak{g}$ is acting on a quantum field theory on a manifold X. Then the quantum analogue of Lemma 12.2.4.2, which we will prove below, shows that we acquire an action of the local dg Lie algebra $\Omega^*_X \otimes \mathfrak{g}$ on the quantum field theory. The quantum Noether theorem ensures we have a central extension of $\Omega^*_X \otimes \mathfrak{g}$, given by a class $\alpha \in H^1(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]))[[\hbar]]$, and a map

$$\mathbb{U}^{BD}(\Omega^*_X \otimes \mathfrak{g}) \to \mathrm{Obs}^q$$

from the twisted enveloping BD algebra of this central extension to the observables of our field theory.

We want to understand this map in terms of conserved currents.

Hence, pick an oriented codimension 1 submanifold $N \subset X$. (We assume for simplicity that X is also oriented.) Choose an identification of a tubular neighbourhood of N with $N \times \mathbb{R}$, and let $\pi : N \times \mathbb{R} \to \mathbb{R}$ denote the projection map to $\mathbb{R}$.

Let us assume, for the moment, that the central extension vanishes. The pushforward $\pi_* \mathbb{U}^{BD}(\Omega^*_X \otimes \mathfrak{g})$ is a locally constant factorization algebra on $\mathbb{R}$, and so encodes a homotopy associative algebra. Moreover, there is thus an isomorphism of associative algebras

$$H^* \left(\pi_* \mathbb{U}^{BD}(\Omega^*_X \otimes \mathfrak{g}) \right) \cong \mathrm{Rees}(U(H^*(N) \otimes \mathfrak{g})),$$

where the right hand side is the Rees algebra for the universal enveloping algebra of the graded Lie algebra $H^*(N) \otimes \mathfrak{g}$, given by de Rham cohomology with values in $\mathfrak{g}$. This Rees algebra is a $\mathbb{C}[[\hbar]]$-algebra that specializes at $\hbar = 0$ to the completed symmetric algebra of $H^*(N) \otimes \mathfrak{g}$, but is non-commutative when $\hbar$ is generic (i.e., invertible).

As a concrete consequence of Noether's theorem, we thus have a map

$$\mathrm{Rees}(U\mathfrak{g}) \to H^0(\pi_* \mathrm{Obs}^q)$$

of factorization algebras on $\mathbb{R}$. This map is closely related to the traditional formulation of Noether's theorem: we are saying that every symmetry (i.e., element of $\mathfrak{g}$) gives rise to an observable on every codimension 1 manifold (that is, a current). The operator product between these observables agrees with the product in the universal enveloping algebra.

Now consider the case when the central extension is non-zero. The group classifying possible central extensions can be identified as

$$\begin{aligned} H^1(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]))[[\hbar]] &= H^{d+1}(X, C^*_{red}(\mathfrak{g}))[[\hbar]] \\ &= \bigoplus_{i+j=d+1} H^i(X) \otimes H^j_{red}(\mathfrak{g})[[\hbar]], \end{aligned}$$

where d is the real dimension of X, and $C^*_{red}(\mathfrak{g})$ is viewed as a constant sheaf of cochain complexes on X. (A version of this computation can be found in Section 14.3 of Chapter 5 in Costello (2011b) and easily amended to prove this statement.)

Suppose for simplicity that X is of the form $N \times \mathbb{R}$, where N is compact and oriented. Then the cocycle α can be integrated over N to yield an element in $H^2_{red}(\mathfrak{g})[[\hbar]]$, which can be viewed as an ordinary, unshifted central extension of the Lie algebra $\mathfrak{g}$ (but dependent on $\hbar$). Form the twisted universal enveloping algebra $U_\alpha \mathfrak{g}$, obtained by taking the universal enveloping algebra of the central extension of $\mathfrak{g}$ and then setting the central parameter to 1. This twisted enveloping algebra admits a filtration, so that we can form its Rees algebra. Our formulation of Noether's theorem then produces a map

$$\mathrm{Rees}(U_\alpha \mathfrak{g}) \to H^0(\pi_* \mathrm{Obs}^q)$$

of factorization algebras on $\mathbb{R}$. In other words, every symmetry determines a current, up to taking a central extension, which is familiar theme of quantum mechanical symmetries.

13.2 Actions of a local L_∞-algebra on a quantum field theory

Let us now build toward the proof of the quantum version of Noether's theorem. As in the setting of classical field theories, the first thing we need to pin down is what it means for a local L_∞ algebra to act on a quantum field theory. Again, there are two variants of the definition we need to consider: one for a field theory with an $\mathcal{L}$-action, and one for a field theory with an *inner* $\mathcal{L}$-action. Just as in the classical story, the central extension that appears in our formulation of Noether's theorem appears as the obstruction to lifting an action to an inner action.

Throughout this book we have used the definition of quantum field theory given in Costello (2011b). The concept of field theory with an action of a local L_∞-algebra $\mathcal{L}$ relies on a refined definition of field theory, also given in Costello (2011b): the concept of a field theory with background fields. (See Section 13 of Chapter 2.) Let us explain this definition.

13.2.1 Equivariant BV quantization, effectively

We recall some notation and terminology from Chapter 7 and 8 before explaining the variant we need.

Fix a classical field theory, defined by a local L_∞ algebra $\mathcal{M}$ on X with an invariant pairing of cohomological degree -3. We use Q to denote the linear differential operator on $\mathcal{M}$. Choose a gauge fixing operator Q^{GF} on $\mathcal{M}$, as discussed in Section 7.2, so that we have a generalized Laplacian $[Q, Q^{GF}]$. As explained in Section 7.2, these choices lead to the following data:

(i) A propagator $P(\Phi) \in \overline{\mathcal{M}}[1]^{\otimes 2}$, defined for every parametrix Φ. If Φ, Ψ are parametrices, then $P(\Phi) - P(\Psi)$ is smooth.

(ii) A kernel $K_\Phi \in \mathcal{M}[1]^{\otimes 2}$ for every parametrix Φ, satisfying

$$Q(P(\Phi) - P(\Psi)) = K_\Psi - K_\Phi.$$

These kernels determine the key operators on the space $\mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]]$, the functionals with proper support, smooth first derivative, and at least cubic modulo $\hbar$. Namely, we obtain the RG flow operator $W(P(\Phi) - P(\Psi), -)$ and BV Laplacian $\triangle_\Phi$, associated to parametrices Φ and Ψ. There is also a BV bracket $\{-,-\}_\Phi$ that satisfies the usual relation with the BV Laplacian $\triangle_\Phi$.

The role of these operators is as follows.

(i) For every parametrix Φ, we have the structure of 1-shifted differential graded Lie algebra on $\mathscr{O}(\mathcal{M}[1])[[\hbar]]$. The Lie bracket is $\{-,-\}_\Phi$, and the differential is

$$Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi.$$

The subspace $\mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]]$ is a nilpotent sub-dg Lie algebra. The Maurer-Cartan equation in this space is called the *quantum master equation* (QME).

(ii) The map $W(P(\Phi) - P(\Psi), -)$ takes solutions to the QME with parametrix Ψ to solutions with parametrix Φ. Equivalently, the Taylor terms of this map define an L_∞ isomorphism between the dg Lie algebras associated to the parametrices Ψ and Φ.

Now we describe how to incorporate background fields.

If $\mathcal{L}$ is a local L_∞ algebra, then the functionals $\mathscr{O}(\mathcal{L}[1])$, with the Chevalley-Eilenberg differential, form a dg commutative algebra. The space $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$ of functionals on $\mathcal{L}[1] \oplus \mathcal{M}[1]$ can be identified with the completed tensor product

$$\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1]) = \mathscr{O}(\mathcal{L}[1]) \,\widehat{\otimes}_\pi\, \mathscr{O}(\mathcal{M}[1]).$$

One can thus extend to this algebra, by $\mathscr{O}(\mathcal{L}[1])$-linearity, the operations $\triangle_\Phi$, $\{-,-\}_\Phi$ and $\partial_{P(\Phi)}$ associated to a parametrix on $\mathcal{M}$. For instance, the operator $\partial_{P(\Phi)}$ is associated to the kernel

$$P(\Phi) \in (\mathcal{M}[1])^{\otimes 2} \subset (\mathcal{M}[1] \oplus \mathcal{L}[1])^{\otimes 2}.$$

If $\mathrm{d}_\mathcal{L}$ denotes the Chevalley-Eilenberg differential on $\mathscr{O}(\mathcal{L}[1])$, then we can form an operator $\mathrm{d}_{\mathcal{L}} \otimes 1$ on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$. Similarly, the linear differential Q on $\mathcal{M}$ induces a derivation of $\mathscr{O}(\mathcal{M}[1])$, which we also denote by Q; we can form a derivation $1 \otimes Q$ of $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$. The operators $\triangle_\Phi$ and ∂_Φ both commute with $\mathrm{d}_{\mathcal{L}} \otimes 1$ and satisfy the same relation described above with the operator $1 \otimes Q$.

Let

$$\mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]] \subset \mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$$

denote the space of those functionals that satisfy the following conditions:

(i) They are at least cubic modulo $\hbar$ when restricted to be functions just on $\mathcal{M}[1]$. That is, we allow functionals that are quadratic as long as they are either quadratic in $\mathcal{L}[1]$ or linear in both $\mathcal{L}[1]$ and in $\mathcal{M}[1]$, and we allow linear functionals as long as they are independent of $\mathcal{M}[1]$. Further, we work modulo the constants $\mathbb{C}[[\hbar]]$. (This clause is related to the superscript $+$ in the notation.)
(ii) We require our functionals to have proper support as functionals on $\mathcal{L}[1] \oplus \mathcal{M}[1]$, in the sense of Definition 7.2.4.1.
(iii) We require our functionals to have smooth first derivative, in the sense of Section 5.4. Note that this condition involves differentiation by elements of both $\mathcal{L}[1]$ and $\mathcal{M}[1]$.

We extend the renormalization group flow operator $W(P(\Phi) - P(\Psi), -)$ on the space $\mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]]$ to an $\mathscr{O}(\mathcal{L})$-linear operator on $\mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$. It is defined by the equation

$$W(P(\Phi) - P(\Psi), I) = \hbar \log \exp(\hbar \partial_{P(\Phi)} - \hbar \partial_{P(\Psi)}) \exp(I/\hbar),$$

as before.

13.2.1.1 Definition. *An element*

$$I \in \mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

solves the $\mathcal{L}$-equivariant quantum master equation *for the parametrix* Φ *if it satisfies*

$$\mathrm{d}_{\mathcal{L}} I + QI + \{I, I\}_\Phi + \hbar \triangle_\Phi I = 0.$$

Here $\mathrm{d}_{\mathcal{L}}$ *indicates the Chevalley differential on* $\mathscr{O}(\mathcal{L}[1])$*, extended by tensoring with* 1 *to an operator on* $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$*, and* Q *is the extension of the linear differential on* $\mathcal{M}[1]$.

As usual, the renormalization group flow takes solutions to the quantum master equation for the parametrix Φ to those for the parametrix Ψ.

13.2.2 Equivariant quantum field theory

There are two different versions of quantum field theory with an action of a Lie algebra that we consider: an action and an inner action. We will follow the format of the classical situation, modified to take into account that in the setting of effective theories, we must work over the space of parametrices.

Suppose we have a quantum field theory on X, with space of fields $\mathcal{M}[1]$, and let $\mathcal{L}$ be a local L_∞ algebra on X. To talk about an action of $\mathcal{L}$ on $\mathcal{M}$, we consider the following class of functionals,

$$\mathrm{Act}^q_{P,sm}(\mathcal{L},\mathcal{M}) = \mathscr{O}^+_{P,sm}(\mathcal{L}[1]\oplus\mathcal{M}[1])/\mathscr{O}_{P,sm}(\mathcal{L}[1])[[\hbar]].$$

It consists of functionals modulo those that only depend on $\mathcal{L}$. Compare with Definition 12.2.2.4. Note that for each parametrix Φ, there is a shifted Lie bracket given by $\{-,-\}_\Phi$, in parallel with that on $\mathrm{Act}(\mathcal{L},\mathcal{M})$.

Now we can define our notion of a quantum field theory acted on by the local L_∞ algebra $\mathcal{L}$.

13.2.2.1 Definition. *Let*

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]]$$

denote the collection of effective interactions on $\mathcal{M}[1]$ describing the quantum field theory . These satisfy the renormalization group equation, BV master equation, and locality axiom, as detailed in Section 7.2.9.1.

An action *of $\mathcal{L}$ on this quantum field theory is a collection of functionals*

$$I^{\mathcal{L}}[\Phi] \in \mathrm{Act}^q_{P,sm}(\mathcal{L},\mathcal{M})$$

for every parametrix Φ satisfying the following properties.

(i) *It satisfies the renormalization group equation*

$$W(P(\Phi)-P(\Psi), I^{\mathcal{L}}[\Psi]) = I^{\mathcal{L}}[\Phi].$$

(ii) *Each $I[\Phi]$ satisfies the quantum master equation (or Maurer-Cartan equation) for the dg Lie algebra structure associated to the parametrix Φ. We can explicitly write out the various terms in the quantum master equation as follows:*

$$\mathrm{d}_{\mathcal{L}} I^{\mathcal{L}}[\Phi] + Q I^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{I^{\mathcal{L}}[\Phi], I^{\mathcal{L}}[\Phi]\}_\Phi + \hbar \triangle_\Phi I^{\mathcal{L}}[\Phi] = 0.$$

Here $\mathrm{d}_{\mathcal{L}}$ refers to the Chevalley-Eilenberg differential on $\mathscr{O}(\mathcal{L}[1])$, and Q to the linear differential on $\mathcal{M}[1]$. As above, $\{-,-\}_\Phi$ is the Lie bracket on $\mathscr{O}(\mathcal{M}[1])$ that is extended in the natural way to a Lie bracket on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$.

(iii) *The locality axiom, as explained in Section 7.2.9.1, holds: the support of $I^{\mathcal{L}}[\Phi]$ converges to the diagonal as the support of Φ tends to zero, with the same bounds explained in Section 7.2.9.1.*

(iv) *Under the natural quotient map*

$$\mathrm{Act}^q_{P,sm}(\mathcal{L}, \mathcal{M}) \to \mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]],$$

given by restricting to functions just of $\mathcal{M}[1]$, the image of $I^{\mathcal{L}}[\Phi]$ is the original action functional $I[\Phi]$ defining the underlying, non-equivariant theory.

We will often refer to a quantum theory with an action of a local L_∞ algebra as an *equivariant quantum field theory*.

Remark: One should interpret this definition as a variant of the definition of a family of theories over a pro-nilpotent base ring $\mathcal{A}$. Indeed, if we have an $\mathcal{L}$-action on a theory on X, then the functionals $I^{\mathcal{L}}[\Phi]$ define a family of theories over the dg base ring $C^*(\mathcal{L}(X))$ of cochains on the L_∞ algebra $\mathcal{L}(X)$ of global sections of $\mathcal{L}$. In the case that X is compact, the L_∞ algebra $\mathcal{L}(X)$ often has finite-dimensional cohomology, so that we have a family of theories over a finitely-generated pro-nilpotent dg algebra.

Standard yoga from homotopy theory tells us that a $\mathfrak{g}$-action on any mathematical object (if $\mathfrak{g}$ is a homotopy Lie algebra) is the same as a family of such objects over the base ring $C^*(\mathfrak{g})$ that restrict to the given object at the central fibre. Thus, our definition of an action of the sheaf $\mathcal{L}$ of L_∞ algebras on a field theory on X gives rise to an action (in this homotopical sense) of the L_∞ algebra $\mathcal{L}(X)$ on the field theory.

However, our definition of action is stronger than this. The locality axiom we impose on the action functionals $I^{\mathcal{L}}[\Phi]$ involves both fields in $\mathcal{L}$ and in $\mathcal{M}$. As we will see later, this means that we have a homotopy action of $\mathcal{L}(U)$ on observables of our theory on U, for every open subset $U \subset X$, in a compatible way. $\Diamond$

13.2.3 Inner actions at the quantum level

The notion of inner action is parallel, except that we enlarge our space of functionals, in parallel with definition 12.2.2.4. Hence, let

$$\mathrm{InnerAct}^{q}_{P,sm}(\mathcal{L}, \mathcal{M}) = \mathscr{O}^{+}_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

Each parametrix Φ equips it with a shifted Lie bracket by $\{-,-\}_{\Phi}$.

13.2.3.1 Definition. *An* inner action *of $\mathcal{L}$ on the quantum field theory associated to the $I[\Phi]$ on $\mathcal{M}$, as above, is defined by a collection of functionals*

$$I^{\mathcal{L}}[\Phi] \in \mathrm{InnerAct}^{q}_{P,sm}(\mathcal{L}, \mathcal{M})$$

that satisfy the same four conditions as for an action, except in this larger space of functionals.

13.3 Obstruction theory for quantizing equivariant theories

As we explained in Section 7.5, the main result of Costello (2011b) is that we can construct quantum field theories from classical ones by obstruction theory, order by order in $\hbar$. More specifically, for a classical field theory described by an elliptic L_∞ algebra $\mathcal{M}$, the obstruction-deformation complex is the reduced, local Chevalley-Eilenberg cochain complex $C^{*}_{red,loc}(\mathcal{M})$, which by definition is the complex of local functionals on $\mathcal{M}[1]$ equipped with the Chevalley-Eilenberg differential. Note that this result means that the *classical* theory determines the obstruction-deformation complex used in constructing quantizations. (The obstruction cocycle, however, depends on the quantization.)

A similar result holds in the equivariant context, so that the classical problem determines the obstruction theory. Suppose a classical field theory is equipped with an action of a local L_∞ algebra $\mathcal{L}$, which we encode as a semi-direct product local L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$, where $\mathcal{M}$ is an elliptic L_∞ algebra. Thus, we can form the local Chevalley-Eilenberg cochain complex

$$\mathscr{O}_{loc}((\mathcal{L} \ltimes \mathcal{M})[1]) = C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M}),$$

but it does not provide the only relevant obstruction-deformation complex. In fact, there are several distinct complexes, depending upon which situation is of interest.

First, consider

$$C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L}) = C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})/C^*_{red,loc}(\mathcal{L})$$

consisting of local functional on $\mathcal{L} \ltimes \mathcal{M}$ except those depending just on $\mathcal{L}$. It provides the correct obstruction-deformation complex for quantizing a classical field theory with an action of $\mathcal{L}$ into a quantum field theory with an action of $\mathcal{L}$. To see this, note that it allows one to modify both the action functional of $\mathcal{M}$ as well as how $\mathcal{L}$ acts on $\mathcal{M}$, but it does not allow one to deform $\mathcal{L}$ itself.

Alternatively, one might wish to deform the action of $\mathcal{L}$ on $\mathcal{M}$, while fixing the classical theory. We denoted the relevant obstruction-deformation by $\mathrm{Act}(\mathcal{L}, \mathcal{M})$ earlier (see Definition 12.2.2.4). Note that it fits into an exact sequence

$$0 \to \mathrm{Act}(\mathcal{L}, \mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L}) \to C^*_{red,loc}(\mathcal{M}) \to 0,$$

since deforming the whole situation except $\mathcal{L}$ itself (the middle term) can be viewed as deforming the underlying theory (the base, just depending on $\mathcal{M}$) as well as how $\mathcal{L}$ acts on the theory (the fiber). A similar remark at the quantum level. If we fix a non-equivariant quantization of our original $\mathcal{L}$-equivariant classical theory $\mathcal{M}$, then we can ask to lift this quantization to an $\mathcal{L}$-equivariant quantization. In this case, the obstruction-deformation complex is also $\mathrm{Act}(\mathcal{L}, \mathcal{M})$.

Now, consider the inner versions of the deformation problems just discussed. For instance, the problem of quantizing a classical field theory with an inner $\mathcal{L}$-action into a quantum field theory with an inner $\mathcal{L}$-action has obstruction-deformation complex $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$. If, instead, we fix a non-equivariant quantization of the original classical theory $\mathcal{M}$, we can ask for the obstruction-deformation complex for lifting it to a quantization with an inner $\mathcal{L}$-action. Here the relevant obstruction-deformation complex is $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ of Section 12.2.3. We note that $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ fits into a short exact sequence

$$0 \to \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M}) \to C^*_{red,loc}(\mathcal{M}) \to 0,$$

in parallel with the short exact sequence involving $\mathrm{Act}(\mathcal{L}, \mathcal{M})$.

13.3.1 Spaces of equivariant theories

A more formal statement of these results about the obstruction-deformation complexes is the following.

Fix a classical field theory $\mathcal{M}$ with an action of a local L_∞ algebra $\mathcal{L}$. Let $\mathcal{T}^{(n)}$ denote the simplicial set of quantizations without any $\mathcal{L}$-equivariance condition. The simplicial structure is defined in Chapter 7.2: an n-simplex is a family of theories over the base ring $\Omega^*(\triangle^n)$ of differential forms on the n-simplex. Similarly, let $\mathcal{T}^{(n)}_{\mathcal{L}}$ denote the simplicial set of $\mathcal{L}$-equivariant quantizations of this field theory defined modulo $\hbar^{n+1}$. We will use DK to denote the Dold-Kan functor from non-positively graded cochain complexes to simplicial sets.

Theorem. *The simplicial sets $\mathcal{T}^{(n)}_{\mathcal{L}}$ are Kan complexes. In particular, there is an obstruction map of simplicial sets*

$$\mathcal{T}^{(n)}_{\mathcal{L}} \to DK\left(C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})[1]\right).$$

sitting in a homotopy fibre diagram

$$\begin{tikzcd} \mathscr{T}^{(n+1)}_{\mathcal{L}} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathscr{T}^{(n)}_{\mathcal{L}} \arrow[r] & DK\left(C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})[1]\right). \end{tikzcd}$$

Furthermore, the natural map

$$\mathscr{T}^{(n)}_{\mathcal{L}} \to \mathscr{T}^{(n)},$$

obtained by forgetting the $\mathcal{L}$-equivariance data in the quantization, is a fibration of simplicial sets.

Finally, there is a homotopy fibre diagram

$$\begin{tikzcd} \mathscr{T}^{(n+1)}_{\mathcal{L}} \arrow[r] \arrow[d] & \mathscr{T}^{(n+1)} \times_{\mathscr{T}^{(n)}} \mathscr{T}^{(n)}_{\mathcal{L}} \arrow[d] \\ \mathscr{T}^{(n)}_{\mathcal{L}} \arrow[r] & DK\left(\mathrm{Act}(\mathcal{L}, \mathcal{M})[1]\right). \end{tikzcd}$$

We interpret the second fibre diagram as follows. The simplicial set $\mathscr{T}^{(n+1)} \times_{\mathscr{T}^{(n)}} \mathscr{T}^{(n)}_{\mathcal{L}}$ describes pairs consisting of an $\mathcal{L}$-equivariant quantization modulo $\hbar^{n+1}$ and a non-equivariant quantization modulo $\hbar^{n+2}$, which agree as non-equivariant quantizations modulo $\hbar^{n+1}$. The obstruction-deformation group to lifting such a pair to an equivariant quantization modulo $\hbar^{n+2}$ is the group $\mathrm{Act}(\mathcal{L}, \mathcal{M})$. That is, a lift exists if the obstruction class in $H^1(\mathrm{Act}(\mathcal{L}, \mathcal{M}))$ is zero, and the simplicial set of such lifts is a torsor for the simplicial Abelian group associated to the cochain complex $\mathrm{Act}(\mathcal{L}, \mathcal{M})$). At the level of zero-simplices, the set of lifts is a torsor for $H^0(\mathrm{Act}(\mathcal{L}, \mathcal{M}))$.

This result implies, for instance, that if we fix a non-equivariant quantization to all orders, then the obstruction-deformation complex for making this into an equivariant quantization is $\mathrm{Act}(\mathcal{L}, \mathcal{M})$).

Further elaborations, as detailed in Chapter 7.2, continue to hold in this context. For example, we can work with families of theories over a dg base ring, and everything is fibred over the (typically contractible) simplicial set of gauge fixing conditions. In addition, all of these results hold when we work with translation-invariant objects on $\mathbb{R}^n$ and impose "renormalizability" conditions, as discussed in Chapter 9.

A proof of this theorem in this generality is contained in Costello (2011b) (see Section 13, Chaper 5), and it is essentially the same as the proof of the corresponding non-equivariant theorem. In Costello (2011b), the term "field theory with background fields" is used instead of talking about a field theory with an action of a local L_∞ algebra.

For theories with an inner action, the same result continues to hold, except that the obstruction-deformation complex for the first statement is $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$, and in the second case is $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$.

13.3.2 Lifting actions to inner actions

Given a field theory with an action of $\mathcal{L}$, we can try to lift it to one with an inner action. For classical field theories, we have seen that the obstruction class lives in $H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))$ (with, of course, the Chevalley-Eilenberg differential).

A similar result holds in the quantum setting.

13.3.2.1 Proposition. *Suppose we have a quantum field theory with an action of $\mathcal{L}$. Then there is a cochain*

$$\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1])[[\hbar]] = C^*_{red,loc}(\mathcal{L})$$

of cohomological degree 1 that is closed under the Chevalley-Eilenberg differential, such that trivializing α is the same as lifting $\mathcal{L}$ to an inner action.

Proof This result follows immediately from the obstruction-deformation complexes for constructing the two kinds of $\mathcal{L}$-equivariant field theories. Let us explain explicitly, however, how to calculate this obstruction class, because it will be useful later. Indeed, let us fix a theory with an action of $\mathcal{L}$, defined by functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]].$$

It is always possible to lift $I[\Phi]$ to a collection of functionals

$$\widetilde{I}^{\mathcal{L}}[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

that satisfy the RG flow and locality axioms, but may not satisfy the quantum master equation. The space of ways of lifting is a torsor for the graded abelian group $\mathscr{O}_{loc}(\mathcal{L}[1])[[\hbar]]$ of local functionals on $\mathcal{L}$. The failure of the lift $\widetilde{I}^{\mathcal{L}}[\Phi]$ to satisfy the quantum master equation is independent of Φ, as explained in Section 11, Chapter 5 of Costello (2011b), and therefore the obstruction is a local functional $\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1])$. That is, we have

$$\alpha = \mathrm{d}_{\mathcal{L}}\widetilde{I}^{\mathcal{L}}[\Phi] + Q\widetilde{I}^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{\widetilde{I}^{\mathcal{L}}[\Phi], \widetilde{I}^{\mathcal{L}}[\Phi]\}_{\Phi} + \hbar\triangle_{\Phi}\widetilde{I}^{\mathcal{L}}[\Phi].$$

Note that functionals just of $\mathcal{L}$ are in the centre of the Poisson bracket $\{-,-\}_{\Phi}$, and are also acted on trivially by the BV operator $\triangle_{\Phi}$.

We automatically have $\mathrm{d}_{\mathcal{L}}\alpha = 0$. To lift $I^{\mathcal{L}}[\Phi]$ to a functional $\widetilde{I}^{\mathcal{L}}[\Phi]$ that satisfies the quantum master equation is clearly equivalent to making α exact in $C^*_{red,loc}(\mathcal{L})[[\hbar]]$. □

13.4 The factorization algebra of an equivariant quantum field theory

In this section, we will describe the observables of an equivariant quantum field theory. As above, let $\mathcal{M}$ denote the elliptic L_∞ algebra on a manifold X describing a classical field theory, which is acted on by a local L_∞-algebra $\mathcal{L}$. Recall from Section I.6 that there is a factorization algebra $C^*(\mathcal{L})$ that assigns to an open subset $U \subset X$, the Chevalley-Eilenberg cochains $C^*(\mathcal{L}(U))$. (As usual, we use the appropriate completion of cochains.) Note that $C^*(\mathcal{L})$ is a factorization algebra valued in complete filtered differentiable dg commutative algebras on X.

We will give a brief sketch of the following result.

13.4.0.1 Proposition. *Suppose we have a quantum field theory equipped with an action of a local Lie algebra $\mathcal{L}$. Then there is a factorization algebra $\mathrm{Obs}^q_{\mathcal{L}}$ of equivariant quantum observables, quantizing $\mathrm{Obs}^{cl}_{\mathcal{L}}$, the factorization algebra of equivariant classical observables constructed in Proposition 12.3.0.2. It is a $C^*(\mathcal{L})$-module in factorization algebras.*

Proof The construction is exactly parallel to the non-equivariant version that was explained in Chapter 7.2, so we will only sketch the details. Let $\mathcal{M}$ denote the elliptic L_∞ algebra encoding the corresponding classical field theory.

We define an element of $\mathrm{Obs}^q_{\mathcal{L}}(U)$ of cohomological degree k to be a family of degree k functionals $O[\Phi]$ on the space $\mathcal{L}(X)[1] \oplus \mathcal{M}(X)[1]$ of fields of the theory, with one functional for every parametrix Φ. We require that if ϵ is a square-zero parameter of cohomological degree $-k$, then $I^{\mathcal{L}}[\Phi] + \epsilon O[\Phi]$ satisfies the renormalization group equation

$$W(P(\Phi) - P(\Psi), I^{\mathcal{L}}[\Psi] + \epsilon O[\Psi]) = I^{\mathcal{L}}[\Phi] + \epsilon O[\Phi].$$

Furthermore, we require the same locality axiom detailed in Section 8.4, saying roughly that $O[\Phi]$ is supported on U for sufficiently small parametrices Φ.

The differential on the complex $\mathrm{Obs}^q_{\mathcal{L}}(U)$ is defined by

$$(\mathrm{d}O)[\Phi] = \mathrm{d}_{\mathcal{L}} O[\Phi] + QO[\Phi] + \{I^{\mathcal{L}}[\Phi], O[\Phi]\}_\Phi + \hbar \triangle_\Phi O[\Phi],$$

where Q is the linear differential on $\mathcal{M}[1]$, and $\mathrm{d}_{\mathcal{L}}$ corresponds to the Chevalley-Eilenberg differential on $C^*(\mathcal{L})$.

We make $\mathrm{Obs}^q_{\mathcal{L}}(U)$ into a module over $C^*(\mathcal{L}(U))$ as follows. If $O \in \mathrm{Obs}^q_{\mathcal{L}}(U)$ and $\alpha \in C^*(\mathcal{L}(U))$, we define a new observable $\alpha \cdot O$ by

$$(\alpha \cdot O)[\Phi] = \alpha \cdot (O[\Phi]).$$

This formula makes sense, because α is a functional on $\mathcal{L}(U)[1]$ and so can be made a functional on $\mathcal{M}(U)[1] \oplus \mathcal{L}(U)[1]$. The multiplication on the right hand side is simply multiplication of functionals on $\mathcal{M}(U)[1] \oplus \mathcal{L}(U)[1]$.

It is easy to verify that $\alpha \cdot O$ satisfies the renormalization group equation; indeed, the infinitesimal renormalization group operator is given by differentiating with respect to a kernel in $\mathcal{M}[1]^{\otimes 2}$, and so commutes with multiplication by functionals of $\mathcal{L}[1]$. Similarly, we have

$$\mathrm{d}(\alpha \cdot O) = (\mathrm{d}\alpha) \cdot O + \alpha \cdot \mathrm{d}O$$

where $\mathrm{d}O$ as discussed above and where $\mathrm{d}\alpha$ is the Chevalley-Eilenberg differential applied to $\alpha \in C^*(\mathcal{L}(U)[1])$.

As usual, at the classical level we can discuss observables at scale 0. The differential at the classical level is $\mathrm{d}_{\mathcal{L}} + Q + \{I^{\mathcal{L}}, -\}$, where $I^{\mathcal{L}} \in \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1])$ is the classical equivariant action. This differential is the same as the differential on the Chevalley-Eilenberg differential on the cochains of the semi-direct product L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$. Thus, it is quasi-isomorphic, at the classical level, to the one discussed in Proposition 12.3.0.2. $\square$

13.5 The quantum Noether theorem *redux*

We can now explain Noether's theorem at the quantum level. As above, suppose we have a quantum field theory on a manifold X with space of fields $\mathcal{M}[1]$. Let $\mathcal{L}$ be a local L_∞ algebra that acts on this field theory. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ denote the obstruction to lifting this action to an inner action.

Recall from Section I.3.6 that the enveloping factorization algebra

$\mathbb{U}\mathcal{L}$ of the local L_∞ algebra $\mathcal{L}$ is the factorization algebra whose value on an open subset $U \subset X$ is the Chevalley-Eilenberg chain complex $C_*(\mathcal{L}_c(U))$. Given a cocycle $\beta \in H^1(C^*_{red,loc}(\mathcal{L}))$, we can form a shifted central extension

$$0 \to \mathbb{C}[-1] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$

of the precosheaf $\mathcal{L}_c$ of L_∞ algebras on X. Central extensions of this form have already been discussed in Section 12.4.

We can also form the *twisted* enveloping factorization algebra $\mathbb{U}_\beta\mathcal{L}$, which is the factorization algebra on X whose value on an open subset $V \subset X$ is

$$\mathbb{U}_\beta\mathcal{L}(V) = C_*(\widetilde{\mathcal{L}}_c(V)) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}.$$

Here we note that the center $\mathbb{C}[-1]$ has chains $C_*(\mathbb{C}[-1]) = \mathbb{C}[c]$, where c is the central parameter; it is thus a commutative algebra as well as a coalgebra. The chains $C_*(\widetilde{\mathcal{L}}_c(V))$ is a module over $\mathbb{C}[c]$, so the formula makes sense.

There is a $\mathbb{C}[[\hbar]]$-linear version of the twisted enveloping factorization algebra construction too: if our cocycle α is in $H^1(\mathbb{C}^*_{red,loc}(\mathcal{L}))[[\hbar]]$, then we can form a central extension of the form

$$0 \to \mathbb{C}[[\hbar]][-1] \to \widetilde{\mathcal{L}}_c[[\hbar]] \to \mathcal{L}_c[[\hbar]] \to 0,$$

which is an exact sequence of precosheaves of L_∞ algebras on X in the category of $\mathbb{C}[[\hbar]]$-modules. By performing the $\mathbb{C}[[\hbar]]$-linear version of the construction above, one finds the twisted enveloping factorization algebra $\mathbb{U}^\hbar_\alpha\mathcal{L}$. This factorization algebra assigns to an open subset $V \subset X$, the $\mathbb{C}[[\hbar]]$-module

$$\mathbb{U}^\hbar_\alpha\mathcal{L}(V) = C_*(\widetilde{\mathcal{L}}_c[[\hbar]]) \otimes_{\mathbb{C}[[\hbar]][c]} \mathbb{C}[[\hbar]]_{c=1}.$$

Here Chevalley-Eilenberg chains are taken in the $\mathbb{C}[[\hbar]]$-linear sense.

Our version of Noether's theorem will relate this enveloping factorization algebra of $\mathcal{L}_c$ — twisted by the cocycle α — to the factorization algebra of quantum observables of the field theory on X. The main theorem is the following.

13.5.0.1 Theorem. *Suppose that the local $\mathcal{L}_\infty$-algebra $\mathcal{L}$ acts on a field theory on X such that the obstruction to lifting this action to an inner action is a local* *cocycle* $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$*. There is then a* $\mathbb{C}((\hbar))$*-linear map*

$$\mathbb{U}^{\hbar}_{\alpha}\mathcal{L}[\hbar^{-1}] \to \mathrm{Obs}^q[\hbar^{-1}]$$

of factorization algebras. (Note that on both sides we have inverted $\hbar$*.)*

We prove this result in Section 13.5.3, but before that, we address the natural question of how this statement relates to Noether's theorem for classical field theories.

13.5.1 Dequantizing this construction

In order to provide such a relationship, we need to present a version of quantum Noether's theorem that holds without inverting $\hbar$. For every open subset $V \subset X$, we define the *Rees module* of this twisted enveloping factorization algebra

$$\mathrm{Rees}\,\mathbb{U}^{\hbar}_{\alpha}\mathcal{L}(V) \subset \mathbb{U}^{\hbar}_{\alpha}\mathcal{L}(V)$$

as the submodule spanned by elements of the form $\hbar^k\gamma$, where $\gamma \in \mathrm{Sym}^{\leq k}(\mathcal{L}_c(V))$. This sub-$\mathbb{C}[[\hbar]]$-module forms a sub-factorization algebra. One can check that $\mathrm{Rees}\,\mathbb{U}^{\hbar}_{\alpha}\mathcal{L}(V)$ is a free $\mathbb{C}[[\hbar]]$-module and that

$$\left(\mathrm{Rees}\,\mathbb{U}^{\hbar}_{\alpha}\mathcal{L}(V)\right)[\hbar^{-1}] = \mathbb{U}^{\hbar}_{\alpha}\mathcal{L}(V)$$

upon inverting $\hbar$.

Remark: The reason for the terminology is that when $\alpha = 0$ (or more generally when α is independent of $\hbar$), this module $\mathrm{Rees}\,\mathbb{U}_{\alpha}\mathcal{L}(V)$ is the Rees module for the filtered chain complex $C^{\alpha}_*(\mathcal{L}_c(V))$, in the usual sense. ◊

With this terminology in place, we assert the following variant of our first version of quantum Noether's theorem.

13.5.1.1 Theorem. *The quantum Noether map*

$$\mathbb{U}^{\hbar}_{\alpha}\mathcal{L}[\hbar^{-1}] \to \mathrm{Obs}^q[\hbar^{-1}]$$

of factorization algebras over $\mathbb{C}((\hbar))$ *refines to a map*

$$\mathrm{Rees}\,\mathbb{U}^{\hbar}_{\alpha}\mathcal{L} \to \mathrm{Obs}^q$$

of factorization algebras over $\mathbb{C}[[\hbar]]$*.*

We would like to compare this statement to the classical version of Noether's theorem. Let α^0 denote the reduction of α modulo $\hbar$. Let $\widetilde{\mathcal{L}}_c$ denote the central extension of $\mathcal{L}_c$ arising from α^0. The classical Noether's theorem states that there is a map of precosheaves of L_∞ algebras

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1],$$

where on the right hand side $\widetilde{\mathrm{Obs}}^{cl}[-1]$ is endowed with the structure of a dg Lie algebra via the shifted Poisson bracket on $\widetilde{\mathrm{Obs}}^{cl}$. Furthermore, this map sends the central element in $\widetilde{\mathcal{L}}_c$ to the unit element in $\widetilde{\mathrm{Obs}}^{cl}[-1]$.

We will not use the fact that this arises from an L_∞ map in what follows. It will suffice that the classical Noether map gives rise to a map of precosheaves of cochain complexes $\widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}$. We then note that because Obs^{cl} is a commutative factorization algebra, we automatically get a map of commutative prefactorization algebras

$$\mathrm{Sym}(\widetilde{\mathcal{L}}_c[1]) \to \mathrm{Obs}^{cl}.$$

Furthermore, because the Noether map sends the central element to the unit observable, we get a map of commutative factorization algebras

$$\mathrm{Sym}(\widetilde{\mathcal{L}}_c[1]) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1} \to \mathrm{Obs}^{cl}. \tag{†}$$

In this form, the classical Noether map is similar to the quantum Noether map as expressed in terms of the Rees module $\mathrm{Rees}\, \mathbb{U}_\alpha(\mathcal{L})$. In particular, when we specialize $\hbar$ to zero, we can identify

$$\mathrm{Rees}\, \mathbb{U}_\alpha \mathcal{L}(V) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_{\hbar=0} = \mathrm{Sym}(\widetilde{\mathcal{L}}_c(V)) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}.$$

Hence, we obtain the following dequantization result.

13.5.1.2 Lemma. *The quantum Noether map* $\mathrm{Rees}\, \mathbb{U}_\alpha^\hbar \mathcal{L} \to \mathrm{Obs}^q$ *of factorization algebras becomes, upon setting $\hbar = 0$, the map in equation (†).*

13.5.2 Remarks on twisting by cocycles

Before we begin our proof of quantum Noether's theorem, it may be helpful to discuss, in geometric terms, the meaning of the Chevalley-Eilenberg chains and cochains of an L_∞ algebra twisted by a cocycle.

Recall that if $\mathfrak{g}$ is an L_∞ algebra, then $C^*(\mathfrak{g})$ should be thought of as functions on the formal moduli problem $B\mathfrak{g}$ associated to $\mathfrak{g}$, and $C_*(\mathfrak{g})$ should be viewed as the space of distributions on $B\mathfrak{g}$. Given a cohomology class $\alpha \in H^1(C^*(\mathfrak{g}))$, we then interpret α as specifying a line bundle on $B\mathfrak{g}$ or, equivalently, a rank 1 homotopy representation of $\mathfrak{g}$. Sections of this line bundle are $C^*(\mathfrak{g})$ with the differential $\mathrm{d}_{\mathfrak{g}} - \alpha$, i.e., we change the differential by adding a term given by multiplication by $-\alpha$. We call it a *twisted* differential. Since α is closed and of odd degree, it is automatic that this differential squares to zero. We will often denote this complex by $C^*_\alpha(\mathfrak{g})$.

Similarly, we can define $C_{*,\alpha}(\mathfrak{g})$ to be $C_*(\mathfrak{g})$ with a differential $\mathrm{d}_{\mathfrak{g}} - \iota_\alpha$, given by adding the operator of contracting with $-\alpha$ to the usual differential. Think of $C_{*,\alpha}(\mathfrak{g})$ as the distributions on $B\mathfrak{g}$ twisted by the line bundle associated to α, i.e., the distributions that pair with sections of this line bundle.

Let $\widetilde{\mathfrak{g}}$ be the shifted central extension of $\mathfrak{g}$ associated to α. The center $\mathbb{C}[-1]$ has chains $C_*(\mathbb{C}[-1]) = \mathbb{C}[c]$, where c is the central parameter; it is a commutative algebra as well as a coalgebra. Hence, $C_*(\widetilde{\mathfrak{g}})$ is a module over $\mathbb{C}[c]$, and so we can identify

$$C_*(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1} = C_{*,\alpha}(\mathfrak{g}).$$

A parallel remark holds for cochains.

In particular, for $\mathcal{L}$ a local L_∞ algebra on a manifold X and $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ a local cocycle, we have

$$\mathbb{U}_\alpha \mathcal{L}(V) = C_{*,\alpha}(\mathcal{L}_c(V))$$

for any open subset $V \subset X$. This formula gives a succinct expression for the twisted enveloping factorization algebra.

13.5.3 Proof of the quantum Noether theorem

Now we will turn to the proof of Theorems 13.5.0.1 and 13.5.1.1 and Lemma 13.5.1.2. We begin by describing the map open-by-open, before verifying it respects the full structure of the factorization algebras.

The map on each open subset

The first step is to produce for every open subset $V \subset X$, a cochain map
$$F(V) : C_*^\alpha(\mathcal{L}_c(V)) \to \mathrm{Obs}^q(V)[\hbar^{-1}].$$
And before considering the differentials, we describe a map of graded vector spaces.

Note that a linear map
$$f : \mathrm{Sym}^*(\mathcal{L}_c(V)[1]) \to \mathrm{Obs}^q(V)[\hbar^{-1}]$$
is the same as a collection of linear maps, parametrized by parametrices Φ,
$$f[\Phi] : \mathrm{Sym}^*(\mathcal{L}_c(V)[1]) \to \mathscr{O}(\mathcal{M}(X)[1])((\hbar))$$
that satisfy the renormalization group equation and the locality axiom. It is crucial to note the change of manifold from V to X on the codomain of the map. The locality axiom assures that the maps define a map into observables supported in V.

These $f[\Phi]$, in turn, are the same as a collection of functionals
$$O_f[\Phi] \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(X)[1])((\hbar))$$
satisfying the renormalization group equation and the locality axiom. This identification uses the natural pairing between the symmetric algebra of $\mathcal{L}_c(V)[1]$ and the space of functionals on $\mathcal{L}_c(V)[1]$ to identify a linear map $f[\Phi]$ with a functional $O_f[\Phi]$.

Our goal is thus to find the desired collection $O_f[\Phi]$. Our starting point is the data specifying an action of the local L_∞ algebra $\mathcal{L}$ on our theory, namely the functionals
$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}(\mathcal{L}_c(X)[1] \oplus \mathcal{M}_c(X)[1])[[\hbar]],$$
which satisfy the renormalization group equation and the quantum master equation
$$(\mathrm{d}_{\mathcal{L}} + Q)I^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{I^{\mathcal{L}}[\Phi], I^{\mathcal{L}}[\Phi]\}_\Phi + \hbar\triangle_\Phi\left(I^{\mathcal{L}}[\Phi]\right) = \alpha.$$
(Note that these functionals are supported everywhere on X, so one issue is to whittle the $\mathcal{L}$-dependence down to V.)

It will be convenient to lift functionals on $\mathcal{L}_c(X)[1]$ or $\mathcal{M}_c(X)[1]$ to the larger space $\mathcal{L}_c(X)[1] \oplus \mathcal{M}_c(X)[1]$. To do this, we simply pull back along the projection maps. We will use the notation F to denote the pull-back of some F to a function on all the fields in $\mathcal{L}_c(X)[1] \oplus \mathcal{M}_c(X)[1]$. In particular, if $I[\Phi]$ denotes the effective action of our quantum field theory, which is simply a function of the fields in $\mathcal{M}_c(X)[1]$, we view $I[\Phi]$ as a function on $\mathcal{L}_c(X)[1] \oplus \mathcal{M}_c(X)[1]$.

Thus, consider the collection of functionals

$$\widehat{I}^{\mathcal{L}}[\Phi] = I^{\mathcal{L}}[\Phi] - I[\Phi]$$

in $\mathscr{O}(\mathcal{L}_c(X)[1] \oplus \mathcal{M}_c(X)[1])[[\hbar]]$. This functional satisfies

$$(\mathrm{d}_{\mathcal{L}} + Q)\widehat{I}^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{\widehat{I}^{\mathcal{L}}[\Phi], \widehat{I}^{\mathcal{L}}[\Phi]\}_{\Phi} + \{I[\Phi], \widehat{I}^{\mathcal{L}}[\Phi]\}_{\Phi} + \hbar\triangle_{\Phi}\left(\widehat{I}^{\mathcal{L}}[\Phi]\right) = \alpha, \tag{‡}$$

by construction. Note that this equation is equivalent to the statement that

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_{\Phi})\exp\left(I[\Phi]/\hbar + \widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) = 0,$$

which is an α-twisted and $\mathcal{L}$-equivariant version of the quantum master equation. To properly interpret this assertion, we emphasize that

$$\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) \in \mathscr{O}(\mathcal{L}_c(X)[1] \oplus \mathcal{M}(X)[1])((\hbar)),$$

i.e., negative powers of $\hbar$ appear in a very controlled way. Indeed, although $\hbar^{-1}$ appears in the exponent on the left hand side, each Taylor term of this functional only involves finitely many negative powers of $\hbar$, which is the requirement to live in the space specified on the right hand side.

This exponentiated term also satisfies the renormalization group equation

$$\begin{aligned}\exp\left(\hbar\partial_{P(\Phi)} - \hbar\partial_{P(\Psi)}\right)\exp\left(I[\Psi]/\hbar\right)\exp\left(\widehat{I}^{\mathcal{L}}[\Psi]/\hbar\right)\\ = \exp\left(I[\Phi]/\hbar\right)\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right).\end{aligned}$$

Compare this equality to the renormalization group equation for an observable $\{O[\Phi]\}$ in $\mathrm{Obs}^q(X)$:

$$\exp\left(\hbar\partial_{P(\Phi)} - \hbar\partial_{P(\Psi)}\right)\exp\left(I[\Psi]/\hbar\right)O[\Psi] = \exp\left(I[\Phi]/\hbar\right)O[\Phi].$$

In sum, we have seen that the collection $\exp\left(I[\Psi]/\hbar\right)$ determines a kind of equivariant observable, albeit with Laurent series in $\hbar$.

We wish to use this exponentiated term to produce the desired map $F(V)$ by sending $\ell \in C_*^{\alpha}(\mathcal{L}_c(V))$ to

$$\left\langle \ell, \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\rangle,$$

where $\langle -, - \rangle$ indicates the duality pairing between $C_*^{\alpha}(\mathcal{L}_c(V))$ and $C^*_{\alpha}(\mathcal{L}_c(V))$. This formula is well-defined, but we need to show that the image lands in $\mathrm{Obs}^q(V)[\hbar^{-1}]$, as desired. At the moment, so far as we know, this formula may produce an observable with support throughout the manifold X.

Consider the Taylor components of the functional $\widehat{I}^{\mathcal{L}}$,

$$\widehat{I}^{\mathcal{L}}_{i,k,m}[\Phi] : \mathcal{L}_c(X)^{\otimes k} \times \mathcal{M}_c(X)^{\otimes m} \to \mathbb{C},$$

where the index i denote the coefficients of $\hbar^i$. Such a Taylor term is zero unless $k > 0$, by definition, and, moreover, this term has proper support, which can be made as close as we like to the diagonal by making Φ small. The proper support condition implies that this Taylor term extends to a functional

$$\mathcal{L}_c(X)^{\otimes k} \times \mathcal{M}(X)^{\otimes m} \to \mathbb{C},$$

because we only need one of the inputs has to have compact support and we can choose it to be an $\mathcal{L}$-input. Hence, by restricting to elements of $\mathcal{L}_c(V)$, we obtain a functional

$$\widehat{I}^{\mathcal{L}}[\Phi] \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(X))[[\hbar]],$$

to abuse notation.

We now observe that the renormalization group equation satisfied by $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$ is precisely the one necessary to define, as Φ varies, an element that we denote

$$\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) \in C^*_{\alpha}(\mathcal{L}_c(V), \mathrm{Obs}^q(X))[\hbar^{-1}].$$

The locality property for the functionals $\widehat{I}^{\mathcal{L}}[\Phi]$ tells us that for Φ small these functionals are supported arbitrarily close to the diagonal, and hence

$$\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) \in C^*_{\alpha}(\mathcal{L}_c(V), \mathrm{Obs}^q(V))[\hbar^{-1}].$$

Thus, we have produced a linear map

$$F(V) : C_*^{\alpha}(\mathcal{L}_c(V)) \to \mathrm{Obs}^q(V)[\hbar^{-1}]$$

via the formula

$$F(V)[\Phi](\ell) = \left\langle \ell, \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\rangle,$$

as desired.

We still need to verify that F is a cochain map. Since the duality pairing between Chevalley-Eilenberg chains and cochains of $\mathcal{L}_c(V)$ is a cochain map, even when twisted by α, it suffices to check that the element $\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right)$ is itself closed. But being closed is equivalent to saying that, for each parametrix Φ, the following equation holds:

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_\Phi + \{I[\Phi], -\}_\Phi)\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) = 0.$$

This equation is equivalent to

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_\Phi)\exp\left(I[\Phi]/\hbar + \widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) = 0,$$

which we have already verified. Thus, we have produced a cochain map $F(V)$ from $C_*^\alpha(\mathcal{L}_c(V))$ to $\mathrm{Obs}^q(V)[\hbar^{-1}]$.

Why the construction intertwines with structure maps

It remains to show that this cochain map defines a map of factorization algebras. By construction the map F is a map of precosheaves, as it is compatible with the maps coming from inclusions of open sets $V \subset W$. It remains to check that it is compatible with the structure maps involving disjoint opens.

Let V_1, V_2 be two disjoint subsets of X, but contained in some open W. We need to verify that the following diagram commutes:

$$\begin{tikzcd}
C_*^\alpha(\mathcal{L}_c(V_1)) \times C_*^\alpha(\mathcal{L}_c(V_2)) \arrow[r] \arrow[d, "F(V_1)\times F(V_2)"] & C_*^\alpha(\mathcal{L}_c(W))F(W) \arrow[d] \\
\mathrm{Obs}^q(V_1)[\hbar^{-1}] \times \mathrm{Obs}^q(V_2)[\hbar^{-1}] \arrow[r] & \mathrm{Obs}^q(W)[\hbar^{-1}]
\end{tikzcd},$$

where the horizontal arrows are the structure maps for the factorization algebras. From hereon, we use simply F and suppress explicit dependence on the open set.

Recall that if the O_i are observables for the open sets V_i, respectively, then the factorization product $O_1 O_2 \in \mathrm{Obs}^q(W)$ of these observables is defined by

$$(O_1 O_2)[\Phi] = O_1[\Phi] \cdot O_2[\Phi]$$

for Φ sufficiently small, where $\cdot$ indicates the obvious product on the space of functions on $\mathcal{M}(X)[1]$. (Strictly speaking, we need to check that for each Taylor term this identity holds for sufficiently small parametrices, but we have already discussed this technicality at length and will not belabour it now.)

Let $\ell_i \in C_*^\alpha(\mathcal{L}_c(V_i))$ for $i = 1, 2$. Let $\cdot$ denote the factorization product on the factorization algebra $C_*^\alpha(\mathcal{L}_c)$. It is simply the product in the symmetric algebra on each open set, coupled with the maps coming from the inclusions of open sets. We need to verify that, for Φ sufficiently small,

$$F(\ell_1)[\Phi] \cdot F(\ell_2)[\Phi] = F(\ell_1 \cdot \ell_2)[\Phi]$$

in $\mathscr{O}(\mathcal{M}(X)[1])((\hbar))$.

By choosing a sufficiently small parametrix, we can assume that $\widehat{I}^{\mathcal{L}}[\Phi]$ is supported as close to the diagonal as we like. We can further assume, without loss of generality, that each ℓ_i is a product of elements in $\mathcal{L}_c(V_i)$. Let us write $\ell_i = m_{1i} \cdots m_{k_i i}$ for $i = 1, 2$ and each $m_{ji} \in \mathcal{L}_c(V_i)$. (To extend from this special case to the case of general ℓ_i requires a small functional analysis argument using the fact that F is a smooth map. Since we restrict attention to this special case only for notational convenience, we will not give more details on this point.)

We can explicitly write the map F applied to the elements ℓ_i by the formula

$$F(\ell_i)[\Phi] = \left\{ \frac{\partial}{\partial m_{1i}} \cdots \frac{\partial}{\partial m_{k_i i}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) \right\} \Big|_{0 \times \mathcal{M}(X)[1]}.$$

In other words, we apply the product of all partial derivatives by the elements $m_{ji} \in \mathcal{L}_c(V_i)$ to the function $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$, which is a function on $\mathcal{L}_c(X)[1] \oplus \mathcal{M}(X)[1]$, and then restrict all the $\mathcal{L}_c(V_i)$ variables to zero.

To show

$$F(\ell_1 \cdot \ell_2)[\Phi] = F(\ell_1)[\Phi] \cdot F(\ell_2)[\Phi]$$

for sufficiently small Φ, it suffices to verify that

$$\left\{\frac{\partial}{\partial m_{11}}\cdots\frac{\partial}{\partial m_{k_1 1}}\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}\left\{\frac{\partial}{\partial m_{12}}\cdots\frac{\partial}{\partial m_{k_2 2}}\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}$$
$$= \frac{\partial}{\partial m_{11}}\cdots\frac{\partial}{\partial m_{k_1 1}}\frac{\partial}{\partial m_{12}}\cdots\frac{\partial}{\partial m_{k_2 2}}\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right).$$

Since we are taking derivatives of an exponential, each side can be expanded as a sum of terms, each term of which is a product of factors of the form

$$\frac{\partial}{\partial m_{j_1 i_1}}\cdots\frac{\partial}{\partial m_{j_r i_r}}\widehat{I}^{\mathcal{L}}[\Phi], \tag{†}$$

all multiplied by an overall factor of $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$. In the difference between the two sides, all terms cancel except those which contain a factor of the form expressed in equation (†) with $i_1 = 1$ and $i_2 = 2$. Now, for sufficiently small parametrices,

$$\frac{\partial}{\partial m_{j_1 1}}\frac{\partial}{\partial m_{j_2 2}}\widehat{I}^{L}[\Phi] = 0$$

because $\widehat{I}^{\mathcal{L}}[\Phi]$ is supported as close as we like to the diagonal and $m_{j_1 1} \in \mathcal{L}_c(V_1)$ and $m_{j_2 2} \in \mathcal{L}_c(V_2)$ have disjoint support.

Thus, we have constructed a map

$$F : \mathbb{U}_\alpha \mathcal{L} \to \mathrm{Obs}^q[\hbar^{-1}]$$

of factorization algebras.

Finishing the proof

It remains to check the content of Theorem 13.5.1.1 and of Lemma 13.5.1.2. Let be V any open subset X. For Theorem 13.5.1.1, we need to verify that if $\ell \in \mathrm{Sym}^k(\mathcal{L}_c(V))$, then $F(\ell) \in \hbar^{-k}\mathrm{Obs}^q(V)$. That is, we need to check that for each parametrix Φ, we have

$$F(\ell)[\Phi] \in \hbar^{-k}\mathscr{O}(\mathcal{M}(X)[1])[[\hbar]].$$

Let us assume, for simplicity, that $\ell = m_1 \cdots m_k$ where $m_i \in \mathcal{L}_c(V)$. Then the explicit formula

$$F(\ell)[\Phi] = \left\{\partial_{m_1}\cdots\partial_{m_k}\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}\Big|_{\mathcal{M}(X)[1]}$$

makes it clear that $\hbar^{-k}$ is the largest negative power of $\hbar$ to appear. (Note that $\widehat{I}^{\mathcal{L}}[\Phi]$ is zero when restricted to a function of just $\mathcal{M}(X)[1]$.)

Finally, we need to check Lemma 13.5.1.2, which states that the classical limit of our quantum Noether map is the classical Noether map we constructed earlier. Let $\ell \in \mathcal{L}_c(V)$. Then the classical limit of our quantum Noether map sends ℓ to the classical observable

$$\begin{aligned}\lim_{\hbar\to 0} \hbar F(\ell) &= \lim_{\Phi\to 0}\lim_{\hbar\to 0}\left\{\hbar\partial_\ell \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}\Big|_{\mathcal{M}(X)[1]}\\ &= \lim_{\Phi\to 0}\left\{\partial_\ell I^{\mathcal{L}}_{classical}[\Phi]\right\}\Big|_{\mathcal{M}(X)[1]}\\ &= \left\{\partial_\ell I^{\mathcal{L}}_{classical}\right\}\Big|_{\mathcal{M}(X)[1]}.\end{aligned}$$

Note that $I^{\mathcal{L}}_{classical}[\Phi]$ means the scale Φ version of the functional on $\mathcal{L}[1]\oplus\mathcal{M}[1]$ defining the inner action (at the classical level) of $\mathcal{L}$ on our classical theory, and by $I^{\mathcal{L}}_{classical}$ we mean the scale zero version. Hence, our classical Noether map is the map appearing in the last line of the above displayed equation.

13.6 Trivializing the action on factorization homology

Let X be a closed manifold and on it lives a field theory with an action of a local L_∞ algebra $\mathcal{L}$. Suppose that some class $\alpha \in H^1(C^*_{loc}(\mathcal{L}))[[\hbar]]$ encodes the obstruction to lifting this action to an inner action. Because $\mathcal{L}$ acts on observables, its global sections $\mathcal{L}(X)$ have an L_∞ action on the global quantum observables $\mathrm{Obs}^q(X)$. Moreover, the global sections $\mathcal{L}(X)$ have an L_∞ action on $\mathbb{C}[[\hbar]]$ by the class α; we use $\mathbb{C}_\alpha$ to denote this representation of $\mathcal{L}(X)$.

13.6.0.1 Lemma. *If $\mathcal{L}$ is elliptic, then the action of $\mathcal{L}(X)$ on $\mathrm{Obs}^q(X)\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}_{-\alpha}$ is canonically trivial, once we invert $\hbar$.*

Proof Let $\mathrm{Obs}^q(X)^{\mathcal{L}(X)}$ denote the equivariant observables. We need to show that there is a quasi-isomorphism of $C^*(\mathcal{L}(X))$-modules

$$\mathrm{Obs}^q(X)^{\mathcal{L}(X)}\otimes\mathbb{C}_{-\alpha}\simeq \mathrm{Obs}^q(X)\otimes C^*(\mathcal{L}(X)).$$

Note that issues of completion and topological tensor products are not so important here because the cohomology of $\mathcal{L}(X)$ is finite dimensional. It is only at this point that we use the ellipticity of $\mathcal{L}$.

Tensoring both sides by $\mathbb{C}_{-\alpha}$, we see we need to show that

$$\mathrm{Obs}^q(X)^{\mathcal{L}(X)} \simeq \mathrm{Obs}^q(X) \otimes C^*(\mathcal{L}(X)) \otimes \mathbb{C}_\alpha.$$

Choose a parametrix Φ with its associated BV bracket $\{-,-\}_\Phi$, BV Laplacian $\triangle_\Phi$, interaction $I[\Phi]$, and equivariant interaction $I^{\mathcal{L}}[\Phi]$ (retaining only those terms that depend on $\mathcal{L}$). The representative $\alpha[\Phi]$ of α computed using the parametrix Φ is the failure of $I[\Phi] + I^{\mathcal{L}}[\Phi]$ to satisfy the quantum master equation:

$$\begin{aligned}\alpha[\Phi] = (Q + \mathrm{d}_{\mathcal{L}})(I[\Phi] + I^{\mathcal{L}}[\Phi]) + \tfrac{1}{2}\{I[\Phi] + I^{\mathcal{L}}[\Phi], I[\Phi] + I^{\mathcal{L}}[\Phi]\}_\Phi \\ + \hbar\triangle_\Phi(I[\Phi] + I^{\mathcal{L}}[\Phi]).\end{aligned}$$

Here Q is the term in the differential on $\mathrm{Obs}^q(X)$ coming from the kinetic term in the Lagrangian, and $\mathrm{d}_{\mathcal{L}}$ is the Chevalley-Eilenberg differential for Lie algebra cochains of $\mathcal{L}$.

Note that by assumption $\alpha[\Phi]$ is a function only of elements of $\mathcal{L}(X)$.

In the desired quasi-isomorphism, the differential on the left hand side is

$$Q + \mathrm{d}_{\mathcal{L}} + \{I[\Phi], -\}_\Phi + \{I^{\mathcal{L}}[\Phi], -\}_\Phi + \triangle_\Phi.$$

The differential on the right hand side is

$$Q + \mathrm{d}_{\mathcal{L}} + \{I[\Phi], -\}_\Phi + \triangle_\Phi + \alpha[\phi],$$

where we are multiplying by $\alpha[\phi]$ as part of the differential. The two differentials are related by conjugating by $e^{I^{\mathcal{L}}[\phi]/\hbar}$, which therefore provides a quasi-isomorphism of cochain complexes once we have inverted $\hbar$. Note that $e^{I^{\mathcal{L}}[\phi]/\hbar}$ makes sense, because at order k as a function on $\mathcal{L}(X)$, there are only finitely many negative powers of $\hbar$. $\square$

13.7 Noether's theorem and the local index theorem

In this section we will explain how Noether's theorem gives rise to a definition of the *local index* of an elliptic complex with an action of a local L_∞ algebra. Our analysis follows very closely the two beautiful papers Rabinovich (2019, 2020), and we refer to these works for details not presented here.

13.7.1 Rephrasing the local index

Let us explain what we mean by the local index. Suppose we have an elliptic complex on a closed manifold X. Let $\mathscr{E}(U)$ denote the cochain complex of sections of this elliptic complex on an open subset $U \subset X$. The cohomology of $\mathscr{E}(X)$ is finite-dimensional, and the index of our elliptic complex is defined to be the Euler characteristic of this cohomology. We write

$$\mathrm{Ind}(\mathscr{E}(X)) = \mathrm{STr}_{H^*(\mathscr{E}(X))}(\mathrm{Id}),$$

that is, the index is the super-trace (or graded trace) of the identity operator on cohomology.

More generally, if $\mathfrak{g}$ is a Lie algebra acting on global sections of our elliptic complex $\mathscr{E}(X)$, then we can consider the character of $\mathfrak{g}$ for this representation $H^*(\mathscr{E}(X))$. If $x \in \mathfrak{g}$ is any element, the character is

$$\mathrm{Ind}(x, \mathscr{E}(X)) = \mathrm{STr}_{H^*(\mathscr{E}(X))}(x).$$

The usual index is the special case when $\mathfrak{g}$ is the one-dimensional Lie algebra acting on $\mathscr{E}(X)$ by scaling: $x \in \mathbb{C}$ maps to $x\,\mathrm{Id}$.

This approach admits a natural reformulation, relating a graded trace to a graded determinant. Recall that the *determinant* of the cohomology $H^*(\mathscr{E}(X))$ means

$$\det H^*(\mathscr{E}(X)) = \bigotimes_i \left\{ \det H^i(\mathscr{E}(X)) \right\}^{(-1)^i},$$

which is a super-line: it is even or odd depending on whether the Euler characteristic of $H^*(\mathscr{E}(X))$ is even or odd.

For any endomorphism T of $H^*(\mathscr{E}(X))$, the super-trace of T is the same as the super-trace of T acting on the determinant of $H^*(\mathscr{E}(X))$. It follows that the character of $\mathscr{E}(X)$ as a representation of a Lie algebra $\mathfrak{g}$ is encoded entirely in the natural action of $\mathfrak{g}$ on the determinant

$\det H^*(\mathscr{E}(X))$. In other words, the character of the $\mathfrak{g}$ action on $\mathscr{E}(X)$ is the same data as the one-dimensional $\mathfrak{g}$-representation $\det H^*(\mathscr{E}(X))$.

Now suppose that $\mathfrak{g}$ is global sections of a sheaf $\mathcal{L}$ of dg Lie algebras (or L_∞ algebras) on X. We will further assume that $\mathcal{L}$ is a local L_∞ algebra and that the action of $\mathfrak{g} = \mathcal{L}(X)$ on $\mathscr{E}(X)$ arises from a local action of the sheaf $\mathcal{L}$ of L_∞ algebras on the sheaf $\mathscr{E}$ of cochain complexes.

One can then ask the following question: can the character of the $\mathcal{L}(X)$ action on $\mathscr{E}(X)$ be expressed in a *local* way on the manifold? Given that the character of this $\mathcal{L}(X)$-representation is entirely expressed in the $\mathcal{L}(X)$-action on its determinant, this question is equivalent to the following one: can we express the determinant $\det H^*(\mathscr{E}(X))$ in a local way on the manifold X, indeed, in an $\mathcal{L}$-equivariant way?

As $\mathscr{E}(X)$ is a sheaf, we can certainly describe $\mathscr{E}(X)$ in a way local on X. Informally, we might imagine $\mathscr{E}(X)$ as a direct sum of its fibres over the points in X. More formally, if we choose a cover $\mathfrak{U}$ of X, then the Čech double complex for $\mathfrak{U}$ with coefficients in the sheaf $\mathscr{E}$ produces for us a cochain complex quasi-isomorphic to $\mathscr{E}(X)$. This double complex is an additive expression describing $\mathscr{E}(X)$ in terms of sections of $\mathscr{E}$ in the open cover $\mathfrak{U}$ of X. Heuristically, the Čech double complex gives a formula of the form

$$\mathscr{E}(X) \sim \sum_i \mathscr{E}(U_i) - \sum_{i,j} \mathscr{E}(U_i \cap U_j) + \sum_{i,j,k} \mathscr{E}(U_i \cap U_j \cap U_k) - \cdots,$$

which we can view as the analog of the inclusion-exclusion formula from combinatorics. If $\mathfrak{U}$ is a finite cover and each $\mathscr{E}(U)$ has finite-dimensional cohomology, this formula becomes an identity upon taking Euler characteristics.

Since X is compact, one can also view $\mathscr{E}(X)$ as the global sections of the cosheaf of compactly supported sections of $\mathscr{E}$, and then Čech homology gives us a similar expression.

The determinant functor on the category of graded vector spaces sends direct sums to tensor products. We thus could imagine that the determinant $\det H^*(\mathscr{E}(X))$ can likewise be expressed in a local way on the manifold X, but where the direct sums that appear in sheaf theory are replaced by tensor products.

Factorization algebras have the feature that the value on a disjoint union is a tensor product (rather than a direct sum as appears in sheaf theory). That is, factorization algebras are multiplicative versions of cosheaves. It is therefore natural to express that the determinant of the cohomology of $\mathscr{E}(X)$ can be realized as global sections of a factorization algebra, just as $\mathscr{E}(X)$ is global sections of a cosheaf.

It turns out that this is the case: we can indeed "factorize" the determinant.

13.7.1.1 Lemma. *Let $\mathscr{E}$ be any elliptic complex on a closed manifold X. Consider the free cotangent theory to the Abelian elliptic Lie algebra $\mathscr{E}[-1]$, whose elliptic complex of fields is $\mathscr{E} \oplus \mathscr{E}^![-1]$. Let $\mathrm{Obs}^q_{\mathscr{E}}$ denote the factorization algebra of observables of this theory.*

There is an isomorphism

$$H^*(\mathrm{Obs}^q_{\mathscr{E}}(X)) \cong \det H^*(\mathscr{E}(X))[d]$$

where d is equal to the Euler characteristic of $H^*(\mathscr{E}(X))$ modulo 2.

This lemma states that the cohomology of global observables of the theory is the determinant of the cohomology of $\mathscr{E}(X)$, with its natural $\mathbb{Z}/2$ grading. The proof of this lemma, although easy, will be given at the end of this section. A more refined version of this statement has been proved in Rabinovich (2020): Rabinovich shows that for any family of Dirac operators over a smooth base B, this construction yields a dg vector bundle quasi-isomorphic to the Quillen determinant line of that family.

This lemma shows that the factorization algebra $\mathrm{Obs}^q_{\mathscr{E}}$ provides a local version of $\det H^*(\mathscr{E}(X))$, the determinant of the cohomology of $\mathscr{E}(X)$. With this interpretation in hand, we can ask for a local version of the index living in $\mathrm{Obs}^q_{\mathscr{E}}$.

Let $\mathcal{L}$ be a local L_∞ algebra on X that acts linearly on $\mathscr{E}$. Then $\mathcal{L}$ acts on the corresponding free field theory and hence on the factorization algebra that localizes the determinant. This $\mathcal{L}$-action may not be inner: there is a class

$$\alpha \in H^1(C^*_{loc}(\mathcal{L}))$$

that captures the obstruction to having an inner action. (Note that here, we are setting $\hbar = 1$, which works because we are considering the action on a free field theory.)

When X is compact, there is a map

$$C^*_{loc}(\mathcal{L}) \to C^*(\mathcal{L}(X))$$

under which the class α gets sent to some element

$$\alpha(X) \in H^1(C^*(\mathcal{L}(X))).$$

The class $\alpha(X)$ is the same as a rank one representation of $\mathcal{L}(X)$, up to isomorphism.

13.7.1.2 Lemma. *The class $\alpha(X)$ is represented by the action of $\mathcal{L}(X)$ on the determinant of $H^*(\mathscr{E}(X))$.*

Proof This claim follows from lemma 13.7.1.1, which states that the cohomology of $\mathrm{Obs}^q_{\mathscr{E}}$ is the determinant of $\mathscr{E}(X)$, and from lemma 13.6.0.1, stating that the character of the action of $\mathcal{L}(X)$ on $\mathrm{Obs}^q_{\mathscr{E}}(X)$ is given by $\alpha(X)$. □

This result justifies the following definition.

13.7.1.3 Definition. *In this situation, the* local index *is the class $\alpha \in C^*_{loc}(\mathcal{L})$ that controls the central extension in Noether's theorem.*

13.7.2 The local index for a Dirac operator

So far, we have shown how the determinant line of an elliptic complex can be expressed locally as a factorization algebra. Furthermore, the action of the global sections of a sheaf of Lie algebras $\mathcal{L}$ on the determinant is determined by the cocycle α giving the central extension. To demonstrate how this approach relates to more traditional versions of index theory, we turn to a setting where we can demonstrate the cocycle α is the standard index density.

First, we need to recall the set-up of Berline et al. (1992), where a general class of Dirac operators is defined. Let M be an oriented Riemannian manifold of dimension $2n$. Let Cl_M denote the Clifford algebra on $TM \otimes_{\mathbb{R}} \mathbb{C}$, and let $E = E^0 \oplus E^1$ be a $\mathbb{Z}/2$-graded Cl_M-module with a

Clifford connection and compatible Hermitian structure. As in Berline et al. (1992), this data defines a Dirac operator $\not{\partial} : E \to E$, which is odd for the $\mathbb{Z}/2$-grading. The square $\not{\partial}^2$ is a generalized Laplacian.

We use $\mathscr{E}$ to denote the two-term elliptic complex

$$\mathscr{E}^0 = C^\infty(X, E^0) \xrightarrow{\not{\partial}} C^\infty(X, E^1) = \mathscr{E}^1.$$

As before, we consider the free cotangent field theory whose fields are $\mathcal{F} = \mathscr{E} \oplus \mathscr{E}^![-1]$. In this theory, a field of degree 0 is a section of $E^0 \oplus (E^1)^\vee \otimes \omega_X$, while a field of degree 1 is a section of $E^1 \oplus (E^0)^\vee \otimes \omega_X$. We use ϕ_0 for an element of $\mathscr{E}^0(X)$ and ϕ_1 for an element of $(\mathscr{E}^1)^!(X)$, and in degree 1 we use ψ_0 for an element of $(\mathscr{E}^0)^!(X)$ andψ_1 for an element of $\mathscr{E}^1(X)$. This free theory has

$$\int \mathrm{Tr}_E\, \phi_1 \not{\partial} \phi_0$$

as its action functional.

We make the scaling symmetry on $\mathscr{E}$ local by working with $\mathcal{L} = \Omega^*_X$, the de Rham resolution of the constant sheaf $\underline{\mathbb{C}}$ of Abelian Lie algebras. The scaling action of $\mathbb{C}$ determines a homotopy action of its resolution $\mathcal{L}$. We encode the action of $\mathcal{L}$ via an action functional that depends both on the fields of our free theory as well as on an element $\rho \in \mathcal{L}[1]$.

Let ρ^0 denote the 0-form component of ρ, and ρ^1 the 1-form component of ρ. The classical Lagrangian incorporating the background field ρ is

$$\int \mathrm{Tr}_E\, \phi_1 \not{\partial}_{\rho^1} \phi_0 + \int \rho^0\, \mathrm{Tr}_E\, (\phi_0\psi_0 + \phi_1\psi_1)$$

where $\not{\partial}_{\rho^1}$ is the covariantized version of the Dirac operator. It is straightforward to show that this action solves the equivariant master equation relevant for defining an action of $\mathcal{L}$ on the free theory. (It is important to note that all Feynman diagrams that are not trees have the fields ρ on their external lines. These Feynman diagrams do not contribute to the master equation for an action, but they will when we ask if we have an inner action.)

The obstruction to lifting this action to an inner action will be some cocycle $\alpha \in C^1_{red,loc}(\mathcal{L})$. That is, α will be a local functional of cohomological degree 1 depending solely on the field $\rho \in \Omega^*_X[1]$. Because we are dealing with an Abelian Lie algebra, the cocycle condition on α simply means that α is closed under the de Rham operator.

Let $\alpha^{(1)}$ denote the component of α that is linear as a function of ρ. As $\alpha^{(1)}$ is of cohomological degree 1, it is a functional only of ρ^0, and so we can interpret $\alpha^{(1)}$ as an n-form. (The components of α that are non-linear in ρ can be shown to vanish in cohomology.)

13.7.2.1 Theorem (Rabinovich (2020)). *In this situation, the obstruction cocycle $\alpha^{(1)}$ is the index density defined using the heat kernel, as in Berline et al. (1992).*

Remark: One aspect of this story, and indeed of the local index theorem in general, is that $\alpha^{(1)}$ is most naturally viewed as a degree 1 element of the cochain complex $\Omega^*(X)[n-1]$. As such, if X is non-compact, it is possible to trivialize the obstruction. It remains true, however, that computing the representative of $\alpha^{(1)}$ that naturally arises from the field theory analysis gives precisely the local index density. ◊

The proof follows that presented in Rabinovich (2020), and we refer the reader there for further details. We will assume familiarity with the techniques developed in Costello (2011b) and in the first volume of this work, but we briefly review what we need before embarking on the proof proper.

The differential on the space of fields is a sum of two terms:

$$\begin{array}{rcl} Q : \mathscr{E}^0 & \xrightarrow{\not{\partial}} & \mathscr{E}^1 \\ Q : (\mathscr{E}^1)^! & \xrightarrow{\not{\partial}^!} & (\mathscr{E}^0)^! \end{array}$$

where $\not{\partial}^!$ is the formal adjoint to the Dirac operator. We choose the gauge fixing operator

$$\begin{array}{rcl} Q^{GF} : \mathscr{E}^1 & \xrightarrow{\not{\partial}} & \mathscr{E}^0 \\ Q^{GF} : (\mathscr{E}^0)^! & \xrightarrow{\not{\partial}^!} & (\mathscr{E}^1)^! \end{array}$$

The commutator $[Q, Q^{GF}]$ is the generalized Laplacian $D = (\not{\partial})^2$ on $\mathscr{E}^0$ and $\mathscr{E}^1$, and its formal adjoint on $(\mathscr{E}^0)^\vee$ and $(\mathscr{E}^1)^\vee$.

Associated to the generalized Laplacian D is its heat kernel

$$K_t \in \mathscr{E}^0 \widehat{\otimes} (\mathscr{E}^0)^! \oplus \mathscr{E}^1 \widehat{\otimes} (\mathscr{E}^1)^!.$$

Letting $\mathcal{F}$ denote the total space of fields, we note that the heat kernel K_t is an element in degree 1 in $\mathcal{F}\widehat{\otimes}\mathcal{F}$. A parametrix for the Laplacian is given by $\int_0^L K_t \mathrm{d}t$. The corresponding propagator is $\int_0^L (Q^{GF} \otimes 1) K_t \mathrm{d}t$, and the BV Laplacian associated to the parametrix is $\triangle_L = K_L$.

Proof Our goal is to study the obstruction to quantizing this system at linear order in ρ. First, we need to compute the effective interaction. Since we are only interested in expressions linear in ρ, we need only consider Feynman diagrams whose external lines contain exactly one ρ. There are two kinds of these diagrams: the tree-level interactions appearing in the Lagrangian above, and a one-loop diagram with a single external line labelled by ρ^1.

Let $I^{(1)}[L]$ denote the effective interaction at scale L, to linear order in the external fields ρ, and expand

$$I^{(1)}[L] = I_0^{(1)}[L] + \hbar I_0^{(1)}[L].$$

Our goal is to calculate $I^{(1)}[L]$ explicitly, and check whether or not it satisfies the quantum master equation. From the classical Lagrangian, we see that the tree level part is

$$I_0^{(1)}[L] = \int \mathrm{Tr}_E\, \phi_1 \rho_1 \phi_0 + \int \rho^0\, \mathrm{Tr}_E\, (\phi_0 \psi_0 + \phi_1 \psi_1)\,,$$

where we have implicitly used Clifford multiplication with the one-form ρ_1 to define $\mathrm{Tr}_E\, \phi_1 \rho_1 \phi_0$.

The one-loop term $I_1^{(1)}[L]$ is a little more tricky to compute. We need to use the method of counter-terms as developed in Costello (2011b). By introducing a cut-off at length scale ϵ, we can form a regularized version of $I_1^{(1)}[L]$ by

$$I_1^{(1)}[\epsilon, L](\rho_1) = \int_{t=\epsilon}^{L} \int_{x \in X} \mathrm{STr}_E\, \rho_1(x) \left(\partial\!\!\!/_x K_t(x,y)\right)_{y=x} \mathrm{d}t.$$

As in Costello (2011b), we define a counter-term

$$I^{CT}[\epsilon](\rho_1) = \mathrm{Sing}_\epsilon\, I_1^{(1)}[\epsilon, L](\rho_1)$$

to be the singular part as a function of ϵ. (An explicit expression for I^{CT} can in principle be derived from the asymptotic expansion of the heat kernel, but the explicit form of the counter-term plays no role in our argument.) We then define

$$I_1^{(1)}[L](\rho_1) = \lim_{\epsilon\to 0}\left(I_1^{(1)}[\epsilon, L](\rho_1) - I^{CT}[\epsilon](\rho_1)\right).$$

We now want to check whether the equivariant quantum master equation holds.

The relevant quantum master equation is

$$\mathrm{d}_{dR} I_1^{(1)}[L] + \triangle_L I_0^{(1)} = 0,$$

where d_{dR} is the differential induced from the de Rham operator on $\mathcal{L} = \Omega_X^*$. Thanks to the compatibility between the BV Laplacian and the RG flow,

$$\mathrm{d}_{dR} I^{(1)}[\epsilon, L] = \triangle_\epsilon I_0^{(1)} - \triangle_L I_0^{(1)}.$$

Therefore, the obstruction to solving the master equation is

$$\alpha^{(1)}(\rho_0) = \lim_{\epsilon\to 0}\left(\triangle_\epsilon I_0^{(1)}(\rho_0) - I^{CT}[\epsilon](\mathrm{d}\rho_0)\right).$$

Now,

$$\triangle_\epsilon I_0^{(1)}(\rho_0) = \int_{x\in X} \rho_0(x)\,\mathrm{STr}_E\, K_\epsilon(x,x).$$

According to results explained in Berline et al. (1992) (originally due to Patodi (1971) and Gilkey (1995)), this expression has a well-defined $\epsilon \to 0$ limit known as the index density:

$$\lim_{\epsilon\to 0} \triangle_\epsilon I_0^{(1)}(\rho_0) = (2\pi\imath)^{-n/2}\int \rho^0 \widehat{A}(M)\,\mathrm{ch}(E/S).$$

Here we follow the notation of chapter 4 in Berline et al. (1992),: $\widehat{A}(M)$ is the usual A-hat class, and $\mathrm{ch}(E/S)$ is the “relative” version of the Chern character of the bundle E that appears in the index theorem.

We have almost finished identifying the obstruction with the local index, except that we have an extra term $I^{CT}[\epsilon](\mathrm{d}\rho_0)$. Thankfully, this term must vanish because it is a purely singular function of ϵ (in this context, it is a linear combination of $\log\epsilon$ and ϵ^{-k} for $k > 0$). Furthermore, the $\epsilon \to 0$ limit appearing in the calculation of the obstruction exists, as part of the general framework of Costello (2011b). Finally, the results of Patodi and Gilkey tell us that $\lim_{\epsilon\to 0} I_0^{(1)}(\rho_0)$ exists, which completes the proof. □

13.7.3 Proof of Lemma 13.7.1.1

Before we give the (simple) proof, we should clarify some small points.

Recall that for a free theory, there are two different versions of quantum observables we can consider. We can take our observables to be polynomial functions on the space of fields, and not introduce the formal parameter $\hbar$; or we can take our observables to be formal power series on the space of fields, in which case one needs to introduce the parameter $\hbar$. These two objects encode the same information: the second construction is obtained by applying the Rees construction to the first construction. We will give the proof for the first (polynomial) version of quantum observables. A similar statement holds for the second (power series) version, but one needs to invert $\hbar$ and tensor the determinant of cohomology by $\mathbb{C}((\hbar))$.

Globally, polynomial quantum observables can be viewed as the space $P(\mathscr{E}(X))$ of polynomial functions on $\mathscr{E}(X)$, with a differential that is the sum of the linear differential Q on $\mathscr{E}(X)$ with the BV operator. Let us compute the cohomology by a spectral sequence associated to a filtration of $\mathrm{Obs}^q_{\mathscr{E}}(X)$. The filtration is the obvious increasing filtration obtained by declaring that

$$F^i \mathrm{Obs}^q_{\mathscr{E}}(X) = \mathrm{Sym}^{\leq i}(\mathscr{E}(X) \oplus \mathscr{E}^!(X)[-1])^\vee.$$

The first page of this spectral sequence is cohomology of the associated graded complex. This associated graded is simply the symmetric algebra

$$H^* \mathrm{Gr}\, \mathrm{Obs}^q_{\mathscr{E}}(X) = \mathrm{Sym}^* \left(H^*(\mathscr{E}(X))^\vee \oplus H^*(\mathscr{E}^!(X)[-1])^\vee \right).$$

The differential on this page of the spectral sequence comes from the BV operator associated to the non-degenerate pairing between $H^*(\mathscr{E}(X))$ and $H^*(\mathscr{E}^!(X))[-1]$. Note that $H^*(\mathscr{E}^!(X))$ is the dual to $H^*(\mathscr{E}(X))$.

It remains to show that the cohomology of this secondary differential yields the determinant of $H^*(\mathscr{E}(X))$, with a shift.

We show this claim by treating a more general problem. Given any finite-dimensional graded vector space V, we can give the algebra $P(V \oplus V^*[-1])$ of polynomial functions on $V \oplus V^*[-1]$ a BV operator $\triangle$ arising from the pairing between V and $V^*[-1]$. We want to produce an isomorphism

$$H^*(P(V \oplus V^*[-1]), \triangle) \cong \det(V)[d]$$

where the shift d is equal modulo 2 to the Euler characteristic of V.

Sending V to $H^*(P(V \oplus V^*[-1]), \triangle)$ is a functor from the groupoid of finite-dimensional graded vector spaces and isomorphisms between them, to the category of graded vector spaces. It sends direct sums to tensor products. It follows that to check whether or not this functor recovers the determinant functor, one needs to check that it does in the case that V is a graded line.

Thus, let us assume that $V = \mathbb{C}[k]$ for some $k \in \mathbb{Z}$. We will check that our functor returns $V[1]$ if k is even and V^* if k is odd. Thus, viewed as a $\mathbb{Z}/2$ graded line, our functor returns $\det V$ with a shift by the Euler characteristic of V.

To check this, note that

$$P(V \oplus V^*[-1]) = \mathbb{C}[x, y]$$

where x is of cohomological degree k and y is of degree $-1-k$. The BV operator is

$$\triangle = \frac{\partial}{\partial x}\frac{\partial}{\partial y}.$$

A simple calculation shows that the cohomology of this complex is 1 dimensional, spanned by x if k is odd and by y if k is even. Since x is a basis of V^* and y is a basis of V, this completes the proof.

Remark: It is possible to push this BV approach to the determinant functor farther: Gwilliam and Haugseng (2018) lifts it to a natural level of generality in derived algebraic geometry. $\Diamond$

13.8 The partition function and the quantum Noether theorem

Our formulation of the quantum Noether theorem goes beyond a statement just about symmetries, in the traditional sense of the word. It also involves *deformations*, which are symmetries of cohomological degree

1, as well as symmetries of other cohomological degree. Thus, it has important applications when we consider families of field theories.

The first application we will explain is that the quantum Noether theorem leads to a definition of the *partition function* of a perturbative field theory.

Suppose we have a family of field theories that depends on a formal parameter c, the coupling constant. (Everything we say below will work when the family depends on a number of formal parameters, or indeed on a pro-nilpotent dg algebra.) For example, we could start with a free theory and deform it to an interacting theory. An example of such a family of scalar field theories is given by the action functional

$$S(\phi) = \int \phi(\triangle + m^2)\phi + c\phi^4.$$

We can view such a family of theories as being a *single* theory — in this case the free scalar field theory — with an action of the Abelian L_∞ algebra $\mathbb{C}[1]$. Indeed, by definition, an action of an L_∞ algebra $\mathfrak{g}$ on a theory is a family of theories over the dg ring $C^*(\mathfrak{g})$ that specializes to the original theory upon reduction by the maximal ideal $C^{>0}(\mathfrak{g})$.

We have seen in Lemma 12.2.4.2 that actions of $\mathfrak{g}$ on a theory are the same thing as actions of the local Lie algebra $\Omega^*_X \otimes \mathfrak{g}$. In this way, we see that a family of theories over the base ring $\mathbb{C}[[c]]$ is the same thing as a single field theory with an action of the local abelian L_∞ algebra $\Omega^*_X[-1]$.

We will formulate our definition of the partition function in the quite general context of a field theory acted on by a local L_∞ algebra. Afterwards, we will analyze what it means for a family of field theories depending on a formal parameter c.

A partition function is only defined for field theories with some special properties. Geometrically, we need to be perturbing around an isolated solution to the equations of motion on a closed manifold X. This situation holds, for instance, with a massive scalar field theory. In the setting of our book, given a classical theory described by a local L_∞ algebra $\mathcal{M}$ with an invariant pairing of degree -3, we require X to be a closed manifold and that the global sections of the underlying elliptic complex have trivial cohomology, i.e., $H^*(\mathcal{M}(X)) = 0$. (In other words, the linearized equations of motion have an isolated solution.)

These assumptions imply that $H^*(\mathrm{Obs}^q(X)) \cong \mathbb{C}[[\hbar]]$. There is thus a preferred $\mathbb{C}[[\hbar]]$-linear isomorphism that sends the identity observable $1 \in H^0(\mathrm{Obs}^q(X))$ to a basis vector of $\mathbb{C}[[\hbar]]$. These conditions hold, for instance, if we have a free massive scalar field theory on closed manifold X, which is a good example to bear in mind.

Now suppose that we also have an inner action of a local L_∞ algebra $\mathcal{L}$ on our theory. The Noether map gives us a cochain map

$$C_*(\mathcal{L}(X)) \to \mathrm{Obs}^q(X) \simeq \mathbb{C}[[\hbar]],$$

or, equivalently, an element of $C^*(\mathcal{L}(X))[[\hbar]]$.

13.8.0.1 Definition. *The* partition function *is the element in* $C^*(\mathcal{L}(X))[[\hbar]]$ *given by the Noether map.*

Recall that an action of $\mathcal{L}$ on our theory is the same as a family of theories over the sheaf of formal moduli problem $B\mathcal{L}$. Functions on the global sections of this sheaf of formal moduli problems are $C^*(\mathcal{L}(X))$. Thus, the partition function is, as we would expect, a function on the moduli space $B\mathcal{L}$ of parameters of our theory. In our definition, the partition function is normalized so that it takes value 1 at the base point of the formal moduli problem $B\mathcal{L}(X)$.

The concept of partition function is most useful when the local L_∞ algebra $\mathcal{L}$ is the Abelian Lie algebra $\Omega^*_X[-1]$. The corresponding formal moduli problem is the sheaf of cochain complexes Ω^*_X, which at the level of global sections is equivalent to $H^*(X)$. The partition function in this case is a function on this formal moduli problem of cohomological degree 0, and so a function on $H^0(X)$. Assuming that X is connected, the partition function is then a series in a single variable c, which is the coupling constant.

14
Examples of the Noether theorems

This chapter demonstrates with explicit examples how to use the factorization Noether theorems developed in the preceding chapters. We stick with free theories, where computations are much simpler, and examine how angular momentum appears in a one-dimensional theory and how the Virasoro symmetry appears in a chiral conformal field theory. Moving beyond ordinary Lie algebras, we also examine a kind of extended, higher symmetry in a simple class of topological field theories.

More sophisticated examples of these constructions are also available in the literature. We have already mentioned Rabinovich (2019), which analyzes systematically the axial anomaly for fermionic theories following the style of this book. Gwilliam et al. (2020) applies the Noether theorems in the setting of Gelfand-Kazhdan geometry, showing how these techniques recover chiral differential operators by a globalization process pioneered by Kontsevich (2003) and Cattaneo et al. (2002). Noether's theorems admit interesting applications to higher-dimensional holomorphic field theories Gwi (n.d.); Saberi and Williams (2020, 2019).

14.1 Examples from mechanics

We will focus here on a simple free theory to exhibit the basic techniques, but we will comment along the way about how adding interactions would affect the situation.

Consider a free scalar field living on the real line. Its fields consist of smooth maps $\phi : \mathbb{R} \to V$, where V is a real vector space with inner product $(-,-)$. A free theory is specified by the action functional

$$S_0(\phi) = \int (\dot{\phi}, \dot{\phi}) \mathrm{d}t,$$

whose equation of motion is then $\ddot{\phi} = 0$. It is also natural to consider adding a mass term to this scalar field

$$S_m(\phi) = \int (\dot{\phi}, \dot{\phi}) \mathrm{d}t + m^2 \int (\phi, \phi) \mathrm{d}t,$$

which, in the language of mechanics, corresponds to a harmonic oscillator with equation $\ddot{\phi} = m^2 \phi$. (Hence the mass m is a spring constant.) For simplicity, we will restrict to the massless case, but it is straightforward to modify our arguments to the massive case.

In Chapter I.4 we analyzed these theories and their observables, both classical and quantum. There we used an equivalent action functional

$$S'(\phi) = -\int (\phi, \ddot{\phi}) \mathrm{d}t,$$

which has the same equation of motion. (Note that S' can be obtained from S_0 by applying integration by parts and then disregarding the irrelevant boundary term.)

It is often more convenient to study this theory in its first-order formulation, treats the position q and momentum p as independent fields, both mapping from $\mathbb{R}$ to V. The first-order action functional is

$$S(q, p) = \int (p, \dot{q}) \mathrm{d}t + \frac{1}{2} \int (p, p) \mathrm{d}t,$$

whose equations of motion are $\dot{p} = 0$ and $p = \dot{q}$.

This procedure is analogous to Hamilton's maneuver in doubling the variables to reduce Newton's law to a first-order system of differential equations. Indeed, one can interpret this field theory as a σ-model from

$\mathbb{R}$ to the symplectic manifold $T^*V = V \oplus V^*$. It is this functional that we will use in the constructions below, but the reader might find it illuminating to convert our arguments into the more conventional, second-order formulation, which often makes it easier to recognize well-known formulas from physics.

There are two natural kinds of symmetries to consider here. There are automorphisms of the target space V that preserve the theory, and there are automorphisms of the source space $\mathbb{R}$ that preserve the theory. The first lead to various kinds of momenta, and the second include the energy function. In this section we will focus first on rotational symmetries of the target. In the next section we analyze time-translation symmetry (a symmetry of the source) by a slightly different method.

Before starting, however, we emphasize that we will work with the classical BV theory. Thus, the naive fields $q \in C^\infty(\mathbb{R}) \otimes V$ and $p \in C^\infty(\mathbb{R}) \otimes V$ are upgraded to elements $Q, P \in \Omega^*(\mathbb{R}) \otimes V$. The zero-form components Q^0, P^0 of these $\mathbb{Z}$-graded fields are the fields q, p we had before, and the one-form components provide the antifields. The BV symplectic pairing is

$$\langle Q, P \rangle = \int (Q \wedge P),$$

which indicates that we use the wedge product on the differential form component and use the inner product on the V-component. We thus write

$$S(Q, P) = \int (P \wedge \mathrm{d}Q) + \frac{1}{2} \int (P \wedge P) \wedge \mathrm{d}t,$$

which recovers the earlier first-order action when restricted to "naive" fields in degree zero.

14.1.1 Classical symmetries on the target

The orthogonal group $O(V)$, consisting of linear transformations that preserve the inner product, manifestly acts as symmetries of the classical theory: for any $T \in O(V)$,

$$S(TQ, TP) = S(Q, P)$$

for any fields Q and P. Hence its Lie algebra $\mathfrak{o}(V)$ acts by infinitesimal symmetries.

This action is inner, via the Lie algebra map

$$J^{cl} : \mathfrak{o}(V) \to \mathscr{O}_{loc}[-1] \tag{14.1.1.1}$$

where

$$J^{cl}_a(Q) = \int (P, a \cdot Q) \tag{14.1.1.2}$$

for $a \in \mathfrak{o}(V)$. (In Section 14.1.1 we discuss where this formula comes from.) We now unpack what this formula says, bearing in mind that J^{cl}_a acts via the BV bracket.

At first, this notation may seem cryptic, because for q an ordinary smooth function, the putative integrand is still a function and not a one-form, and hence cannot be integrated. It is crucial to recall here that the BV theory involves *anti*fields as well, which for this theory are a copy of densities $\Omega^1(\mathbb{R}) \otimes V$ in cohomological degree one. When we expand our fields Q, P into 0 and 1-form fields Q^0, Q^1, P^0, P^1, we have

$$J^{cl}_a(Q^0, Q^1, P^0, P^1) = \int (P^0, a \cdot Q^1) + \int (P^1, a \cdot Q^0).$$

One consequence is that we can see J^{cl}_a has cohomological degree -1.

This feature is crucial, as we wish to understand the derivation $\{J^{cl}_a, -\}$ acting on functionals. The bracket amounts to contracting the two functionals using the δ-function along the diagonal $\mathbb{R} \subset \mathbb{R}^2$, but this δ-function is a distributional *one*-form on $\mathbb{R}^2$ and hence it plugs nicely into the two-fold tensor representing J^{cl}_a. In particular, for F a functional of cohomological degree zero, the bracket $\{J^{cl}_a, F\}$ is again degree zero.

Symmetries acting on Obs^{cl}

Let us now consider how this formula defines an action of $\mathfrak{o}(V)$ on the factorization algebra Obs^{cl}, even before constructing a map of factorization algebras. The key idea is encoded in the following very simple lemma.

14.1.1.1 Lemma. *For every $a \in \mathfrak{o}(V)$, the operator $\{J^{cl}_a, -\}$ is a derivation of the factorization algebra Obs^{cl}. In particular, we have a map*

$$J^{cl} : \mathfrak{o}(V) \to \mathrm{Der}(\mathrm{Obs}^{cl})$$

of Lie algebras.

Proof First, we have seen that $\{J_a^{cl}, -\}$ acts as a derivation on the commutative algebra of functionals. It is the derivation determined by the action of a on the fields, which is the space on which the functionals are functions. This action on fields is pointwise in the source manifold $\mathbb{R}$, and hence is manifestly local. In particular, it is a straightforward computation to verify that the action of $\{J_a^{cl}, -\}$ intertwines with the structure maps of Obs^{cl}, since the structure maps are maps of commutative algebras. Thus $\{J_a^{cl}, -\}$ is a derivation of the factorization algebra.

It remains to show that

$$\{J_{[a,b]}^{cl}, -\} = \{J_a^{cl}, \{J_b^{cl}, -\}\} - \{J_b^{cl}, \{J_a^{cl}, -\}\}$$

for all $a, b \in \mathfrak{o}(V)$. This property is a consequence of the fact that we're simply witnessing the action of $\mathfrak{o}(V)$ on functionals induced by the action of fields, where we know we have a Lie algebra action. But our explicit formula for $\{J_a^{cl}, -\}$ can also be used to verify this property directly. □

This lemma says that we have an action of symmetries of the target upon the observables of the classical theory, where we organize those observables as experienced by the worldline itself. An immediate consequence of this factorization algebra-level statement is a statement at the level of Poisson algebras, since this action is invariant under time translation (i.e., under shifting the worldline).

14.1.1.2 Corollary. *The map J^{cl} induces a map of Lie algebras*

$$J^{cl,Lie} : \mathfrak{o}(V) \to \mathrm{Der}_{Poisson}(\mathrm{Sym}(V \oplus V^*))$$

such that $J_a^{cl,Lie}$ is a derivation of $\mathrm{Sym}(V \oplus V^*)$ that preserves the Poisson bracket.

This map sends $a \in \mathfrak{o}(V)$ to the symplectic vector field on T^*V induced by the vector field on V itself determined by a.

Proof The first statement is immediate from the preceding lemma and the results of Section I.4.3, where we showed that $H^0 \mathrm{Obs}^{cl}$ for this system is isomorphic to the locally constant factorization algebra arising from the Poisson algebra $\mathrm{Sym}(V \oplus V^*)$ of functions on solutions to the equation of motion.

The second statement amounts to unraveling the consequences of J^{cl} using the correspondence between the classical observables and $\mathrm{Sym}(V \oplus V^*)$. We have shown that J^{cl} is simply the action of $\mathfrak{o}(V)$ on functionals induced by the action on fields themselves. We know that this Lie algebra preserves the subspace of solutions to the equation of motion, and we know its action on solutions — the symplectic space T^*V — is determined by its action on the underlying space of configurations, the target space V. Hence, we obtain the claim. □

Making the symmetry local

The usual formulation of the current in a field theory is that it is an operator of cohomological degree 0. The current J^{cl} we have built is instead something of cohomological degree 1. In this section we explain how, by making the symmetry local on the source (i.e. treating it as a background gauge field) we can recover the standard current.

The local version of the symmetry algebra is given by the local Lie algebra

$$\mathcal{L} = \Omega^* \otimes \mathfrak{o}(V),$$

for which the inclusion of constant functions into de Rham forms determines a map of dg Lie algebras

$$i : \mathfrak{o}(V) \to \mathcal{L}$$

that is a quasi-isomorphism on every open set in $\mathbb{R}$. (We will examine soon what happens when the source manifold is a circle.)

Like all local Lie algebras, this $\mathcal{L}$ can be viewed as a sheaf on $\mathbb{R}$ of formal moduli spaces. The formal moduli space corresponding to $\mathcal{L}(U)$ is the space of solutions to the Maurer-Cartan equation modulo gauge. In this example, the formal moduli problem is that of principal $\mathfrak{o}(V)$-bundles with a connection.

Giving an action of this local Lie algebra on a theory is the same as giving a family of theories over the corresponding formal moduli problem, i.e., giving a theory that includes a $\mathfrak{o}(V)$-gauge field as a background field. To start, we introduce a background gauge field $A \in$

$\Omega^1(\mathbb{R}, \mathfrak{o}(V))$ into our mechanical system in a natural way, via the action functional

$$S_A(q, p) = \int (p, \mathrm{d}_A q) + \frac{1}{2}\int (p, p)\mathrm{d}t,$$

where $\mathrm{d}_A = \mathrm{d} + A$ is the associated connection. This action is known as the minimal coupling, since it does the minimum necessary: it just promotes d to the connection d_A. (The reader might enjoy verifying it is equivalent to the theory determined by the minimally coupled second-order action.) Everything here makes sense viewing q and p as "naive" fields, concentrated in degree zero.

The action S_A encodes the idea that this particle is charged, because it couples to this gauge field, and this coupling changes the particle's notion of a "straight line" (i.e., parallel transport). Alternatively but equivalently, we can view A as picking out a point in the base space $B\mathcal{L}_c$, which parametrizes a space of equations of motion. The fiber over each point is a one-dimensional field theory.

This functional is invariant under $\mathfrak{o}(V)$-gauge symmetries that act on q, p, and A. To encode the gauge symmetry in an action functional, we use the BV formalism. Again, we lift q and p to elements Q and P of $\Omega^*(\mathbb{R}, V)$, and we now let $\alpha \in \mathcal{L}(\mathbb{R})[1]$ denote a background field living in the dg Lie algebra of symmetries. Then the equivariant action functional is

$$S^{eq}(Q, P, \alpha) = \int (P, \mathrm{d}Q) + \int (P, \alpha \cdot Q) + \frac{1}{2}\int (P, P)\mathrm{d}t \qquad (14.1.1.3)$$

This action S^{eq} satisfies the equivariant classical master equation

$$\mathrm{d}_{\mathcal{L}} S^{eq} + \tfrac{1}{2}\{S^{eq}, S^{eq}\} = 0, \qquad (14.1.1.4)$$

where $\mathrm{d}_{\mathcal{L}}$ is the differential on $\mathcal{L}(\mathbb{R})[1]$.

Note that when α is a one-form A, we recover S_A from above. When α is a zero-form, it encodes the gauge symmetry action of $\Omega^0(\mathbb{R}, \mathfrak{o}(V))$ on the fields q, p of the original mechanical system. If the zero-form component is constant with value $a \in \mathfrak{o}(V)$, then $S^{eq}(Q, P, a)$ is precisely the current J_a^{cl} we introduced above.

Via the functional S^{eq} of (14.1.1.3), we obtain a map

$$\begin{array}{rccc} J^{cl}: & \mathcal{L}(\mathbb{R}) & \to & \mathscr{O}_{loc}[-1] \\ & \alpha & \mapsto & \frac{\partial}{\partial \alpha}(S^{eq}). \end{array}$$

(We continue to use the notation J^{cl} for this map, because on $H^0(\mathcal{L}(\mathbb{R})) = \mathfrak{o}(V)$ it is the map (14.1.1.1).) It is a map of dg Lie algebras because S^{eq} satisfies the equivariant classical master equation (14.1.1.4). That is, we have an *inner* action of $\mathcal{L}(\mathbb{R})$ on our theory. In particular, restricting to $\mathfrak{o}(V) \subset \mathcal{L}(\mathbb{R})$ gives us the inner action of $\mathfrak{o}(V)$ on $\mathscr{O}_{loc}[-1]$.

One can see immediately that if we restrict J^{cl} to compactly-supported sections, we obtain a map of precosheaves of dg Lie algebras

$$\begin{array}{rccc} J^{cl}: & \mathcal{L}_c & \to & \mathrm{Obs}^{cl}[-1] \\ & \alpha & \mapsto & \int \langle P, \alpha \cdot Q \rangle . \end{array}$$

We must be a little careful when defining the dg Lie algebra structure on the right hand side. The BV bracket, which is the relevant Lie bracket on the right hand side, is well-defined on the image of this map, but not on all of $\mathrm{Obs}^{cl}[-1]$. It is, however, well-defined on the quasi-isomorphic sub-factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$ whose elements are observables with smooth first derivative. We will suppress this minor issue, as it is examined in detail in Chapter 5.

Note that for any interval $I \subset \mathbb{R}$, the Poincaré lemma for compactly supported forms tells us that the cohomology of $\mathcal{L}_c(I)$ is $\mathfrak{o}(V)$ concentrated in degree 1. A basis for this cohomology is given by an element of $\mathfrak{o}(V)$ multiplied by $f(t)\mathrm{d}t$, where $f(t)$ is a function of compact support on the interval whose integral is 1. It follows that at the level of cohomology, the map J^{cl} determines a map

$$\begin{array}{rccc} J^{cl}: & H^1(\mathcal{L}_c(I)) = \mathfrak{o}(V) & \to & H^0(\mathrm{Obs}^{cl}(I)) \\ & a & \mapsto & \int (P, a \cdot Q) f(t) \mathrm{d}t. \end{array}$$

The observable on the right hand side only depends on the 0-form components $q = Q^0$, $p = P^0$ of the BV fields Q, P. Hence, we can write it as

$$J^{cl}(a) = \int (p, a \cdot q) f(t) \mathrm{d}t.$$

By using more singular observables, we can also represent the image of an element $a \in \mathfrak{o}(V)$ by

$$J^{cl}(a) = \langle p(t_0), a \cdot q(t_0) \rangle$$

by taking $f(t)\mathrm{d}t = \delta_{t=t_0}$.

Recovering angular momentum

Let us now quickly indicate how the formula (14.1.1) relates to the conventional approach to rotational symmetry. Consider a solution to the equation of motion (i.e., an on-shell field):

$$q(t) = tv + x \quad \text{and} \quad p(t) = v$$

where v in V denotes the velocity vector of the particle and x in V denotes the position at time $t = 0$. (As before, we take $q = Q^0$ and $p = P^0$ to be the 0-form components of our fields.) This solution encodes a particle moving in a straight line.

The observable $J^{cl}(a)$ in these terms, measured at time t, is

$$J^{cl}(a,t)(q,p) = (v, t(a \cdot v)) + (v, a \cdot x) = t(v, a \cdot v) + (v, a \cdot x)$$

for each solution. Note that $(v, a \cdot v) = 0$ since by the definition of $\mathfrak{o}(V)$, $(w, a \cdot w') = -(a \cdot w, w')$ for all $w, w' \in V$. Hence

$$J^{cl}(a,t)(q,p) = (v, a \cdot x),$$

which is manifestly independent of time and hence a conserved quantity.

Note that this observable is the a-component of the usual angular momentum. It may help here to specialize to the case $V = \mathbb{R}^3$, and to use the standard identification of the Lie algebra $\mathfrak{o}(3) = \mathfrak{so}(3)$ with the cross product on $\mathbb{R}^3$. In that case, we see

$$(v, a \cdot x) = (v, a \times x) = (a, x \times v)$$

by the scalar triple product identity. That last expression is precisely the a-component of the angular momentum $x \times v$.

Obtaining the moment map

By taking the enveloping P_0 factorization algebra of $\mathcal{L}$, we obtain a map of factorization algebras

$$J^{cl,fact} : \mathbb{U}^{P_0}\mathcal{L} = \mathrm{Sym}(\mathcal{L}_c[1]) \to \mathrm{Obs}^{cl}.$$

By the universal property of a enveloping P_0 factorization algebra, this map is determined by its restriction to the generators $\mathcal{L}_c[1]$, where it is simply a shift of the map J^{cl} just defined. It is, in particular, a map of commutative algebras. At the level of the cohomology, this map becomes familiar.

14.1.1.3 Lemma. *The map $J^{cl,fact}$ induces a map of (unshifted) Poisson algebras*

$$J^{Pois} : \mathrm{Sym}(\mathfrak{o}(V)) \to \mathrm{Sym}(V \oplus V^*).$$

This map is compatible with the map $J^{cl,Lie}$ of Lie algebras: for every $a \in \mathfrak{o}(V)$,

$$\{J_a^{Pois}, -\} = J_a^{cl,Lie},$$

i.e., the Hamiltonian vector field of the function J_a^{Pois} recovers the Poisson derivation $J_a^{cl,Lie}$.

This lemma says that we have lifted $J^{cl,Lie}$ to an inner action of $\mathfrak{o}(V)$ on the algebra of functions on the space of solutions to the equations of motion. But it admits a description in more traditional terms as well.

Note that $\mathrm{Sym}(\mathfrak{o}(V))$ is the enveloping Poisson algebra of $\mathfrak{o}(V)$, which is the Poisson algebra on the coadjoint space $\mathfrak{o}(V)^*$. Hence the map J^{Pois} is the pull back of functions along the moment map $\mu : T^*V \to \mathfrak{o}(V)$.

Proof The first statement is a consequence of the fact that both factorization algebras are locally constant and hence correspond to associative algebras. The factorization algebra map thus determines a map of associative algebras, indeed of commutative algebras in this case.

It remains to verify the compatibility with the Poisson structures. One conceptual way to see it is to rely on the analysis of Noether's theorem at the quantum level. At the quantum level, $\mathrm{Sym}(\mathfrak{o}(V))$ is deformed into the Rees algebra of the universal enveloping algebra $U(\mathfrak{o}(V))$, and $\mathrm{Sym}(V \oplus V^*)$ is deformed into the Rees algebra for the Weyl algebra on $V \oplus V^*$. The quantum version of Noether's theorem says that we have a map from one algebra to the other lifting the map present at the classical level. Working modulo $\hbar^2$ then implies that the map at the classical level is compatible with Poisson brackets. □

Generalizations

There are natural variants of this situation, which we briefly discuss.

First, one might imagine adding a potential, so that we have an interacting theory. That is, we fix a function $U : V \to \mathbb{R}$ and work with the action functional

$$S_U(q, p) = \int (p, \dot{q}) \mathrm{d}t + \frac{1}{2} \int (p, p) \mathrm{d}t + \int U(q(t)) \mathrm{d}t,$$

whose equations of motion are

$$p = -\partial_t q \tag{14.1.1.5}$$
$$\partial_t p = \nabla U(q), \tag{14.1.1.6}$$

where ∇U denotes the gradient of U as a function on the target manifold V.

In this case the relevant symmetries are the vector fields on the space V that fix U (i.e., whose Lie derivative of U vanishes). The massive free theory gave an example using

$$U(v) = -m^2(v, v) = -m^2|v|^2,$$

which manifestly has the orthogonal Lie algebra as symmetries, but there are many other potentials with interesting Lie algebras of symmetries. If one works through a process modeled on the case we have just analyzed, then one will recover familiar constructions from classical mechanics.

A more sophisticated variation is to consider nonlinear σ-models, so that V is a manifold or even something somewhat exotic, like a Lie algebroid. Here it is convenient to use Gelfand-Kazhdan formal geometry, which means that one begins by letting V be a formal disk and one allows formal vector fields as the symmetries. In essence, Fedosov introduced the procedure in his work on deformation quantization. (This viewpoint is highlighted, however, by Kontsevich in his work on deformation quantization Kontsevich (2003) and explained in clarifying

$$Q \longrightarrow P$$

Figure 14.1 The propagator as directed from Q to P

detail by Cattaneo et al. (2002), although those works focus on a two-dimensional σ-model.) For a treatment compatible with the discussion here, see Grady et al. (2017).

14.1.2 Quantized symmetries on the target

We wish now to quantize the example just developed. Our approach is to view the classical theory as living over the base ring $C^*(\mathfrak{o}(V))$ (equivalently, as living over the formal moduli space $B\mathfrak{o}(V)$), and then to attempt to BV quantize over that base ring. To enforce locality in a technically convenient way, we can work with $C^*(\mathcal{L})$, where $\mathcal{L}$ is the local Lie algebra $\Omega^* \otimes \mathfrak{o}(V)$ on the real line.

The Feynman diagrams

Before worrying about any issues of analysis (which will not arise here), let us examine the structure of the diagrammatics.

Recall that action functional in the presence of a background field $\alpha \in \mathcal{L}[1]$ is

$$S^{eq}(Q, P, \alpha) = \int (P, \mathrm{d}Q) + \int (P, \alpha \cdot Q) + \frac{1}{2} \int (P, P)\mathrm{d}t \qquad (14.1.2.1)$$

Here Q is the fundamental field of the model, and P is an auxiliary field.

The first and third term of the equivariant action functional S^{eq} are quadratic and provides the propagator and hence the label for edges. See Figure 14.1. Note that the ends of the edge are labeled by Q and P and do not involve α in any way (i.e., α is a background or "nonpropagating" field). The second term is cubic and provides a trivalent vertex. The α-leg is undirected. See Figure 14.2.

We now ask what kind of graphs can be built from this data; we focus on connected graphs.

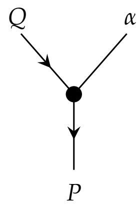

Figure 14.2 The trivalent vertex

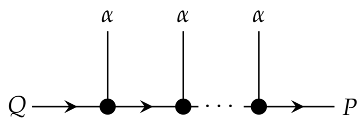

Figure 14.3 The tree with n incoming α-legs

The answer is attractively simple. The only possible trees have two Q, P-legs and arbitrarily many α-legs. These are constructed by joining together trivalent vertices with the edges. See Figure 14.3. Note that the propagator cannot attach to α-legs, because we treat α as a background field.

The only possible graphs that are connected but not simply connected are given by wheels whose external legs are all labelled by α. See Figure 14.4. Because α does not propagate, we can not build any graphs with more loops by gluing these wheels together.

In consequence, the naive form of the quantized equivariant action

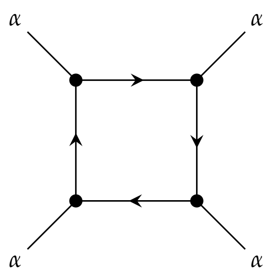

Figure 14.4 A wheel with four vertices

functional is

$$S^q_{naive}[\Psi] = \sum_{\text{trees Y}} I_{\mathrm{Y}}[\Psi](Q, P, \alpha) + \hbar \sum_{\text{wheels } \circlearrowleft} I_{\circlearrowleft}[\Psi](\alpha),$$

where Ψ denotes some parametrix.

Remark: This functional is naive in the sense that it arises via the diagrams determined by the classical action functional, but it may not satisfy the quantum master equation and hence not define a quantum field theory in the sense of this book. We will investigate shortly whether S^q_{naive} satisfies the quantum master equation. Further, in general dimension larger than 1, an expression like the one defining $S^q_{naive}[\Psi]$ might be ill-defined, and would need to be renormalized using the techniques of Costello (2011b). In the case of quantum mechanics, as we will carefully check below, there are no such difficulties. $\Diamond$

The points we wish to emphasize are that

(i) there are no powers of $\hbar$ beyond the first, and
(ii) the $\hbar$-dependent term is independent of the fields P, Q and simply a function of the background field α.

For these reasons, the situation is particularly accessible.

Recall that, in our context, we have two separate notions of an action of a local Lie algebra on a field theory: an action versus an inner action. To give an action, we require a solution $S^q[\Psi](P, Q, \alpha)$ to the quantum master equation that has no terms which depend *only* on α. If we drop the terms from $S^q_{naive}[\Psi]$ which depend only on α, we are left with only diagrams which are trees. In that case, the quantum master equation reduces to the classical master equation, which will hold automatically, because it holds for $S^{cl}(P, Q, \alpha)$. Thus, we find immediately that we have an action of $\mathcal{L}$ on our theory.

The more interesting question is whether we can lift this to an *inner* action. To check, we need to determine whether the quantum master equation holds for $S^q[\Psi](P, Q, \alpha)$. Applying the BV Laplacian to the one-loop term $I_{\circlearrowleft}[\Psi]$ gives zero, because this quantity only depends on α and the BV Laplacian contracts a Q leg with a P leg. Similarly, $\{I_{\circlearrowleft}[\Psi], -\}_{\Psi}$ is zero. The only term in the quantum master equation that is not already captured by the classical master equation is

$$O[\Psi] = \sum_{\text{trees } Y} \triangle_\Psi I_Y + \mathrm{d}_{\mathcal{L}} I_{\circlearrowleft}[\Psi].$$

Note that this expression depends only on α, and is an element of $C^1(\mathcal{L})$. As is always true for obstructions to solving the quantum master equation, this expression is independent of Ψ so that we can take Ψ to be arbitrarily small, giving us a cohomology class in $H^1(C_{loc}(\mathcal{L}))$.

To sum up, we have found that

(i) We automatically have an action of $\mathcal{L}$ on the quantum theory.
(ii) The obstruction to finding an inner action is a quantity described by one-loop Feynman diagrams, and lives in $H^1(C_{loc}(\mathcal{L})$.

As we have seen, an element of $H^1(C_{loc}(\mathcal{L})$ gives rise to a shifted central extension, of $\mathcal{L}$, which we can in turn use to form the twisted enveloping factorization algebra.

In the case at hand, however, the class in $H^1(C_{loc}(\mathcal{L}))$ usually vanishes for cohomological reasons. Indeed, the local Lie algebra cohomology of $\mathcal{L}$ coincides with the Lie algebra cohomology of $\mathfrak{o}(V)$, by the results of section 12.2.4. By Whitehead's lemma, assuming that $\dim V > 2$, the Lie algebra cohomology of $\mathfrak{o}(V)$ has vanishing H^1.

Now let us explain why there are no analytic difficulties. The key point is that there exists a parametrix whose corresponding propagator is a continuous function with bounded derivative. In this context, as usual, we mean a parametrix for the Laplacian ∂_t^2. For any choice of a smooth function $f(t)$ with $f(t) = 0$ for $|t| \gg 0$ and $f(t) = 1$ in a neighbourhood of $t = 0$, then the parametrix is simply

$$\tfrac{1}{2}\,|t_1 - t_2|\, f(t_1 - t_2). \tag{14.1.2.2}$$

(It is an excercise to check that applying $\partial_{t_1}^2$ to this expression gives $\delta_{t_1=t_2}$ plus a smooth function). The corresponding propagator is the derivative of this parametrix:

$$|t_1 - t_2|\, f'(t_1 - t_2) + f(t_1 - t_2)(\delta_{t_1>t_2} - \delta_{t_1<t_2}). \tag{14.1.2.3}$$

Clearly, this function is bounded, even if not continuous.

Remark: The fact that the parametrix for the Laplacian in one dimension is continuous is closely related to the fact that the Wiener measure for random paths is supported on continuous paths. In higher dimensions, the analog of the Wiener measure — known as the Gaussian free field measure — has the feature that continuous functions have measure zero. Correspondingly, the parametrix is a distribution, and not continuous. ♢

Given this propagator, when our interaction contains no derivatives (as in our example), the amplitude of the Feynman diagram is given by an absolutely convergent integral, and no regularization is required. Thanks to this feature, our expression for S^q_{naive} makes sense, and the cohomological argument given above shows that in this case there is an inner action of $\mathcal{L}$ on our theory.

The quantum Noether theorem applies immediately.

14.1.2.1 Corollary. *There is a map of factorization algebras*

$$J^{q,fact} : \mathbb{U}^{BD}\mathcal{L} = C^{\hbar}_*(\mathcal{L}_c) \to \mathrm{Obs}^q,$$

since no twisting cocycle is required as the obstruction for an inner action vanished.

Both factorization algebras are locally constant, as is the map. Hence by taking cohomology, we obtain a map of associative algebras

$$J^{alg} : U^{\hbar}\mathfrak{o}(V) \to \mathrm{Weyl}(V \oplus V^*)$$

such that modulo $\hbar$, it recovers J^{Pois}.

This map J^{alg} is a *quantum moment map*, as it is a deformation quantization of the moment map described in Lemma 14.1.1.3.

Remark: Since Obs^q is locally constant and corresponds to the Weyl algebra, the locally constant derivations (including the image of $J^{q,Lie}$) correspond to derivations of Weyl algebras. All such derivations are inner as $HH^1(\mathrm{Weyl}) = 0$, which implies that up to the ambiguity of constants, there is a lift of $J^{q,Lie}$ to a map into the Weyl algebra. The map J^{alg} is such a map.

Direct computation shows that J^{Pois} sends an element of $\mathfrak{o}(V)$ to a quadratic function on T^*V, asseen already in the quadratic nature of J^{cl}_x.

Something similar holds at the quantum level. J^{alg} will be an expression of degree at most 2 in the generators of the Weyl algebra. If we equip the Weyl algebra $\mathrm{Weyl}(T^*V) = \mathrm{Diff}(V)$ with the filtration where the generators in $V \oplus V^*$ have weight 1, then J^{alg} sits in $F^2\,\mathrm{Weyl}$. Hence the associatied graded of J^{alg} in $F^2\,\mathrm{Weyl}\,/\,F^1\,\mathrm{Weyl}$ is the classical moment map.

In many situations, the constant term of the quantum moment map (sitting in $F^0\,\mathrm{Weyl}(V)$) is ambiguous; we can add the identity to $J^{alg}(a)$ without changing the commutator $[J^{alg}(a), -]$. In the case at hand, however, there is no such ambiguity because $J^{alg} : \mathfrak{o}(V) \to F^2\,\mathrm{Weyl}(V)$ must be an $\mathfrak{o}(V)$-equivariant map. If $\dim V > 2$, then $\mathfrak{o}(V)$ has no rank 1 representations, which tells us that we cannot adjust the constant term in $J^{alg}(a)$ in an $\mathfrak{o}(V)$-equivariant way.

This result is closely related to the cohomological argument for vanishing of the obstruction to an inner action. That anomaly is a cocycle in $C^2(\mathfrak{o}(V))$, and it can by eliminated by choosing a bounding cochain in $C^1(\mathfrak{o}(V))$. Equivalence classes of bounding cochains are a torsor for $H^1(\mathfrak{o}(V))$, which vanishes precisely because $\mathfrak{o}(V)$ has no rank one representations. $\Diamond$

14.1.3 Noether's theorem and partition functions

We have seen in section 13.8 how Noether's theorem allows us to define and compute partition functions. In this section, we will examine how this procedure works for the harmonic oscillator.

The harmonic oscillator is the deformation of the topological quantum mechanics analyzed above, and it is obtained by adding the term $\int (P^2 + Q^2)\mathrm{d}t$ to the Lagrangian. Noether's theorem provides a nice interpretation where we give topological quantum mechanics an inner action of the Abelian local Lie algebra $\mathcal{L} = \Omega^*$ via the coupled action

$$\int P\mathrm{d}Q + (P^2 + Q^2)\alpha$$

with background field $\alpha \in \Omega^*[1]$. That is, we are making the one-dimensional Abelian Lie algebra $\mathbb{C}$ act by the current $P^2 + Q^2$. The Poisson bracket with this current is $2P\partial_Q - 2Q\partial_P$, so that the symmetry is associated to this vector field on the target.

Just as in the case of the $\mathfrak{o}(V)$ action on topological quantum mechanics discussed above, this action lifts to an inner action at the quantum level. Indeed, as we saw above, the propagator is a bounded function, so that there are no analytical difficulties. There is no cohomological obstruction to the quantum master equation for an inner action holding because $H^2(C^*_{loc}(\mathcal{L})) = 0$. As $H^1(C^*_{loc}(\mathcal{L})) = \mathbb{C}$, there is not, however, a unique way to solve the quantum master equation. Different ways of doing so are related by adding on the one-loop term $\int \alpha$.

Let us recall how Noether's theorem allows us to analyze the partition function. We replace the real line by a circle, a natural compactification of the line. The global Noether current is a map

$$J^{global} : C_*(\mathcal{L}(S^1)) \to \mathrm{Obs}^q(S^1).$$

On the left hand side, we have the Lie algebra chain complex of the global sections of the Lie algebra $\mathcal{L}$ of symmetries. Dually, we can view the global Noether current as an elemen

$$J^{global} \in C^*(\mathcal{L}(S^1), \mathrm{Obs}^q(S^1)).$$

Let us make the assumption that the identity operator gives us a quasi-isomorphism

$$\mathrm{Id} : \mathbb{C} \to \mathrm{Obs}^q(S^1).$$

It is convenient here to set $\hbar = 1$, which we can do because only one-loop Feynman diagrams appear in our analysis.

When our assumption is true, then we can replace $\mathrm{Obs}^q(S^1)$ with $\mathbb{C}$, so that

$$J^{global} \in C^*(\mathcal{L}(S^1))$$

In the case of interest, there is a quasi-isomorphism

$$H^*(\mathcal{L}(S^1)) \cong H^*(S^1)$$

where both sides are Abelian Lie algebras, and so

$$H^0(C^*(\mathcal{L}(S^1))) = \mathbb{C}[[c]]$$

for a formal variable c.

By this isomorphism, we can interpret J^{global} as a function of a single variable c. This function can be understood as the partition function of the quantum mechanical system with Lagrangian

$$\int P\mathrm{d}Q + c\int (P^2+Q^2)\mathrm{d}t.$$

Our goal in this section is thus to compute this partition function, both via our approach with Noether's theorem and Feynman diagrams but also via the standard Hamiltonian approach. We will find that the answers coincide, but only after deploying some fun number-theoretic identities.

Before we turn to the computation, we first need to deal with a problem: for topological quantum mechanics with Lagrangian $\int P\mathrm{d}Q$, the map

$$\mathrm{Id} : \mathbb{C} \to \mathrm{Obs}^q(S^1) \tag{14.1.3.1}$$

is *not* a quasi-isomorphism. Indeed, the right hand side is the Hochschild homology of the Weyl algebra, which is concentrated in a non-zero degree.

We can fix this problem, however, by asking that the fields p, q are not sections of the trivial flat bundle on S^1, but sections of a flat bundle with nontrivial monodromy. Here we will consider the case where the monodromy is -1, which means that p, q are anti-periodic.

Remark: Studying topological mechanics when p, q are anti-periodic is equivalent to studying the harmonic oscillator when p, q are periodic but where the coefficient of the $(p^2+q^2)\mathrm{d}t$ term is near $\pi/2$. The Hamiltonian flow generated by p^2+q^2 sends $p \to -p, q \to -q$ after time $\pi/2$. For discussion, see Section I.8.1.2, where we examined the quantum harmonic oscillator and computed the quantum observables over a circle; there is an interesting dependence on the mass and the circumference of the circle. ◊

For the BV theory, the fields P and Q live in the de Rham complex with coefficients in this flat bundle with monodromy -1. This twisted de Rham cohomology over S^1 is trivial, and so for this theory, the space of classical observables $\mathrm{Obs}^{cl}(S^1)$ has cohomology $\mathbb{C}$ in degree 0, spanned by the identity operator. A spectral sequence then tells us that the cohomology of $\mathrm{Obs}^q(S^1)$ is also spanned by the identity operator.

We want something more explicit than a spectral sequence, though.

Recall that a quantum observable $O \in \mathrm{Obs}^q(S^1)$ is a functional $O[\Psi](P,Q)$ of the fields P, Q, for every parametrix Ψ. The differential is a sum of two terms: the first term is induced by the de Rham operator acting on P, Q, and the second is the BV Laplacian $\triangle_\Psi$ associated to the matrix.

There is a special parametrix at length scale ∞, which is the Green's function on the circle for an anti-periodic scalar field. With this parametrix, the BV operator $\triangle_\infty$ is zero. Indeed, the BV operator is the harmonic representative of the delta-function on the diagonal in $S^1 \times S^1$, and there are no harmonic forms with coefficients in the flat bundle of monodromy -1. Thus, the differential on observables at scale ∞ arises simply from the de Rham operator. There is then an explicit quasi-isomorphism

$$\begin{array}{rccc} \langle - \rangle : & \mathrm{Obs}^q(S^1) & \to & \mathbb{C} \\ & O & \mapsto & O[\infty](0) \end{array}$$

obtained by evaluating $O[\infty]$ at $P = Q = 0$.

The global Noether current can be expressed in terms of Feynman diagrams, once we choose a parametrix Ψ. The quantum effective action that couples the fields P, Q with the background field α is

$$S^q[\Psi](P,Q,\alpha) = \sum_{\text{trees } Y} I_Y[\Psi](Q,P,\alpha) + \sum_{\text{wheels } \circlearrowleft} I_\circlearrowleft[\Psi](\alpha).$$

The global Noether current, as a function of $\alpha \in \Omega^*(S^1)[1]$, is the expression

$$J^{global}(\alpha, P, Q)[\Psi] = \exp(S^q[\Psi](P,Q,\alpha)).$$

(We view this expression as a formal series in α.) If α is a one-form, so that α is closed, this current automatically satisfies the RG flow equation and the quantum master equation that defines a quantum observable.

We are interested in the expectation value of this observable, when $\alpha = c\,\mathrm{d}\theta$ is a harmonic one-form. This expectation value is the partition function $Z^N(c)$ of the harmonic oscillator, defined using Noether's theorem.

From the definition of the expectation value, we find

$$\log Z^N(c) = \log \left\langle J^{global}(c\mathrm{d}\theta) \right\rangle = \sum_{\text{wheels } \circlearrowleft} I_\circlearrowleft[\infty](c\mathrm{d}\theta).$$

Now, following Gwilliam and Grady (2014), we can express it in terms of a trace on the space of anti-periodic functions on S^1. We call this space $C^\infty_{odd}(S^1)$. The operator ∂_θ is invertible on this space. The amplitude for a wheel with n external lines is proportional to

$$\operatorname{Tr}_{C^\infty_{odd}(S^1)} \partial_\theta^{-n} = \sum_k \frac{1}{((2k+1)\pi i)^n}.$$

When n is odd, we view this putative sum as vanishing; the transformation $k \mapsto -k-1$ pairs off terms that cancel. (This kind of maneuver is familiar when dealing with series like this one.) We thus only consider even n even. We need to carefully calculate the coefficient of each such diagram.

Each vertex in the wheel must come from the cP^2 or cQ^2 interaction. The Feynman rules tell us that contribution at each vertex involves the derivatives of the quantity P^2 or Q^2 at the vertex. Thus, we get an extra factor of 2 for each vertex, giving us an overall factor of 2^n.

Because the propagator connects P to Q, the vertices labelled by P^2 and those labelled by Q^2 alternate. The propagator connecting P to Q is ∂_θ^{-1}, whereas that connecting Q to P is $-\partial_\theta^{-1}$. This pattern gives us an overall sign of $(-1)^{n/2} = i^n$.

Finally, each such expression comes with a fractional weight counting the number of automorphisms of the diagram. The symmetry group of our wheel diagram, preserving the labelling of each vertex by P^2 or Q^2, is the dihedral group $D_{n/2}$, which has order n. The overall factor for each diagram is then $(2ic)^n/n$. In sum, we find

$$\log Z^N(c) = \sum_{n \text{ even}} \frac{(2ic)^n}{n} \sum_k \frac{1}{((2k+1)\pi i)^n},$$

expressing the partition function of this system in terms of diagrams.

Let us compare this formula to the Hamiltonian computation of the partition function of the harmonic oscillator that one finds in standard quantum mechanics textbooks. The Hamiltonian is $c(P^2+Q^2)$ where $[P, Q] = 1$. We can write

$$c(P^2+Q^2) = \tfrac{c}{2}\left(\rho\rho^\dagger + \rho^\dagger\rho\right)$$

where

$$\rho = P + iQ,$$
$$\rho^\dagger = P - iQ.$$

These satisfy the relations

$$[P^2 + Q^2, \rho] = 2i\rho,$$
$$[P^2 + Q^2, \rho^\dagger] = -2i\rho^\dagger.$$

A basis for the Hilbert space $\mathcal{H}$ of the system is obtained by applying the raising operator ρ repeatedly to the vacuum vector $|0\rangle$. If we represent P, Q as the operators ∂_x, x on the space of functions of a single variable, then the vacuum vector is $e^{ix/2}$, which is an eigenfunction of eigenvalue i.

The commutation relations between the Hamiltonian $c(P^2 + Q^2)$ and ρ tells us that the eigenvalue of $\rho^n |0\rangle$ is $2i(n + \frac{1}{2})c$. For our purposes, it is convenient to assume that $\operatorname{Im} c < 0$, so that the real part of the eigenvalues of the Hamiltonian are negative. The partition function, as obtained in the Hamiltonian framework, is then the twisted trace of e^H in the Hilbert space, where "twisted trace" means we count states $\rho^n |0\rangle$ with a sign of $(-1)^n$. In formulae, if R denotes the operator which acts as $(-1)^n$ on $\rho^n |0\rangle$, the partition function is

$$\begin{aligned} Z^H(c) &= \operatorname{Tr}_{\mathcal{H}} Re^H \\ &= \sum_{n \geq 0} (-1)^n e^{(2n+1)ic} = \frac{e^{ic}}{1 + e^{2ic}}. \end{aligned}$$

This formula uses the operator-theoretic approach.

To check that these partition functions agree, we need to know that

$$\log \frac{e^{ic}}{1 + e^{2ic}} = \sum_{n \text{ even}} \frac{(2ic)^n}{n} \sum_k \frac{1}{((2k+1)\pi i)^n}.$$

This identity turns out to be true, but it is nontrivial (although amusing) to prove. (This identity holds only up to a constant: the Noether's theorem approach recovers the partition function up to a multiplicative constant, and so its logarithm up to an additive constant).

We outline one approach. First, recall the identity

$$\sum_{k \text{ odd}} \frac{1}{k^n} = (1 - 2^{-n}) 2\zeta(n)$$

where ζ is the Riemann zeta function (see e.g. Gould and Shonhiwa (2008)). In terms of the zeta function we have

$$\log Z^N(c) = \sum_{n \text{ even}} \frac{(2ic)^n (1 - 2^{-n})}{n(\pi i)^n} 2\zeta(n).$$

There is then an equivalence of power series

$$\log\left(\frac{x}{1 - e^{-x}}\right) - \frac{x}{2} = \sum_{n \text{ even}} 2\zeta(n) \frac{x^n}{n(2i\pi)^n},$$

which gives

$$\begin{aligned}
\log Z^N(c) &= -\log\left(\frac{2ic}{1 - e^{-2ic}}\right) + \log\left(\frac{4ic}{1 - e^{-4ic}}\right) - ic \\
&= \log\left(\frac{2(1 - e^{-2ic})}{1 - e^{-4ic}}\right) - ic \\
&= \log 2 + \log\left(\frac{1}{1 + e^{-2ic}}\right) - ic.
\end{aligned}$$

The $\log 2$ is irrelevant, as $\log Z^N(c)$ is only defined up to an additive constant. We find

$$\begin{aligned}
Z^N(c) &= \frac{e^{-ic}}{1 + e^{-2ic}} \\
&= \frac{e^{ic}}{1 + e^{2ic}} \\
&= Z^H(c),
\end{aligned}$$

proving that the diagrammatic partition function via Noether's theorem and the Hamiltonian partition function coincide.

14.2 Examples from chiral conformal field theory

We will consider here a chiral analogue of the one-dimensional theory just considered, namely the free $\beta\gamma$ system on a Riemann surface Σ, which we examined in Example 12.6.2. Fix a complex vector space V.

The fields consist of $\gamma : \Sigma \to V$ and $\beta \in \Omega^{1,0}(\Sigma) \otimes V^*$. The action functional is

$$S(\beta, \gamma) = \int \langle \beta, \overline{\partial} \gamma \rangle$$

where $\langle -, - \rangle$ denotes the evaluation pairing between V and its dual V^* extended linearly over the Dolbeault complex $\Omega^{0,*}(\Sigma)$. In Section I.5.4, we examined the factorization algebra of this theory and computed its associated vertex algebra.

In this section, we will apply our formulation of Noether's theorem to show that the Virasoro vertex algebra, at a particular level, embeds into the vertex algebra of the $\beta\gamma$ system. We will also explain how the Kac-Moody vertex algebra associated to $\mathfrak{gl}(V)$, again at a particular level, embeds into the factorization algebra of the $\beta\gamma$ system. Both of these computations are standard in the physics literature. The novelty of our presentation is that the Virasoro and Kac-Moody factorization algebras are presented in quite a geometric way. (These arguments are easily applied to the chiral fermion, which provided a running example in Chapter 12, starting with Example 12.2.2.)

Remark: The examples developed here are explored in a more sophisticated way in Gwilliam et al. (2020), where it is shown that one can recover the sheaf of vertex algebras known as chiral differential operators (CDOs) from a sheaf of factorization algebras. The classical field theory is known as the curved $\beta\gamma$ system, and it amounts to replacing the vector space V here by a complex manifold X. The Noether theorems play a key role in constructing the sheaves, as one must understand how diffeomorphisms of the target X act as symmetries of the curved $\beta\gamma$ system. In this example, the obstruction element is nontrivial and corresponds to the second Chern character $\mathrm{ch}_2(T_X)$ of the target X. ♢

14.2.1 Some useful analytic facts

Our goal is to explain the structure of these computations, not to unravel the intricacies of various graph integrals. We thus depend upon general, and very helpful, results about graph integrals for chiral theories on $\mathbb{C}$, due to Li (2012). These results show that if we deform the $\beta\gamma$ term as a holomorphic theory, by adding an appropriate interaction, then we will not need counterterms in producing a quantized action.

Further, Li's results allow one to compute the obstruction completely explicitly. Hence, the analytical challenges to quantizing a symmetry are at a relative minimum, and we can focus on structural aspects.

Li's results apply for Feynman diagrams at all loops. However, the calculations we will need only involve Feynman diagrams of at most one loop, so we will not need the full power of Li's results.

To state the relevant version of Li's result, we will introduce some notation that is needed to discuss the graph integrals relevant to our situations.

Let

$$\triangle = -4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$$

denote the Laplacian with which we are concerned, let

$$K_t(z,\bar{z}) = \frac{1}{4\pi t}e^{-|z|^2/4t}$$

denote the associated heat kernel, and let

$$H_{\epsilon<L}(z,\bar{z}) = \int_{\epsilon}^{L} \mathrm{d}t\, K_t(z,\bar{z}).$$

The propagator $P_{\epsilon<L}$ for the $\beta\gamma$ system has integral kernel $\bar{\partial}^* H_{\epsilon<L}$.

Given a graph Γ, let $V(\Gamma)$ denote the set of vertices and let $E(\Gamma)$ denote the set of edges. Each edge e has a head $h(e)$ and tail $t(e)$ vertex (which are possibly the same). The graph is allowed to have a collection of external legs (or "half-edges"), and we will say there are n_Γ of them. Let $\nu : E(\Gamma) \to \mathbb{Z}_{>0}$ assign a positive integer to each edge. We define

$$W_{\Gamma,\nu}(H_{\epsilon<L},\Phi) = \prod_{v\in V(\Gamma)} \int_{\mathbb{Z}} \mathrm{d}^2 z_v \left(\prod_{e\in E(\Gamma)} \partial_{z_e}^{\nu(e)} H_{\epsilon<L}(z_e,\bar{z}_e) \right) \Phi$$

where $z_e = z_{h(e)} - z_{t(e)}$ and $\Phi \in C_c^\infty(\mathbb{C}^{n_\gamma})$ is a function of n_Γ variables. This formula specifies a family of distributions on $\mathbb{C}^{n_\Gamma}$, depending on the parameters ϵ and L.

14.2.1.1 Proposition (Prop. B.1, Li (2012)). *The limit*

$$\lim_{\epsilon\to 0} W_{\Gamma,\nu}(H_{\epsilon<L},\Phi)$$

exists for any $L > 0$ and any Φ.

It should be clear to those familiar with Feynman diagrams that this kind of integral shows up naturally in questions about the $\beta\gamma$ system. (Consider the case of $\nu \equiv 1$. Proposition 9.5 of Gwilliam et al. (2020) examines this case for one-loop graphs in great detail, without invoking Li's general result.)

This result will ensure that no counterterms are necessary in the constructions we pursue below.

There is closely related result, first articulated in Costello (2011a) at one-loop and proved in general in Li (2012).

14.2.1.2 Proposition. *Only diagrams with two vertices contribute to obstructions.*

All the computations we perform in this section will involve Feynman diagrams with at most one loop, and we will need only analyze one-loop obstructions.

14.2.2 Symmetries on the target

We let $B \in \Omega^{1,*}(\Sigma, K_\Sigma \otimes V)$ and $\Gamma \in \Omega^{0,*}(\Sigma, V)$ be the fields of the $\beta\gamma$ system in the BV formalism. The components of cohomological degree 0 will be β, γ. The BV symplectic pairing is $\int \langle B, \Gamma \rangle$ and the action functional is $\int \left\langle B, \overline{\partial}\Gamma \right\rangle$.

Linear symmetries of the target space V lift to symmetries of the $\beta\gamma$ system. It then extends naturally to an action of the local Lie algebra $\mathcal{L} = \Omega^{0,*} \otimes \mathfrak{gl}(V)$, which acts on the fields B, Γ by combining the natural action of $\mathfrak{gl}(V)$ on V and V^* with the wedge product of forms.

When we unravel the Noether theorems for this case, there are compelling chiral analogues to the one-dimensional situations analyzed earlier, where we obtained maps of algebras $U\mathfrak{g} \to A$. Here, by Section I.5.4, we know that the twisted enveloping factorization algebra of $\mathcal{L}$ recovers a Kac-Moody vertex algebra. Hence the quantized symmetry will lead to map of vertex algebras

$$\mathrm{KM}_\lambda \to V_{\beta\gamma}$$

for some level λ. The factorization algebra-level statement gives global versions of this situation, explaining how chiral currents determine observables and yield Ward identities.

The action of $\mathcal{L}$ on Obs^{cl} is encoded by the local functional

$$J^{cl}_{\alpha}(B,\Gamma) = \int \langle B, \alpha \cdot \Gamma \rangle$$

with $\alpha \in \mathcal{L}_c$ and $\alpha \cdot \Gamma$ means the pointwise action. By the same argument as for Lemma 14.1.1.1, we see that for $\{J^{cl}_{\alpha}, -\}$ is a derivation of the factorization algebra Obs^{cl}.

The equivariant action functional is

$$S^{eq}(B,\Gamma,\alpha) = S(B,\Gamma) + J^{cl}_{\alpha}(\Gamma, B),$$

which satisfies the classical master equation over $\mathbb{C}$ and hence determines an inner action.

The anomaly to a quantized inner action

The form of this equivariant action functional is similar to that in the one-dimensional case. We can ask if this action can be lifted to the quantum level. As in the one-dimensional case, the only Feynman diagrams that can appear are

- trees with a single Γ-leg, a single B-leg, and arbitrarily many α-legs,
- wheels, with only α-legs,

just as in Section 14.1.2. By Proposition 14.2.1.1, we find that the graph integrals associated to these wheels have no divergences in the $\epsilon \to 0$ limit, which is analogous to the fact that no divergences appear in the mechanical system.

We can ask whether, at the quantum level, we have an *action* of the local Lie algebra or an *inner action*. Because all the diagrams at one loop have only α-legs, they do not contribute to the action of $\mathcal{L}$, only to the inner action. In consequence, we know the following.

14.2.2.1 Lemma. *The equivariant classical $\beta\gamma$ system admits an equivariant quantization at the quantum level, given an action of $\mathcal{L}$ on the $\beta\gamma$ system.*

To determine whether the quantized action of $\mathcal{L}$ on Obs^q is inner, we need to identify the obstruction element. Thanks to Lemma 14.2.1.2, we only need to examine the wheel with two vertices. The obstruction will be a local cocycle on $\mathcal{L}$ of degree 1.

14.2.2.2 Proposition. *The obstruction to the action of $\mathcal{L}$ becoming an inner action is the cocycle*

$$\int_\Sigma \mathrm{Tr}_V\, \alpha \partial \alpha.$$

Here we view cochains of $\mathcal{L}$ as the space of functions on $\mathcal{L}[1] = \Omega^{0,*}_\Sigma \otimes \mathfrak{gl}(V)[1]$, and we view α as an element of $\mathcal{L}[1]$. This local cocycle is of degree 1, as it eats one $(0,1)$-form and one $(0,0)$-form, which are in degrees 0 and -1, respectively.

Proof One can derive this claim from the general results of Li (2012) or Costello (2011a), but it is instructive to calculate it directly. Since this computation is well-documented elsewhere, we will not present every detail.

We work on flat space $\Sigma = \mathbb{C}$. Let us choose a basis for V, and view α as a matrix α^i_j of $(0,*)$-forms. The propagator is

$$P^L_\epsilon = \int_{t=\epsilon}^L \partial_z \frac{1}{4\pi t} e^{-|z-z'|^2/4t}.$$

(Here, we have implicitly trivialized the canonical bundle on $\mathbb{C}$ so that we can treat both B and Γ as $(0,*)$-forms.) The amplitude for the wheel with two vertices, with external lines labelled by $\alpha, \widetilde{\alpha}$ is

$$\lim_{\epsilon\to 0} \int_{z,z'} P^L_\epsilon(z,z')^2 \alpha^i_j(z) \widetilde{\alpha}^j_i(z').$$

This expression is a linear map $(\Omega^{0,*}(\mathbb{C}) \otimes \mathfrak{gl}_N)^{\otimes 2} \to \mathbb{C}$. It is a tensor product of an algebraic factor $\mathrm{Tr} : \mathfrak{gl}_N^{\otimes 2} \to \mathbb{C}$, with an algebraic factor

$$W_{an}[L](\alpha, \alpha') = \lim_{\epsilon \to 0} \int_{z,z'} P^L_\epsilon(z,z')^2 \alpha(z) \widetilde{\alpha}(z'),$$

where α means here as an element of $\Omega^{0,*}(\mathbb{C})$. This expression is a symmetric linear functional on $(\Omega^{0,*}(\mathbb{C})[1])^{\otimes 2}$. We can ask if it is a cochain map. Integration by parts shows this functional fails to be a cochain map by applying $\overline{\partial}$ to $P(z,z')$:

$$(\overline{\partial} W_{an}[L])(\alpha,\widetilde{\alpha}) = \lim_{\epsilon\to 0}\int_{z,z'} \overline{\partial} P_\epsilon^L(z,z')^2\alpha(z)\widetilde{\alpha}(z').$$

Since we have

$$\overline{\partial} P(\epsilon, L) = K_L - K_\epsilon,$$

we have

$$(\overline{\partial} W_{an}[L])(\alpha,\widetilde{\alpha}) = 2\lim_{\epsilon\to 0}\int_{z,z'} P_\epsilon^L(z,z')(K_L(z,z') - K_\epsilon(z,z'))\alpha(z)\widetilde{\alpha}(z').$$

One of the terms in the quantum master equation for one-loop diagrmas applies $\triangle_L$ to a graph that is a tree. Since $\triangle_L$ is obtained by contracting with K_L, it will not be surprising that this term cancels with the term in $\overline{\partial} W_{an}$ that involves K_L. We find that the anomaly — the failure to satisfy the equivariant quantum master equation — is

$$O[L](\alpha,\widetilde{\alpha}) = -2\lim_{\epsilon\to 0}\int_{z,z'} P_\epsilon^L(z,z')K_\epsilon(z,z')\alpha(z)\widetilde{\alpha}(z').$$

We will calculate this anomaly up to an overall factor, into which we will absorb the factor of -2. Inserting the explicit expressions for $P(\epsilon, L)$ and K_ϵ, we find the anomaly is computed by the integral

$$-\int_{t=\epsilon}^{L}\int_{z,z'}(\overline{z}-\overline{z}').\frac{1}{16\pi t^2}e^{-|z-z'|^2/4t}\frac{1}{4\pi\epsilon}e^{-|z-z'|^2/4\epsilon}\alpha(z)\widetilde{\alpha}(z')\mathrm{d}z\mathrm{d}z'.$$

Changing variables to $u = (z-z')/2$, $v = (z+z')/2$, and dropping some multiplicative factors, we obtain

$$\int_{t=\epsilon}^{L}\int_v\int_u \frac{1}{t^2\epsilon}e^{-|u|^2(\frac{1}{t}+\frac{1}{\epsilon})}\overline{u}\alpha(u+v)\widetilde{\alpha}(v-u)\mathrm{d}u\mathrm{d}v.$$

The integral over the u-plane is a Gaussian integral, and so can be computed in series in the parameter $(t^{-1}+\epsilon^{-1})^{-1}$, according to Wick's lemma. The leading order term in this expansion gives us the value of the integrand at $u=0$, multiplied by $(t^{-1}+\epsilon^{-1})^{-2}$. This term vanishes, because the integrand has a factor of $\overline{u}$. The next term in the expansion gives us $\partial_u\partial_{\overline{u}}$ applied to $\overline{u}\alpha(u+v)\widetilde{\alpha}(v-u)$ and evaluated at zero:

$$\begin{aligned}\int_u \frac{1}{t^2\epsilon}e^{-|u|^2(\frac{1}{t}+\frac{1}{\epsilon})}&\overline{u}\alpha(u+v)\widetilde{\alpha}(v-u)\mathrm{d}u\\ &\simeq \frac{1}{t^2\epsilon}(t^{-1}+\epsilon^{-1})^{-2}\partial_{\overline{u}}\partial_u\left(\overline{u}\alpha(v+u)\widetilde{\alpha}(v-u)\right)\\ &= \frac{t^2\epsilon^2}{t^2\epsilon(t+\epsilon)^2}\left(\partial_v\alpha(v)\widetilde{\alpha}(v) - \alpha(v)\partial_v\widetilde{\alpha}(v)\right).\end{aligned}$$

It turns out that the remaining terms in the expansion in the parameter $(t^{-1} + \epsilon^{-1})^{-1}$ do not contribute to the $\epsilon \to 0$ limit. It is easy to see that

$$\lim_{\epsilon \to 0} \int_{t=\epsilon}^{L} \frac{t^2 \epsilon^2}{t^2 \epsilon (t+\epsilon)^2} \mathrm{d}t = \tfrac{1}{2}$$

so that, up to a non-zero constant, the anomaly is

$$\begin{aligned} O[L] &= \int_z \partial_z \alpha(z) \widetilde{\alpha}(z) \mathrm{d}z - \int_z \alpha(z) \partial_z \widetilde{\alpha}(z) \mathrm{d}z \\ &= 2 \int \alpha(z) \partial \widetilde{\alpha}(z) \end{aligned}$$

as desired. □

14.2.3 Symmetries on the source

In this section we use the formalism to identify how the Virasoro algebra acts on the observables of the free $\beta\gamma$ system. The arguments are borrowed from Williams (2017), where Williams provides a systematic analysis of the Virasoro algebra from the perspective developed in these books.

The Virasoro factorization algebra

Consider the local Lie algebra on a Riemann surface Σ given by

$$\mathcal{T} = \Omega^{0,*}(-, T^{1,0}),$$

which is how one describes holomorphic vector fields in a manner convenient for complex differential geometry. It has a direct relationship with polynomial vector fields on the circle: for any annulus $A = \{r < |z| < R\}$, there is a natural dense inclusion

$$\mathbb{C}[z, z^{-1}]\partial_z \hookrightarrow H^*(\mathcal{T}(A)) \cong \mathcal{O}(A)\partial_z,$$

since Laurent polynomials are dense in the holomorphic functions $\mathcal{O}(A)$ on an annulus and hence the same applies to vector fields. It is through this relationship that we will recognize more standard approaches to the Virasoro algebra.

Remark: This local Lie algebra also admits a nice interpretation as a formal moduli space. Consider global sections $\mathcal{T}(\Sigma)$ over a closed Riemann surface Σ. It is well-known that $H^1(\Sigma, T^{1,0})$ describes first-order deformations of complex structure, and more generally that $\mathcal{T}(\Sigma)$ describes the formal neighborhood in the moduli of curves of the point represented by Σ. That is, the Maurer-Cartan functor of this dg Lie algebra corresponds to the formal moduli functor obtained by completing the stack of curves there. Hence, $\mathcal{T}$ is a local-to-global object encoding deformations of complex structure. Compare to our discussion in the first remark of section I.5.5.1. ◊

There is a well-known Gelfand-Fuks cocycle for these polynomial vector fields

$$\omega_{\mathrm{GF}}(f(z)\partial_z, g(z)\partial_z) = \frac{1}{12}\operatorname{Res}_{z=0}(f'''g\,\mathrm{d}z),$$

which determines a central extension

$$0 \to \mathbb{C}c \to \mathrm{Vir} \to \mathbb{C}[z, z^{-1}]\partial_z \to 0$$

known as the *Virasoro Lie algebra*. One often studies representations of this Lie algebra where the action of the center is fixed to some scalar value λ called the *central charge*.

There is an analogous central extension $\widetilde{\mathcal{T}}_c$ of the cosheaf $\mathcal{T}_c$ using the cocycle

$$\omega(f_0 + f_1\mathrm{d}\bar{z}, g_0 + g_1\mathrm{d}\bar{z}) = \frac{1}{12}\frac{1}{2\pi i}\int \mathrm{d}^2z\,(\partial_z^3 f_0)g_1 + (\partial_z^3 f_1)g_0$$

where the f_i and g_i denote compactly-supported smooth functions. This object has a naturally associated factorization algebra.

14.2.3.1 Definition (2.5.1, Williams (2017)). *The* Virasoro factorization algebra $\mathcal{V}\mathrm{ir}$ *is the twisted enveloping factorization algebra*

$$\mathbb{U}_\omega\mathcal{T} = C_*(\widetilde{\mathcal{T}}_c).$$

It is a factorization algebra with values in modules over $\mathbb{C}[c]$, where the central element c has cohomological degree 0.

We use $\mathcal{V}\mathrm{ir}_{c=\lambda}$ to denote the factorization algebra obtained by specializing c to $\lambda \in \mathbb{C}$.

The main theorem of Williams (2017) is that this factorization algebra recovers the vertex algebra Vir typically known as the Virasoro vertex algebra. (See Section 2.5 of Frenkel and Ben-Zvi (2004) or 4.1 of Williams

(2017) for more on this vertex algebra.) It is a natural analogue of the results of Chapter I.5, where other vertex algebras are recovered from holomorphic factorization algebras via the functorial construction of Theorem I.5.3.3.

14.2.3.2 Theorem (Thm 9, Williams (2017)). *The vertex algebra recovered from the factorization algebra $\mathcal{V}\mathrm{ir}$ is isomorphic to the Virasoro vertex algebra* Vir. *Moreover, this isomorphism specializes to any choice of central charge, so that the vertex algebra for $\mathcal{V}\mathrm{ir}_{c=\lambda}$ is isomorphic to $\mathrm{Vir}_{c=\lambda}$.*

Remark: This construction and theorem is for the Riemann surface $\Sigma = \mathbb{C}$, so it is an interesting to ask what the analogous result would be on an arbitrary Riemann surface. The cocycle ω explicitly uses the choice of coordinate z and hence does not admit an immediate extension. Instead, there is a somewhat subtle construction of a cocycle depending upon a choice of projective connection. In the end this choice is irrelevant, as changing the projective connection changes the cocycle but not its cohomology class. See Section 5 of Williams (2017) or Section 8.2 of Frenkel and Ben-Zvi (2004) for detailed discussions. $\Diamond$

Virasoro symmetry of the $\beta\gamma$ system

We now turn to examining how the Lie algebra of holomorphic vector fields acts on the free $\beta\gamma$ system. The action on the fields themselves is simple: given a vector field $x = f(z)\partial_z$ and a field $\gamma(z)$, we have

$$x \cdot \gamma = f \partial_z \gamma,$$

and similarly for the field β. This action is realized by the local functional

$$J^{cl}_\alpha(\beta, \gamma) = \int \langle \beta, \alpha \cdot \gamma \rangle$$

where

$$\alpha \in \mathcal{T}_c = \Omega^{0,*}_c(T^{1,0}).$$

This formula determines a map of local Lie algebras

$$J^{cl} : \mathcal{T}_c \to \mathrm{Obs}^{cl}[-1].$$

We now examine how it interacts with the BV quantization of the free $\beta\gamma$ system.

14.2.3.3 Lemma (Lemma 4, Williams (2017)). *The obstruction cocycle has the form*

$$\begin{aligned}\mathrm{Ob}(f\,\partial_z, g\,\mathrm{d}\overline{z}\,\partial_z) &= \dim(V)\frac{1}{2\pi}\frac{1}{12}\int \mathrm{d}^2z\,(\partial_z^3 f)g \\ &= \dim(V)\omega(f\,\partial_z, g\,\mathrm{d}\overline{z}\,\partial_z)\end{aligned}$$

where f, g are smooth, compactly supported functions on $\mathbb{C}$.

We follow Williams' proof.

Proof The obstruction cocycle is the limit $\lim_{L\to 0}\mathrm{Ob}[L]$ of a scale-dependent obstruction cocycle, which is determined by the two-vertex wheel.

Before examining the explicit integral, we note that the linear dependence on $\dim(V)$ is simple: the vector fields $f\,\partial_z$ and $g\,\partial_z$ act diagonally on the fields and so the obstruction cocycle is given by $\dim(V)$ copies of the case with $V \cong \mathbb{C}$. (More explicitly, if we fix a basis for V, we get fields γ_i and β_i, which are γ and β in coordinates. The obstruction cocycle is just a sum over the index for the basis.) Hence, assume $\dim(V) = 1$ for the rest of the argument.

Fix $L > 0$. Then $\mathrm{Ob}[L]$ corresponds to the two-edge wheel with one edge labeled by a propagator P and the other edge labeled by the heat kernel K. More accurately, we take the $\epsilon \to 0$ limit with the propagator $P_{\epsilon<L}$ and heat kernel K_ϵ inserted:

$$\mathrm{Ob}[L](f\,\partial_z, g\,\mathrm{d}\overline{z}\,\partial_z) = \int_{\mathbb{C}^2}\mathrm{d}^2z\,\mathrm{d}^2w\, f(z,\overline{z})\,(\partial_z P_{\epsilon<L}(z,w))\,g(w,\overline{w})\,(\partial_w K_\epsilon(w,z))\,.$$

To obtain this integral, we are expressing the contraction of tensors implicit in the diagram in terms of the description of the tensors via integral kernels. Hence each vertex corresponds to a copy of $\mathbb{C}$ over which we integrate. Note that we label the vertex receiving the g input with the variable w and, perhaps abusively, use z for the vertex receiving f as an input.

Now

$$\partial_w K_\epsilon(w,z) = \frac{1}{4\pi\epsilon}\frac{\overline{z}-\overline{w}}{4\epsilon}e^{-|z-w|^2/4\epsilon}$$

and so

$$\partial_z P_{\epsilon<L}(z,w) = \int_\epsilon^L \mathrm{d}t\,\frac{1}{4\pi t}\frac{\overline{z}-\overline{w}}{4t}e^{-|z-w|^2/4t}.$$

We change coordinates by shearing: set $y = z - w$. Then

$$\mathrm{Ob}[L] = \int_{\mathbb{C}^2} \mathrm{d}^2 y\, \mathrm{d}^2 w\, f g \frac{1}{(16\pi)^2} \int_\epsilon^L \mathrm{d}t\, \frac{1}{(\epsilon t)^2} \overline{y}^3 \exp\left(-\frac{1}{4}\left(\frac{1}{\epsilon}+\frac{1}{t}\right)|y|^2\right).$$

By integration by parts, one knows that for any compactly supported ϕ and $a > 0$,

$$\int_{\mathbb{C}} \mathrm{d}^2 y\, \phi(y) \overline{y}^k e^{-a|y|^2} = \frac{(-1)^k}{a^k} \int_{\mathbb{C}} \mathrm{d}^2 y\, \left(\partial_y^k \phi\right) e^{-a|y|^2}.$$

Hence we find

$$\mathrm{Ob}[L] = \frac{1}{16\pi^2} \int_{\mathbb{C}^2} \mathrm{d}^2 y\, \mathrm{d}^2 w\, \partial_y^3 (fg) \int_\epsilon^L \mathrm{d}t\, \frac{\epsilon t}{(\epsilon + t)^3} \exp\left(-\frac{1}{4}\left(\frac{1}{\epsilon}+\frac{1}{t}\right)|y|^2\right).$$

Now consider taking the integral over y, but replacing $\partial_y^3(fg)$ with its partial Taylor expansion. The constant term of the Taylor expansion contributes

$$\frac{1}{2\pi}\left(\int_{\mathbb{C}} \mathrm{d}^2 w\, (\partial_w^3 f) g\right) \int_\epsilon^L \mathrm{d}t\, \frac{\epsilon^2 t}{(\epsilon+t)^4},$$

and this integral over t converges in the $\epsilon \to 0$ limit to $1/12$. (Note the surprising fact that this integral is independent of L.) The higher order Taylor terms contribute extra factors of $\epsilon t/(\epsilon + t)$, which means their $\epsilon \to 0$ limit is zero. □

As an immediate consequence of this obstruction computation and Theorem 14.2.3.2, one obtains the following result.

14.2.3.4 Proposition (Prop. 14, Williams (2017)). *The obstruction to an inner action is given by* $\dim(V)\,\omega$*, so that we obtain a map of factorization algebras*

$$\mathcal{V}\mathrm{ir}_{c=\dim(V)} \to \mathrm{Obs}^q.$$

This map induces a map of vertex algebras

$$\mathrm{Vir}_{c=\dim(V)} \to V_{\beta\gamma},$$

and this map is given by the usual conformal vector for the $\beta\gamma$ vertex algebra.

Remark: There are natural consequences of this map at the level of global sections on a closed Riemann surface. Williams explains a general recursion formula for n-point functions, and he unpacks the consequences with explicit examples for $\Sigma = \mathbb{CP}^1$. The map of factorization algebras thus identifies these formulas with certain computations of observables for the $\beta\gamma$ system, which can be computed using Wick's lemma and a propagator. $\Diamond$

14.3 An example from topological field theory

A particularly pleasant class of topological field theories are of BF-type, which we now describe. Let M be an orientable smooth manifold M of dimension n and without boundary. Let $\mathfrak{g}$ be a Lie algebra. The A-fields are elements of $\Omega^*(M) \otimes \mathfrak{g}[1]$ and the B-fields are elements of $\Omega^*(M) \otimes \mathfrak{g}^*[n-2]$, where $\mathfrak{g}^*$ denotes the linear dual to $\mathfrak{g}$. The action functional is

$$S(A, B) = \int \langle B, \mathrm{d}A + \frac{1}{2}[A, A] \rangle = \int \langle B, F \rangle,$$

where F denotes the curvature of the connection $\nabla_A = \mathrm{d} + A$ on the trivial $\mathfrak{g}$-bundle. Here $\langle -, - \rangle$ denotes the evaluation pairing between $\mathfrak{g}$ and its linear dual $\mathfrak{g}^*$ extended to forms with valued therein, and hence is a pairing on the fields with values in de Rham forms. (To work with non-orientable manifolds, we let the B-fields be de Rham forms twisted by the orientation line bundle, so that we have a version of Poincaré duality.)

One can view BF theory as the cotangent theory for the moduli space of flat $\mathfrak{g}$-connections, which is encoded in the dg Lie algebra $\Omega^*(M) \otimes \mathfrak{g}$ associated to the A-fields. The equations of motion, after all, are

$$F = \mathrm{d}A + \frac{1}{2}[A, A] = 0$$

and

$$\nabla_A B = \mathrm{d}B + [A, B] = 0.$$

Hence, in the classical theory, we are looking for a flat connection and a horizontal section in the coadjoint bundle. As no metric plays a role in this classical theory, it is natural to view a BF theory as a nice example of a classical topological field theory and to expect it quantizes to a topological field theory as well.

Remark: There is an extensive literature on BF theories, among which we point out Cattaneo and Rossi (2001); Baez (1996); Cattaneo et al. (1998a) as places to start that we found particularly helpful. We focus here on quite elementary facets of the theory and hence do what we need from scratch. ◊

For simplicity, we will stick to $\mathfrak{g}$ an Abelian Lie algebra, as the theory is then free and hence easy to quantize. To emphasize that A and B are forms with values in vector spaces, we use V and V^*: $A \in \Omega^*(M) \otimes V[1]$ and $B \in \Omega^*(M) \otimes V^*[n-2]$. Hence, the equations of motion pick out closed V-valued one-forms and closed V^*-valued $n-2$-forms.

We will enliven the situation, however, by letting V be a representation of a non-Abelian Lie algebra $\mathfrak{g}$ and viewing V^* as the dual representation. This extra structure puts us into a situation where the Noether framework is applicable. As usual we promote $\mathfrak{g}$ to the local Lie algebra

$$\mathcal{L} = \Omega^* \otimes \mathfrak{g},$$

and work with the current

$$J^{cl} : \Omega^*_c(M) \otimes \mathfrak{g} \to \mathrm{Obs}^{cl}(M)$$

where

$$J^{cl}_\alpha = \int \langle B, \alpha A \rangle.$$

The equivariant action functional is

$$\begin{aligned} S^{eq}(A, B, \alpha) &= S(A, B) + J^{cl}_\alpha(A, B) \\ &= \int \langle B, \mathrm{d}A + \alpha A \rangle. \end{aligned}$$

In other words, the fields are charged with respect to a background gauge field α.

Our goal is to analyze the quantization of this system. Let ω denote the obstruction cocycle, which may be trivial. The outcome of equivariant quantization will be a map of factorization algebras

$$J^q : \mathbb{U}^{BD}_\omega \mathcal{L} \to \mathrm{Obs}^q_{BF},$$

where $\mathbb{U}^{BD}_\omega \mathcal{L}$ is the locally constant factorization algebra given by taking the ω-twisted BD envelope of $\mathcal{L}$. Since this factorization algebra is locally constant, it corresponds to an E_n algebra. In fact, it corresponds to the $\hbar$-filtered E_n enveloping algebra of $\mathfrak{g}$, and hence is a natural generalization of the universal enveloping algebra of $\mathfrak{g}$. (See Chapter I.3 and Knudsen (2018) for further discussion.) The observables of this free BF

theory are the enveloping factorization algebra of another local Lie algebra, in this case a Heisenberg Lie algebra, as explained in Chapter I.4. Hence, Obs^q_{BF} can be interpreted as an E_n algebra analogue of the Weyl algebra. In this sense the map J^q determines an E_n algebra map from a twisted E_n enveloping algebra of $\mathfrak{g}$ to a Weyl E_n algebra.

Remark: When $n = 1$, this construction is a close cousin of the construction in Section 14.1.2, where $\mathfrak{g} = \mathfrak{o}(V)$ and we found a map of associative algebras from $U\mathfrak{g}$ to the ordinary Weyl algebra for $V \oplus V^*$. Indeed, if one modifies the action there to

$$S_c(\phi, \psi) = \int (\psi, \mathrm{d}\phi) + c \int (\psi, \psi)\mathrm{d}t$$

with c a constant, then we obtain a one-parameter family of theories such that at $c = 1$, we have the massless free field (in first-order formulation) and at $c = 0$, we get Abelian BF theory by viewing ϕ as A and ϕ as B. $\Diamond$

14.3.1 The diagrammatics

When one constructs the Feynman diagrams arising from this classical action functional, only trees and wheels appear. The naive quantized action is thus

$$S^q_{naive}[\Psi] = \sum_{\text{trees } Y} I_Y[\Psi](\phi, \alpha) + \hbar \sum_{\text{wheels } \circlearrowleft} I_{\circlearrowleft}[\Psi](\alpha),$$

where a tree interaction I_Y has an A-leg, a B-leg, and the remainder are α-legs and where a wheel interaction $I_{\circlearrowleft}$ only depends on background α-fields.

No counterterms are necessary for this theory. We will sketch the argument, which relies on the configuration space method introduced by Axelrod and Singer (1992, 1994) and Kontsevich (1994). A discussion compatible with our definitions here is seen in Costello (2007).

14.3.1.1 Proposition. *The naive action S^q_{naive} is well-defined without counterterms.*

Proof sketch The main idea is a kind of point-splitting regularization. A propagator behaves as a partial inverse d^{-1}, and its integral kernel P

would be smooth away from the diagonal

$$\Delta : M \hookrightarrow M \times M.$$

We now explain the crucial property of this situation.

Consider the real blowup along the diagonal, which replaces the diagonal by the sphere bundle associated to the normal bundle. It is a manifold with boundary, where the boundary is the sphere bundle. The complement of this boundary is just $M^2 \setminus \Delta(M)$, and so the integral kernel defines a smooth section on this open set. The important property is that this integral kernel has a natural smooth extension $\overline{P}$ to the boundary.

This property ensures that we can avoid divergences: instead of using the propagator with its singularity along the diagonal, use $\overline{P}$ instead and integrate over the real blowup.

The configuration space method allows one to do this systematically and prove that no divergences appear. For each Feynman diagram Γ, the putative integral over $M^{|V(\Gamma)|}$ is replaced by a blowup along diagonals, so that the singular support of the putative distributions is avoided. $\square$

14.3.2 The obstruction cocycle

We now turn to analyzing the obstruction cocycle, which has degree one in $C^*_{loc,red}(\mathcal{L})$. As we have seen,

$$C^*_{loc,red}(\mathcal{L}) \simeq \Omega^*(M)[n] \otimes C^*_{red}(\mathfrak{g}).$$

We can examine this obstruction locally, i.e., take $M = \mathbb{R}^n$. In that case, the obstruction cocycle determines a cohomology class in $H^{n+1}(\mathfrak{g})$.

This observation has immediate consequences, by exploiting facts about Lie algebra cohomology. Note that we assume here that $\mathfrak{g}$ is a finite-dimensional Lie algebra. For dg Lie and L_∞ algebras, one must use more sophisticated arguments.

14.3.2.1 Lemma. *The obstruction group vanishes if*

- $n = \dim(M) > \dim(\mathfrak{g})$, *or*

- $\mathfrak{g}$ *is semisimple and* $n+1 = 1, 2,$ *or* 4, *or*
- $n+1 = 5$ *if* $\mathfrak{g}$ *is semisimple and contains no factors* $\mathfrak{sl}_d$ *for* $d > 2$.

We will not compute any explicit obstruction cocycles in the cases where it need not vanish. (It would be interesting to explore these cases.) Our focus here is on examining the map of factorization algebras and understanding its basic behavior. A nontrivial obstruction cocycle creates a central extension but the flavor of the enveloping factorization algebra is similar, and so we will proceed as if the obstruction vanishes.

14.3.3 The local situation

Let us consider $M = \mathbb{R}^n$ and examine the factorization algebras $\mathbb{U}^{BD}_{\omega}\mathfrak{g}$ and Obs^q_{BF}. As a first pass at getting a feel for them, consider their cohomology when evaluated on a disk.

14.3.3.1 Lemma. *We have*

$$H^*\mathbb{U}^{BD}_{\omega}\mathfrak{g}(\mathbb{R}^n) \cong \mathrm{Sym}(\mathfrak{g}[1-n])[\hbar]$$

and

$$H^*\,\mathrm{Obs}^q_{BF}(\mathbb{R}^n) \cong \mathrm{Sym}(V^*[-1] \oplus V[2-n])[\hbar]$$

as isomorphisms of graded vector spaces.

Proof These are both enveloping factorization algebras and hence are given by a Lie algebra chain complex (or rather the $\hbar$-weighted version that is the enveloping BD algebra). Consider the filtration by symmetric powers. The first page is computed by using the differentials that preserve the symmetric powers. In this case that means the de Rham differential d acting on compactly supported de Rham forms. Hence the first page is

$$\mathrm{Sym}(\mathfrak{g}[1-n])[\hbar]$$

and

$$\mathrm{Sym}(V^*[-1] \oplus V[2-n])[\hbar].$$

In the first case, there are no further differentials because the Lie bracket on

$$H^*(\Omega^*_c(\mathbb{R})) \otimes \mathfrak{g} = \mathfrak{g}[-n]$$

is trivial. In the second case, the induced BV Laplacian is trivial as the linear generators $V^*[-1] \oplus V[2-n]$ bracket together trivially. ◻

We emphasize the the cohomology in degree zero is simply $\mathbb{C}[\hbar]$. Hence there are no nontrivial currents or observables with support in a disk.

Nonetheless, there is interesting information in this situation. Nontrivial currents and observables appear in degree zero if one works with the dimensional reduction of the theory along a closed manifold of dimension $n-1$. In other words, if M has interesting topology, then interesting phenomena appears with a more traditional flavor. Compare with section I.4.5, which examines the dimensional reduction of Abelian Chern-Simons theory and where cycles in a surface correspond to meaningful observables.

We remark that, from the perspective of homotopical algebra, there is interesting information even on a disk. These graded vector spaces are equipped with the structure of algebras over the operad H_*E_n, since they are the cohomology of E_n-algebras. Recall that the operad H_*E_n describes shifted Poisson algebras with a degree $1-n$ Poisson bracket. Hence both $H^*\mathbb{U}^{BD}_{\omega}\mathfrak{g}(\mathbb{R}^n)$ and $H^*\,\mathrm{Obs}^q_{BF}(\mathbb{R}^n)$ are shifted Poisson algebras, and the current map induces a map H^*J^q of such shifted Poisson algebras.

Remark: The case $n = 2$ may be the most familiar: a -1-shifted Poisson algebra is often known as a Gerstenhaber algebra. Our construction thus produces a map of Gerstenhaber algebras for two-dimensional BF theory.

We note that the topological B-model can be viewed as a two-dimensional BF theory, if one encodes the target space as a kind of Lie algebra object. See Li and Li (2016) for an extensive treatment of this theory in such terms. Our methods here indicate what happens when the target admits an action of $\mathfrak{g}$ as symmetries: one will obtain a map of Gerstenhaber algebras from $\mathrm{Sym}(\mathfrak{g}[-1])$ into the polyvector fields of the target, as a shadow of the more refined construction using factorization algebras. ◊

Appendix A
Background

Lie algebras, and their homotopical generalization L_∞ algebras, appear throughout this book in a variety of contexts. It might surprise the reader that we never use their representation theory or almost any aspects emphasized in textbooks on Lie theory. Instead, we primarily use dg Lie algebras as a convenient language for formal derived geometry. In the first section of this appendix, we overview homological constructions with dg Lie algebras, and in the following section, we overview deformation theory, its relationship with derived geometry, and the use of dg Lie algebras in modeling deformations.

A.1 Lie algebras and L_∞ algebras

We use these ideas in the following settings.

- We use the Chevalley-Eilenberg complex to construct a large class of factorization algebras, via the *factorization envelope* of a sheaf of dg Lie algebras. This class includes the observables of free field theories and the Kac-Moody vertex algebras.
- We use the Lie-theoretic approach to deformation functors to motivate our approach to classical field theory.
- We introduce the notion of a *local* Lie algebra to capture the symmetries of a field theory and prove generalizations of Noether's theorem.

We also use Lie algebras in the construction of gauge theories in the usual way.

A.1.1 Differential graded Lie algebras and L_∞ algebras

We now quickly extend and generalize homologically the notion of a Lie algebra. Our base ring will now be a commutative algebra R over a characteristic zero field $\mathbb{K}$, and we encourage the reader to keep in mind the simplest case: where $R = \mathbb{R}$ or $\mathbb{C}$. Of course, one can generalize the setting considerably, with a little care, by working in a symmetric monoidal category (with a linear flavor); the cleanest approach is to use operads.

Before introducing L_∞ algebras, we treat the simplest homological generalization.

A.1.1.1 Definition. *A* dg Lie algebra *over R is a $\mathbb{Z}$-graded R-module $\mathfrak{g}$ such that*

(1) there is a differential

$$\cdots \xrightarrow{d} \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^0 \xrightarrow{d} \mathfrak{g}^1 \to \cdots$$

making $(\mathfrak{g}, d)$ into a dg R-module;

(2) there is a bilinear bracket $[-,-] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ such that

- $[x, y] = -(-1)^{|x||y|}[y, x]$ *(graded antisymmetry),*
- $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$ *(graded Leibniz rule),*
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$ *(graded Jacobi rule),*

where $|x|$ denotes the cohomological degree of $x \in \mathfrak{g}$.

In other words, a dg Lie algebra is an algebra over the operad *Lie* in the category of dg R-modules. In practice — and for the rest of the section — we require the graded pieces $\mathfrak{g}^k$ to be projective R-modules so that we do not need to worry about the tensor product or taking duals.

Here are several examples.

(a) We construct the dg analog of $\mathfrak{gl}_n$. Let (V, d_V) be a cochain complex over $\mathbb{K}$. Let $\mathrm{End}(V) = \oplus_n \mathrm{Hom}^n(V, V)$ denote the graded vector space where Hom^n consists of the linear maps that shift degree by n, equipped with the differential

$$d_{\mathrm{End}\,V} = [d_V, -] : f \mapsto d_V \circ f - (-1)^{|f|} f \circ d_V.$$

The commutator bracket makes $\mathrm{End}(V)$ a dg Lie algebra over $\mathbb{K}$.

(b) For M a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra (such as $su(2)$), the tensor product $\Omega^*(M) \otimes_{\mathbb{R}} \mathfrak{g}$ is a dg Lie algebra where the differential is simply the exterior derivative and the bracket is

$$[\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x, y].$$

We can view this dg Lie algebra as living over $\mathbb{K}$ or over the commutative dg algebra $\Omega^*(M)$. This example appears naturally in the context of gauge theory.

(c) For X a simply-connected topological space, let $\mathfrak{g}_X^{-n} = \pi_{1+n}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and use the Whitehead product to provide the bracket. Then $\mathfrak{g}_X$ is a dg Lie algebra with zero differential. This example appears naturally in rational homotopy theory.

We now introduce a generalization where we weaken the Jacobi rule on the brackets in a systematic way. After providing the (rather convoluted) definition, we sketch some motivations.

A.1.1.2 Definition. *An* L_∞ algebra over R *is a* $\mathbb{Z}$*-graded, projective* R*-module* $\mathfrak{g}$ *equipped with a sequence of multilinear maps of cohomological degree* $2 - n$

$$\ell_n : \underbrace{\mathfrak{g} \otimes_R \cdots \otimes_R \mathfrak{g}}_{n \text{ times}} \to \mathfrak{g},$$

with $n = 1, 2, \ldots$, *satisfying the following properties.*

(i) *Each bracket* ℓ_n *is graded-antisymmetric, so that*

$$\ell_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = -(-1)^{|x_i||x_{i+1}|} \ell_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

for every n*-tuple of elements and for every* i *between* 1 *and* $n - 1$.

(ii) *Each bracket* ℓ_n *satisfies the* n*-Jacobi rule, so that*

$$0 = \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1 < \cdots < i_k \\ j_{k+1} < \cdots < j_n \\ \{i_1, \ldots, j_n\} = \{1, \ldots, n\}}} (-1)^{\epsilon} \ell_{n-k+1}(\ell_k(x_{i_1}, \ldots, x_{i_k}), x_{j_{k+1}}, \ldots, x_{j_n}).$$

Here $(-1)^\varepsilon$ denotes the sign for the permutation

$$\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_{k+1} & \cdots & j_n \end{pmatrix}$$

acting on the element $x_1 \otimes \cdots \otimes x_n$ given by the alternating-Koszul sign rule, where the transposition $ab \mapsto ba$ acquires sign $-(-1)^{|a||b|}$.

For small values of n, we recover familiar relations. For example, the 1-Jacobi rule says that $\ell_1 \circ \ell_1 = 0$. In other words, ℓ_1 is a differential! Momentarily, let's denote ℓ_1 by d and ℓ_2 by the bracket $[-,-]$. The 2-Jacobi rule then says that

$$-[dx_1, x_2] + [dx_2, x_1] + d[x_1, x_2] = 0,$$

which encodes the graded Leibniz rule. Finally, the 3-Jacobi rule rearranges to

$$\begin{aligned} &[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] \\ &\qquad = d\ell_3(x_1, x_2, x_3) + \ell_3(dx_1, x_2, x_3) + \ell_3(dx_2, x_3, x_1) + \ell_3(dx_3, x_1, x_2). \end{aligned}$$

In short, $\mathfrak{g}$ does not satisfy the usual Jacobi rule *on the nose* but the failure is described by the other brackets. In particular, at the level of cohomology, the usual Jacobi rule *is* satisfied.

Example: There are numerous examples of L_∞ algebras throughout the book, but many are simply dg Lie algebras spiced with analysis. We describe here a small, algebraic example of interest in topology and elsewhere. (See, for instance, Henriques (2008), Baez and Crans (2004), Baez and Rogers (2010).) The *String Lie 2-algebra* $string(n)$ is the graded vector space $so(n) \oplus \mathbb{R}\beta$, where β has degree -1, equipped with two nontrivial brackets:

$$\ell_2(x, y) = \begin{cases} [x, y], & x, y \in so(n) \\ 0, & x = \beta \end{cases}$$
$$\ell_3(x, y, z) = \mu(x, y, z)\beta \quad x, y, z \in so(n),$$

where μ denotes $\langle -, [-,-] \rangle$, the canonical (up to scale) 3-cocycle on $so(n)$ arising from the Killing form. This L_∞ algebra arises as a model for the "Lie algebra" of $String(n)$, which itself appears in various guises (as a topological group, as a smooth 2-group, or as a more sophisticated object in derived geometry). $\lozenge$

There are two important cochain complexes associated to an L_∞ algebra, which generalize the two Chevalley-Eilenberg complexes we defined earlier.

A.1.1.3 Definition. *For $\mathfrak{g}$ an L_∞ algebra, the* Chevalley-Eilenberg complex for homology $C_*\mathfrak{g}$ *is the dg cocommutative coalgebra*

$$\mathrm{Sym}_R(\mathfrak{g}[1]) = \bigoplus_{n=0}^{\infty} \left((\mathfrak{g}[1])^{\otimes n}\right)_{S_n}$$

equipped with the coderivation d whose restriction to cogenerators d_n $: \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ *are precisely the higher brackets ℓ_n.*

We sometimes call this complex $C_*\mathfrak{g}$ the *Lie algebra chains* of $\mathfrak{g}$ as it models Lie algebra homology.

Remark: The coproduct $\Delta : C_*\mathfrak{g} \to C_*\mathfrak{g} \otimes_R C_*\mathfrak{g}$ is given by running over the natural ways that one can "break a monomial into two smaller monomials." Namely,

$$\Delta(x_1 \cdots x_n) = \sum_{\sigma \in S_n} \sum_{1 \leq k \leq n-1} (x_{\sigma(1)} \cdots x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \cdots x_{\sigma(n)}).$$

A coderivation respects the coalgebra analog of the Leibniz property, and so it is determined by its behavior on cogenerators. ◊

This coalgebra $C_*\mathfrak{g}$ conveniently encodes all the data of the L_∞ algebra $\mathfrak{g}$. The coderivation d puts all the brackets together into one operator, and the equation $d^2 = 0$ encodes all the higher Jacobi relations. It also allows for a concise definition of a map between L_∞ algebras.

A.1.1.4 Definition. *A* map of L_∞ algebras $F : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ *is given by a map of dg cocommutative coalgebras $F : C_*\mathfrak{g} \to C_*\mathfrak{h}$.*

This definition encodes a *homotopy-coherent map* of Lie algebras. Note that a map of L_∞ algebras is *not* determined just by its behavior on $\mathfrak{g}$, which is why we use $\rightsquigarrow$ to denote such a morphism. Unwinding the definition above, one discovers that such a morphism consists of a linear map $\mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{h}$ for each n, satisfying compatibility conditions ensuring that we get a map of coalgebras.

To define the other Chevalley-Eilenberg complex $C^*\mathfrak{g}$, we use the graded linear dual of $\mathfrak{g}$,

$$\mathfrak{g}^\vee = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_R(\mathfrak{g}^n, R)[n],$$

which is the natural notion of dual in this context.

A.1.1.5 Definition. *For $\mathfrak{g}$ an L_∞ algebra, the* Chevalley-Eilenberg complex for cohomology $C^*\mathfrak{g}$ *is the dg commutative algebra*

$$\widehat{\mathrm{Sym}}_R(\mathfrak{g}[1]^\vee) = \prod_{n=0}^{\infty} \left((\mathfrak{g}[1]^\vee)^{\otimes n}\right)_{S_n}$$

equipped with the derivation d whose Taylor coefficients $d_n : \mathfrak{g}[1]^\vee \to \mathrm{Sym}^n(\mathfrak{g}[1]^\vee)$ are dual to the higher brackets ℓ_n.

We sometimes call this complex $C^*\mathfrak{g}$ the *Lie algebra cochains* of $\mathfrak{g}$ as it models Lie algebra cohomology.

We emphasize that this dg algebra is *completed* with respect to the filtration by powers of the ideal generated by $\mathfrak{g}[1]^\vee$. This filtration will play a crucial role in the setting of deformation theory.

Now that we have a dg commutative algebra $C^*\mathfrak{g}$, we can ask about derivations. In other words, if we view $C^*\mathfrak{g}$ as the ring of functions on some space, we want to describe the vector fields on that space. For a free commutative algebra $\mathrm{Sym}(V^*)$, the derivations are $\mathrm{Sym}(V^*) \otimes V$, where an element $v \in V$ determines a constant-coefficient differential operator ∂_v by extending the evaluation pairing $\mathrm{Sym}^1(V^*) \otimes V \to k$ by the Leibniz rule. Since the underlying graded commutative algebra of $C^*\mathfrak{g}$ is free, it is easy to do this in our situation, and the definition is the homotopically correct answer.

A.1.1.6 Definition. *For $\mathfrak{g}$ an L_∞ algebra, its* derivations *are*

$$Der(\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}[1])$$

where we view $\mathfrak{g}[1]$ as a $\mathfrak{g}$-module by a shift of the adjoint action.

Note that this complex is naturally a dg Lie algebra and acts canonically on $C^*\mathfrak{g}$ by the Lie derivative.

A.1.2 References

We highly recommend Getzler (2009) for an elegant and efficient treatment of L_∞ algebras, as well as simplicial sets and how these constructions fit together with deformation theory. The book Kontsevich and Soibelman (n.d.) provides a wealth of examples, motivation, and context.

A.2 Derived deformation theory

In physics, one often studies very small perturbations of a well-understood system, wiggling an input infinitesimally or deforming an operator by a small amount. Asking questions about how a system behaves under small changes is ubiquitous in mathematics, too, and there is an elegant formalism for such problems in the setting of algebraic geometry, known as *deformation theory*. Here we will give a very brief sketch of *derived* deformation theory, where homological ideas are mixed with classical deformation theory.

A major theme of this book is that perturbative aspects of field theory — both classical and quantum — are expressed cleanly and naturally in the language of derived deformation theory. In particular, many constructions from physics, like the the Batalin-Vilkovisky formalism, obtain straightforward interpretations. Moreover, derived deformation theory suggests how to rephrase standard results in concise, algebraic terms and also suggests how to generalize these results substantially (see, for instance, the discussion on Noether's theorem).

In this section, we begin with a quick overview of formal deformation theory in algebraic geometry. We then discuss its generalization in derived algebraic geometry. Finally, we explain the powerful relationship between deformation theory and L_∞ algebras, which we exploit throughout the book.

A.2.1 The formal neighborhood of a point

Let $\mathcal{S}$ denote some category of spaces, such as smooth manifolds or complex manifolds or schemes. The Yoneda lemma implies we can understand any particular space $X \in \mathcal{S}$ by understanding how other spaces $Y \in \mathcal{S}$ map into X. That is, the functor represented by X, namely

$$\begin{array}{rccc} h_X : & \mathcal{S}^{op} & \to & \text{Sets} \\ & Y & \mapsto & \mathcal{S}(Y, X) \end{array}$$

knows everything about X as a space of type $\mathcal{S}$. We call h_X the *functor of points of* X, and this functorial perspective on geometry will guide our work below. Although abstract at first acquaintance, this perspective is especially useful for thinking about general features of geometry.

Suppose we want to describe what X looks like near some point $p \in X$. Motivated by the perspective of functor of points, we might imagine describing "X near p" by some kind of functor. The input category ought to capture all possible "small neighborhoods of a point" permitted in $\mathcal{S}$, so that we can see how such models map into X near p. We now make this idea precise in the setting of algebraic geometry.

Let $\mathcal{S} = Sch_{\mathbb{C}}$ denote the category of schemes over $\mathbb{C}$. Every such scheme X consists of a topological space X_{top} equipped with a sheaf of commutative $\mathbb{C}$-algebras $\mathscr{O}_X$ (satisfying various conditions we will not specify). We interpret the algebra $\mathscr{O}_X(U)$ on the open set U as the "algebra of functions on U." Every commutative $\mathbb{C}$-algebra R determines a scheme $\operatorname{Spec} R$ where the prime ideals of R provide the set of points of the topological space $(\operatorname{Spec} R)_{top}$ and where the stalk of $\mathscr{O}$ at a prime ideal $\mathfrak{P}$ is precisely the localization of R with respect to $R - \mathfrak{P}$. We call such a scheme $\operatorname{Spec} R$ an *affine scheme*. By definition, every scheme admits an open cover by affine schemes.

It is a useful fact that the functor of points h_X of a scheme X is determined by its behavior on the subcategory $\mathit{Aff}_{\mathbb{C}}$ of affine schemes. By construction, $\mathit{Aff}_{\mathbb{C}}$ is the opposite category to $\mathrm{CAlg}_{\mathbb{C}}$, the category of commutative $\mathbb{C}$-algebras. Putting these facts together, we know that every scheme X provides a functor from $CAlg_{\mathbb{C}}$ to Sets. Here are two examples.

Example: Consider the polynomial $q(x,y) = x^2 + y^2 - 1$. The functor

$$\begin{array}{rccc} h_X : & \mathrm{CAlg}_{\mathbb{C}} & \to & \mathrm{Sets} \\ & R & \mapsto & \{(a,b) \in R^2 \mid 0 = q(a,b) = a^2 + b^2 - 1\} \end{array}$$

corresponds to the affine scheme $\operatorname{Spec} S$ for the algebra $S = \mathbb{C}[x,y]/(q)$. This functor simply picks out solutions to the equation $q(x,y) = 0$ in the algebra R, which we might call the "unit circle" in R^2. Generalizing, we see that any system of polynomials (or ideal in an algebra) defines a similar functor of "solutions to the system of equations." $\Diamond$

Example: Consider the scheme SL_2, viewed as the functor

$$\begin{array}{rccc} SL_2 : & \mathrm{CAlg}_{\mathbb{C}} & \to & \mathrm{Sets} \\ & R & \mapsto & \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{c} a,b,c,d \in R \text{ such} \\ \text{that } 1 = ad - bc \end{array} \right\}. \end{array}$$

Note that $SL_2(\mathbb{C})$ is precisely the set that we usually mean. One can check as well that this functor factors through the category of groups. $\Diamond$

The notion of "point" in this category is given by $\operatorname{Spec}\mathbb{C}$, which is the locally ringed space given by a one-point space $\{*\}$ equipped with $\mathbb{C}$ as its algebra of functions. A *point in the scheme X* is then a map $p : \operatorname{Spec}\mathbb{C} \to X$. Every point is contained in some affine patch $U \cong \operatorname{Spec} R \subset X$, so it suffices to understand points in affine schemes. It is now possible to provide an answer to the question, "What are the affine schemes that look like small thickenings of a point?"

A.2.1.1 Definition. *A commutative $\mathbb{C}$-algebra A is* artinian *if A is finite-dimensional as a $\mathbb{C}$-vector space. A* local algebra *A with unique maximal ideal $\mathfrak{m}$ is artinian if and only if there is some integer n such that $\mathfrak{m}^n = 0$.*

Any local artinian algebra $(A, \mathfrak{m})$ provides a scheme $\operatorname{Spec} A$ whose underlying topological space is a point but whose scheme structure has "infinitesimal directions" in the sense that every function $f \in \mathfrak{m}$ is "small" because $f^n = 0$ for some n. Let $\mathrm{Art}_{\mathbb{C}}$ denote the category of local artinian algebras, which we will view as the category encoding "small neighborhoods of a point."

Remark: Hopefully it seems reasonable to choose $\mathrm{Art}_{\mathbb{C}}$ as a model for "small neighborhoods of a point." There are other approaches imaginable but this choice is quite useful. In particular, the most obvious topology for schemes — the Zariski topology — is quite coarse, so that open sets are large and hence do not reflect the idea of "zooming in near the point." Instead, we use schemes whose space is just a point but have interesting but tractable algebra. $\Diamond$

A point $p : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} R$ corresponds to a map of algebras $P : R \to \mathbb{C}$. Every local artinian algebra $(A, \mathfrak{m})$ has a distinguished map $Q : A \to A/\mathfrak{m} \cong \mathbb{C}$. Given a point p in $\operatorname{Spec} R$, we obtain a functor

$$h_p : \begin{array}{ccc} \mathrm{Art}_{\mathbb{C}} & \to & \mathrm{Sets} \\ (A, \mathfrak{m}) & \mapsto & \{F : R \to A \mid P = Q \circ F\} \end{array}.$$

Geometrically, this condition on ϕ means p is the composition $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A \overset{\operatorname{Spec} F}{\to} \operatorname{Spec} R$. The map F thus describes some way to "extend infinitesimally" away from the point p in X. A concrete example is in order.

Example: Our favorite point in SL_2 is given by the identity element $\mathbb{1}$. Let $h_{\mathbb{1}}$ denote the associated functor of artinian algebras. We can describe the tangent space $T_{\mathbb{1}}SL_2$ using it, as follows. Consider the artinian algebra $\mathbb{D} = \mathbb{C}[\epsilon]/(\epsilon^2)$, often called the *dual numbers*. Then a matrix in $h_{\mathbb{1}}(\mathbb{D})$ has the form

$$M = \begin{pmatrix} 1 + s\epsilon & t\epsilon \\ u\epsilon & 1 + v\epsilon \end{pmatrix}$$

and it must satisfy

$$\begin{aligned} \det(M) &= (1 + s\epsilon)(1 + v\epsilon) - tu\epsilon^2 \\ &= 1 + (s + v)\epsilon = 1 \end{aligned}$$

so that $s + v = 0$. In other words, writing $M = \mathbb{1} + \epsilon N$, we see $\det(M) = 1$ if and only if $\operatorname{Tr}(N) = 0$. Thus,

$$\begin{aligned} h_{\mathbb{1}}(\mathbb{D}) &= \left\{ M = 1 + \epsilon \begin{pmatrix} s & t \\ u & v \end{pmatrix} \;\middle|\; \begin{array}{c} s, t, u, v \in \mathbb{C} \text{ and} \\ 0 = s + v \end{array} \right\} \\ &\cong \{N \in M_2(\mathbb{C}) \mid \operatorname{Tr} N = 0\} \\ &= \mathfrak{sl}_2(\mathbb{C}). \end{aligned}$$

where the isomorphism is given by $M = \mathbb{1} + \epsilon N$. Thus, we have recovered the underlying set of the Lie algebra. $\Diamond$

For any point p in a scheme X, the set $h_p(\mathbb{D})$ is the tangent space to p in X. By considering more complicated artinian algebras, one can study the "higher order jets" at p. We say that h_p describes the *formal neighborhood* of p in X. The following proposition motivates this terminology.

A.2.1.2 Proposition. *Let $P : R \to \mathbb{C}$ be a map of algebras (i.e., we have a point $p : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} R$). Then*

$$h_p(A) = \mathrm{CAlg}_{\mathbb{C}}(\widehat{R}_p, A),$$

where

$$\widehat{R}_p = \varprojlim R/\mathfrak{m}_p^n$$

is the completed local ring given by the inverse limit over powers of $\mathfrak{m}_p = \ker P$, the maximal ideal given by the functions vanishing at p.

In other words, the functor h_p is not represented by a local artinian algebra (unless R is artinian), but it is represented inside the larger category $CAlg_{\mathbb{C}}$. When R is noetherian, the ring $\widehat{R}_p$ is given by an inverse system of local artinian algebras, so we say h_p is *pro*-represented. When R is a regular ring (such as a polynomial ring over $\mathbb{C}$), $\widehat{R}_p$ is isomorphic to formal power series. This important example motivates the terminology of *formal neighborhood*.

There are several properties of such a functor h_p that we want to emphasize, as they guide our generalization in the next section. First, by definition, $h_p(\mathbb{C})$ is simply a point, namely the point p. Second, we can study h_p in stages, by a process we call *artinian induction*. Observe that every local artinian algebra $(A, \mathfrak{m})$ is equipped with a natural filtration

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n = 0.$$

Thus, every local artinian algebra can be constructed iteratively by a sequence of *small* extensions, namely a short exact sequence of vector spaces

$$0 \to I \hookrightarrow B \xrightarrow{f} A \to 0$$

where $f : B \to A$ is a surjective map of algebras and I is an ideal in B such that $\mathfrak{m}_B I = 0$. We can thus focus on understanding the maps $h_p(f) : h_p(B) \to h_p(A)$, which are simpler to analyze. In summary, h_p is completely determined by how it behaves with respect to small extensions.

A third property is categorical in nature. Consider a pullback of local artinian algebras

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

and note that $B \times_A C$ is local artinian as well. Then the natural map

$$h_p(B \times_A C) \to h_p(B) \times_{h_p(A)} h_p(C)$$

is surjective — in fact, it is an isomorphism. (This property will guide us in the next subsection.)

As an example, we describe how to study small extensions for the model case. Let $(R, \mathfrak{m}_R)$ be a complete local ring with residue field $R/\mathfrak{m}_R \cong \mathbb{C}$ and with finite-dimensional *tangent space* $T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$. Consider the functor $h_R : A \mapsto CAlg(R, A)$, which describes the formal neighborhood of the closed point in Spec R. The following proposition provides a tool for understanding the behavior of h_R on small extensions.

A.2.1.3 Proposition. *For every small extension*

$$0 \to I \hookrightarrow B \xrightarrow{f} A \to 0,$$

there is a natural exact sequence of sets

$$0 \to T_R \otimes_{\mathbb{C}} I \to h_R(B) \xrightarrow{f\circ-} h_R(A) \xrightarrow{ob} O_R \otimes I,$$

where exact *means that a map $\phi \in h_R(A)$ lifts to a map $\tilde{\phi} \in h_R(B)$ if and only if $ob(\phi) = 0$ and the space of liftings is an affine space for the vector space $T_R \otimes_{\mathbb{C}} I$.*

Here *ob* denotes the *obstruction* to lifting maps, and O_R is a set where an obstruction lives. An obstruction space O_R only depends on the algebra R, not on the small extension. One can construct an obstruction space as follows. If $d = \dim_{\mathbb{C}} T_R$, there is a surjection of algebras

$$r : S = \mathbb{C}[[x_1, \dots, x_d]] \to R$$

such that $J = \ker r$ satisfies $J \subset \mathfrak{m}_S^2$, where $\mathfrak{m}_S = (x_1, \dots, x_d)$ is the maximal ideal of S. In other words, Spec R can be embedded into the formal neighborhood of the origin in $\mathbb{A}^d$, and minimally, in some sense.

Then O_R is $(J/\mathfrak{m}_S J)^\vee$. For a proof of the proposition, see Theorem 6.1.19 of Fantechi et al. (2005).

This proposition hints that something homotopical lurks behind the scenes, and that the exact sequence of sets is the truncation of a longer sequence. For a discussion of these ideas and the modern approach to deformation theory, we highly recommend the 2010 ICM talk Lurie (2010).

References

The textbook Eisenbud and Harris (2000) is a lovely introduction to the theory of schemes, full of examples and motivation. There is an extensive discussion of the functor of points approach to geometry, carefully compared to the locally ringed space approach. For an introduction to deformation theory, we recommend the article of Fantechi and Göttsche in Fantechi et al. (2005). Both texts provide extensive references to the literature.

A.2.2 Formal moduli spaces

The functorial perspective on algebraic geometry suggests natural generalizations of the notion of a scheme by changing the source and target categories. For instance, stacks arise as functors from $CAlg_{\mathbb{C}}$ to the category of groupoids, allowing one to capture the idea of a space "with internal symmetries." It is fruitful to generalize even further, by enhancing the source category from commutative algebras to dg commutative algebras (or simplicial commutative algebras) and by enhancing the target category from sets to simplicial sets. (Of course, one needs to simultaneously adopt a more sophisticated version of category theory, namely ∞-category theory.) This generalization is the subject of derived algebraic geometry, and much of its power arises from the fact that it conceptually integrates geometry, commutative algebra, and homotopical algebra. As we try to show in this book, the viewpoint of derived geometry provides conceptual interpretations of constructions like Batalin-Vilkovisky quantization.

We now outline the derived geometry version of studying the formal neighborhood of a point. Our aim to pick out a class of functors that capture our notion of a formal derived neighborhood.

A.2.2.1 Definition. *An* artinian dg algebra *A is a dg commutative algebra over $\mathbb{C}$ such that*

(1) *each component A^k is finite-dimensional, $\dim_{\mathbb{C}} A^k = 0$ for $k << 0$ and for $k > 0$, and*
(2) *A has a unique maximal ideal $\mathfrak{m}$, closed under the differential, and $A/\mathfrak{m} = \mathbb{C}$.*

Let $\mathrm{dgArt}_{\mathbb{C}}$ denote the category of artinian algebras, where morphisms are simply maps of dg commutative algebras.

Note that, as we only want to work with local rings, we simply included it as part of the definition. Note as well that we require A to be concentrated in nonpositive degrees. (This second condition is related to the Dold-Kan correspondence: we want A to correspond to a simplicial commutative algebra.)

We now provide an abstract characterization of a functor that behaves like the formal neighborhood of a point, motivated by our earlier discussion of functors h_p.

A.2.2.2 Definition. *A* formal moduli problem *is a functor*

$$F : \mathrm{dgArt}_{\mathbb{C}} \to \mathrm{sSet}$$

such that

(i) *$F(\mathbb{C})$ is a contractible Kan complex,*
(ii) *F sends a surjection of dg artinian algebras to a fibration of simplicial sets, and*
(iii) *for every pullback diagram in* dgArt

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

the map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is a weak homotopy equivalence.

Note that since surjections go to fibrations, the strict pullback $F(B) \times_{F(A)} F(C)$ agrees with the homotopy pullback $F(B) \times^h_{F(A)} F(C)$.

We now describe a large class of examples. Let R be a commutative dg algebra over $\mathbb{C}$ whose underlying graded algebra is $\widehat{\mathrm{Sym}} V$, where V is a $\mathbb{Z}$-graded vector space, and whose differential d_R is a degree 1 derivation. It has a unique maximal ideal generated by V. Let h_R denote the functor into simplicial sets whose n-simplices are

$$h_R(A)_n = \{f : R \to A \otimes \Omega^*(\triangle^n) \mid f \text{ a map of unital dg commutative algebras}\}$$

and whose structure maps arise from those between the de Rham complexes of simplices. Then h_R is a formal moduli problem.

References

We are modeling our approach on Lurie's, as explained in Lurie (2010) and chapter 13 of Lurie (n.d.). For a discussion of these ideas in our context of field theory, see Costello (2013a).

A.2.3 The role of L_∞ algebras in deformation theory

There is another algebraic source of formal moduli functors — L_∞ algebras — and, perhaps surprisingly, formal moduli functors arising in geometry often manifest themselves in this form. We begin by introducing the Maurer-Cartan equation for an L_∞ algebra $\mathfrak{g}$ and explaining how it provides a formal moduli functor. *This construction is at the heart of our approach to classical field theory.* We then describe several examples from geometry and algebra.

A.2.3.1 Definition. *Let* $\mathfrak{g}$ *be an* L_∞ *algebra. The* Maurer-Cartan equation *(or MC equation) is*

$$\sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0,$$

where α *denotes a degree* 1 *element of* $\mathfrak{g}$.

Note that when we consider the dg Lie algebra $\Omega^*(M) \otimes \mathfrak{g}$, with M

a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra, the MC equation becomes the equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

A $\mathfrak{g}$-connection $\alpha \in \Omega^1 \otimes \mathfrak{g}$ on the trivial principal G-bundle on M is *flat* if and only if it satisfies the MC equation. (This is the source of the name Maurer-Cartan.)

There are two other perspectives on the MC equation. First, observe that a map of commutative dg algebras $\underline{\alpha} : C^*\mathfrak{g} \to \mathbb{C}$ is determined by its behavior on the generators $\mathfrak{g}^\vee[-1]$ of the algebra $C^*\mathfrak{g}$. Hence $\underline{\alpha}$ is a linear functional of degree 0 on $\mathfrak{g}^\vee[-1]$ — or, equivalently, a degree 1 element α of $\mathfrak{g}$ — that commutes with differentials. This condition $\underline{\alpha} \circ d = 0$ is precisely the MC equation for α. The second perspective uses the coalgebra $C_*\mathfrak{g}$, rather than the algebra $C^*\mathfrak{g}$. A solution to the MC equation α is equivalent to giving a map of cocommutative dg coalgebras $\tilde{\alpha} : \mathbb{C} \to C_*\mathfrak{g}$.

Now observe that L_∞ algebras behave nicely under base change: if $\mathfrak{g}$ is an L_∞ algebra over $\mathbb{C}$ and A is a commutative dg algebra over $\mathbb{C}$, then $\mathfrak{g} \otimes A$ is an L_∞ algebra (over A and, of course, $\mathbb{C}$). Solutions to the MC equation go along for the ride as well. For instance, a solution α to the MC equation of $\mathfrak{g} \otimes A$ is equivalent to both a map of commutative dg algebras $\underline{\alpha} : C^*\mathfrak{g} \to A$ and a map of cocommutative dg coalgebras $\tilde{\alpha} : A^\vee \to C_*\mathfrak{g}$. Again, we simply unravel the conditions of such a map restricted to (co)generators. As maps of algebras compose, solutions play nicely with base change. Thus, we can construct a functor out of the MC solutions.

A.2.3.2 Definition. *For an L_∞ algebra $\mathfrak{g}$, its* Maurer-Cartan functor

$$\mathrm{MC}_{\mathfrak{g}} : \mathrm{dgArt}_{\mathbb{C}} \to \mathrm{sSet}$$

sends $(A, \mathfrak{m})$ to the simplicial set whose n-simplices are solutions to the MC equation in $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\triangle^n)$.

We remark that tensoring with the nilpotent ideal $\mathfrak{m}$ makes $\mathfrak{g} \otimes \mathfrak{m}$ is nilpotent. This condition then ensures that the simplicial set $\mathrm{MC}_{\mathfrak{g}}(A)$ is a Kan complex, by Hinich (2001); Getzler (2009). In fact, their work shows the following.

A.2.3.3 Theorem. *The Maurer-Cartan functor* $\mathrm{MC}_{\mathfrak{g}}$ *is a formal moduli problem.*

In fact, every formal moduli problem is represented — up to a natural notion of weak equivalence — by the MC functor of an L_∞ algebra.

References

For a clear, systematic introduction with an expository emphasis, we highly recommend the lecture Manetti (2009), which carefully explains how dg Lie algebras relate to deformation theory and how to use them in algebraic geometry. The unpublished book Kontsevich and Soibelman (n.d.) contains a wealth of ideas and examples; it also connects these ideas to many other facets of mathematics. The article Hinich (2001) is the original published treatment of derived deformation theory, and it provides one approach to necessary higher category theory. For the relation with L_∞ algebras, we recommend Getzler (2009), which contains elegant arguments for many of the ingredients, too. Finally, see Lurie (n.d.) for a proof that every formal moduli functor is described by a dg Lie algebra (equivalently, L_∞ algebra).

Appendix B
Functions on spaces of sections

Our focus throughout the book is on the "observables of a field theory," where for us the fields of a field theory are sections of a vector bundle and the observables are polynomial (or power series) functions on these fields. In this appendix, we will use the setting introduced in appendix I.B to give a precise meaning to this notion of observable.

B.1 Classes of functions on the space of sections of a vector bundle

Let M be a manifold and E a graded vector bundle on M. Let $U \subset M$ be an open subset. In this section we will introduce some notation for various classes of functionals on sections $\mathscr{E}(U)$ of E on U. These spaces of functionals will all be differentiable cochain complexes (or pro-cochain complexes) as described in appendix I.C. (In this appendix, however, the differential will always be trivial, so that it is natural to think of these spaces of functionals as differentiable pro-graded vector spaces.)

Recall the following notations:

- $\mathscr{E}(M)$ denotes the vector space of smooth sections of E over M,
- $\mathscr{E}_c(M)$ denotes the vector space of compactly supported smooth sections of E over M,
- $\overline{\mathscr{E}}(M)$ denotes the vector space of distributional sections of E over M,

- $\overline{\mathscr{E}}_c(M)$ denotes the vector space of compactly supported distributional sections of E over M.

We can view these spaces as living in LCTVS, BVS, CVS, or DVS, as suits us, thanks to the discussion in appendix I.B. In LCTVS, there is a standard isomorphism between the continuous linear dual $\mathscr{E}(M)^*$, equipped with the strong topology, and $\overline{\mathscr{E}}^!_c(M)$, the compactly supported distributional sections of the bundle $E^! = E^\vee \otimes \mathrm{Dens}_M$. Likewise, there is an isomorphism between $\mathscr{E}_c(M)^*$ and $\overline{\mathscr{E}}^!(M)$.

One goal of this section to explain and justify our notation $\mathscr{O}(\mathscr{E}(U))$ for a graded commutative algebra of functions on $\mathscr{E}(U)$, and various variants. As seen in Lemma B.1.1.1, one finds the same answer whether working with topological or convenient vector spaces.

B.1.1 Functions

Given an ordinary vector space V, the symmetric algebra $\mathrm{Sym}\, V^*$ on the dual space V^* provides a natural class of functions on V. Similarly, the completed symmetric algebra $\widehat{\mathrm{Sym}}\, V^*$ describes the formal power series centered at the origin, which is interpreted as functions on the formal neighborhood of the origin in V. We wish to describe the analogs of these constructions when the vector space is $\mathscr{E}(U)$, and hence we need to be careful in our choice of tensor products and ambient category. In the end, we will show that two natural approaches coincide and thus provide our definition.

From the point of view of topological vector spaces, a natural approach is use the completed projective tensor product $\widehat{\otimes}_\pi$ and follow the general recipe for constructing symmetric algebras. Because we will consider other approaches as well, we will call this construction the π-symmetric powers and define it as

$$\mathrm{Sym}^n_\pi\, \mathscr{E}^!_c(U) = \left(\mathscr{E}^!_c(U)^{\widehat{\otimes}_\pi n}\right)_{S_n},$$
$$\mathrm{Sym}^n_\pi\, \overline{\mathscr{E}}^!_c(U) = \left(\overline{\mathscr{E}}^!_c(U)^{\widehat{\otimes}_\pi n}\right)_{S_n},$$

where the subscript S_n denotes the coinvariants with respect to the action of this symmetric group. Then we define the uncompleted π-symmetric algebra as

$$\mathrm{Sym}_\pi\, \mathscr{E}_c^!(U) = \bigoplus_{n=0}^{\infty} \mathrm{Sym}_\pi^n\, \mathscr{E}_c^!(U)$$

and the completed π-symmetric algebra

$$\widehat{\mathrm{Sym}}_\pi \mathscr{E}_c^!(U) = \prod_{n=0}^{\infty} \mathrm{Sym}_\pi^n\, \mathscr{E}_c^!(U).$$

Using the same formulas, one defines $\mathrm{Sym}_\pi\, \overline{\mathscr{E}}^!_c(U)$ and $\widehat{\mathrm{Sym}}_\pi \overline{\mathscr{E}}^!_c(U)$.

If one views $\mathscr{E}_c^!(U)$ and $\overline{\mathscr{E}}^!_c(U)$ as convenient vector spaces, the natural choice is to work with the tensor product $\widehat{\otimes}_\beta$ and then to follow the standard procedure for constructing symmetric algebras. In short, we define the uncompleted β-symmetric algebra as

$$\mathrm{Sym}_\beta\, \mathscr{E}_c^!(U) = \bigoplus_{n=0}^{\infty} \left(\mathscr{E}_c^!(U)^{\widehat{\otimes}_\beta n} \right)_{S_n}$$

and the completed β-symmetric algebra

$$\widehat{\mathrm{Sym}}_\beta \mathscr{E}_c^!(U) = \prod_{n=0}^{\infty} \left(\overline{\mathscr{E}}^!_c(U)^{\widehat{\otimes}_\beta n} \right)_{S_n}.$$

Using the same formulas, one defines $\mathrm{Sym}_\beta\, \overline{\mathscr{E}}^!_c(U)$ and $\widehat{\mathrm{Sym}}_\beta \overline{\mathscr{E}}^!_c(U)$.

Thankfully, these two constructions provide the same differentiable vector spaces, via proposition I.B.7.3.

B.1.1.1 Lemma. *As graded differentiable vector spaces, there are isomorphisms*

$$\begin{aligned}
\mathrm{Sym}_\pi\, \mathscr{E}_c^!(U) &\cong \mathrm{Sym}_\beta\, \mathscr{E}_c^!(U), \\
\mathrm{Sym}_\pi\, \overline{\mathscr{E}}^!_c(U) &\cong \mathrm{Sym}_\beta\, \overline{\mathscr{E}}^!_c(U), \\
\widehat{\mathrm{Sym}}_\pi \mathscr{E}_c^!(U) &\cong \widehat{\mathrm{Sym}}_\beta \mathscr{E}_c^!(U), \\
\widehat{\mathrm{Sym}}_\pi \overline{\mathscr{E}}^!_c(U) &\cong \widehat{\mathrm{Sym}}_\beta \overline{\mathscr{E}}^!_c(U).
\end{aligned}$$

In light of this lemma, we can write $\mathscr{O}(\mathscr{E}(U))$ for $\widehat{\mathrm{Sym}}_\pi \overline{\mathscr{E}}^!_c(U)$, as it is naturally interpreted as the algebra of formal power series on $\mathscr{E}(U)$.

(This notation emphasizes the role of the construction rather than its inner workings.) Similarly, we write $\mathscr{O}(\overline{\mathscr{E}}(U))$ for $\widehat{\mathrm{Sym}}_\pi \mathscr{E}_c^!(U)$ and so on for $\mathscr{O}(\mathscr{E}_c(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$.

These completed spaces of functionals are all products of the differentiable vector spaces of symmetric powers, and so they are themselves differentiable vector spaces. We will equip all of these spaces of functionals with the structure of a differentiable pro-vector space, induced by the filtration

$$F^i \mathscr{O}(\mathscr{E}(U)) = \prod_{n \geq i} \mathrm{Sym}^i \overline{\mathscr{E}}_c^!(U)$$

(and similarly for $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$).

The natural product $\mathscr{O}(\mathscr{E}(U))$ is compatible with the differentiable structure, making $\mathscr{O}(\mathscr{E}(U))$ into a commutative algebra in the multicategory of differentiable graded pro-vector spaces. The same holds for the spaces of functionals $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$.

B.1.2 One-forms

Recall that for V is a vector space, we view the formal neighborhood of the origin as having the ring of functions $\mathscr{O}(V) = \widehat{\mathrm{Sym}}(V^\vee)$. Then we likewise define the space of one-forms on this formal scheme as

$$\Omega^1(V) = \mathscr{O}(V) \otimes V^\vee.$$

There is a universal derivation, called the exterior derivative map,

$$\mathrm{d} : \mathscr{O}(V) \to \Omega^1(V).$$

In components the exterior derivative is just the composition

$$\mathrm{Sym}^{n+1} V^\vee \to (V^\vee)^{\otimes n+1} \to \mathrm{Sym}^n(V^\vee) \otimes V^\vee,$$

where the maps are the inclusion followed by the natural projection, up to an overall combinatorial constant. (As a concrete example, note that $\mathrm{d}(xy) = y\mathrm{d}x + x\mathrm{d}y$ can be computed by taking the tensor representative $(x \otimes y + y \otimes x)/2$ for xy and then projecting off the last factor.)

This construction extends naturally to our context. We define

$$\Omega^1(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U)) \widehat{\otimes}_\beta \overline{\mathscr{E}}_c^!(U),$$

where we take the associated differentiable vector space. In concrete terms,

$$\Omega^1(\mathscr{E}(U)) = \prod_n \mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!_c(U).$$

In this way, $\Omega^1(\mathscr{E}(U))$ becomes a differentiable pro-cochain complex, where the filtration is defined by

$$F^i\Omega^1(\mathscr{E}(U)) = \prod_{n\geq i-1} \mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!_c(U).$$

Further, $\Omega^1(\mathscr{E}(U))$ is a module for the commutative algebra $\mathscr{O}(\mathscr{E}(U))$, where the module structure is defined in the multicategory of differentiable pro-vector spaces.

In a similar way, define the exterior derivative

$$\mathrm{d} : \mathscr{O}(\mathscr{E}(U)) \to \Omega^1(\mathscr{E}(U))$$

by saying that on components it is given by the same formula as in the finite-dimensional case.

B.1.3 Other classes of sections of a vector bundle

Before we introduce our next class of functionals — those with proper support — we need to introduce some further notation concerning classes of sections of a vector bundle.

Let $f : M \to N$ be a smooth fibration between two manifolds. Let E be a vector bundle on M. We say a section $s \in \Gamma(M, E)$ has *compact support over* f if the map

$$f : \mathrm{Supp}(s) \to N$$

is proper. We let $\Gamma_{c/f}(M, E)$ denote the space of sections with compact support over f. It is a differentiable vector space: if X is an auxiliary manifold, a smooth map $X \to \Gamma_{c/f}(M, E)$ is a section of the bundle $\pi_M^* E$ on $X \times M$ that has compact support relative to the map

$$M \times X \to N \times X.$$

(It is straightforward to write down a flat connection on $C^\infty(X, \Gamma_{c/f}(M, E))$, using arguments of the type described in section I.B.5.1.)

Next, we need to consider spaces of the form $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N)$, where M and N are manifolds and E, F are vector bundles on the manifolds M, N, respectively. We want a more geometric interpretation of this tensor product.

We will view $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N)$ as a subspace

$$\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N) \subset \overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \overline{\mathscr{F}}(N).$$

It consists of those elements D with the property that, if $\phi \in \mathscr{E}_c^!(M)$, then map

$$\begin{array}{cccc} D(\phi): & \mathscr{F}_c^!(N) & \to & \mathbb{R} \\ & \psi & \mapsto & D(\phi \otimes \psi) \end{array}$$

comes from an element of $\mathscr{F}(N)$. Alternatively, $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N)$ is the space of continuous linear maps from $\mathscr{E}_c^!(M)$ to $\mathscr{F}(N)$.

We can similarly define $\overline{\mathscr{E}}_c(M)\widehat{\otimes}_\beta \mathscr{F}(N)$ as the subspace of those elements of $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N)$ that have compact support relative to the projection $M \times N \to N$.

These spaces form differentiable vector spaces in a natural way: a smooth map from an auxiliary manifold X to $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N)$ is an element of $\overline{\mathscr{E}}(N)\widehat{\otimes}_\beta \mathscr{F}(N)\widehat{\otimes}_\beta C^\infty(X)$. Similarly, a smooth map to $\overline{\mathscr{E}}_c(M)\widehat{\otimes}_\beta \mathscr{F}(N)$ is an element of $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta \mathscr{F}(N)\widehat{\otimes}_\beta C^\infty(X)$ whose support is compact relative to the map $M \times N \times X \to N \times X$.

B.1.4 Functions with proper support

Recall that

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U).$$

We can thus define a subspace

$$\mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U) \subset \Omega^1(\mathscr{E}_c(U)).$$

The Taylor components of elements of this subspace are in the space

$$\mathrm{Sym}^n(\overline{\mathscr{E}}_c^!(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U),$$

which in concrete terms is the S_n-coinvariants of

$$\overline{\mathscr{E}}_c^!(U)^{\widehat{\otimes}_\beta n}\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U).$$

B.1.4.1 Definition. *A function* $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ *has* proper support *if*

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U) \subset \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U).$$

The reason for the terminology is as follows. Let $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ and let

$$\Phi_n \in \mathrm{Hom}(\mathscr{E}_c(U)^{\widehat{\otimes}_\beta n}, \mathbb{R})$$

be the nth term in the Taylor expansion of Φ. Then Φ has proper support if and only if, for all n, the composition with any projection map

$$\mathrm{Supp}(\Phi_n) \subset U^n \to U^{n-1}$$

is proper.

We will let

$$\mathscr{O}^P(\mathscr{E}_c(U)) \subset \mathscr{O}(\mathscr{E}_c(U))$$

be the subspace of functions with proper support. Note that functions with proper support are *not* a subalgebra.

Because $\mathscr{O}^P(\mathscr{E}_c(U))$ fits into a fiber square

$$\begin{array}{ccc} \mathscr{O}^P(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta \mathscr{E}_c(U)^\vee \\ \downarrow & & \downarrow \\ \mathscr{O}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \mathscr{E}_c(U)^\vee \end{array}$$

it has a natural structure of a differentiable pro-vector space.

B.1.5 Functions with smooth first derivative

B.1.5.1 Definition. *A function* $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ *has* smooth first derivative *if* $\mathrm{d}\Phi$*, which is* a priori *an element of*

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U),$$

is an element of the subspace

$$\mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \mathscr{E}^!(U).$$

In other words, the 1-form $\mathrm{d}\Phi$ can be evaluated on a distributional tangent vectors from $\overline{\mathscr{E}}$, and not just smooth tangent vectors.

Note that we can identify, concretely, $\mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \mathscr{E}^!(U)$ with the space

$$\prod_n \mathrm{Sym}^n \overline{\mathscr{E}}^!(U)\widehat{\otimes}_\beta \mathscr{E}^!(U)$$

and

$$\mathrm{Sym}^n \overline{\mathscr{E}}^!(U)\widehat{\otimes}_\beta \mathscr{E}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta n}\widehat{\otimes}_\beta \mathscr{E}^!(U).$$

(Spaces of the form $\mathscr{E}(U)\widehat{\otimes}_\beta \overline{\mathscr{E}}(U)$ were described concretely above.)

Thus $\mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \mathscr{E}^!(U)$ is a differentiable pro-vector space. It follows that the space of functionals with smooth first derivative is a differentiable pro-vector space, since it is defined by a fiber diagram of such objects.

An even more concrete description of the space $\mathscr{O}^{sm}(\mathscr{E}_c(U))$ of functionals with smooth first derivative is as follows.

B.1.5.2 Lemma. *A functional $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ has smooth first derivative if each of its Taylor components*

$$D_n\Phi \in \mathrm{Sym}^n \overline{\mathscr{E}}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta n}$$

lies in the intersection of all the subspaces

$$\overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta k}\widehat{\otimes}_\beta \mathscr{E}^!(U)\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta n-k-1}$$

for $0 \leq k \leq n-1$.

The proof is a simple calculation.

Note that the space of functions with smooth first derivative is a subalgebra of $\mathscr{O}(\mathscr{E}_c(U))$. We will denote this subalgebra by $\mathscr{O}^{sm}(\mathscr{E}_c(U))$. Again, the space of functions with smooth first derivative is a differentiable pro-vector space, as it is defined as a fiber product.

Similarly, we can define the space of functions on $\mathscr{E}(U)$ with smooth first derivative, $\mathscr{O}^{sm}(\mathscr{E}(U))$ as those functions whose exterior derivative lies in $\mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta \mathscr{E}_c^!(U) \subset \Omega^1(\mathscr{E}(U))$.

B.1.6 Functions with smooth first derivative and proper support

We are particularly interested in those functions which have both smooth first derivative and proper support. We will refer to this subspace as $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$. The differentiable structure on $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$ is, again, given by viewing it as defined by the fiber diagram

$$\begin{array}{ccc} \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta \mathscr{E}^!(U) \\ \downarrow & & \downarrow \\ \mathscr{O}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta \overline{\mathscr{E}}^!(U). \end{array}$$

We have inclusions

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) \subset \mathscr{O}^{sm}(\mathscr{E}_c(U)),$$

where each inclusion has dense image.

B.2 Derivations

As before, let M be a manifold, E a graded vector bundle on M, and U an open subset of M. In this section we will define derivations of algebras of functions on $\mathscr{E}(U)$.

To start with, recall that for V a finite dimensional vector space, which we treat as a formal scheme, the algebra of function is $\mathscr{O}(V) = \prod \mathrm{Sym}^n V^\vee$, the formal power series on V. We then identify the space of continuous derivations of $\mathscr{O}(V)$ with $\mathscr{O}(V) \otimes V$. We view these derivations as the space of vector fields on V and use the notation $\mathrm{Vect}(V)$.

In a similar way, we define the space of vector fields $\mathrm{Vect}(\mathscr{E}(U))$ of vector fields on $\mathscr{E}(U)$ as

$$\mathrm{Vect}(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta \mathscr{E}(U) = \prod_n \left(\mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U))\widehat{\otimes}_\beta \mathscr{E}(U)\right).$$

We have already seen (section B.1) how to define the structure of differentiable pro-vector space on spaces of this nature.

In this section we will show the following.

B.2.0.1 Proposition. $\mathrm{Vect}(\mathscr{E}(U))$ *has a natural structure of Lie algebra in the multicategory of differentiable pro-vector spaces. Further,* $\mathscr{O}(\mathscr{E}(U))$ *has an action of the Lie algebra* $\mathrm{Vect}(\mathscr{E}(U))$ *by derivations, where the structure map* $\mathrm{Vect}(\mathscr{E}(U)) \times \mathscr{O}(\mathscr{E}(U)) \to \mathscr{O}(\mathscr{E}(U))$ *is smooth.*

Proof To start with, let's look at the case of a finite-dimensional vector space V, to get an explicit formula for the Lie bracket on $\mathrm{Vect}(V)$, and the action of $\mathrm{Vect}(V)$ on $\mathscr{O}(V)$. Then, we will see that these formulae make sense when $V = \mathscr{E}(U)$.

Let $X \in \mathrm{Vect}(V)$, and let us consider the Taylor components $D_n X$, which are multilinear maps

$$V \times \cdots \times V \to V.$$

Our conventions are such that

$$D_n(X)(v_1, \dots, v_n) = \left(\frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_n} X \right)(0) \in V$$

Here, we are differentiating vector fields on V using the trivialization of the tangent bundle to this formal scheme arising from the linear structure.

Thus, we can view $D_n X$ as living in the endomorphism operad of the vector space V.

If $A : V^{\times n} \to V$ and $B : V^{\times m} \to V$, let us define

$$A \circ_i B(v_1, \dots, v_{n+m-1}) = A(v_1, \dots, v_{i-1}, B(v_i, \dots, v_{i+m-1}), v_{i+m}, \dots, v_{n+m-1}).$$

If A, B are symmetric (under S_n and S_m, respectively), then define

$$A \circ B = \sum_{i=1}^{n} A \circ_i B.$$

Then, if X, Y are vector fields, the Taylor components of $[X, Y]$ satisfy

$$D_n([X, Y]) = \sum_{k+l=n+1} c_{k,l} \left(D_k X \circ D_l Y - D_l Y \circ D_k X \right)$$

where $c_{k,l}$ are combinatorial constants whose values are irrelevant for our purposes.

Similarly, if $f \in \mathscr{O}(V)$, the Taylor components of f are multilinear maps

$$D_n f : V^{\times n} \to \mathbb{C}.$$

In a similar way, if X is a vector field, we have

$$D_n(Xf) = \sum_{k+l=n+1} c'_{k,l} D_k(X) \circ D_k(f).$$

Thus, we see that in order to define the Lie bracket on $\mathrm{Vect}(\mathscr{E}(U))$, we need to give maps of differentiable vector spaces

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta m}, \mathscr{E}(U)) \to \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta (n+m-1)}, \mathscr{E}(U))$$

where here Hom indicates the space of continuous linear maps, treated as a differentiable vector space. Similarly, to define the action of $\mathrm{Vect}(\mathscr{E}(U))$ on $\mathscr{O}(\mathscr{E}(U))$, we need to define a composition map

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta m}) \to \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta n+m-1}).$$

We will treat the first case; the second is similar.

Now, if X is an auxiliary manifold, a smooth map

$$X \to \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta m}, \mathscr{E}(U))$$

is the same as a continuous multilinear map

$$\mathscr{E}(U)^{\times m} \to \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X).$$

Here, "continuous" means for the product topology.

This is the same thing as a continuous $C^\infty(X)$-multilinear map

$$\Phi : (\mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X))^{\times m} \to \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X).$$

If

$$\Psi : (\mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X))^{\times n} \to \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X).$$

is another such map, then it is easy to define $\Phi \circ_i \Psi$ by the usual formula:

$$\Phi \circ_i \Psi(v_1, \dots, v_{n+m-1}) = \Phi(v_1, \dots v_{i-1}, \Psi_i(v_i, \dots, v_{m+i-1}), \dots, v_{n+m-1})$$

if $v_i \in \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X)$. This map is $C^\infty(X)$-linear. $\square$

Appendix C
A formal Darboux lemma

Our goal here is to articulate and prove a kind of Darboux lemma for formal moduli spaces equipped with a shifted symplectic structure in the sense of Pantev et al. (2013b). Recall that the Darboux lemma in symplectic geometry says that for any point in a symplectic manifold of dimension $2n$, there exists a local coordinate patch exhibiting a symplectomorphism with an open in the standard symplectic space $T^*\mathbb{R}^n$. It is not possible to make such a statement in formal derived geometry (e.g., coordinate patches make little sense), but one can formulate a close analogue as follows.

First, note that every formal space $\mathcal{X}$ admits a closely related linear space, namely the fiber of the tangent complex $\mathbb{T}_{\mathcal{X}}$ over the basepoint x of $\mathcal{X}$. Second, as discussed in Appendix A.2.3, every formal space is equivalent to the formal space $B\mathfrak{g}$ associated to some L_∞ algebra, with the underlying cochain complex of $\mathfrak{g}$ is equivalent to $\mathbb{T}_{\mathcal{X},x}[-1]$. In other words, any "nonlinearity" of $\mathcal{X}$ is encoded in the nontrivial bracket structure of $\mathfrak{g}$. This relationship can be understood as a replacement for a local coordinate patch. We can then ask how to rephrase a shifted symplectic structure on $\mathcal{X}$ in terms of $\mathfrak{g}$, so that it looks like a linear structure, much as the model symplectic space is determines by a symplectic pairing on a vector space. Thus, as a third step, we consider invariant bilinear pairings on $\mathfrak{g}$ and ask which correspond to shifted symplectic forms. Loosely speaking, our Darboux lemma says there is an equivalence between

(i) non-degenerate invariant pairings on $\mathfrak{g}$ of degree k, and

(ii) symplectic forms of degree $k+2$ on the formal derived space $B\mathfrak{g}$.

We now turn to introducing the terminology needed to state and prove a precise version.

C.1 Shifted symplectic structures

We will assume throughout that $\mathfrak{g}$ is an L_∞ algebra whose underlying graded vector space is bounded and of total finite dimension. One can work around this constraint, but it makes the arguments simpler and clearer.

This L_∞ algebra has an associated formal derived space denoted by $B\mathfrak{g}$, and we interpret its Chevalley-Eilenberg cochains $C^*(\mathfrak{g})$ as the completed dg commutative algebra of functions $\mathscr{O}(B\mathfrak{g})$ on $B\mathfrak{g}$. Hence, the k-forms $\Omega^k(B\mathfrak{g})$ are identified with the dg $C^*(\mathfrak{g})$-module $C^*(\mathfrak{g}, \mathrm{Sym}^k(\mathfrak{g}^*[-1]))$. A 2-*form* means a cocycle in $\Omega^2(B\mathfrak{g})$, which can have cohomological degree k. (Compare with ordinary differential geometry, where a 2-form always lives in degree 0.) Such a cocycle ω corresponds to a cocycle in $C^*(\mathfrak{g}, \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^*))$, and hence determines a map $\omega^\sharp$ from the tangent complex of $B\mathfrak{g}$ to its cotangent complex. We call a 2-form *nondegenerate* if the map $\omega^\sharp$ is a quasi-isomorphism.

We now turn to explaining what the condition *closed* means in this setting, which requires us to explain the de Rham differential. As in ordinary geometry, the universal derivation

$$C^*(\mathfrak{g}) = \mathscr{O}(B\mathfrak{g}) \xrightarrow{\mathrm{d}_{dR}} \Omega^1(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}^*[-1])$$

extends to higher forms in a natural way and hence yields a de Rham complex $\Omega^*(B\mathfrak{g})$, obtained by totalizing the double complex

$$\Omega^0(B\mathfrak{g}) \xrightarrow{\mathrm{d}_{dR}} \Omega^1(B\mathfrak{g}) \xrightarrow{\mathrm{d}_{dR}} \Omega^2(B\mathfrak{g}) \xrightarrow{\mathrm{d}_{dR}} \cdots .$$

We define the closed 2-forms to be the truncated de Rham complex

$$\Omega^2_{cl}(B\mathfrak{g}) = \mathrm{Tot}\left(\Omega^2(B\mathfrak{g}) \xrightarrow{\mathrm{d}_{dR}} \Omega^3(B\mathfrak{g}) \xrightarrow{\mathrm{d}_{dR}} \cdots\right).$$

A *closed 2-form* on $B\mathfrak{g}$ then means a cocycle in this complex, which can have cohomological degree k. There is natural projection map $\tau : \Omega^2_{cl}(B\mathfrak{g}) \to \Omega^2(B\mathfrak{g})$ by discarding the rest of the truncated de Rham complex.

We can now describe one side of our correspondence.

C.1.0.1 Definition. *A* k-shifted symplectic structure *on a formal derived space $B\mathfrak{g}$ is a closed 2-form ω of cohomological degree k such that the underlying 2-form $\tau(\omega)$ is nondegenerate.*

C.2 Rephrasing the problem

We discussed in section 4.2 what we mean by an invariant pairing, namely a bilinear form such that if one postcomposes it with an L_∞ bracket, the composite is antisymmetric in the appropriate sense. It is straightforward to relate this notion to shifted symplectic structures.

C.2.0.1 Lemma. *A degree $k-2$ invariant pairing b naturally defines a k-shifted symplectic structure ω_b.*

Proof Observe that the pairing b is an element of $\mathrm{Sym}^2(\mathfrak{g}^*[-1])$ and hence of $\Omega^2(B\mathfrak{g})$ by the map of graded vector spaces

$$\mathrm{Sym}^2(\mathfrak{g}^*[-1]) \xrightarrow{1\otimes \mathrm{id}} \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}^2(\mathfrak{g}^*[-1])$$

As it is invariant, it is annihilated by the Chevalley-Eilenberg differential and hence determines a 2-form. This element is also annihilated by the de Rham differential, since it is a constant-coefficient 2-form. Hence this element determines a cocycle ω_b in $\Omega^2_{cl}(B\mathfrak{g})$. (Its "underlying k-form" is zero for $k > 2$.) □

This construction is at the level of explicit cocycles and hence is not particularly homotopically meaningful as stated. We want an identification of ∞-groupoids between invariant pairings and shifted symplectic structures. In fact, we will show something a little stronger. Given an L_∞ algebra $\mathfrak{g}$ with invariant pairing b, there is a natural formal moduli space $\mathcal{M}_{\mathfrak{g},b}$ describing deformations of it as L_∞ algebra with invariant pairings. There is also a natural formal moduli space $\mathcal{M}_{B\mathfrak{g},\omega_b}$ describing deformations of the pair $(B\mathfrak{g}, \omega_b)$ as a shifted symplectic formal space. We will exhibit an equivalence between these two formal moduli spaces.

Thus, we need to describe these spaces precisely.

C.2.1 Deformations of L_∞ algebras with pairings

For an L_∞ algebra $\mathfrak{g}$, there are two distinct kinds of deformations to consider:

- deformations as a possibly-curved L_∞ algebra, which is described by the dg Lie algebra of derivations $C^*(\mathfrak{g}, \mathfrak{g}[1])$, equipped with the commutator bracket, or
- deformations as an (uncurved) L_∞ algebra, which is described by the sub-dg Lie algebra $C^*(\mathfrak{g}, \mathfrak{g}[1])/\mathfrak{g}[1]$ of derivations that preserve the augmentation of the algebra $C^*(\mathfrak{g})$.

In both cases, the point is that all the brackets of an L_∞ algebra are encoded in the differential of the dg commutative algebra $C^*(\mathfrak{g})$, so it suffices to study how to deform that derivation, and the dg Lie algebra of derivations is hence the natural model for that formal moduli space. Moreover, the key difference between the two cases is whether one preserves the augmentation of $C^*(\mathfrak{g})$, which encodes the basepoint of the formal space $B\mathfrak{g}$. To start, we will focus on the first class of deformations, as it is mildly easier to analyze.

When $\mathfrak{g}$ is equipped with a pairing b, it is natural to restrict to the derivations that preserve the pairing. Since we will use the pairing to produce a shifted symplectic structure, this condition amounts to working with the symplectic derivations. In the setting of ordinary symplectic geometry, every symplectic derivation is locally equivalent to a Hamiltonian vector field, which is unique up to a constant. This identification uses the fact that the map $\omega^\sharp$ can be inverted. If we try to mimic this approach in our setting, we run into the issue that we only have a quasi-isomorphism

$$\omega_b^\sharp : C^*(\mathfrak{g}, \mathfrak{g}[1]) \to C^*(\mathfrak{g}, \mathfrak{g}^*[-1])[k]$$

at the cochain level between derivations and (shifted) 1-forms.

Here it is convenient to restrict to *minimal* L_∞ algebras, namely those L_∞ algebras whose ℓ_1 bracket (i.e., differential) are zero. This restriction is no constraint for us, since one can transfer the L_∞ structure of an L_∞ algebra $\mathfrak{h}$ to the graded vector space $H^*(\mathfrak{h}, \ell_1)$ of its cohomology and thus obtain a minimal model for $\mathfrak{h}$. (Indeed, for this reason, our arguments in this appendix can be amended to encompass L_∞ algebras with finite-dimensional cohomology.) Hence, up to homotopy, we may as well assume $\mathfrak{g}$ is minimal.

When $\mathfrak{g}$ is minimal, however, the nondegeneracy condition on the pairing b implies that it is nondegenerate on the underlying graded vector space of $\mathfrak{g}$. Hence the map $\omega_b^\sharp$ is an isomorphism and not just a quasi-isomorphism, and it admits a unique inverse. In consequence, we see that we have a natural cochain map

$$Ham_b : C^*(\mathfrak{g})[k] \to C^*(\mathfrak{g}, \mathfrak{g}[1])$$

sending a function on $B\mathfrak{g}$ to its Hamiltonian vector field by the composition

$$Ham_b = (\omega_b^\sharp)^{-1} \circ \mathrm{d}_{dR},$$

just as in ordinary symplectic geometry. This map determines a $-k$-shifted Poisson bracket on $C^*(\mathfrak{g})$, so that $C^*(\mathfrak{g})[k]$ is a dg Lie algebra.

Direct inspection shows that any derivation that (strictly) preserves b is Hamiltonian. (This claim is analogous to the familiar fact that in ordinary symplectic geometry, a symplectic vector field is represented locally by a Hamiltonian function). On the other hand, the kernel of Ham_b consists of the constant functions. Hence we see that the dg Lie algebra $C^*_{red}(\mathfrak{g})[k]$ models the symplectic derivations, where the dependence on b is wrapped up in the shifted Poisson bracket.

So far we have described deformations as a possibly-curved L_∞ algebra with pairing. To eliminate the appearance of curving, we need to restrict to basepoint-preserving symplectic derivations. These are given by the intersection of all symplectic derivations with $C^*(\mathfrak{g}, \mathfrak{g}[1])/\mathfrak{g}[1]$, the Lie algebra of basepoint-preserving derivations. Using the map Ham_b, one can see straightforwardly that it is equivalent to work with the subspace

$$C^*_{bpp}(\mathfrak{g}) = (\mathrm{Sym}^{\geq 2}(\mathfrak{g}^*[-1]), \mathrm{d}_{CE})$$

of $C^*(\mathfrak{g})$, as it is the linear term of a Hamiltonian function that produces the constant term of the Hamiltonian vector field.

C.2.2 Deformations of shifted symplectic formal spaces

We know that deformations of $B\mathfrak{g}$ as a formal derived space are equivalent to deformations of $\mathfrak{g}$ as an L_∞ algebra, by the fundamental theorem of derived deformation theory. Hence we know that the derivations $C^*(\mathfrak{g}, \mathfrak{g}[1])$ model deformations as an unpointed formal space, and that basepoint-preserving derivations model deformations as a pointed formal space. We will focus on unpointed deformations here, as it should be clear in light of our preceding example how to modify our argument to deal with pointed deformations.

On the other hand, if we fix the space $B\mathfrak{g}$, we can ask how to model deformations of the k-shifted symplectic structure around a shifted symplectic structure ω. It should be clear that $\Omega^2_{cl}(B\mathfrak{g})[k-1]$ models such deformations: for any dg Artinian algebra $(R, \mathfrak{m})$, we consider deforming ω to $\omega + \omega'$, where ω' is a degree 1 element of $\mathfrak{m} \otimes \Omega^2_{cl}(B\mathfrak{g})$, and ask whether it satisfies the conditions of being a shifted symplectic form over the base algebra R. Nondegeneracy holds automatically, since we only modify ω in a nilpotent direction. Hence the condition of being shifted symplectic is simply that ω' is closed in $\mathfrak{m} \otimes \Omega^2_{cl}(B\mathfrak{g})[k-1]$.

If we ask about deforming the space and its symplectic structure simultaneously, then we are deforming $\mathfrak{g}$ as well as the symplectic form. Deforming $\mathfrak{g}$ modifies the differential on $C^*(\mathfrak{g})$ — and hence also on $\Omega^2_{cl}(B\mathfrak{g})[k-1]$ — and this modified differential determines the relevant condition on the symplectic form. To be explicit, for any dg Artinian algebra $(R, \mathfrak{m})$, we ask for a derivation $D \in \mathfrak{m} \otimes C^*(\mathfrak{g}, \mathfrak{g}[1])$ and $\omega' \in \mathfrak{m} \otimes \Omega^2_{cl}(B\mathfrak{g})[k-1]$ such that D satisfies the Maurer-Cartan equation and ω' satisfies

$$\mathrm{d}_{tot}\omega' + L_D(\omega + \omega') = 0, \qquad (\dagger)$$

where d_{tot} denotes the total differential in $\mathfrak{m} \otimes \Omega^2_{cl}(B\mathfrak{g})[k-1]$ and L_D denotes the Lie derivative.

Cartan's formula tells us that

$$L_D\alpha = [\mathrm{d}_\Omega, \iota_D]\alpha,$$

where we are using d_Ω to denote the total differential of the de Rham complex. We know $\mathrm{d}_\Omega \omega = 0$, so equation (†) becomes

$$\mathrm{d}_{tot}\omega' + L_D\omega' + \mathrm{d}_\Omega(\iota_D\omega) = 0. \tag{‡}$$

This new equation makes clear how a deformation D of the space $B\mathfrak{g}$ affects the condition of deforming the symplectic structure, namely one modifies the differential on Ω^2_{cl} by both the deformation of the space and by a term depending on the symplectic form, namely $\mathrm{d}_\Omega(\iota_D\omega)$, which is an exact 1-form, loosely speaking.

In sum, we have identified the dg Lie algebra describing deformations of the shifted symplectic formal space $(B\mathfrak{g}, \omega)$ as

$$\Omega^2_{cl}(B\mathfrak{g})[k-1] \rtimes_\omega C^*(\mathfrak{g}, \mathfrak{g}[1]),$$

where equation (‡) describes the action of derivations on $\Omega^2_{cl}(B\mathfrak{g})$. We use the notation $\rtimes_\omega$ to indicate how the extension depends on the symplectic form ω.

Under the assumption that $\mathfrak{g}$ is minimal, we know that the map $\tau(\omega)^\sharp$ is an isomorphism. (Recall that $\tau(\omega)$ is the underlying 2-form, which is what provides the identification of derivations and k-shifted 1-forms.) Hence, we can replace the dg Lie algebra $C^*(\mathfrak{g}, \mathfrak{g}[1])$ with the k-shifted 1-forms $\Omega^1(B\mathfrak{g})[k]$ and transfer the Lie bracket. The dg Lie algebra describing deformations of the shifted symplectic formal space $(B\mathfrak{g}, \omega)$ is then the cochain complex

$$\Omega^1_{cl}(B\mathfrak{g})[k]$$

equipped with a nontrivial Lie bracket depending on ω. (For the basepoint-preserving case, replace the component $\Omega^1(B\mathfrak{g})[k]$ by the image of the basepoint-preserving derivations under the map $\tau(\omega)^\sharp$.)

C.3 Relating the two formal moduli spaces

In light of these descriptions of the formal moduli spaces, it should be clear that there is a natural map: the de Rham differential

$$\mathrm{d}_{dR} : C^*(\mathfrak{g}) = \mathscr{O}(B\mathfrak{g}) \to \Omega^1_{cl}(B\mathfrak{g})$$

defines a map of cochain complexes, so we take the k-fold shift to obtain a map

$$\mathrm{d}_{dR}[k] : C^*_{red}(\mathfrak{g})[k] \to \Omega^1_{cl}(B\mathfrak{g})[k]$$

between the underlying cochain complexes of the relevant dg Lie algebras.

Note that we took the reduced cochains on the left hand side, which is fine since the de Rham differential annihilates constants. This is the correct thing to do because Hamiltonian vector fields taken up to an additive constant match with symplectic vector fields, and so define deformations of the L_∞ algebra with an invariant pairing.

The map (with reduced cochains) is a quasi-isomorphism, since the cone of this map is a shift of the reduced de Rham complex $\Omega^*_{red}(B\mathfrak{g})$, which is acyclic.

For $\mathfrak{g}$ a minimal L_∞ algebra and b a nondegenerate invariant pairing of degree k, it is a direct computation to verify that this map preserves the Lie bracket. Hence, we have obtained the following.

C.3.0.1 Proposition. *There is an equivalence of formal moduli spaces between $\mathcal{M}_{\mathfrak{g},b}$ and $\mathcal{M}_{B\mathfrak{g},\omega_b}$ realized by the dg Lie algebra map*

$$\mathrm{d}_{dR}[k] : C^*_{red}(\mathfrak{g})[k] \to \Omega^1_{cl}(B\mathfrak{g})[k],$$

when $\mathfrak{g}$ is a minimal L_∞ algebra whose underlying graded vector space has finite total dimension.

This result implies that if we start with a shifted symplectic structure on $B\mathfrak{g}$ that comes from an invariant pairing on $\mathfrak{g}$, then infinitesimally nearby symplectic structures can also be modeled by invariant pairings.

Now let us consider a small variant of this result. Suppose that $\mathfrak{g}$ is minimal. Consider the two variants of the concept of shifted symplectic structure, both of which preserve a framing on the tangent space of the base point (which is $\mathfrak{g}[1]$), as well as the symplectic form on this graded vector space. For the strict version (modelled by L_∞ algebras with an invariant pairing), the Lie algebra controlling deformations that preserve the framing on the tangent space is $C^{\geq 3}(\mathfrak{g})[k] = \mathrm{Sym}^{\geq 3}(\mathfrak{g}[-1])[k]$, which consists of Hamiltonian functions that are at least cubic. (They are at least cubic because quadratic Hamiltonians can rotate the tangent space at the origin.)

For the lax version of shifted symplectic structure, this constraint is implemented by asking that the component in Ω^1 has coefficients that are at least quadratic (corresponding to a vector field that has quadratic coefficients and so does not rotate the tangent space at the origin), and the component in Ω^2 has at least linear coefficients (so that we do not change the symplectic form on the tangent space). We will let $\Omega^1_{+,cl}(B\mathfrak{g})$ denote this sub-complex of $\Omega^1_{cl}(B\mathfrak{g})$. As before, the map

$$C^{\geq 3}(\mathfrak{g})[k] \to \Omega^1_{+,cl}(B\mathfrak{g})[k] \tag{C.1}$$

is a quasi-isomorphism of dg Lie algebras.

Now, each of the two dg Lie algebras appearing in this equation is pro-nilpotent, so that the Maurer-Cartan equation makes sense without inputting a nilpotent Artinian ring.

C.3.1 Completing the proof

Suppose that $\mathfrak{g}$ is a minimal L_∞ algebra with an invariant pairing b. We let $(\mathfrak{g}_0, b_0)$ be the underlying Lie algebra with invariant pairing (obtained by dropping the higher terms of the L_∞ algebra). The deformation of $(\mathfrak{g}_0, b_0)$ to $(\mathfrak{g}, b)$ is then given by a a Maurer-Cartan elements of $C^{\geq 3}(\mathfrak{g})[k]$.

Similarly, suppose we have a minimal L_∞ algebra $\mathfrak{g}$ with a non-degenerate and homotopically closed two-form ω. We can drop the higher terms in the L_∞ structure to get a Lie algebra $\mathfrak{g}_0$,and the constant-coefficient part of the two-form ω gives an invariant pairing b_0 on $\mathfrak{g}_0$. The deformation from (g_0, b_0) to (g, ω) is represented by a Maurer-Cartan element in $\Omega^1_{+,cl}(B\mathfrak{g})$.

Because the map in equation C.1 is a quasi-isomorphism, and because both Lie algebras appearing in equation C.1 are pro-nilpotent, we conclude that every Maurer-Cartan element in $\Omega^1_{+,cl}(B\mathfrak{g})$ is equivalent (in a way canonical up to contractible choice) to one in the image of $C^{\geq 3}(\mathfrak{g})[k]$. The latter represents an L_∞ structure on $\mathfrak{g}$ with invariant pairing, as desired.

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