1. Let \((M, \omega)\) be a symplectic manifold with vanishing 1st de Rham cohomology group (e.g. \(M\) is simply connected). Show that for any smooth flow \(\theta_t : M \to M\) which preserves the symplectic structure, i.e., \(\theta_t^* \omega = \omega\), the following is true: (1) there is a smooth function \(H : M \to \mathbb{R}\) such that \(dH = i_X \omega\) where \(X\) denotes the smooth vector field which generates the flow \(\theta_t\) (\(H\) is called a Hamiltonian function of the flow, uniquely determined up to a constant), (2) \(\theta_t\) preserves the level sets of \(H\).

2. (cont. of #1) Consider \(S^2\), the unit sphere in \(\mathbb{R}^3\), equipped with a symplectic form which is the area form induced from the Euclidean metric on \(\mathbb{R}^3\). Let \(\theta_t : S^2 \to S^2\) be the flow which is given by a rotation of angle \(t\) counterclockwise about the \(z\)-axis. (1) Show that \(\theta_t\) preserves the symplectic form on \(S^2\), (2) Determine the smooth vector field \(X\) on \(S^2\) which generates \(\theta_t\), (3) Find a Hamiltonian function \(H\) of the flow.

3. Let \((M, g)\) be a Riemannian manifold. Prove that for any \(p \in M\), there is an \(\epsilon_0 > 0\) such that for any \(\epsilon \leq \epsilon_0\), the geodesic ball of radius \(\epsilon\) centered at \(p\), \(B_\epsilon(p)\), is geodesically convex. More precisely, for any \(q_1, q_2 \in B_\epsilon(p)\), there is a unique geodesic \(\gamma\) connecting \(q_1, q_2\) which lies inside the geodesic ball (i.e. \(\gamma \subset B_\epsilon(p)\)).

4. Let \(\nabla\) be a connection in a smooth vector bundle \(E\), and let \(e_i\) be a local frame of \(E\). Suppose \(\nabla = d + A\) with respect to \(e_i\), where \(A\) is a 1-form valued matrix. Then with respect to the local frame \(e_i\), the curvature of \(\nabla\) is given by a 2-form valued matrix \(\Omega_A\), where
\[
\Omega_A = dA + [A, A].
\]
Prove that if \(e'_i\) is another local frame, \(e'_i = \sum_j e_i s_{ij}\) for some non-singular matrix \(S = (s_{ij})\), and write \(\nabla = d + A'\) with respect to \(e'_i\), then the 2-form valued matrices \(\Omega_A\) and \(\Omega_{A'}\) are related by
\[
\Omega_{A'} = S^{-1} \Omega_A S.
\]

5. Consider the Riemannian manifold \((M, g)\), where \(M = \{(x, y) \in \mathbb{R}^2 | y > 0\}\), \(g = y^{-4} (dx^2 + dy^2)\).
(1) Compute the Gaussian curvature of \((M, g)\).
(2) Verify that for any vertical line \(x = x_0\), a suitable parametrization of it satisfies the geodesic equations.