Do all 8 problems. Each problem is worth 10 points.

1. Find all complex solutions \( z \in \mathbb{C} \) of the equation: \( z^4 - z^3 + iz - i = 0 \).
HINT: One solution can be seen by inspection.

2.(a) Exhibit the Taylor series about \( z = 0 \) of the function
\[
f(z) = \frac{1 - \cos z}{z^2},
\]
and determine its radius of convergence.
(b) Using part (a), calculate
\[
\frac{d^2}{dz^2} \left. \frac{1 - \cos z}{z^2} \right|_{z=0}.
\]

3.(a) For any complex number, \( c = a + ib \), establish the identity
\[
z^c = e^{a \ln r - b \theta} \left[ \cos(b \ln r + a \theta) + i \sin(b \ln r + a \theta) \right] \quad (z = re^{i\theta}).
\]
(b) What is the principal branch of the analytic function \( w = z^c \)?
(c) Determine the image in the \( w \)-plane of the unit circle \(|z| = 1\), \( \text{Arg } z \neq \pm \pi \).

4. Consider the contour integral,
\[
I = \int_C \frac{e^{-z}}{z - \frac{a_i}{2}} \, dz,
\]
where \( C \) is the positively-oriented boundary of the square of side length 4 centered at \( z = 0 \), so that its vertices are at \( \pm 2 \pm 2i \).
(a) Express \( I \) as a sum of integrals with respect to real parameters for the four sides; do not evaluate these integrals.
(b) Evaluate \( I \) using the Cauchy Integral Formula.
5. Determine the radius of convergence of the power series

(a) \[ \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)} z^n , \]
(b) \[ \sum_{n=0}^{\infty} 2^{n/2} \cos \frac{n\pi}{2} z^n . \]

6. Find the residue at \( z = 0 \) of the functions:

(a) \( f(z) = \frac{1}{z + z^2} \), \hspace{1em} (b) \( f(z) = \frac{1}{z^2 + 5z^3} \), \hspace{1em} (c) \( f(z) = \frac{\sin z}{z^2} \).

7. Use residue theory to deduce the value of the definite integral

\[ \int_0^{2\pi} \frac{\sin^2 \theta}{a + b \cos \theta} \, d\theta = \frac{\pi}{b^2} \left( a - \sqrt{a^2 - b^2} \right) \]
for real constants with \( a > |b| > 0 \).

8. Use residue theory to calculate the improper integral

\[ \int_{-\infty}^{+\infty} \frac{\cos kx}{x^2 + 1} \, dx , \]
for any real constant \( k > 0 \).